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Article

A Formal Proof of Castigliano Theorem and a Related Generalization

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Abstract

This short communication develops a formal proof of Castigliano Theorem in a elasticity context. The results are base on standard tools of applied functional analysis and calculus of variations. It is worth mentioning such results here presented may be easily extended to a non-linear elasticity context. Finally, in the last section we present a numerical example in order to illustrate the results applicability.

Keywords: Castigliano theorem; generalized Castigliano theorem; virtual work principle; elasticity theory; numerical example

MSC: 74G05

1. Introduction

In this article version we present some corrections and improvements concerning the previous article version [1].

In the next section we present the mathematical formalism of a result in elasticity theory known as the Castigliano's Theorem.

We also present a generalization of such an theorem and its connection with the principle of virtual work in a elasticity theory context.

The results are obtained through an application of basic tools of functional analysis and calculus variations to a solid mechanics theory.

Our main reference in solid mechanics is [2]. Similar results have been presented in section 40 of preprint [3].

For basic topics on the applied functional analysis and calculus of variations, please see [4,5].

For the Sobolev spaces involved, we would cite [6].

Remark 1. *In this text we have adopted the Einstein convention of summing up repeated indices, unless otherwise indicated.*

2. A Formal Proof of Castigliano Theorem

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipischitzian) boundary denoted by $\partial\Omega$.

In a context of linear elasticity, consider the functional $J : V \rightarrow \mathbb{R}$ where

$$J(u) = E_{in} - \langle u_i, f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j) P_{ij},$$

$u = (u_1, u_2, u_3) \in W_0^{1,2}(\Omega; \mathbb{R}^3) \equiv V, f = (f_1, f_2, f_3) \in L^2(\Omega; \mathbb{R}^3), Y = Y^* = L^2(\Omega; \mathbb{R}^3)$, and

$$P_{ij} \in \mathbb{R}, \forall i \in \{1, 2, 3\}, j \in \{1, \dots, N\}$$

for some $N \in \mathbb{N}$.

Here we have denoted

$$E_{in} = \frac{1}{2} \int_{\Omega} H_{ijkl} e_{ij}(u) e_{kl}(u) dx,$$

$$e_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Moreover H_{ijkl} is a fourth order positive definite and constant tensor.

Observe that the variation of J in u_i give us the following Euler-Lagrange equation

$$-(H_{ijkl} e_{kl}(u))_{,j} - f_i - \sum_{j=1}^N P_{ij} \delta(x_j) = 0, \text{ in } \Omega. \quad (1)$$

Symbolically such a system stands for

$$\frac{\partial J(u)}{\partial u_i} = 0, \forall i \in \{1, 2, 3\},$$

so that

$$\frac{\partial(E_{in} - \langle u_i, f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j) P_{ij})}{\partial u_i} = 0, \forall i \in \{1, 2, 3\}. \quad (2)$$

We denote $u \in V$ solution of (1) by $u = u(f, P)$, so that multiplying the concerning extremal equation by u_i and integrating by parts, we get

$$\begin{aligned} H_1(u(f, P), f, P) &= 2E_{in}(u(f, P)) - \langle u_i(f, P), f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j, f, P) P_{ij} \\ &= 0, \forall f \in Y^*, P \in \mathbb{R}^{3N}. \end{aligned} \quad (3)$$

Therefore

$$\frac{d}{dP_{ls}} (H_1(u(f, P), f, P)) = 0,$$

so that

$$2 \frac{dE_{in}}{dP_{ls}} - \frac{d}{dP_{ls}} \left(\langle u_i(f, P), f_i \rangle_{L^2} + \sum_{j=1}^N u_i(x_j, f, p) P_{ij} \right) = 0, \quad (4)$$

where we recall that, from the Implicit Function Theorem, we have

$$\frac{dE_{in}}{dP_{ls}} = \frac{\partial E_{in}}{\partial u_k} \frac{\partial u_k}{\partial P_{ls}}.$$

Hence, from this and (4), we obtain

$$\frac{dE_{in}}{dP_{ls}} + \frac{d}{dP_{ls}} \left(E_{in} - \langle u_i(f, P), f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j, f, p) P_{ij} \right) = 0,$$

so that

$$\begin{aligned} &\frac{dE_{in}}{dP_{ls}} + \left\langle \frac{\partial(E_{in} - \langle u_i, f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j) P_{ij})}{\partial u_k}, \frac{\partial u_k}{\partial P_{ls}} \right\rangle_{L^2} \\ &\quad - \frac{\partial}{\partial P_{ls}} \left(\langle u_i, f_i \rangle_{L^2} + \sum_{j=1}^N u_i(x_j) P_{ij} \right) \\ &= 0. \end{aligned} \quad (5)$$

From this, recalling that

$$\frac{\partial(E_{in} - \langle u_i, f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j)P_{ij})}{\partial u_k} = 0, \quad \forall k \in \{1, 2, 3\}$$

we obtain

$$\frac{dE_{in}}{dP_{ls}} - u_l(x_s) = 0,$$

so that

$$u_l(x_s) = \frac{dE_{in}}{dP_{ls}} = \frac{d}{dP_{ls}} \left(\frac{1}{2} \int_{\Omega} H_{ijkl} e_{ij}(u(f, P)) e_{kl}(u(f, P)) dx \right),$$

$$\forall l \in \{1, 2, 3\}, \forall s \in \{1, \dots, N\}.$$

With such results in mind, we have proven the following theorem.

Theorem 1 (Castigliano). *Considering the notations and definitions in this section, we have*

$$u_l(x_s) = \frac{dE_{in}}{dP_{ls}} = \frac{d}{dP_{ls}} \left(\frac{1}{2} \int_{\Omega} H_{ijkl} e_{ij}(u(f, P)) e_{kl}(u(f, P)) dx \right),$$

$$\forall l \in \{1, 2, 3\}, \forall s \in \{1, \dots, N\}.$$

2.1. A Generalization of Castigliano Theorem

In this subsection, denoting by $\delta(x_k)$ a standard Dirac delta function in a distributional sense, we present a more general version of the Castigliano theorem.

Considering the context of last section, we recall that

$$\begin{aligned} H_1(u(f, P), f, P) &= 2E_{in}(u(f, P)) - \langle u_i(f, P), f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j, f, P) P_{ij} \\ &= 0, \quad \forall f \in Y^*, P \in \mathbb{R}^{3N}. \end{aligned} \quad (6)$$

Therefore, here denoting the Gâteaux derivative of H_1 related to f_l by

$$\frac{d}{df_l}(H_1(u(f, P), f, P)),$$

from the extremal equation

$$\frac{d}{df_l}(H_1(u(f, P), f, P)) = 0,$$

for $x_k \in \Omega$ such that

$$x_k \neq x_j, \quad \forall j \in \{1, \dots, N\},$$

we have, in an appropriate distributional sense,

$$\left\langle \frac{d}{df_l}(H_1(u(f, P), f, P)), \delta(x_k) \right\rangle_{L^2} = 0,$$

so that

$$\begin{aligned} & 2 \left\langle \frac{d}{df_l} (E_{in}(u(f, P))), \delta(x_k) \right\rangle_{L^2} \\ & - \left\langle \frac{d}{df_l} \left(\langle u_i(f, P), f_i \rangle_{L^2} + \sum_{j=1}^N u_i(x_j, f, p) P_{ij} \right), \delta(x_k) \right\rangle_{L^2} \\ & = 0, \end{aligned} \quad (7)$$

that is

$$\begin{aligned} & \left\langle \frac{d}{df_l} (E_{in}(u(f, P))), \delta(x_k) \right\rangle_{L^2} \\ & + \left\langle \frac{\partial}{\partial u_s} \left(E_{in}(u(f, P)) - \langle u_i(f, P), f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j, f, p) P_{ij} \right) \frac{\partial u_s}{\partial f_l}, \delta(x_k) \right\rangle_{L^2} \\ & - \left\langle \frac{\partial}{\partial f_l} \left(\langle u_i(f, P), f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j, f, p) P_{ij} \right), \delta(x_k) \right\rangle_{L^2} \\ & = 0. \end{aligned} \quad (8)$$

From such results, we may obtain

$$\left\langle \frac{d}{df_l} (E_{in}(u(f, P))), \delta(x_k) \right\rangle_{L^2} - \langle u_l(x), \delta(x_k) \rangle_{L^2} = 0,$$

so that

$$\left\langle \frac{d}{df_l} (E_{in}(u(f, P))), \delta(x_k) \right\rangle_{L^2} - u_l(x_k) = 0,$$

that is

$$u_l(x_k) = \left\langle \frac{d}{df_l} (E_{in}(u(f, P))), \delta(x_k) \right\rangle_{L^2},$$

$\forall l \in \{1, 2, 3\}, \forall x_k \in \Omega$ such that $x_k \neq x_j, \forall j \in \{1, \dots, N\}$.

With such results in mind, we have proven the following theorem.

Theorem 2 (The Generalized Castigliano Theorem). *Considering the notations and definitions in this section, here again denoting the Gâteaux derivative of H_1 related to f_i by*

$$\frac{d}{df_i} (H_1(u(f, P), f, P)),$$

we have

$$u_i(x_k) = \left\langle \frac{d}{df_i} (E_{in}(u(f, P))), \delta(x_k) \right\rangle_{L^2},$$

$\forall i \in \{1, 2, 3\}, \forall x_k \in \Omega$ such that $x_k \neq x_j, \forall j \in \{1, \dots, N\}$.

2.2. The Virtual Work Principle

Considering the definitions, results and statements of the previous section and subsection, we may easily prove the following theorem.

Theorem 3 (The virtual work principle). *Let $x_l \in \Omega$ such that $x_l \neq x_j, \forall j \in \{1, \dots, N\}$.*

For a virtual constant load $\hat{P} \in \mathbb{R}$ on x_l at the direction of $u_k(x_l)$, define now $J : V \rightarrow \mathbb{R}$ where

$$J(u) = E_{in} - \langle u_i, f_i \rangle_{L^2} - \sum_{j=1}^N u_i(x_j) P_{ij} - \hat{P} u_k(x_l).$$

Under such hypotheses,

$$u_k(x_l) = \left(\frac{d E_{in}(u(f, P, \hat{P}))}{d \hat{P}} \right)_{\hat{P}=0},$$

$\forall k \in \{1, 2, 3\}$, $\forall x_l \in \Omega$ such that $x_l \neq x_j$.

Proof. The proof is exactly the same as in the Castigliano Theorem in the previous section except by setting the virtual load $\hat{P} = 0$ in the end of this calculation and will not be repeated. $\square$

2.3. A Numerical Example Related to the Castigliano Theorem

Let $\Omega = [0, 1] \subset \mathbb{R}$ be the axis of a straight beam with a rectangular cross section of dimensions $b \times h$, where units in this subsections refer to the international system.

Let

$$I = \frac{bh^3}{12}$$

and denote by $E > 0$ the Young modulus for a steel beam.

Assume such a beam is subject to a vertical load $P > 0$ uniformly distributed on Ω .

Assume also the beam is clamped at $x = 0$ and simply supported at $x = 1$.

Denoting by $w \in V = W^{2,2}(\Omega)$ the vertical field of displacements results from the action of P , the related boundary conditions are given by

$$w(0) = w(1) = 0, \quad w_{,x}(0) = 0, \quad w_{,xx}(1) = 0.$$

The total beam energy is defined by $J : V \rightarrow \mathbb{R}$, where

$$J(w) = \frac{EI}{2} \int_{\Omega} w_{,xx}^2 dx - \int_{\Omega} Pw dx.$$

In order to apply the results of the previous section, we free the rotations of the beam at $x = 0$, considering a moment load M on $x = 0$ with general work

$$Mw_{,x}(0).$$

Hence, we define

$$V_1 = \{w \in W^{2,2}(\Omega) : w(0) = w(1) = w_{,xx}(1) = 0\},$$

and define $J_1 : V_1 \rightarrow \mathbb{R}$, by

$$J_1(w) = \frac{EI}{2} \int_{\Omega} w_{,xx}^2 dx - \int_{\Omega} Pw dx - M w_{,x}(0).$$

We emphasize again, through the methods of the previous section, we intend to obtain the value of M which corresponds to

$$w_{,x}(0) = 0.$$

Let $\varphi \in W^{2,2}$. The variation

$$\delta J_1(w; \varphi),$$

stands for

$$\delta J_1(w; \varphi) = EI \int_{\Omega} w_{,xx} \varphi_{,xx} \, dx - \int_{\Omega} P \varphi \, dx - M \varphi_{,x}(0).$$

Here assuming, w is smooth enough, integrating by parts and recalling that $\varphi \in V_1$, we obtain

$$\delta J_1(w; \varphi) = EI \int_{\Omega} w_{,xxxx} \varphi \, dx - EI w_{,xx}(0) \varphi_{,x}(0) - \int_{\Omega} P \varphi \, dx - M \varphi_{,x}(0),$$

so that the extremal condition

$$\delta J_1(w; \varphi) = 0, \, \forall \varphi \in V_1,$$

provide us the following natural boundary condition

$$M = -EI w_{,xx}(0)$$

and the equation

$$EI w_{,xxxx} - P = 0, \text{ in } \Omega.$$

Here we recall the remaining essential boundary conditions,

$$w(0) = w(1) = w_{,xx}(1) = 0.$$

A particular solution of such an equation

$$EI w_{,xxxx} - P = 0, \text{ in } \Omega$$

stands for

$$w_p(x) = ax^4,$$

where

$$a = \frac{P}{4! \, EI}.$$

The concerning general solution stands for

$$w(x) = w_p(x) + bx^3 + cx^2 + dx + e.$$

The boundary condition $w(0) = 0$ implies that $e = 0$.

The boundary condition

$$M = -EI w_{,xx}(0),$$

stands for

$$M = -EI(2c),$$

so that

$$c = -\frac{M}{2EI}.$$

Moreover, from

$$w_{,xx}(1) = 0,$$

we obtain

$$12a + 6b + 2c = 0.$$

From

$$w(1) = 0$$

we have

$$a + b + c + d = 0.$$

From these last two equations, we obtain

$$b = -\frac{c}{3} - 2a \equiv b(c),$$

and

$$d = a - \frac{2}{3}c.$$

From the Castigliano Theorem, $M \in \mathbb{R}$ must be such that

$$w_x(0) = \frac{dE_{in}}{dM} = 0,$$

where

$$E_{in} = \frac{EI}{2} \int_{\Omega} w_{,xx}^2 \, dx.$$

Observe that

$$\frac{dE_{in}}{dM} = \frac{dE_{in}}{dc} \frac{dc}{dM} = 0,$$

so that in fact, it suffices to obtain,

$$\frac{dE_{in}}{dc} = 0.$$

Moreover, observe that

$$\begin{aligned} E_{in} &= \frac{EI}{2} \int_{\Omega} w_{,xx}^2 \, dx \\ &= \frac{EI}{2} \int_0^1 (12ax^2 + 6b(c)x + 2c)^2 \, dx \\ &= \frac{EI}{2} \int_0^1 (144a^2x^4 + 36b(c)^2x^2 + 4c^2 \\ &\quad + 144ab(c)x^3 + 48acx^2 + 24xb(c)c) \, dx \\ &= \frac{EI}{2} \left[\frac{144a^2x^5}{5} + 36\frac{b(c)^2x^3}{3} + 4c^2x \right. \\ &\quad \left. + \frac{144ab(c)x^4}{4} + \frac{48acx^3}{3} + \frac{24b(c)cx^2}{2} \right]_0^1 \\ &= \frac{EI}{2} \left(\frac{144a^2}{5} + \frac{36b(c)^2}{3} \right. \\ &\quad \left. + 4c^2 + \frac{144ab(c)}{4} + \frac{48ac}{3} + \frac{24cb(c)}{2} \right). \end{aligned} \tag{9}$$

Hence, we must have

$$\begin{aligned} \frac{dE_{in}}{dc} &= \frac{\partial E_{in}}{\partial b} \frac{db}{dc} + \frac{\partial E_{in}}{\partial c} \\ &= 0 \end{aligned} \tag{10}$$

so that,

$$12(2)b(c) \frac{db(c)}{dc} + 8c + 36a \frac{db(c)}{dc} + 16a + 12b(c) + 12c \frac{db(c)}{dc} = 0. \tag{11}$$

Recalling that

$$\frac{db(c)}{dc} = -1/3,$$

we have got

$$-8b(c) + 8c - 12a + 16a + 12b(c) - 4c = 0.$$

From such a result and recalling that

$$b(c) = -\frac{c}{3} - 2a,$$

we have got

$$\frac{8c}{3} + 16a - 12a + 16a - 4c - 24a - 4c = 0,$$

so that

$$\frac{8c}{3} - 4a = 0,$$

which has a solution

$$c = \frac{3a}{2}.$$

Thus,

$$c = \frac{3 P}{(2) 4! EI},$$

so that

$$M = -2cEI = -\frac{3 P}{4!}.$$

2.4. *Checking this Last Result for M by Solving the Concerning Ordinary Differential Equation*

In this section we check the result obtained for M by solving the following ODE,

$$EIw_{xxxx} - P = 0, \text{ in } \Omega = [0, 1],$$

with the boundary conditions,

$$w(0) = w(1) = w_{,x}(0) = w_{,xx}(1) = 0.$$

We recall the general solution stands for

$$w(x) = ax^4 + bx^3 + cx^2 + dx + e,$$

where

$$a = \frac{P}{4!EI}.$$

From $w(0) = 0$, we obtain $e = 0$.

From $w_{,x}(0) = 0$, we obtain $d = 0$.

From $w(1) = 0$, we have

$$a + b + c = 0.$$

From $w_{,xx}(1) = 0$, we have

$$12a + 6b + 2c = 0.$$

From such results, we obtain

$$b = -\frac{5a}{2},$$

and

$$c = \frac{3a}{2}$$

Observe that

$$M(x) = -EIw_{,xx},$$

so that in particular

$$M = M(0) = -EIw_{,xx}(0) = -EI(2c) = -\frac{3 P}{4!}.$$

This value for M here obtained coincide with the one obtained in the previous subsection, as expected.

The objective of this section is complete.

3. Conclusion

In this article, we have presented a formal proof Castigliano Theorem in a elasticity theory context.

We have also presented a generalization of such a result and a numerical example to exemplify its applicability.

Conflicts of Interest: The author declares no conflict of interest concerning this article.

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