Review

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Developments in Modular Space Fixed Point Theory

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Developments in Modular Space Fixed Point Theory

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Abstract

This survey article offers a snapshot view of the present state of Fixed Point Theory within modular spaces, highlighting fundamental principles and their applications. The discussion primarily revolves around operators and their semigroups that adhere to pointwise asymptotic nonexpansive and contractive conditions in the modular sense in a way that they may be directly applied also to Banach spaces. Utilizing the framework of regular and super-regular modular spaces, our research generalizes several established results concerning fixed points of nonlinear operators, applicable to both Banach spaces and modular function spaces. The study seeks to identify and discuss current challenges, knowledge gaps, and unresolved questions, providing insights into potential of future research opportunities.

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MSC: 47H09; 47H10; 47H20; 46A80

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1. Introduction

This paper will consistently refer to X as a real vector space. We will begin by recalling the definitions of convex modulars and modular spaces.

Definition 1. *[6] A functional $\rho : X \to [0, \infty]$ is called a convex modular if*

1. $\rho(x) = 0$ *if and only if* $x = 0$
2. $\rho(-x) = \rho(x)$
3. $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ *for any* $x, y \in X$*, and* $\alpha, \beta \ge 0$ *with* $\alpha + \beta = 1$

The vector space $X_\rho = \{x \in X : \rho(\lambda x) \to 0,\ as\ \lambda \to 0\}$ *is called a modular space.*

Since the seminal works [1,2], it has been widely recognized that modular equivalents of norm concepts frequently appear in fixed point and approximation theories due to their practical applications. Moreover, modular techniques often yield results unattainable within normed spaces (refer to [2] and to Remark 1 for further commentary). This has led to the development of fixed point theory in modular function spaces as a dynamic research field, as illustrated in [3] and subsequent studies. Interestingly, fixed point results in modular function spaces share similarities with those derived for operators acting in Banach spaces. However, since not all Banach spaces qualify as modular function spaces, there is a need for a framework encompassing both normed and modular function spaces. To address this, we utilize the framework of regular modular spaces introduced in [4], with essential assumptions such as the convexity of the modular, the closedness of modular balls, and the modular completeness. While many results in Banach spaces rely on weak topology, which parallels the convergence ρ-a.e. in modular function spaces, we also employ the framework of super-regular spaces [5], introducing L-convergence aligned with the modular structure. These conditions are satisfied in Banach spaces, where norms are examples of super-regular modulars, and in modular function spaces with the Fatou property. It is important to recognize that super-regular modular spaces encompass spaces beyond

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Banach or modular function spaces, as illustrated by modular spaces defined by ϕ-variations. For an example, see Example 4.

This survey article is is organized as follows: Section 2 'Modular Spaces' offers a summary of key foundational facts of the theory of modular spaces in the Subsection 2.1, followed by necessary background information about regular modular spaces in the Subsection 2.2 and super-regular modular spaces in 2.3. Section 3 deals with the fixed point theorems for nonlinear operators acting in such spaces, with Subsection 3.1 devoted to contractive and nonexpansive mappings and the subsequent Subsection 3.2 developing the foundations of the theory for asymptotic and pointwise asymptotic mappings. Section 4 provides reference to the key results related to existence of common fixed points for semigroups of nonlinear mappings operating in regular and super-regular modular spaces. This section is followed by Section 5, where we discuss the fixed point construction processes in the context of super-regular modular spaces. The final Section 6 'Discussion' summarizes the contents of this survey paper, pointing to the main notions and results discussed in the article. It also discusses gaps, open questions and identifies future research opportunities.

2. Modular Spaces

2.1. Preliminaries

It is important to highlight that, despite certain superficial similarities noted in Definition 1, modulars are fundamentally different from norms and metrics. Firstly, $\rho(x)$ can take on an infinite value for elements $x \in X_\rho$. Furthermore, modulars are not obliged to adhere to the triangle inequality. Such characteristics often arise in various important contexts, particularly within the theory of Orlicz spaces and their numerous generalizations. The unique attributes of modular spaces, coupled with their significant applications, have prompted the development of specialized methods and tools in modular-based fixed point theory over the past 35 years. Therefore, the findings of fixed point theory in modular spaces cannot be simply replaced with those from general metric space theory (see, for instance, [7,8]) or from generalized metric space theories, such as the theory of (E)-metric spaces (see, for example, [9,10]).

The concepts outlined in Definitions 2, 3, and 4 below are well-established in the theory of modular spaces and their applications (see, for example, [3,4]).

Definition 2. *Let ρ be a modular defined on a vector space X.*

1. *We say that $\{x_n\}$, a sequence of elements of X_ρ, is ρ-convergent to x, and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$.*
2. *A sequence $\{x_n\}$ where $x_n \in X_\rho$ is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.*
3. *X_ρ is called ρ-complete if every ρ-Cauchy is ρ-convergent to an $x \in X_\rho$.*
4. *A set $B \subset X_\rho$ is called ρ-closed (or simply closed) if for any sequence of $x_n \in B$, the convergence $x_n \xrightarrow{\rho} x$ implies that x belongs to B.*
5. *A set $B \subset X_\rho$ is called ρ-bounded (or simply bounded) if its ρ-diameter defined as*

$$\mathrm{diam}_\rho(B) = \sup\{\rho(x - y) : x \in B, y \in B\}$$

   is finite.
6. *A set $B \subset X_\rho$ is called strongly ρ-bounded (or simply, strongly bounded) if there exists a $\beta > 1$ such that βA is bounded.*
7. *A set $K \subset X_\rho$ is called ρ-compact if for any sequence $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ and an $x \in K$ such that $\rho(x_{n_k} - x) \to 0$.*
8. *A ρ-ball $B_\rho(x, r)$ is defined by $B_\rho(x, r) = \{y \in X_\rho : \rho(x - y) \le r\}$.*

The concept of ρ-convergence plays a significant role in fixed point theory and approximation theory within modular spaces. It is clear that if the ρ-limit of a sequence in a modular space exists, it is uniquely defined. Moreover, every subsequence of a ρ-convergent sequence also converges to the same limit. In addition, if $x_n \xrightarrow{\rho} x$, it follows that $x_n - y \xrightarrow{\rho} x - y$. Furthermore, if $x_n \xrightarrow{\rho} x$, $y_n \xrightarrow{\rho} y$, and $x_n - y_n \xrightarrow{\rho} 0$, then it can be concluded that $x = y$ (see [[11, Prop. 2.1]]). Similar to metric spaces, the ρ-compactness of a set $C \subset X$ ensures that C is ρ-closed.

However, it is essential to proceed with caution, as some standard properties of convergence in topological vector spaces do not directly transfer to ρ-convergence. For instance, $x_n \xrightarrow{\rho} x$ does not generally imply that $\lambda x_n \xrightarrow{\rho} \lambda x$ for $\lambda > 1$. Additionally, the ρ-compactness of a set does not necessarily guarantee its ρ-boundedness. Moreover, ρ-balls are not always ρ-closed, as illustrated in the following example.

Example 1. *Let $X = \mathbb{R}$ and define $\rho(x) = |x|$ when $|x| < 1$; otherwise, let $\rho(x) = +\infty$. Observe that $x_n = 1 - \dfrac{1}{2n} \in B_\rho(0,1)$, and that $x_n \xrightarrow{\rho} 1$. However, $1 \notin B_\rho(0,1)$ because $\rho(1) = +\infty$. Therefore, the ρ-ball $B_\rho(0,1)$ is not ρ-closed.*

Nevertheless, in many noteworthy cases, ρ-balls are indeed ρ-closed. This underscores the importance of regular modular spaces, which guarantee that ρ-balls are always ρ-closed.

2.2. Regular Modular Spaces

Definition 3. *[4, Def.2.3] A convex modular is called BC-regular if every ρ-ball $B_\rho(x,r)$, where $x \in X_\rho$, and $r > 0$, is ρ-closed. In this context, we also refer to the modular space X_ρ as possessing property (BC).*

Definition 4. *[4, Def.2.4] A modular space X_ρ is called a regular modular space if ρ is a convex, (BC)-regular modular, and X_ρ is ρ-complete.*

The following result offers an important characterization of BC-regularity.

Proposition 1. *[11, Prop. 2.2] The following two conditions are equivalent*

1. *ρ is BC-regular;*
2. *$\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ provided $x_n \xrightarrow{\rho} x$.*

Modulars that satisfy condition (2) of Proposition 1 are frequenty referred to as ρ-lower semicontinuous, or equivalently as possessing the Fatou property.

The class of regular modular spaces includes all real Banach spaces (where ρ denotes a norm) and all modular function spaces that possess the Fatou property, such as Lebesgue spaces, Orlicz spaces, Musielak-Orlicz spaces, and variable exponent Lebesgue spaces.

It is well known that every convex modular space can be equipped with a norm, referred to as the Luxemburg norm, which is defined as follows,

$$\|x\|_\rho = \inf\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \leq 1\}.$$

Remark 1. *It is clear that the convergence of a sequence in this norm implies its ρ-convergence; however, the reverse is generally not true. This norm is not directly defined and can often be challenging to compute, whereas modulars are usually represented by explicit formulas, making calculations easier. Additionally, as established in [2], while nonexpansiveness concerning this norm ensures nonexpansiveness in the modular sense, there are mappings that are nonexpansive in the modular sense but not with respect to the corresponding Luxemburg norm. Moreover, in numerous instances, an operator or a collection of operators can give rise to a modular space where these operators exhibit essential characteristics such as continuity, boundedness, or nonexpansiveness. For further details, see [3], as well as Example 5 in Section 5 of the current article. Consequently, due to these and similar considerations, standard practice in fixed point theory within modular spaces is to express all conditions placed on operators—such as various forms of nonexpansiveness or uniform convexity—exclusively in modular*

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terms. It is noteworthy that some regular modular spaces are neither normed spaces nor modular function spaces, see Example 4.

The literature presents multiple not always equivalent definitions of modular uniform convexity (see, for example, [3,4,12]). In the context of regular modular spaces, the strongest version - often referred to as the $(UUC1)$ property - is commonly employed because, in case when ρ is a norm, it is equivalent to the standard definition of uniform convexity in normed spaces. In this paper, we adopt this perspective, as outlined in the following definition.

Definition 5. *Let X_ρ be a regular modular space. Let $r > 0, \varepsilon > 0$. Set*

$$D_1(r,\varepsilon) = \{(x,y) : x, y \in X_\rho, \rho(x) \le r, \rho(y) \le r, \rho(x-y) \ge \varepsilon r\}.$$

Let

$$\delta_1(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\rho\left(\frac{x+y}{2}\right) : (x,y) \in D_1(r,\varepsilon)\right\}, if D_1(r,\varepsilon) \neq \emptyset,$$

$\delta_1(r,\varepsilon) = 1$ if $D_1(r,\varepsilon) = \emptyset$. We say that X_ρ is uniformly convex if, for every $s \ge 0, \varepsilon > 0$, there exists

$$\eta_1(s,\varepsilon) > 0,$$

depending only on s and ε such that

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0 \; for \; r > s.$$

The significance of the above definition of modular uniform convexity is exemplified by the following dscussion drawn from the theory of Orlicz spaces.

Example 2. *[13, Example 3] It is known that in Orlicz spaces, the Luxemburg norm is uniformly convex if and only φ is uniformly convex and Δ_2 property holds. Furthermore, it is recognized that, under appropriate conditions, modular uniform convexity in Orlicz spaces is equivalent to the very convexity of the Orlicz function [14,15]. Note that a function φ is termed very convex if, for every $\varepsilon > 0$ and any $x_0 > 0$, there exists a $\delta > 0$ such that*

$$\varphi\left(\frac{1}{2}(x-y)\right) \ge \frac{\varepsilon}{2}(\varphi(x) + \varphi(y)) \ge \varepsilon\varphi(x_0),$$

implies

$$\varphi\left(\frac{1}{2}(x+y)\right) \le \frac{1}{2}(1-\delta)(\varphi(x) + \varphi(y)).$$

Typical examples of Orlicz functions that do not satisfy the Δ_2 condition but are very convex are: $\varphi_1(t) = e^{|t|} - |t| - 1$ and $\varphi_2(t) = e^{t^2} - 1$, [16,17]. Therefore, these are the examples of regular modular spaces that are not uniformly convex in the Luxemburg norm sense and hence the classical Banach space fixed point theorems cannot be easily applied. However, these spaces are uniformly convex in the modular sense, and respective modular fixed point results can be applied.

We will also employ a weaker variant of uniform convexity, referred to as uniformly convexity in every direction (UCED), which serves as a generalization of this concept within Banach spaces.

Definition 6. *[2, Def. 3.9] [18, Def. 3.6] For any nonzero $u \in X_\rho$ and $r > 0$, we define the r-modulus of uniform convexity of ρ in the direction of u as*

$$\delta(r,u) = \inf\left\{1 - \frac{1}{r}\rho\left(y + \frac{1}{2}u\right)\right\},$$

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where the infimum is taken over all $y \in X_\rho$ such that $\rho(y) \leq r$ and $\rho(y+u) \leq r$.
We say that X_ρ is uniformly convex in every direction $(UCED)$ if $\delta(r,u) > 0$ for every nonzero $u \in X_\rho$ and all $r > 0$.

By a straightforward calculation we find that $\delta(r,u) \geq \delta_1(r,\varepsilon)$ for any $\varepsilon > 0$ such that $\rho(u) \geq r\varepsilon$. Therefore, uniform convexity of a regular modular spaces implies that it is uniformly convex in every direction. As known, even in the Banach space case, the converse is not true.

2.3. Super-regular Modular Spaces

In regular modular spaces, there is no equivalent to the weak topology found in Banach spaces, nor is there a counterpart to ρ-almost everywhere convergence that plays a similar role in modular function spaces. Both concepts are fundamental to fixed point theory in their respective contexts, necessitating the development of an analogous framework for studying the existence of fixed points and the convergence of fixed point approximation processes in regular modular spaces. This brings us to the concept of super-regular modular spaces, introduced in [5]. These spaces are defined as modular spaces equipped with a sequential convergence structure, as proposed by the author in [19,20], and based on the framework of L-spaces developed by Kisyński in [21] (see also [22]), which builds upon earlier work by Fréchet [23] and Urysohn [24]. This setting allows for the utilization of convergence types that are not inherently tied to a topology, with convergence almost everywhere serving as a notable example.

Definition 7. *[19, Def. 2.4] Let X be any nonempty set. A relation ζ between sequences $\{x_n\}_{n=1}^{\infty}$ of elements of X and elements x of X, denoted by $x_n \xrightarrow{\zeta} x$, is called a sequential convergence on X if*

1. *if $x_n = x$ for all $n \in \mathbb{N}$ then $x_n \xrightarrow{\zeta} x$,*
2. *if $x_n \xrightarrow{\zeta} x$ and $\{x_{n_k}\}$ is a proper subsequence of $\{x_n\}$, then $x_{n_k} \xrightarrow{\zeta} x$.*

The pair (X,ζ) (or shortly X) is called a convergence space.

Given a sequential convergence ζ on X we can introduce notions of closed and sequentially compact sets.

Definition 8. *[19, Def. 2.5] Let (X,ζ) be a convergence space. A set $K \subset X$ is called closed if whenever $x_n \in K$ all $n \in \mathbb{N}$ and $x_n \xrightarrow{\zeta} x$, then $x \in K$. Similarly, K is called sequentially compact if from every sequence $\{x_n\}$ of elements of K we can choose a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{\zeta} x$ for an $x \in K$.*

Definition 9. *[19, Def. 2.6] A sequential convergence ζ is called an L-convergence on X if*

1. *if $x_n \xrightarrow{\zeta} x$ and $x_n \xrightarrow{\zeta} y$, then $x = y$.*

The pair (X,ζ) (or shortly X) is called an L-space.

Let us define LTI-convergence, LTI-spaces and modulated LTI-spaces.

Definition 10. *[19, Def. 2.8] Let X be a real vector space and let ζ be an L-convergence on X. We say that ζ is an LTI-convergence (translation invariant convergence) if $x_n \xrightarrow{\zeta} x$ implies that $x_n - y \xrightarrow{\zeta} x - y$ for any $y \in X$. In this case, the pair (X,ζ) is called an LTI-space.*

Definition 11. *[19, Def. 2.9] Let ρ be a modular defined on X and let ζ be an L-convergence on X_ρ. The triplet (X_ρ,ρ,ζ) is called a modulated LTI-space if (X_ρ,ζ) is an LTI-space and the following two conditions are satisfied*

1. $x_n \xrightarrow{\zeta} x \Rightarrow \rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$,

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2. *if* $x_n \xrightarrow{\rho} x$ *then there exists a sub-sequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $x_{n_k} \xrightarrow{\zeta} x$*, where* $x,\ x_n \in X$*.*

The statements in the following Proposition are straightforward consequences of the relevant definitions.

Proposition 2. *[5, Proposition 1] Let* (X_ρ, ρ, ζ) *be a modulated* LTI*-space. Then the following assertions are true.*

1. *Every* ζ*-closed set is also* ρ*-closed.*
2. *Every* ρ*-compact set is also sequentially* ζ*-compact.*
3. *Every* ρ*-ball* $B_\rho(x, r)$ *is* ζ*-closed (and hence also* ρ*-closed).*
4. *Every sequentially* ζ*-compact set is* ζ*-closed.*
5. *Every* ζ*-closed subset of a sequentially* ζ*-compact set is sequentially* ζ*-compact.*

In view of item 3 of Proposition 2, every ρ-complete modulated LTI-space is regular. For the sake of simplicity let us introduce the following definition.

Definition 12. *[5, Def.10] By a super-regular modular space* (X_ρ, ζ) *we will understand a* ρ*-complete modulated* LTI*-space* (X_ρ, ρ, ζ)*.*

Example 3. *Typical examples of super-regular modular spaces include:*

(a) *Banach spaces where* ρ *is a norm and* ζ *represents convergence in the weak topology;*
(b) *Modular function spaces with the Fatou property and* ζ *being* ρ*-almost everywhere convergence;*
(c) *Lebesgue spaces, variable exponent Lebesgue spaces, Orlicz spaces, Musielak-Orlicz spaces with* ζ *corresponding to almost everywhere convergence with respect to a measure;*
(d) *Orlicz-Sobolev spaces with* ζ *corresponding to almost everywhere convergence with respect to a measure of all involved generalized derivatives.*

The next example merits special attention as it demonstrates that the class of super-regular modular spaces encompasses certain classical modular spaces that are neither normed nor categorized as modular function spaces. It revisits the notion of φ-variation, which was introduced by Musielak and Orlicz in [25] as a generalization of the classical quadratic variation defined by Wiener over a century ago [26]. The ρ-convergence illustrated here, referred to in the literature as convergence in φ-variation, has been utilized in numerous applications.

Example 4. *[13, Example 2] [6] [25] Let* $\varphi : [0, \infty) \to [0, \infty)$ *be a convex function such that* $\varphi(t) = 0$ *if and only if* $t = 0$*. Let* X *be a space of all real-valued functions defined in the interval* $[a, b]$ *and vanishing at* $t = a$*. Musielak and Orlicz defined in [25] a* φ*-variation of a function* $x \in X$ *as follows,*

$$\rho(x) = \sup_{\Pi} \sum_{i=1}^{\infty} \varphi(|x(t_i) - x(t_{t-1})|),$$

where the supremum is taken over all partitions $\Pi : a = t_0 < t_1 < \ldots < t_m = b$ *of the interval* $[a, b]$*. It is easy to see that* ρ *is a convex modular on* X *and that the value of* $\rho(x)$ *may be infinite. Using results of [25] and [6, 10.7], it is straightforward to demonstrate that* (X_ρ, ζ) *is a super-regular modular space, where* ζ *is the pointwise convergence over the interval* $[a, b]$*. The space* X_ρ *is not a modular function space because* φ*-variation is not monotone.*

3. Fixed Points of Nonlinear Operators in Modular Spaces

In this section, we present a comprehensive overview of the significant existing fixed point results for nonlinear operators operating within regular modular spaces.

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3.1. Contractions and Nonexpansaive Mappings in Regular Modular Spaces

First, let us revisit the definitions of Lipschitzian mappings, contractions, and nonexpansive mappings within the context of regular modular spaces. It is important to note that these definitions are direct generalizations of the corresponding concepts from Banach spaces.

Definition 13. *[4, Def. 2.7] Let X_ρ be a regular modular space and let $C \subset X_\rho$ be convex, nonempty, ρ-closed and ρ-bounded. A mapping $T : C \to C$ is called*

(i) Lipschitzian if there exists $\alpha > 0$ such that

$$\rho(T(x) - T(y)) \leq \alpha\rho(x - y) \text{ for any } x, y \in C.$$

(ii) a contraction if it is Lipschitzian with $alpha < 1$.
(iii) nonexpansive if it is Lipschitzian with $\alpha = 1$.

An element $x \in C$ is called a fixed point of T whenever $T(x) = x$. The set of fixed points of T will be denoted by $F(T)$.

As a straightforward corollary to Definition 13, the following characterization of the sets of fixed points for Lipschitzian mappings in regular modular spaces is obtained.

Proposition 3. *[4, Prop. 2.8] $F(T)$ is ρ-closed for every Lipschitzian mapping T.*

Variants of the renowned Banach Contraction Principle are regarded as vital tools for demonstrating the existence and uniqueness of solutions characterized as fixed points of self-mappings defined on metric, normed, and modular function spaces. Additionally, these theorems provide constructive approaches for locating such fixed points. Our next result provides a version of this esteemed theorem tailored to the context of regular modular spaces.

Theorem 1. *[13, Theorem 1] Let X_ρ be a regular modular space, and let $C \subset X_\rho$ be nonempty, ρ-closed and ρ-bounded. Let $T : C \to C$ be a contraction. Then, T has a unique fixed point $\bar{x} \in C$. Moreover, for any $x \in C$, $\rho(T^n(x) - \bar{x}) \to 0$ as $n \to \infty$, where T^n is the n-th iterate of T.*

Since Kirk's seminal 1965 paper [32], the concept of normal structure has been crucial in the study of fixed point theory for nonexpansive operators in Banach spaces. Consequently, it is logical to adapt this technique for proving the existence of fixed points in super-regular modular spaces.

Definition 14. *[18, Def. 3.1]*
Let C be a ρ-bounded subset of X_ρ.

(1) *The quantity $r_\rho(x, C) = \sup\{\rho(x - y) : y \in C\}$ will be called the ρ-*Chebyshev radius of C with respect to x.
(2) *The* ρ-Chebyshev radius of C *is defined by* $R_\rho(C) = \inf\{r_\rho(x, C) : x \in C\}$.

Note that $R_\rho(C) \leq r_\rho(x, C) \leq \mathrm{diam}_\rho(C)$, for any $x \in C$ and any ρ-bounded nonempty subset C of X_ρ.

Let C be a ρ-bounded subset of X_ρ such that $\mathrm{diam}_\rho(C) > 0$. and let $\mathcal{A}$ be a class of subsets of C. With this in mind, let us introduce the following two definitions adapted from [18, Def. 4.2].

Definition 15. *A class $\mathcal{A}$ is said to be ρ-normal if for each $A \in \mathcal{A}$, not reduced to a single point, we have $R_\rho(A) < \mathrm{diam}_\rho(A)$.*

Definition 16. *We will say that $\mathcal{A}$ is countably compact if any decreasing sequence $\{A_n\}_{n\geq 1}$ of nonempty elements of $\mathcal{A}$, has a nonempty intersection.*

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Our next definition extends the standard definition of normal structure to the class of super-regular modular spaces.

Definition 17. *[5, Def. 13] We say that (X_ρ, ζ) possesses the ζ-normal structure if every nonempty, convex, ρ-bounded, ζ-sequentially compact set $C \subset X_\rho$, $\mathrm{diam}_\rho(C) > 0$, has a ρ-nondiametral point $x_0 \in C$, that is,*

$$r_\rho(x_0, C) := \sup\{\rho(x_0 - y) : y \in C\} < \mathrm{diam}_\rho(C).$$

It is well known that weak normal structure—characterized by ρ as a norm in a Banach space and ζ representing convergence in the weak topology—implies the weak fixed point property; however, the converse does not hold true. We will demonstrate that this situation extends to the broader context of super-regular modular spaces.

The proof of the generic existence theorem relies on a technical result that serves as a modular version of Kirk's lemma [28, Lemma 3]. This lemma, in turn, is an abstraction of a result originally established by Gillespie and Williams [29].

Lemma 1. *[27, Lemma 4.3] Let C be a ρ-bounded subset of X_ρ. Let $\mathcal{A}$ be a class of subsets of C which is stable under arbitrary intersections and contains all sets of the form $C \cap B_\rho(x, p)$, where $x \in C$ and $p > 0$. Suppose $T : C \to C$ is ρ-nonexpansive. Then for each $\varepsilon > 0$ there exists $C_\varepsilon \in \mathcal{A}$ such that $T(C_\varepsilon) \subset C_\varepsilon$ and for which*

$$\mathrm{diam}_\rho(C_\varepsilon) \le R_\rho(C) + \varepsilon \,\mathrm{diam}_\rho(C).$$

Using the above technical lemma, we can establish the following result which generalizes a fixed point theorem for modular function spaces [3, Theorem 5.9] and extends Kirk's theorem [28, Theorem I] to super-regular modular spaces. The proof of this result follows an identical path as demonstrated in [27].

Theorem 2. *[27, Theorem 4.4] Let C be a ρ-bounded subset of X_ρ. Let $\mathcal{A}$ be a class of subsets of C which is stable under arbitrary intersections and contains all sets of the form $C \cap B_\rho(x, p)$, where $x \in C$ and $p > 0$. In addition, let us assume that $\mathcal{A}$ is normal and countably compact. If $T : C \to C$ is nonexpansive, then T has a fixed point in C.*

Assume that C is a convex, ρ-bounded and ζ-closed subset of a super-regular modular space (X_ρ, ζ). Denote by $\mathcal{A}$ the class of all nonempty, convex, ρ-bounded and ζ-closed subsets of X_ρ . Obviously $\mathcal{A}$ is stable under arbitrary intersections and contains all sets of the form $C \cap B_\rho(x, p)$, where $x \in C$ and $p > 0$, since ρ-balls are convex and ζ-closed. Therefore, we can restate Theorem 2 in a more classical form.

Theorem 3. *Assume that (X_ρ, ζ) is a super-regular modular space with the ζ-normal structure property. Let C be a convex ρ-bounded and ζ-sequentially compact subset of X_ρ. If $T : C \to C$ is nonexpansive, then T has a fixed point in C.*

An alternative formulation of Theorem 3 is that the ζ-normal structure property ensures the ζ-fixed point property (denoted as ζ-fpp).

As noted by Sims in [30], Banach spaces that are uniformly convex in every direction ($UCED$) possess weak normal structure, meaning that every weakly compact convex set exhibits normal structure. This crucial result stems from the work of Garkavi [31]. Consequently, by virtue of Kirk's theorem [32], $UCED$ Banach spaces possess the weak fixed point property. As expected, the same holds true for super-regular modular spaces.

Theorem 4. *Let (X_ρ, ζ) be a UCED super-regular modular space. Let $C \subset X_\rho$ be convex, ρ-bounded and ζ-sequentially compact. If $T : C \to C$ is nonexpansive, then T has a fixed point.*

Theorem 4, being an extension of the Browder fixed point theorem [33, Theorem 1] to super-regular modular spaces, directly follows from Theorem 3 and the following technical result.

Proposition 4. *[18, Prop.3.7] Let a modular space X_ρ be UCED, and let $C \subset X_\rho$ be convex, ρ-bounded and not a singleton. Then C has a ρ-nondiametral point.*

Note that, since every uniformly convex space is uniformly convex in every direction, Theorem 4 also applies to uniformly convex super-regular modular spaces.

A Banach space $(X, \|\cdot\|)$ is said to possess a weak uniformly Kadec-Klee if for every $\varepsilon > 0$ there exists a $0 < \eta < 1$ such that for every sequence $\{x_n\}$ in the unit ball, weakly converging to x, we have $\|x\| \leq 1 - \eta$ provided that

$$\mathrm{sep}\{x_k\} := \inf\{\rho(x_n - x_m) : n \neq m\} > \varepsilon.$$

Beginning in the early 1980s, it became clear that the weak uniform Kadec-Klee property is connected to the fixed point property via the normal structure property [34–37]. Since the Kadec-Klee property is inherently sequential rather than topological, this naturally led to an expanded definition that includes various types of sequence convergence, beyond merely weak or weak*-topologies. For example, see [3,38–44]. To explore the possibility of achieving analogous results in super-regular modular spaces, we will follow [5] and recall the following definition of the Kadec-Klee property for such spaces.

Definition 18. *[5, Def.11] We say that a super-regular (X_ρ, ζ) possesses the uniform ζ-KK1 property if for every $\varepsilon > 0$ and every $r > 0$ there exists an $\eta_1 > 0$ such that for every sequence $\{x_n\}$ in $B_\rho(0, r)$, ζ-convergent to x, such that*

$$\mathrm{sep}\{x_k\} := \inf\{\rho(x_n - x_m) : n \neq m\} > r\varepsilon,$$

we have

$$\rho(x) \leq r(1 - \eta_1).$$

The following result demonstrates the connection between the ζ-KK1 property and the ζ-normal structure, thereby extending the previously mentioned Banach space findings to super-regular modular spaces.

Theorem 5. *[5, Theorem 1] Assume that (X_ρ, ζ) possesses the uniform ζ-KK1 property. Then, (X_ρ, ζ) possesses the ζ-normal structure.*

Theorems 5 and 3 collectively lead to the following significant fixed point result.

Theorem 6. *[5, Theorem 3] Assume that (X_ρ, ζ) possesses the uniform ζ-KK1 property. Then, (X_ρ, ζ) possesses the ζ-fpp.*

It is clear that the uniform ζ-KK1 property is equivalent to the standard definition of the uniform Kadec-Klee property for a Banach space $(X, \|\cdot\|)$ with ζ standing for convergence in the weak topology of X. Consequently, Theorems 5 and 6 generalize well known results that provide linkage between the Kadec-Klee property, the normal structure property, and the weak fixed point property in Banach spaces. We will now discuss how these results relate to the fixed point theory in modular function spaces. Before we start, we will need to revisit some important facts from this theory.

Definition 19. *A function modular ρ is said to be orthogonally additive if $\rho(f1_{A\cup B}) = \rho(f1_A) + \rho(f1_B)$ whenever $A \cap B = \emptyset$.*

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Note that many classical function modulars, including Lebesgue, Orlicz, Musielak-Orlicz modulars, are orthogonally additive.

Definition 20. *[3, Definition 3.7] We say that ρ satisfies the Δ_2-type condition if there exists a finite constant M_2 such that $\rho(2f) \leq M_2\rho(f)$ for every $f \in X_\rho$.*

The class of modular spaces satisfying the Δ_2-type condition includes spaces such as Lebesgue variable exponent spaces $L^p(t)$ for $1 \leq p(t) < \infty$ and Orlicz spaces L^φ for φ satisfying the Δ_2-condition. The following result is used in the proof of the Theorem 8.

Theorem 7. *[3, Theorem 4.7] Let ρ be an orthogonally additive function modular. Let $\{f_n\}$ be strongly bounded, ρ-convergent to zero sequence of elements of L_ρ. For any $g \in E_\rho$ it holds*

$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g).$$

Theorem 8. *[5, Theorem 5] Let L_ρ be a a modular function space, where ρ is orthogonally additive. Let ζ denotes the ρ-a.e. convergence. If ρ satisfies the Δ_2-type property, then the modulated LTI-space (L_ρ, ζ) possesses the uniform ζ-KK1 property. Consequently, via Theorem 6, (L_ρ, ζ) possesses the ζ-fpp.*

While there are numerous examples of modular function spaces that satisfy the conditions of Theorem 8 – such as Lebesgue spaces L^p, Orlicz spaces L^φ meeting the Δ_2 condition, and variable exponent Lebesgue spaces $L^{p(\cdot)}$ with $1 \leq p(t) \leq M < \infty$ – it is essential to also consider cases where the function ρ does not exhibit the Δ_2-type property. Keeping this task in mind, let us recall the concept of the strong ζ-Opial property in super-regular modular spaces.

Definition 21. *[5, Def. 17] We say that a super-regular modular space (X_ρ, ζ) has the strong ζ-Opial property if*

$$\rho(x_0) + \liminf_{n\to\infty} \rho(x_n - x_0) \leq \liminf_{n\to\infty} \rho(x_n), \tag{1}$$

provided $x_n \xrightarrow{\zeta} x_0$ and the sequence $\{x_n - x_0\}$ is strongly bounded.

Observe that, by the Fatou property of modular function spaces, it is always true that

$$\rho(x_0) \leq \liminf_{n\to\infty} \rho(x_n), \tag{2}$$

whenever $x_n \xrightarrow{\zeta} x_0$. In spaces exhibiting the strong ζ-Opial property, the inequality (1) provides a better estimate than given by 2.

Remark 2. *While the concept of the strong ζ-Opial property, as articulated in Definition 21 and related to the renowned weak Opial property [64], is relatively new, it has been established since Khamsi's 1996 work [65] (also see [3][Theorem 4.7]) that any Δ_2 modular function space determined by a convex, orthogonally additive modular ρ, has the strong ζ-Opial property. Our Theorem 7 extends this result to spaces lacking the Δ_2 property. These findings encompass a broad spectrum of function spaces possessing the strong ζ-Opial property, including L^p spaces for $p \geq 1$, variable Lebesgue spaces $L^{p(\cdot)}$ with $1 \leq p(t) < +\infty$, as well as Orlicz and Musielak-Orlicz spaces. It is also important to note, as pointed out in [64], that the weak Opial property does not hold in L^p spaces for $1 \leq p \neq 2$.*

As quite usual in modular space settings, a single concept used in normed spaces splits up into two distinct yet interrelated entities. Therefore, we will need the following variant of the Kadec-Klee property.

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Definition 22. *[5, Def. 12] We say that (X_ρ, ζ) possesses the uniform ζ-KK2 property if for every $\varepsilon > 0$ and every $r > 0$ there exists an $\eta_2 > 0$ such that for every strongly bounded sequence $\{x_n\}$ in $B_\rho(0,r)$, ζ-convergent to x, such that*

$$\text{sep}_2\{x_k\} := \inf\left\{\rho\left(\frac{x_n - x_m}{2}\right) : n \neq m\right\} > r\varepsilon$$

we have

$$\rho(x) \leq r(1-\eta_2).$$

The above notions are utilized in the following result, which ensures the existence of numerous examples of super-regular modular spaces exhibiting the strong ζ-Opial property.

Theorem 9. *[5, Theorem 6] If (X_ρ, ζ) has the strong ζ-Opial property then (X_ρ, ζ) possesses the uniform ζ-KK2 property.*

Remark 3. *Following [5], we raise the following open questions:*

(Q1) What is the relationship between ζ-KK1 and ζ-KK2 properties?
(Q2) Does ζ-KK2 imply some form of normal structure?
(Q3) Does ζ-KK2 directly imply some fixed point property?

3.2. Asymptotic Contractions and Asymptotically Nonexpansive Mappings in Regular Modular Spaces

Existence theorems for fixed points of contractions and nonexpansive mappings constitute the foundational elements of classical fixed point theory. In the context of normed, metric and modular spaces, these results are still most valuable whenever an operator T can be easily associated with a suitable space X and its subset C that fulfill the criteria of these existence theorems. However, in practice, identifying such spaces and subsets can frequently prove to be quite challenging, if not outright impossible. This fact is a key reason why numerous scenarios with relaxed assumptions on operators have been explored over the past fifty years. We will discuss one of this scenarios which employs the asymptotic approach.

Based on the definition of asymptotically nonexpansive mappings in Banach spaces established by Goebel and Kirk in [45], and further generalized by Kirk and Xu in [46,47], the analogous (and, if ρ is a norm, identical) concepts within the framework of regular modular spaces were introduced in [4]. We also refer to definitions in the narrower context of the modular function spaces, as discussed in [3,48,49]).

Definition 23. *[4, Def. 2.7] Let X_ρ be a regular modular space and let $C \subset X_\rho$ be convex, nonempty, ρ-closed and ρ-bounded. A mapping $T : C \to C$ is called*

(i) an asymptotic contraction if there exists a sequence of nonnegative numbers $\{\alpha_n\}$ with $\limsup_{n\to\infty} \alpha_n < 1$ such that

$$\rho(T^n(x) - T^n(y)) \leq \alpha_n \rho(x-y)$$

for every $n \in \mathbb{N}$ and all $x, y \in C$.

(ii) asymptotically nonexpansive if there exists a sequence of nonnegative numbers $\{\alpha_n\}$ with $\limsup_{n\to\infty} \alpha_n = 1$ such that

$$\rho(T^n(x) - T^n(y)) \leq \alpha_n \rho(x-y)$$

for every $n \in \mathbb{N}$ and all $x, y \in C$.

Building upon the foundational work of papers [47] for Banach spaces, [50] for metric spaces, and [48,49] for modular function spaces, extensive research has been conducted on the following pointwise variants.

Definition 24. *Let X_ρ be a regular modular space and let $C \subset X_\rho$ be convex, nonempty, ρ-closed and ρ-bounded. A mapping $T : C \to C$ is called*

(i) *a pointwise contraction if for every $x \in C$ there exists an $\alpha(x) \in [0,1)$ such that*

$$\rho(T(x) - T(y)) \leq \alpha(x)\rho(x-y)$$

for every $x \in C$.

(i) *an asymptotic pointwise contraction if for every $x \in C$ and every $n \in \mathbb{N}$ there exists an $\alpha_n(x) \in [0,1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and*

$$\rho(T^n(x) - T^n(y)) \leq \alpha_n \rho(x-y).$$

(iii) *an asymptotic pointwise nonexpansive mapping if for every $x \in C$ and every $n \in \mathbb{N}$ there exists an $\alpha_n(x) \in [0,1)$ such that $\limsup_{n\to\infty} \alpha_n \leq 1$ and*

$$\rho(T^n(x) - T^n(y)) \leq \alpha_n \rho(x-y).$$

Every pointwise contraction is inherently an asymptotic pointwise contraction, which further qualifies as an asymptotic pointwise nonexpansive mapping. Similarly, every contraction qualifies as a pointwise contraction, while every asymptotic contraction is an asymptotic pointwise contraction. Additionally, every nonexpansive mapping can be considered asymptotically nonexpansive, which also makes it an asymptotic pointwise nonexpansive mapping. An interest in pointwise variants stems partly from the following observation due to Kirk.

Proposition 5. *[51, Proposition 2.1] Let A be a bounded open convex subset of a Banach space X and let $T : A \to X$ be continuously Fréchet differentiable on A. Then, T is a pointwise contraction mapping on A if and only if $\|T'_{x_0}\| < 1$ for every $x_0 \in A$.*

Recall that, given a convex open subset A of X and $T : A \to X$, we say that T is continuously Fréchet differentiable on A if the mapping $x \mapsto T'_x$ from A to the space of continuous linear operators on X is continuous. Here, T'_x denotes the Fréchet derivative of T at x, namely

$$T'_x(y) = \lim_{t\to 0} \frac{T(x+ty) - T(x)}{t},$$

provided the limit is uniform for all y with $\|y\| = 1$.

Utilizing Proposition 5, one can easily establish analogous conditions for other types of pointwise contractions and pointwise nonexpansive mappings, including their asymptotic versions. This demonstrates that pointwise type conditions show up quite naturally in Banach spaces. Parallel results in the context of modular spaces are yet to be discovered.

The following result is known in the context of modular function spaces, see [48, Theorem 2.3]. An inspection of the proof of this theorem shows that both the theorem and the proof hold true for the case of regular modular spaces without any real modification.

Theorem 10. *Let X_ρ be a regular modular space and let $C \subset X_\rho$ be nonempty, ρ-closed and ρ-bounded. Assume that $T : C \to C$ is a pointwise contraction or asymptotic pontwise contraction. Then T has at most one fixe point in C. Moreover, if x_0 is a fixed point of T, then the orbit $\{T^n(x)\}$ ρ-converges to x_0 for any $x \in C$.*

Later in this section, we will examine the question of existence of fixed points for these types of mappings.

To deal with asymptotic pointwise nonexpansive mappings, we need to introduce the following concepts.

Definition 25. *[4, Def. 2.9] We say that a regular modular space X_ρ has the Asymptotic Fixed Point Property, or shortly $(AFPP)$-property if every asymptotically nonexpansive mapping $T : C \to C$ defined on any convex, nonempty, ρ-closed and ρ-bounded $C \subset X_\rho$, has a fixed point.*

We extend this concept to the pointwise case.

Definition 26. *We say that a regular modular space X_ρ has the Asymptotic Pointwise Fixed Point Property, or shortly $(APFPP)$-property if every asymptotic pointwise nonexpansive mapping $T : C \to C$ defined on any convex, nonempty, ρ-closed and ρ-bounded $C \subset X_\rho$, has a fixed point.*

Definition 27. *[4, Def. 3.1] Let $C \subset X_\rho$ be convex, nonempty, ρ-closed and ρ-bounded.*

1. *A function $\tau : C \to [0, \infty]$ is called a ρ-type defined in X_ρ (or shortly a type) if there exists a sequence $\{y_m\}$ of elements of C such that for any $z \in C$,*

$$\tau(z) = \limsup_{m\to\infty} \rho(y_m - z).$$

2. *Let τ be a type. A sequence $\{g_n\}$ is called a minimising sequence of τ if*

$$\lim_{n\to\infty} \tau(g_n) = \inf\{\tau(f) : f \in C\}.$$

Note that τ is convex provided ρ is convex.

Definition 28. *[4, Def. 3.2] We say that a regular modular space X_ρ has the Minimising Sequence Property, or shortly (MSP)-property if every minimising sequence of any type is ρ-convergent, and its limit is independent of the minimising sequence.*

We can employ these notions to prove the following fixed point theorem.

Theorem 11. *[4, Theorem 3.3] (MSP)-property implies $(AFPP)$-property.*

By examining the proof of Theorem 11 and considering related findings for modular function spaces, as detailed in proofs of [3, Theorem 5.7] and [49, Theorem 4.1], we can broaden this result to encompass asymptotic pointwise nonexpansive mappings.

Theorem 12. *(MSP)-property implies $(APFPP)$-property.*

Theorem 12 provides a comprehensive framework for establishing fixed point results in regular modular spaces. To apply this theorem effectively, we must identify the conditions under which these spaces possess the (MSP)-property. To achieve this, we will first review the following two properties of regular modular spaces.

Definition 29. *[4, Def. 4.1] The Parallelogram Function Ψ associated with a regular modular space X_ρ is defined as follows:*

$$\Psi(r, s, \varepsilon) = \inf\left\{\frac{1}{2}\rho^2(x) + \frac{1}{2}\rho^2(y) - \rho^2\left(\frac{x+y}{2}\right)\right\},$$

where $0 < s < r$ and $\varepsilon \geq 0$, and the infimum is taken over all $x, y \in X_\rho$ such that $\rho(x) \leq r$, $\rho(y) \leq r$, $\max\{\rho(x), \rho(y)\} \geq s$, and $\rho(x - y) \geq r\varepsilon$.

Remark 4. *[4, Remark 4.2] It easily follows from the above definition that*

1. *$\Psi(r, s, \varepsilon) \geq 0$,*
2. *$\Psi(r, s, 0) = 0$,*
3. *$\Psi(r, s, \varepsilon)$ is a nondecreasing function of ε with $0 < s < r$ fixed,*

4. *if* $\lim_{n\to\infty} \Psi(r,s,t_n) = 0$, *then* $\lim_{n\to\infty} t_n = 0$.

Due to the evident similarity with the Parallelogram Law in Hilbert spaces, we define the following property for regular modular spaces.

Definition 30. *[4, Def. 4.3] We say that a regular modular space X_ρ has the Parallelogram Property (or shortly, (PP)-property) if*

$$\Psi(r,s,\varepsilon) > 0$$

whenever $\varepsilon > 0$.

Following [3, Def. 3.9], let us recall the definition of the property (R) in modular spaces.

Definition 31. *We say that X_ρ has property (R) if and only if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of X_ρ has nonempty intersection.*

Note that in the case ρ is a norm, it follows from the Eberlein- Šmulian theorem that the property (R) is equivalent to reflexivity.

We are now prepared to present the following fundamental result, which extends Theorem 4.6 in [4] to the case of asymptotic pointwise nonexpansive mappings.

Theorem 13. *Let a regular modular space X_ρ possess the properties (PP) and (R). Then, X_ρ also has the (MSP)-property, which, according to Theorem 12, implies the $(APFPP)$-property.*

The proof of Theorem 13 follows the path of the proof of [3, Lemma 4.3]. Note also that the (MSP)-property in abstract modular spaces with the Fatou property was considered in [12]. Browder's original result from 1965, as presented in [33, Theorem 1], along with many subsequent generalizations, is articulated in terms of uniform convexity. Utilizing Theorems 13 and 11, we will now show that this holds true for regular modular spaces as well. To accomplish this, let us first revisit the following two results.

Theorem 14. *[4, Theorem 5.5] Every uniformly convex regular modular space has property (PP).*

Theorem 15. *[4, Theorem 5.6] Every uniformly convex regular modular space has property (R).*

Let X_ρ be a uniformly convex regular modular space. According to Theorem 15, X_ρ possesses property (R), and by Theorem 14, it also exhibits property (PP). With the application of Theorem 13, we can conclude that X_ρ has property $(APFPP)$, which is articulated in the following fundamental fixed point result.

Theorem 16. *Every uniformly convex regular modular space has the $(APFPP)$-property.*

Following the argument presented in the proof of [3, Theorem 5.7], we can demonstrate that the set of common fixed points of an asymptotic pointwise nonexpansive mapping is both ρ-closed and convex. Considering this, we can express our findings in a more traditional manner.

Theorem 17. *Let C be a nonempty, ρ-closed, convex and ρ-bounded subset of a uniformly convex regular modular space X_ρ, and let $T : C \to C$ be an asymptotic pointwise nonexpansive mapping. Then, T has a fixed point. Furthermore, the set of its all fixed points, $F(T)$ is ρ-closed and convex.*

Since every asymptotic pointwise contraction is as an asymptotic pointwise nonexpansive mapping, Theorem 17 addresses the earlier question regarding the existence of fixed points for asymptotic

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pointwise contractions. Specifically, when combined with Theorem 10, it yields the following fixed point result.

Theorem 18. *Let C be a nonempty, ρ-closed, convex and ρ-bounded subset of a uniformly convex regular modular space X_ρ, and let $T : C \to C$ be an asymptotic pointwise contraction. Then, T has a unique fixed point $\bar{x} \in C$. Moreover, for any $x \in C$, $\rho(T^n(x) - \bar{x}) \to 0$ as $n \to \infty$, where T^n is the n-th iterate of T.*

While many, important from applications perspective, regular modular spaces are uniformly convex, this requirement is still quite restrictive. Our next result offers a more general and less restrictive condition.

Theorem 19. *Let C be a nonempty ρ-bounded, ρ-closed subset of a regular modular space X_ρ. Assume $T : C \to C$ to be an asymptotic pointwise contraction. Moreover, assume that there exists an $x \in C$ such that the ρ-type Φ defined for $u \in C$ by $\Phi(u) = \limsup_{k\to\infty} \rho(T^k(x) - u)$ attains its minimum in C. Then, there exists a unique fixed point $z \in F(T)$. Moreover, $\rho(T^k(u) - z) \to 0$ for every $u \in C$.*

Proof. In view of Theorem 10, we need only to prove the existence of a fixed point. By our assumption, there exists $z \in C$ such that $\Phi(z) = \inf\{\Phi(y) : y \in C\}$. We will prove now that $\Phi(z) = 0$. First, note that for any $m, k \in \mathbb{N}$

$$\rho(T^{m+k}(x) - T^m(z)) \le a_m(z)\rho(T^k(x) - z),$$

and that by letting $k \to +\infty$ we get $\Phi(T^m(z)) \le a_m(z)\Phi(z)$, which, after passing with m to infinity, gives

$$\Phi(z) \le \limsup_{m\to+\infty} \Phi(T^m(z)) \le \limsup_{m\to+\infty} a_m(z)\Phi(z).$$

Since $\limsup_{m\to+\infty} a_m(z) < 1$, we conclude that $\Phi(z) = 0$. Consequently, using the definition of Φ, we have

$$\begin{aligned} 0 \le \limsup_{k\to+\infty} \rho(T^k(x) - T(z)) &= \limsup_{k\to+\infty} \rho(T^{k+1}(x) - T(z)) \\ &\le \limsup_{k\to+\infty} \rho(T^k(x) - z) = \Phi(z) = 0. \end{aligned}$$

Hence, $T^k(x) \xrightarrow{\rho} z$ and $T^k(x) \xrightarrow{\rho} T(z)$, which, by the uniqueness of the ρ-limit, implies that $T(z) = z$, as claimed. □

To effectively apply Theorem 19, we need practical methods for determining when ρ-types reach their minimum. In the context of Banach spaces, this is satisfied when C is nonempty, convex, bounded, and weakly compact, as established in sources such as [52,53]. However, this conclusion relies on specific Banach space characteristics, such as the norm's triangle property, and the principle that a closed convex subset of a weakly compact set remains weakly compact. Since these attributes are generally absent in modulated topological vector spaces, we will introduce a powerful method involving uniformly continuous modulars.

Definition 32. *[3, Definition 5.4] A modular ρ is called uniformly continuous if for every $\varepsilon > 0$ and every $0 < L < \infty$ there exists $\delta > 0$ such that*

$$|\rho(x + y) - \rho(x)| < \varepsilon,$$

whenever $x \in X_\rho$, $y \in X_\rho$, $\rho(y) < \delta$, and $\rho(x) \le L$.

The following two lemmas proven originally for function modulars can be easily extended to regular modular spaces.

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Lemma 2. *[3, Lemma 5.1] Let C be a nonempty ρ-bounded, ρ-closed, convex subset of a regular modular space X_ρ, where ρ is uniformly continuous. Then, any ρ-type defined in C is ρ-lower semicontinuous on C.*

Lemma 3. *[3, Lemma 3.3] Let C be a nonempty ρ-bounded, ρ-closed, convex subset of a regular modular space X_ρ, where ρ possesses property (R). If φ is a ρ-lower semicontinuous ρ-type defined in C then φ attains its minimum in C.*

Our next fixed point theorem for asymptotic pointwise contractions follows immediately from Lemma 2, Lemma 3 and Theorem 19.

Theorem 20. *Let C be a nonempty ρ-bounded, ρ-closed, convex subset of a regular modular space X_ρ. Assume that ρ is uniformly continuous and has property (R). Let $T : C \to C$ be an asymptotic pointwise contraction. Then, there exists a unique common fixed point $z \in F(T)$. Moreover, $\rho(T^k(u) - z) \to 0$ for every $u \in C$.*

Every norm is inherently a uniformly continuous modular, which means that in the context of Banach spaces, the only additional requirement is the possession of property (R). In this case, this property equates to reflexivity. This aligns with our earlier observation, as reflexivity is equivalent to the weak compactness of the unit ball in a Banach space.

Generally, for convex modulars, the uniform continuity of ρ is equivalent to ρ satisfying the Δ_2-type condition, see for example [54], hence Theorem 20 can be applied to a wide class of spaces including Lebesgue spaces L^p, Orlicz spaces L^φ with φ satisfying a relevant Δ_2 condition, and variable exponent Lebesgue spaces $L^{p(\cdot)}$ with $1 \le p(t) \le M < \infty$.

4. Semigroups of Nonlinear Operators in Modular Spaces

In this section we will outline fixed point considerations related to semigroups of nonlinear operators acting within modular spaces. Let us start with the following basic definitions.

Definition 33. *[11, Def. 2.10] Let X_ρ be a regular modular space and let $C \subset X_\rho$ be convex, nonempty, ρ-closed and ρ-bounded. Let J will be a parameter semigroup of nonnegative numbers, that is, a subsemigroup of $[0, +\infty)$ with normal addition such that $0 \in J$ and there is $0 < t \in J$. A one-parameter family $\mathcal{T} = \{T_t : t \in J\}$ of mappings from C into itself is called a pointwise Lipschitzian semigroup on C if $\mathcal{T}$ satisfies the following conditions:*

1. *$T_0(x) = x$ for $x \in C$;*
2. *$T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \in J$;*
3. *for each $t \in J$, T_t is a pointwise Lipschitzian mapping, i.e. there exists a function $\alpha_t : C \to [0, \infty)$ such that for every $x, y \in C$,*
$$\rho(T_t(x) - T_t(y)) \le \alpha_t(x)\rho(x - y),$$
4. *for each $x \in C$, the mapping $t \to T_t(x)$ is ρ-continuous at every $t \in J$, i.e,*
$$\rho(T_{t_n}(x) - T_t(x)) \to 0,$$
whenever $t_n \in J$ for every $n \in \mathbb{N}$, and $t_n \to t$.

For each $t \in J$, let $F(T_t)$ denote the set of its fixed points. Define then the set of all common fixed points set for mappings from $\mathcal{T}$ as the intersection
$$F(\mathcal{T}) = \bigcap_{t \in J} F(T_t).$$

Definition 34. *Using assumptions and notations from Definition 33, we say that a pointwise Lipschitzian semigroup $\mathcal{T}$ is asymptotic pointwise nonexpansive, if in addition*

$$\limsup_{t\to\infty} \alpha_t(x) \le 1. \tag{3}$$

Similarly, $\mathcal{T}$ is called asymptotic pointwise contractive if

$$\limsup_{t\to\infty} \alpha_t(x) < 1. \tag{4}$$

Remark 5. *Let us note the following:*

1. *Assumptions on J immediately imply that $+\infty$ is a cluster point of J in the sense of the natural topology inherited by J from $[0,+\infty)$.*
2. *Typical examples of the parameter semigroups J satisfying conditions from Definitions 33 and 34 are: $J = [0,+\infty)$ and ideals of the form $J = \{n\alpha : n = 0,1,2,3,...\}$ for a given $\alpha > 0$.*
3. *In (3) and (4), as in the whole of this section, we use a convention that the notation $t \to \infty$ means that t tends to infinity over J.*
4. *Without loss of generality we may assume $\alpha_t(x) \geq 1$ for any $t \in J$ and any $x \in C$, as well as that $\limsup_{t\to\infty} \alpha_t(x) = \lim_{t\to\infty} \alpha_t(x) = 1$.*

In mathematics and its applications, scenarios involving continuous semigroups of operators are prevalent. For instance, in dynamical systems theory, the space X represents the state space, while the mapping $(t,x) \to T_t(x)$ models the evolution function of a dynamical system. The parameter t can signify either continuous or discrete time, contingent on the nature of the parameter set J. Common fixed points of the semigroup can be interpreted as the system's stationary points that remain unchanged under the transformation T_t for all $t \in J$. The exploration of semigroups of nonlinear operators as resolving semigroups has been a significant area of research in recent decades. Since the state space X can be infinite-dimensional, these findings are relevant to both deterministic and stochastic dynamical systems. In this framework, creating algorithms to identify common fixed points of such semigroups is intrinsically connected to the challenge of solving stochastic evolution equations.

The fixed point theory for semigroups of nonlinear operators operating within normed and metric spaces has been thoroughly developed and documented. For detailed references, see works such as [3,7,53,56,57]. The recent book [58] offers an extensive discussion on the fixed point theory of semigroups of pointwise Lipschitzian operators acting within Banach spaces. This book also provides a comprehensive justification for employing the pointwise asymptotic approach (as presented in this section), supported by a variety of examples.

In the realm of modular function spaces, an analogous theory has been developed across numerous works. To highlight a few, notable references include [2,3,12,59–63]. Some of these results can be adapted to regular modular spaces, while others are contingent upon the specifics of modular function space theory.

Foundations of the semigroup theory for regular modular spaces can be found in [55] and [4]. The existence of common fixed point is established in author's recent paper [11], as is summarised in the following theorem.

Theorem 21. *[11, 3.12] Let X_ρ be uniformly convex regular modular space. Let $\mathcal{T}$ be an asymptotic pointwise nonexpansive semigroup on C, where $C \subset X_\rho$ is convex, nonempty, ρ-closed and ρ-bounded. Then, there exists a common fixed point for $\mathcal{T}$, which means that the set $F(\mathcal{T})$ is nonempty. Moreover, $F(\mathcal{T})$ is convex and ρ-closed.*

This theorem naturally leads to a parallel result for asymptotic pointwise contractive semigroups.

Theorem 22. *Let X_ρ be uniformly convex regular modular space. Let $\mathcal{T}$ be an asymptotic pointwise contractive semigroup on C, where $C \subset X_\rho$ is convex, nonempty, ρ-closed and ρ-bounded. Then, there exists a unique common fixed point $z \in F(\mathcal{T})$. Moreover, for every $u \in C$, $\rho(T_t(u) - z) \to 0$ as $t \to \infty$.*

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For contractive semigroups, a result from [55] can be adapted from the context of modulated topological vector spaces, where the ζ-convergence is defined as the sequential convergence with respect to a linear Hausdorff topology.

Theorem 23. *[55, Theorem 3.5] Let C be a nonempty ρ-bounded subset of a regular modular space X_ρ and let $\mathcal{T} = \{T_t : t \geq 0\}$ be a contractive semigroup on C. Assume that there exists an $x \in C$ such that the ρ-type Φ defined for $u \in C$ by $\Phi(u) = \limsup_{t\to\infty} \rho(T_t(x) - u)$ attains its minimum in C. Then there exists a unique common fixed point $z \in F(\mathcal{T})$. Moreover, $\rho(T_t(u) - z) \to 0$ for every $u \in C$.*

To effectively utilize Theorem 23, it is crucial to identify practical criteria for determining the ρ-types that achieve their minimum for specific sets C. In the context of Banach spaces, it is well-established that this holds when C is a nonempty, convex, bounded, and weakly compact set, as noted in sources such as [52,53]. However, this conclusion relies on certain characteristics unique to Banach spaces, such as the triangle inequality for norms, and the property that closed convex subsets of weakly compact sets remain weakly compact. Given that these properties may be absent in super-regular modular spaces, we will employ the robust techniques of ζ-Opial sets (see Def. 35) modified after the analogue concept used in the theory of modulated topological vector spaces. This approach will enable us to establish our next common fixed point theorem for contractive semigroups and provide a comprehensive list of examples and applications.

Definition 35. *[55, Definition 3.7] Let C be a nonempty subset of a super-regular modular space (X_ρ, ζ). We say that C is a ζ-Opial set if for every $y \in C$ and every sequence $\{x_n\}$ of elements of C with $x_n \xrightarrow{\zeta} x$ for an $x \in C$,*

$$\liminf_{n\to\infty} \rho(x_n - y) = \liminf_{n\to\infty} \rho(x_n - x) + \rho(x - y). \tag{5}$$

We will start by revisiting a standard result expressed in the context of super-regular modular spaces. Here, we denote ζ-l.s.c as lower semi-continuity with respect to the ζ convergence. Specifically, this means that for a given function $\Psi : X \to [0, +\infty)$, the inequality $\Psi(y) \leq \liminf_{k\to\infty} \Psi(y_k)$ holds whenever the sequence y_k of elements of C converges to y with respect to ζ (i.e., $y_k \xrightarrow{\zeta} y$).

Lemma 4. *[55, Lemma 3.6] Let C be a nonempty, sequentially ζ-compact subset of a super-regular modular space (X_ρ, ζ). Let $\Psi : C \to [0, +\infty)$. If Ψ is ζ-l.s.c. then Ψ attains its minimum in C.*

Using this lemma and following the flow of the proof of Theorem 3.8 in [55] one can prove the following important result.

Theorem 24. *Let C be a nonempty, sequentially ζ-compact, ρ-bounded subset of a super-regular modular space (X_ρ, ζ). If C is a ζ-Opial set then every ρ-type Φ on C is sequentially ζ-l.s.c. and attains its minimum in C. Moreover, if $\{y_n\}$ is a sequence of elements of C such that $y_n \xrightarrow{\zeta} y$ then*

$$\Phi(y) + \liminf_{n\to\infty} \rho(y_n - y) \leq \liminf_{n\to\infty} \Phi(y_n). \tag{6}$$

Combining Theorems 23 and 24 we immediately obtain the following fixed point result (compare with the proof of Theorem 3.9 in [55]).

Theorem 25. *[55, Theorem 3.9] Let C be a ρ-bounded and sequentially ζ-compact subset of a super-regular modular space (X_ρ, ζ). Let $\mathcal{T} = \{T_t : t \geq 0\}$ be a contractive semigroup on C. If C is a ζ-Opial set then there exists a unique common fixed point $z \in F(\mathcal{T})$. Moreover, $\rho(T_t(u) - z) \to 0$ for every $u \in C$.*

Remark 6. *Observe that if a super-regular modular space (X_ρ, ζ) has a strong ζ-Opial property then every strongly bounded closed convex set is a ζ-Opial set. In view of Remark (2), this means that there are many* *important examples of scenarios, where Theorem 25 can be applied, including spaces like L^p for $p \geq 1$, variable Lebesgue spaces $L^{p(\cdot)}$ where $1 \leq p(t) < +\infty$, Orlicz and Musielak-Orlicz spaces.*

5. Fixed Point Construction Processes

It is well known that Picard's technique of iterates and orbits, effective in establishing fixed points for contraction mappings (as demonstrated by our Theorems 1, 20, 22, 19, 25, 23) does not typically lead to a converging process in the case of nonexpansive mappings. To address this issue for mappings operating in Hilbert, Banach, metric, and modular function spaces, a vast array of effective algorithms has been developed over the past few decades. Due to space constraints in this brief survey, we cannot do full justice to this extensive body of work; therefore, we will highlight just a few resources that may be beneficial. See , for example, [3,58,63,64,66–81].

It is important to note that many iterative processes, such as the generalized Krasnosel'skii-Mann and Ishikawa processes, which have been thoroughly studied in the context of modular function spaces, can be cautiously adapted to the framework of super-regular modular spaces. In the 2024 paper [13], the author began an investigation into implicit iterative processes in regular modular spaces. Below, we summarize the key findings of that work.

Definition 36. *[13, Def. 8] Let C be a convex, nonempty, ρ-closed, and ρ-bounded subset of a regular modular space X_ρ. Given a ρ-nonexpansive semigroup $\mathcal{T} = \{T_t : t \in [0, \infty)\}$ on C, the implicit iteration process $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is defined by the following formula:*

$$\begin{cases} x_0 \in C \\ x_{k+1} = c_k T_{t_{k+1}}(x_{k+1}) + (1 - c_k)x_k, \ \text{for } k \in \mathbb{N}_0, \end{cases} \tag{7}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the sequence $\{c_k\}_{k \in \mathbb{N}_0}$ of real numbers from $(0, 1)$ is bounded away from 0 and 1, and $\{t_k\}_{k \in \mathbb{N}}$ is a sequence of positive real numbers. We will also say that the sequence $\{x_k\}_{k \in \mathbb{N}_0}$ is generated by the process $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ and write

$$\{x_k\}_{k \in \mathbb{N}^0} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\}). \tag{8}$$

For $k \in \mathbb{N}_0$, $u \in C$, $w \in C$, let us introduce the following notation:

$$P_{k,w}(u) = c_k T_{t_{k+1}}(u) + (1 - c_k)w. \tag{9}$$

Since each $P_{k,w}(u) : C \to C$ is a ρ-contraction, it follows from the Banach Contraction Principle (Theorem 1) that each x_{k+1} in (7) is uniquely defined.

Theorem 26. *[13, Theorem 3] Let X_ρ be a uniformly convex regular modular space, and let $C \subset X_\rho$ be convex, nonempty, ρ-compact, and ρ-bounded. Assume that $\mathcal{T}$ is a ρ-nonexpansive semigroup on C. Let $\{x_k\}_{k \in \mathbb{N}^0} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ be an implicit iteration process, where*

(i) $t_n > 0$ *for every* $n \in \mathbb{N}$
(ii) $\liminf_{n \to \infty} t_n = 0$
(iii) $\limsup_{n \to \infty} t_n > 0$
(iv) $\lim_{n \to \infty} (t_{n+1} - t_n) = 0$.

There then exists a common fixed point $x \in F(\mathcal{T})$, such that $\rho(x_k - x) \to 0$ when $k \to \infty$.

Building upon [13], we would like to offer a few remarks regarding the applicability of Theorem 26. It is straightforward to construct a sequence $\{t_k\}$ that satisfies conditions (i) through (iv). Furthermore, we have previously emphasized that the class of regular modular spaces is quite broad, encompassing all Banach spaces and modular function spaces. Consequently, it includes a variety of significant

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spaces, such as L^p, l^p, variable exponent versions of these spaces, Orlicz spaces, Musielak–Orlicz spaces, and many other important function and sequence spaces. The existence of natural examples of ρ-nonexpansive semigroups within these modular spaces has been established since the work of Khamsi [59], as well as in [82] and the associated literature.

The following example shows how the modular space fixed point results can be employed for construction of a stationary point of a system defined by the Urysohn integral operator. It is also worthwhile noting that the modular itself is constructed from the components of the operator in question emphasizing the flexibility advantage of the modular techniques over a traditional norm-approach, which would necessitate additional constraints on the operator.

Example 5. *[13, Example 4] Consider the Urysohn operator:*

$$T(f)(x) = \int_0^1 k(x,y,|f(y)|)dy + f_0(x), \tag{10}$$

where f_0 is a fixed function and $f : [0,1] \to \mathbb{R}$ is Lebesgue measurable. For the kernel k we assume that

(a) $k : [0,1] \times [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ *is Lebesgue measurable,*
(b) $k(x,y,0) = 0$,
(c) $k(x,y,.)$ *is continuous, convex and increasing to* $+\infty$,
(d) $\int_0^1 k(x,y,t)dx > 0$ *for* $t > 0$ *and* $y \in (0,1)$.

In addition, assume that for almost all $t \in [0,1]$, and any two measurable functions f and g, the following inequality holds:

$$\int_0^1 \Big\{ \int_0^1 k(t,u,|k(u,v,|f(v)|) - k(u,v,|g(v)|)|)dv \Big\} du \le \int_0^1 k(t,u,|f(u) - g(u)|)du.$$

Define

$$\rho(f) = \int_0^1 \Big\{ \int_0^1 k(x,y,|f(y)|)dy \Big\} dx.$$

It is not difficult to to show that ρ is a regular modular, and that $\rho(T(f) - T(g)) \le \rho(f - g)$, that is, T is ρ-nonexpansive. To ensure that the Urysohn operator given by (10) generates a dynamic system, it is essential to verify that for every $f \in C$, where C is a ρ-ball centered at f, the subsequent Initial Value Problem,

$$\begin{cases} u(0) = f \\ u'(t) + (I - T)u(t) = 0, \end{cases} \tag{11}$$

has a solution $u_f : [0,+\infty] \to C$. It can be shown that, under certain technical assumptions, equation (11) has a solution u_f for every initial values of f. Denote $S_t(f) = u_f(t)$. Following the pattern given in [59], we can demonstrate that $\{S_t\}_{t\ge 0}$ constitutes a ρ-nonexpansive semigroup of nonlinear operators on C. Therefore, the results of our theory enable us to construct a common fixed point for the semigroup $\{S_t\}_{t\ge 0}$ by applying the implicit iterative process defined in (7).

6. Discussion

This survey article aimed to solidify the framework of regular and super-regular modular spaces and to utilize it in providing a cohesive overview of significant advancements in Fixed Point Theory as applied to nonlinear operators acting within modular spaces. Our objective was to establish an open framework that could be applicable to both Banach spaces and modular function spaces, without restricting it to just these two cases.This topic is undoubtedly too broad for a paper of this nature to address comprehensively and explore all known or potential directions. Consequently, we have set boundaries for our discussion, focusing primarily on operators and their semigroups that satisfy pointwise asymptotic nonexpansive and contractive conditions within the modular framework.

The core principles of the framework are outlined in Section 2. This section began with foundational definitions and properties of general modular spaces, detailed in Subsection 2.1. It then progressed to an examination of regular modular spaces in Subsection 2.2, which starts with Definition 3. Finally, the discussion advanced to the crucial concepts of super-regular modular spaces in Subsection 2.3, introduced with Definition 12. To revisit the fundamental concept of super-regular modular spaces, it centers on the introduction of convergence that enables results analogous to those derived from weak convergence in Banach spaces and ρ-almost everywhere convergence in modular function spaces. This advancement is made possible through the framework of modulated LTI-spaces (as defined in Definition 10), which were introduced in [19] and are based on the principles of L-convergence (defined in Definition 9), originally presented in [21]. Examples 3, 4, and 5 illustrate that the class of super-regular modular spaces is extremely vast, underscoring the significance of advancing this area for both theoretical exploration and practical applications.

The rich structure of super-regular modular spaces facilitates the application of the ζ-normal structure concept, enabling the generalization and unification of Kirk's Fixed Point Theorem (Theorem 3) and Browder's Fixed Point Theorems (Theorem 4) within the context of super-regular modular spaces. It also gives rise to the concepts of Kadec-Klee properties for these spaces, as outlined in Definitions 18 and 22. In a manner analogous to established findings in Banach spaces, we observe that the uniform $\zeta - KK1$ property entails the ζ-normal structure (as stated in Theorem 5). This, in turn, leads to the conclusion—through Theorem 3—that the analogous ζ-fixed point property also holds, as demonstrated in Theorem 6. In addition to its applicability to Banach spaces, this fixed point result extends to a broad category of orthogonally additive modular function spaces with the Δ_2-type property, as illustrated in Theorem 8 and the accompanying discussion. Despite its broad applicability, we must also take into account scenarios where the function ρ lacks the Δ_2-type property. By employing the innovative concept of the strong ζ-Opial property within super-regular modular spaces (see Definition 21 and Remark 2), we established in Theorem 9 that the strong ζ-Opial property guarantees the uniform $\zeta - KK2$ property. This finding raises pertinent open questions (Remark 3).

Expanding beyond the scope of contractions and nonexpansive mappings in super-regular modular spaces, we drew inspiration from works such as [3,4,45–49] to introduce the concepts of asymptotic contractions and asymptotic nonexpansive mappings, along with their pointwise equivalents, as defined in Definitions 23 and 24. The key results in this area generalize the classical fixed point existence results to this new context, as illustrated in Theorem 17 for asymptotic pointwise nonexpansive mappings and in Theorem 18 for the case of asymptotic pointwise contractions. These theorems are established through a series of non-trivial results pertaining to the geometry of super-regular modular spaces such as modular Uniform Convexity Property, Minimising Sequence Property and Parallelogram Property. For the case of asymptotic pointwise contractions, we reproduced a less restrictive result, where the only assumption was an existence of a special type function which attains its minimum (Theorem 19). This general result can be applied in situations where the modular ρ is uniformly continuous and has property (R), which is a modular equivalent of reflexivity, as demonstrated in Theorem 20. We observe that the uniform continuity of convex modulars is equivalent to the Δ_2-type property. This equivalence, along with its clear relevance to Banach spaces, shows that Theorem 20 is applicable to a broad range of spaces. These include Lebesgue spaces L^p, Orlicz spaces L^φ where the function φ meets an appropriate Δ_2-condition, and variable exponent Lebesgue spaces $L^{p(\cdot)}$ satisfying $1 \leq p(t) \leq M < \infty$.

Due to the significance of semigroups of nonlinear operators in various fields of pure and applied mathematics, including dynamical systems, differential equations, probability, and stochastic evolution equations, research on this topic within the framework of modular spaces has been ongoing since the 1990s. The common fixed points of such a semigroup $\{T_t\}_{t\in J}$ can be understood as the stationary points of the corresponding dynamical system, remaining invariant under each transformation T_t. The theory of super-regular modular spaces provides an opportunity to unify disparate results into a cohesive and comprehensive framework. Theorems 21 and 22, referenced following the recent paper [11], provide foundational results for the existence of common fixed points in the context of asymptotic pointwise nonexpansive and asymptotic pointwise contractive semigroups in uniformly convex regular modular spaces.

Analogous to the situation involving a single operator addressed in Theorem 19, we refer to the common fixed point existence result (Theorem 23), as noted after [55], which does not necessitate uniform convexity. By utilizing the innovative concept of ζ-Opial sets, as defined in Definition 35, we arrive at an additional existence result, outlined in Theorem 25. Furthermore, Remark 6 highlights the broad applicability of these findings to a wide range of significant spaces.

In Section 5, we briefly addressed the issue of fixed point construction processes for operators and semigroups of operators operating within regular and super-regular modular spaces. In contrast to well-established results in Banach, metric and modular function spaces, similar research in the context of regular modular spaces has only recently begun, with a notable example being the implicit iteration process outlined in Theorem 26.

As promised in the Abstract, this survey also aims to identify and discuss current challenges, knowledge gaps, and unresolved questions, thereby providing insights into potential future research opportunities.

As indicated in Remark 3, which highlights specific open questions, the entire field of research concerning Kadec-Klee, Opial, normal structure, and fixed point properties stands to gain significantly from further investigation. This research should particularly focus on exploring the relationships among these concepts from diverse perspectives.

An entire field of research related to monotone nonexpansive operators and operators in modular spaces with graphs is not covered in this survey. Given its wide range of applications, this area, particularly within the context of regular and super-regular modular spaces, is of significant interest. For additional background information, the reader is referred to [86–93] and the literature referenced therein.

The area of fixed point construction processes in regular and super-regular modular spaces is ripe for systematic exploration. This includes a focus on both "strong" ρ-convergence and "weak" ζ-convergence to fixed points, as well as, in the case of semigroups, the study of relevant convergence to common fixed points. Existing methods like the Krasnosel'skii-Mann and Ishikawa processes for modular function spaces can provide a reliable starting point, see, e.g., [3,63,85].

Moreover, the questions surrounding the stability of such processes in the context of regular modular spaces remain largely unresolved. Recent works such as [74–80,83,83] offer valuable, albeit in different contexts, background information on these topics.

Last but certainly not least, the topics concerning the applications of fixed point theory in regular and super-regular modular spaces, frequently mentioned in this study and the cited literature, represent a fascinating area for future research development.

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