Article

Not peer-reviewed version

On *n*-Derivations and *n*-Homomorphisms in Perfect Lie Superalgebras

Shakir Ali *, Amal S. Alali, Mukhtar Ahmad, Md Shamim Akhter

Posted Date: 2 September 2025

doi: 10.20944/preprints202509.0196.v1

Keywords: Lie algebra; perfect Lie superalgebra; *n*-derivation; *n*-homomorphism; enveloping Lie

Article

On n-Derivations and n-Homomorphisms in Perfect Lie Superalgebras

Shakir Ali[1,2,*], Amal S. Alali [3], Mukhtar Ahmad [1] and Md Shamim Akhter [4]

[1] Department of Mathematics, Faculty of Science, Aligarh Muslim University, Aligarh-202002, India
[2] Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Yogyakarta-55598, Indonesia
[3] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh-11671, Saudi Arabia
[4] Department of Computer science & Engineering, Jamia Hamdard, New Delhi-110062, India
* Correspondence: shakir.ali.mm@amu.ac.in

Abstract

Let $n \geq 2$ be a fixed integer. The main aim of this paper is to investigate the properties of n-derivations within the framework of perfect Lie superalgebras over a commutative ring R. The main result shows that, if the base ring contains $\frac{1}{n-1}$, and L is a perfect Lie superalgebra with a center equal to zero, then any n-derivation of L is necessarily a derivation. Additionally, every n-derivation of the derivation algebra $Der(L)$ is an inner derivation. Finally, extend the concept of n-homomorphisms to mappings between Lie superalgebras L and L', and prove that under specific assumptions, homomorphisms, anti-homomorphisms, and their combinations are all n-homomorphisms.

Keywords: Lie algebra; perfect Lie superalgebra, n-derivation; n-homomorphism; enveloping Lie

MSC: 17B35; 16W25; 17B05; 17B10; 17B20

---

1. Introduction

In the context of associative algebras [1–3], the concept of derivation naturally extends into several types of triple derivations, such as Jordan triple derivations, Lie triple derivations, etc. Each of derivation plays a role in understanding the broader mathematical structures, with Lie triple derivations being particularly significant. These derivations are not only important for associative algebras and rings but have broader applications in the study of Lie groups [4] and operator algebras [5–7]. The triple derivation in Lie algebras can be thought of as an extension of the more familiar concept of a derivation, and it serves as an analogy to the corresponding triple derivations in both associative and Jordan algebras. First introduced by Müller [4] under the term "prederivation," this concept has gained importance in the field. A key property is that every derivation in a Lie algebra automatically qualifies as a Lie triple derivation, but the converse does not necessarily hold [8].

The relationships among homomorphisms and their variants such as anti-homomorphisms, Jordan homomorphisms, Lie homomorphisms, and Lie triple homomorphisms have long attracted interest. In particular, Bresar [9] characterized Lie triple isomorphisms on certain associative algebras. At the same time, Jacobson and Rickart's study [10] established a theoretical structure showing that any Jordan homomorphism of a ring must necessarily act as either a standard homomorphism or an anti-homomorphism. These studies have inspired similar investigations in operator algebras [11,12], and a comparable result has been established for perfect Lie algebras [13] and Cohomology theory of Lie groups and Lie algebras [19], broadening the applicability of these homomorphism relations.

Lie superalgebras, which are a natural extension of Lie algebras, have important applications across various fields, including both mathematics and physics. Beyond their applications, they also present intriguing mathematical properties. We are motivated to generalize key results from [8,13] of

Lie superalgebras, with a particular emphasis on triple-derivations and triple-homomorphisms. Our goal is to broaden the scope of these earlier findings and extend their applicability to the context of Lie superalgebras.

By exploring these concepts, we seek to deepen our understanding of their properties and interactions in Lie superalgebras, building upon existing theoretical frameworks and expanding their applications. Our investigation will provide insights into the role of n-derivations and n-homomorphisms in the structure and behavior of Lie superalgebras, contributing to the broader field of algebraic research.

The notion of n-derivations has been studied in various algebraic settings, such as n-derivations for finitely generated graded Lie algebras [14], n-derivations of Lie color algebras [15]. Our investigation builds upon this body of work, aiming to extend n-derivations, n-automorphisms, and Lie n-systems to the Lie algebra of a Lie group, thereby enhancing the understanding of their applications.

2. Preliminaries

Let $n \geq 2$ be a fixed integer. We use L to refer to a Lie superalgebra over a commutative ring R with unity. A Lie superalgebra L is perfect if its derived subalgebra $[L, L]$ equals L. For any subset $S \subseteq L$, we denote $\mathrm{C}_L(S)$ the centralizer of S in L, while the center of L is represented by $\mathrm{Z}(L)$. A Lie superalgebra is called centerless if $\mathrm{Z}(L) = \{0\}$. The algebra of derivations of L is denoted by $Der(L)$ and algebra of Inner derivation of L denoted by $ad(L)$. Definition 1 and Definitions such as, 2, 3, 4 and Definition 5 below are taken from [15], [18] and [19].

Definition 1. *Let L be a direct sum of two components, $L_{\bar{0}}$ and $L_{\bar{1}}$, where L is a $\mathbf{Z}_2$-graded algebra over a commutative ring R that contains a multiplicative identity. We say that L is a Lie superalgebra if the multiplication operation, denoted by [,], adheres to the following set of identities:*

(a) $[v_1, v_2] = -(-1)^{|v_1||v_2|}[v_1, v_2]$; *(graded skew − symmetry)*

(b) $[v_1, [v_2, v_3]] = [[v_1, v_2], v_3] + (-1)^{|v_1||v_2|}[v_2, [v_1, v_3]]$ *(graded Jacobi identity).*

Let $v_1, v_2, v_3 \in hg(L)$, where $hg(L)$ represents the set of all $\mathbf{Z}_2$-homogeneous elements of L. Throughout this paper, whenever $|v|$ appears, we interpret v as a $\mathbf{Z}_2$-homogeneous element, and $|v|$ denotes the $\mathbf{Z}_2$-degree of v.

Definition 2. *For a subset S of L, the enveloping Lie superalgebra of S is the Lie subalgebra of L generated by S. A Lie superalgebra is said to be indecomposable if it cannot be written as a direct sum of two nontrivial ideals.*

Definition 3. *An endomorphism D of an R-module L is called a triple derivation of L if for all $v_1, v_2, v_3 \in L$, D satisfies the following condition:*

$$D([[v_1, v_2], v_3]) = [[D(v_1), v_2], v_3] + (-1)^{|D||v_1|}[[v_1, D(v_2)], v_3] + (-1)^{|D|(|v_1|+|v_2|)}[[v_1, v_2], D(v_3)].$$

More generally, D is called an n-derivation of L, if it satisfies the following identity.

$$\begin{aligned} D\big([\cdots[[v_1, v_2], v_3]\cdots v_n]\big) &= [\cdots[[D(v_1), v_2], v_3]\cdots v_n] \\ &+ (-1)^{|D||v_1|}[\cdots[[v_1, D(v_2)], v_3]\cdots v_n] \\ &+ (-1)^{|D|(|v_1|+|v_2|)}[\cdots[[v_1, v_2], D(v_3)]\cdots v_n] \\ &+ \cdots \\ &+ (-1)^{|D|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[\cdots[[v_1, v_2], v_3]\cdots D(v_n)] \\ &\qquad \textit{for all } \; v_1, v_2, v_3\cdots v_n \in L. \end{aligned}$$

Denote by $\mathrm{nDer}(L)$ *the set of all n-derivations of a Lie algebra L. It is straightforward to verify that* $\mathrm{nDer}(L)$ *forms a Lie algebra under the usual commutator bracket of endomorphisms of the R-module. An n-derivation of a Lie algebra generalizes the usual derivation by satisfying a Leibniz rule for the n-fold Lie bracket. This concept*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2025 doi:10.20944/preprints202509.0196.v1

parallels similar generalizations in associative and Jordan algebras. It was introduced independently in [4] by Müller, where it was referred to as a prederivation *in the specific case of triple derivations. Müller [4] proved that, if G is a Lie group equipped with a bi-invariant semi-Riemannian metric and $\mathfrak{g}$ its Lie algebra, then the Lie algebra of the group of isometries of G fixing the identity element is a subalgebra of* $\mathrm{nDer}(\mathfrak{g})$ *when* $n = 3$. *Therefore, the study of the Lie algebra of n-derivations is of interest not only from an algebraic perspective but also due to its relevance in the geometric theory of Lie groups.*

Definition 4. *Let L and L' be two Lie superalgebras over R. An even R-linear mapping $f : L \to L'$ is called:*
(a) a homomorphism if it satisfies

$$f([v_1, v_2]) = [f(v_1), f(v_2)] \text{ for all } v_1, v_2 \in L.$$

(b) an anti-homomorphism if it meets the condition

$$f([v_1, v_2]) = (-1)^{|v_1||v_2|}[f(v_2), f(v_1)] \text{ for all } v_1, v_2 \in hg(L).$$

(c) a triple homomorphism if it meets the condition

$$f([v_1, [v_2, v_3]]) = [f(v_1), [f(v_2), f(v_3)]] \text{ for all } v_1, v_2, v_3 \in L.$$

(d) a n-homomorphism if it meets the condition

$$\begin{aligned} f\big([v_1, v_2, v_3 \cdots v_n]\big) &= [f(v_1), f(v_2), f(v_3) \cdots f(v_n)] \\ &= [\cdots [[f(v_1), f(v_2)], f(v_3)] \cdots f(v_n)] \\ &\quad \text{for all } v_1, v_2, v_3 \cdots v_n \in L. \end{aligned}$$

Definition 5. *Let L and L' be Lie superalgebras. A map $g : L \to L'$ is said to be the direct sum of maps $g_1, g_2 : L \to L'$ if $g = g_1 + g_2$ and there exist ideals I_1 and I_2 of the enveloping Lie superalgebra of $g(L)$ such that $I_1 \cap I_2 = \{0\}$ with $g_1(L) \subseteq I_1$ and $g_2(L) \subseteq I_2$.*

Proposition 1. *If L is perfect, then $ad(L)$ forms an ideal of the Lie superalgebra $nDer(L)$.*

Proof. Let $D \in nDer(L), v \in hg(L)$. Then, for any $w \in L$, we have

$$\begin{aligned}
&[D, adv](w) = Dadv(w) - (-1)^{|D||v|}adv(D(w)) = D[v,w] - (-1)^{|D||v|}[v, D(w)] \\
=&D\left(\left[\sum_{i\in I}\left[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}\right], w\right]\right) \\
&-(-1)^{|D||v|}\left[\sum_{i\in I}\left[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}\right], D(w)\right] \\
=&\sum_{i\in I} D\left(\left[\left[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}\right], w\right]\right) \\
&\quad -\sum_{i\in I}(-1)^{|D||v|}\left[\left[\left[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}\right], w\right], D(w)\right] \\
=&\sum_{i\in I}\left[\left[\cdots[[D(v_{1i}), v_{2i}], v_{3i}]\cdots v_{(n-1)i}\right], w\right] \\
&\quad +\sum_{i\in I}(-1)^{|D||v_{1i}|}\left[\left[\cdots[[v_{1i}, D(v_{2i})], v_{3i}]\cdots v_{(n-1)i}\right], w\right] \\
&\quad +\sum_{i\in I}(-1)^{|D|(|v_{1i}|+|v_{2i}|)}\left[\left[\cdots[[v_{1i}, v_{2i}], D(v_{3i})]\cdots v_{(n-1)i}\right], w\right] \\
&\quad +\cdots-\sum_{i\in I}(-1)^{|D||v|}\left[\left[\cdots[[v_{1i}, v_{2i}], D(v_{3i})]\cdots v_{(n-1)i}\right], D(w)\right].
\end{aligned}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2025 doi:10.20944/preprints202509.0196.v1

Hence,

$$\begin{aligned}[D, adv](w) =& ad\Big(\sum_{i\in I}[\cdots[[D(v_{1i}), v_{2i}], v_{3i}]\cdots v_{(n-1)i}] \\ &+\sum_{i\in I}(-1)^{|D||v_{i1}|}[\cdots[[v_{1i}, D(v_{2i})], v_{3i}]\cdots v_{(n-1)i}] \\ &+\sum_{i\in I}(-1)^{|D|(|v_{i1}|+|v_{2i}|)}[\cdots[[v_{1i}, v_{2i}], D(v_{3i})]\cdots v_{(n-1)i}]+\cdots \\ &+\sum_{i\in I}(-1)^{|D|(|v_{i1}|+|v_{2i}|+|v_{i3}|+\cdots+|v_{i(n-2)}|)}[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots D(v_{(n-1)i})]\Big)(w).\end{aligned}$$

Since w is arbitrary, it follows that $[D, adv]$ is an inner derivation. Hence, $ad(L)$ is an ideal of $nDer(L)$. □

3. The Proof of the Main Results

We state and prove our first main result of this paper.

Theorem 1. *Let $n \geq 2$ be a fixed integer and let L be a Lie superalgebra over a commutative ring R. If $\frac{1}{n-1} \in R$, and L is perfect with a trivial center, then the following hold:*
(a) $nDer(L) = Der(L)$;
(b) $nDer(Der(L)) = ad(Der(L))$.

We begin the proof of our main result through the following lemmas.

Lemma 1. *[17] For any Lie superalgebra L, if $v \in L$ and $D \in Der(L)$, then $[D, adv] = ad(D(v))$.*

Lemma 2. *For any Lie superalgebra L, the set $nDer(L)$ is invariant under the standard Lie bracket operation.*

Proof. Let $D_1, D_2 \in nDer(L), v_1, v_2, v_2 \cdots v_{n-1} \in hg(L), v_n \in L$."By the definition of n-derivation, we have

$$
\begin{aligned}
&D_1D_2([[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],v_n])\\
=&D_1\big([[\cdots[[D_2(v_1),v_2],v_3]\cdots v_{n-1}],v_n]+(-1)^{|D_2||v_1|}[[\cdots[[v_1,D_2(v_2)],v_3]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_2|(|v_1|+|v_2|)}[[\cdots[[v_1,v_2],D_2(v_3)]\cdots v_{n-1}],v_n]\\
&+\cdots+(-1)^{|D_2|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],D_2(v_n)]\big)\\
=&[[\cdots[[D_1D_2(v_1),v_2],v_3]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_1|(|D_2|+|v_1|)}[[\cdots[[D_2(v_1),D_1(v_2)],v_3]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_1|(|D_2|+|v_1|+|v_2|)}[[\cdots[[D_2(v_1),v_2],D_1(v_3)]\cdots v_{n-1}],v_n]\\
&+\cdots+(-1)^{|D_1|(|D_2|+|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[D_2(v_1),v_2],v_3]\cdots v_{n-1}],D_1(v_n)]\\
&+(-1)^{|D_2||v_1|}[[\cdots[[D_1(v_1),D_2(v_2)],v_3]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_2||v_1|}(-1)^{|D_1||v_1|}[[\cdots[[v_1,D_1D_2(v_2)],v_3]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_2||v_1|}(-1)^{|D_1|(|D_2|+|v_1|+|v_2|)}[[\cdots[[v_1,D_2(v_2)],D_1(v_3)]\cdots v_{n-1}],v_n]+\cdots\\
&+(-1)^{|D_2||v_1|}(-1)^{|D_1|(|D_2|+|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[v_1,D_2(v_2)],v_3]\cdots v_{n-1}],D_1(v_n)]\\
&+(-1)^{|D_2|(|v_1|+|v_2|)}[[\cdots[[D_1(v_1),v_2],D_2(v_3)]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_2|(|v_1|+|v_2|)}(-1)^{|D_1|(|v_1|)}[[\cdots[[v_1,D_1(v_2)],D_2(v_3)]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_2|(|v_1|+|v_2|)}(-1)^{|D_1|(|v_1|+|v_2|)}[[\cdots[[v_1,v_2],D_1D_2(v_3)]\cdots v_{n-1}],v_n]+\cdots\\
&+(-1)^{|D_2|(|v_1|+|v_2|)}(-1)^{|D_1|(|D_2|+|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}\\
&[[\cdots[[v_1,v_2],D_2(v_3)]\cdots v_{n-1}],D_1(v_n)]\\
&+\cdots+(-1)^{|D_2|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[D_1(v_1),v_2],v_3]\cdots v_{n-1}],D_2(v_n)]\\
&+(-1)^{|D_2|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}(-1)^{|D_1||v_1|}[[\cdots[[v_1,D_1(v_2)],v_3]\cdots v_{n-1}],D_2(v_n)]\\
&+(-1)^{|D_2|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}(-1)^{|D_1|(|v_1|+|v_2|)}[[\cdots[[v_1,v_2],D_1(v_3)]\cdots v_{n-1}],D_2(v_n)]\\
&+\cdots+(-1)^{|D_2|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}(-1)^{|D_1|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}\\
&[[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],D_1D_2(v_n)].
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&D_2D_1([[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],v_n])\\
=&D_2\big([[\cdots[[D_1(v_1),v_2],v_3]\cdots v_{n-1}],v_n]+(-1)^{|D_1||v_1|}[[\cdots[[v_1,D_1(v_2)],v_3]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_1|(|v_1|+|v_2|)}[[\cdots[[v_1,v_2],D_1(v_3)]\cdots v_{n-1}],v_n]\\
&+\cdots+(-1)^{|D_1|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],D_1(v_n)]\big)\\
=&[[\cdots[[D_2D_1(v_1),v_2],v_3]\cdots v_{n-1}],v_n]+(-1)^{|D_2|(|D_1|+|v_1|)}\\
&[[\cdots[[D_1(v_1),D_2(v_2)],v_3]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_2|(|D_1|+|v_1|+|v_2|)}[[\cdots[[D_1(v_1),v_2],D_2(v_3)]\cdots v_{n-1}],v_n]\\
&+\cdots+(-1)^{|D_2|(|D_1|+|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[D_1(v_1),v_2],v_3]\cdots v_{n-1}],D_2(v_n)]\\
&+(-1)^{|D_1||v_1|}[[\cdots[[D_2(v_1),D_1(v_2)],v_3]\cdots v_{n-1}],v_n]
\end{aligned}
$$

 doi:10.20944/preprints202509.0196.v1

$$
\begin{aligned}
&+(-1)^{|D_1||v_1|}(-1)^{|D_2||v_1|}[[\cdots[[v_1,D_2D_1(v_2)],v_3]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_1||v_1|}(-1)^{|D_2|(|D_1|+|v_1|+|v_2|)}[[\cdots[[v_1,D_2(v_2)],D_2(v_3)]\cdots v_{n-1}],v_n]\\
&+\cdots+(-1)^{|D_1||v_1|}(-1)^{|D_2|(|D_1|+|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[v_1,D_1(v_2)],v_3]\cdots v_{n-1}],D_2(v_n)]\\
&+(-1)^{|D_1|(|v_1|+|v_2|)}[[\cdots[[D_2(v_1),v_2],D_1(v_3)]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_1|(|v_1|+|v_2|)}(-1)^{|D_2|(|v_1|)}[[\cdots[[v_1,D_2(v_2)],D_1(v_3)]\cdots v_{n-1}],v_n]\\
&+(-1)^{|D_1|(|v_1|+|v_2|)}(-1)^{|D_2|(|v_1|+|v_2|)}[[\cdots[[v_1,v_2],D_2D_1(v_3)]\cdots v_{n-1}],v_n]\cdots\\
&+(-1)^{|D_1(|v_1|+|v_2|)}(-1)^{|D_2|(|D_1|+|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[v_1,v_2],D_1(v_3)]\cdots v_{n-1}],D_2(v_n)]\\
&+\cdots\\
&+(-1)^{|D_1|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[D_2(v_1),v_2],v_3]\cdots v_{n-1}],D_1(v_n)]\\
&+(-1)^{|D_1|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}(-1)^{|D_2||v_1|}[[\cdots[[v_1,D_2(v_2)],v_3]\cdots v_{n-1}],D_1(v_n)]\\
&+(-1)^{|D_1|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}(-1)^{|D_2|(|v_1|+|v_2|)}[[\cdots[[v_1,v_2],D_2(v_3)]\cdots v_{n-1}],D_1(v_n)]\cdots\\
&+(-1)^{|D_1|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}(-1)^{|D_2|(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}\\
&[[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],D_2D_1(v_n)].
\end{aligned}
$$

By simple calculation, we obtain

$$
\begin{aligned}
&[D_1,D_2]([[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],v_n])\\
&=D_1D_2([[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],v_n])-(-1)^{|D_1||D_2|}D_2D_1[[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],v_n]\\
&=[[\cdots[[[D_1,D_2](v_1),v_2],v_3]\cdots v_{n-1}],v_n]+(-1)^{|v_1(|D_2|+|D_1|)}\\
&\quad[[\cdots[[v_1,[D_1,D_2](v_2)],v_3]\cdots v_{n-1}],v_n]\\
&\quad+(-1)^{(|v_1|+|v_1|)(|D_2|+|D_1|)}[[\cdots[[v_1,v_2],[D_1,D_2](v_3)]\cdots v_{n-1}],v_n]+\cdots\\
&\quad+(-1)^{(|D_1|+|D_2|)(|v_1|+|v_2|+|v_2|+\cdots|v_{n-1}|)}[[\cdots[[v_1,v_2],v_3]\cdots v_{n-1}],[D_1,D_2](v_n)].
\end{aligned}
$$

Hence, $[D_1,D_2]\in nDer(L)$, completing the proof of the lemma. □

It is evident that both $ad(L)$ and $Der(L)$ are subalgebras of $nDer(L)$." Since L is perfect, every element $v\in(h\circ g)(L)$ can be written as a finite sum of Lie brackets, that is, there exist a finite index set I such that

$$
v=\sum_{\substack{i\in I\\ |v_{i1}|+|v_{2i}|+|v_{i3}|\cdots+|v_{(n-1)i}|=|v|}}\left[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots v_{(n-1)i}\right],
$$

for some $v_{i1},v_{2i},v_{i3}\cdots v_{(n-1)i}\in L$. In this article, we always put $\sum$ in place of

$$
\sum_{\substack{i\in I\\ |v_{i1}|+|v_{2i}|+|v_{i3}|\cdots+|v_{(n-1)i}|=|v|}}\quad \text{for convenience.}
$$

Lemma 3. *If L is a perfect Lie superalgebra with a trivial center, then there exists an R-module homomorphism $\delta: nDer(L)\to \mathrm{End}(L)$, defined by $\delta(D)=\delta_D$ such that for all $v\in L$ and $D\in nDer(L)$, the following holds:*

$$
[D,adv]=ad\delta_D(v).
$$

Proof. In view of Lemma 1, if L is perfect and L has zero center, $D \in nDer(L)$, then we can construct a module endomorphism δ_D on L such that for any $v = \sum_{i\in I}\left[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}\right] \in hg(L)$,

$$\begin{aligned}\delta_D(v) = &\sum_{i\in I}\Big([\cdots[[D(v_{1i}), v_{2i}], v_{3i}]\cdots v_{(n-1)i}] + (-1)^{|D||v_{i1}|}[\cdots[[v_{1i}, D(v_{2i})], v_{3i}]\cdots v_{(n-1)i}] \\ &+(-1)^{|D|(|v_{i1}|+|v_{2i}|)}[\cdots[[v_{1i}, v_{2i}], D(v_{3i})]\cdots v_{(n-1)i}] + \cdots \\ &+(-1)^{|D|(|v_{i1}|+|v_{2i}|+|v_{i3}|+\cdots+|v_{i(n-2)}|)}[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots D(v_{(n-1)i})]\Big).\end{aligned}$$

In fact, the definition does not depend on the specific expression of v. To prove this, let

$$\begin{aligned}\mathbb{S} = &\sum_{i\in I}\Big([\cdots[[D(v_{1i}), v_{2i}], v_{3i}]\cdots v_{(n-1)i}] + (-1)^{|D||v_{i1}|}[\cdots[[v_{1i}, D(v_{2i})], v_{3i}]\cdots v_{(n-1)i}] \\ &+(-1)^{|D|(|v_{i1}|+|v_{2i}|)}[\cdots[[v_{1i}, v_{2i}], D(v_{3i})]\cdots v_{(n-1)i}] + \cdots \\ &+(-1)^{|D|(|v_{i1}|+|v_{2i}|+|v_{i3}|+\cdots+|v_{i(n-2)}|)}[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots D(v_{(n-1)i})]\Big).\end{aligned}$$

Next, let

$$\begin{aligned}\mathbb{M} = &\sum_{i\in I}\Big([\cdots[[D(u_{j1}), u_{j2}], u_{j3}]\cdots u_{j(n-1)}] + (-1)^{|D||u_{i1}|}[\cdots[[u_{j1}, D(u_{j2})], u_{j3}]\cdots u_{j(n-1)}] \\ &+(-1)^{|D|(|u_{i1}|+|u_{i2}|)}[\cdots[[u_{j1}, u_{j2}], D(u_{j3})]\cdots u_{j(n-1)}] + \cdots \\ &+(-1)^{|D|(|u_{i1}|+|u_{i2}|+|u_{i3}|+\cdots+|u_{i(n-2)}|)}[\cdots[[u_{j1}, u_{j2}], u_{j3}]\cdots D(u_{i(n-1)})]\Big).\end{aligned}$$

Since $D \in nDer(L)$, for all $w \in L$, we have

$$\begin{aligned}&[\mathbb{S}, w] = \\ &\sum_{i\in I}\Big(\Big[[\cdots[[D(v_{1i}), v_{2i}], v_{3i}]\cdots v_{(n-1)i}], w\Big] + (-1)^{|D||v_{i1}|}\Big[[\cdots[[v_{1i}, D(v_{2i})], v_{3i}]\cdots v_{(n-1)i}], w\Big] \\ &+(-1)^{|D|(|v_{i1}|+|v_{2i}|)}\Big[[\cdots[[v_{1i}, v_{2i}], D(v_{3i})]\cdots v_{(n-1)i}], w\Big] + \cdots \\ &+(-1)^{|D|(|v_{i1}|+|v_{2i}|+|v_{i3}|+\cdots+|v_{i(n-2)}|)}\Big[[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots D(v_{(n-1)i})], w\Big]\Big) \\ &=\sum_{i\in I}\Big(\Big(D\Big[[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}], w\Big]\Big) - (-1)^{|D||v_{i1}|}\Big[[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}], D(w)\Big] \\ &=D([v, w]) - (-1)^{|D||v|}[v, D(w)] \\ &=\sum_{i\in I}(D\Big(\Big[\Big[\cdots[[u_{j1}, u_{j2}], u_{j3}]\cdots u_{j(n-1)}\Big], w\Big]\Big) - (-1)^{|D||u|} \\ &\quad\Big[\Big[\cdots[[u_{j1}, u_{j2}], u_{j3}]\cdots u_{j(n-1)}\Big], D(w)\Big]) \\ &=\sum_{i\in I}\Big(\Big[[\cdots[[D(u_{j1}), u_{j2}], u_{j3}]\cdots u_{j(n-1)}], w\Big] \\ &+(-1)^{|D||u_{i1}|}\Big[[\cdots[[u_{j1}, D(u_{j2})], u_{j3}]\cdots u_{j(n-1)}], w\Big] \\ &+(-1)^{|D|(|u_{i1}|+|u_{i2}|)}\Big[[\cdots[[u_{j1}, u_{j2}], D(u_{j3})]\cdots u_{j(n-1)}], w\Big] + \cdots \\ &+(-1)^{|D|(|u_{i1}|+|u_{i2}|+|u_{i3}|+\cdots+|u_{i(n-2)}|)}\Big[[\cdots[[u_{j1}, u_{j2}], u_{j3}]\cdots D(u_{i(n-1)})], w\Big]\Big) \\ &=[\mathbb{M}, w].\end{aligned}$$

Thus, $[\mathbb{S}-\mathbb{M}, w]=0$, which implies that $\mathbb{S}-\mathbb{M}\in Z(L)$. Since "$Z(L)=\{0\}$, it follows that $\mathbb{S}=\mathbb{M}$. Therefore, δ_D is well-defined. The remainder of the lemma is a consequence of the proof of Proposition 1. □

Lemma 4. *If L is a perfect Lie superalgebra with a trivial center, then for every $D\in nDer(L)$, δ_D belongs to $Der(L)$.*

Proof. Suppose $D\in nDer(L)$, from Proposition 1, we have

$$v=\sum_{i\in I}\Big[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots v_{(n-1)i}\Big]\in hg(L),\ w\in L.$$

Then

$$\begin{aligned}&\Big[D, ad\Big(\Big[\sum_{i\in I}[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots v_{(n-1)i}],w\Big]\Big)\Big]\\&=ad\delta_D\Big(\Big[\sum_{i\in I}[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots v_{(n-1)i}],w\Big]\Big).\end{aligned}$$

Alternatively,

$$\begin{aligned}&\Big[D, ad\Big(\Big[\sum_{i\in I}[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots v_{(n-1)i}],w\Big]\Big)\Big]=\Big[D,\Big[\sum_{i\in I}[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots v_{(n-1)i}],w\Big]\Big]\\
&=\sum_{i\in I}\Big[\Big[\cdots[[[D,ad(v_{1i})],ad(v_{2i})],ad(v_{3i})]\cdots ad(v_{(n-1)i})\Big],ad(w)\Big]\\
&\quad+\sum_{i\in I}(-1)^{|D||v_{1i}|}\Big[\Big[\cdots[ad(v_{1i}),[D,ad(v_{2i})],ad(v_{3i})]\cdots ad(v_{(n-1)i})\Big],ad(w)\Big]\\
&\quad+\sum_{i\in I}(-1)^{|D|(|v_{1i}|+|v_{2i}|)}\Big[\Big[\cdots[ad(v_{1i}),[ad(v_{2i}),[D,ad(v_{3i})]]]\cdots ad(v_{(n-1)i})\Big],ad(w)\Big]\\
&\quad+\cdots\\
&\quad+\sum_{i\in I}(-1)^{|D||v|}\Big[\Big[\cdots[[ad(v_{1i}),ad(v_{2i})],ad(v_{3i})]\cdots v_{(n-1)i}\Big],[D,ad(w)]\Big]\\
&=\sum_{i\in I}\Big[\Big[\cdots[[ad\delta_D(v_{1i}),ad(v_{2i})],ad(v_{3i})]\cdots ad(v_{(n-1)i})\Big],ad(w)\Big]\\
&\quad+\sum_{i\in I}(-1)^{|D||v_{1i}|}\Big[\Big[\cdots[[ad(v_{1i}),ad\delta_D(v_{2i})],ad(v_{3i})]\cdots ad(v_{(n-1)i})\Big],ad(w)\Big]\\
&\quad+\sum_{i\in I}(-1)^{|D|(|v_{1i}|+|v_{2i}|)}\Big[\Big[\cdots[ad(v_{1i}),[ad(v_{2i}),ad\delta_D(v_{3i})]]\cdots ad(v_{(n-1)i})\Big],ad(w)\Big]\\
&\quad+\cdots\\
&\quad+\sum_{i\in I}(-1)^{|D||v|}\Big[\cdots[[ad(v_{1i}),ad(v_{2i})],ad(v_{3i})]\cdots[v_{(n-1)i},ad\delta_D(w)]\Big].\end{aligned}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2025 doi:10.20944/preprints202509.0196.v1

Hence,

$$
\begin{aligned}
&= ad\Bigg(\sum_{i\in I}\Big[\Big[\cdots[[\delta_D(v_{1i}),v_{2i}],v_{3i}]\cdots v_{(n-1)i}\Big],w\Big] \\
&\quad+\sum_{i\in I}(-1)^{|D||v_{1i}|}\Big[\Big[\cdots[[v_{1i},\delta_D(v_{2i})],v_{3i}]\cdots v_{(n-1)i}\Big],w\Big] \\
&\quad+\sum_{i\in I}(-1)^{|D|(|v_{1i}|+|v_{2i}|)}\Big[\Big[\cdots[v_{1i},[v_{2i},\delta_D(v_{3i})]]\cdots v_{(n-1)i}\Big],w\Big] \\
&\quad+\cdots+\sum_{i\in I}(-1)^{|D||v|}\Big[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots[v_{(n-1)i},\delta_D(w)]\Big]\Bigg) \\
&= ad\delta_D\left(\left[\sum_{i\in I}[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots v_{(n-1)i}],w\right]\right) \\
&= ad\Bigg(\sum_{i\in I}\Big[\Big[\cdots[[\delta_D(v_{1i}),v_{2i}],v_{3i}]\cdots v_{(n-1)i}\Big],w\Big] \\
&\quad+\sum_{i\in I}(-1)^{|D||v_{1i}|}\Big[\Big[\cdots[[v_{1i},\delta_D(v_{2i})],v_{3i}]\cdots v_{(n-1)i}\Big],w\Big] \\
&\quad+\sum_{i\in I}(-1)^{|D|(|v_{1i}|+|v_{2i}|)}\Big[\Big[\cdots[v_{1i},[v_{2i},\delta_D(v_{3i})]]\cdots v_{(n-1)i}\Big],w\Big] \\
&\quad+\cdots+\sum_{i\in I}(-1)^{|D||v|}\Big[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots[v_{(n-1)i},\delta_D(w)]\Big]\Bigg).
\end{aligned}
$$

Since $Z(L)=\{0\}$, it follows that

$$
\begin{aligned}
&\delta_D\left(\left[\sum_{i\in I}[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots v_{(n-1)i}],w\right]\right) \\
=&\Bigg(\sum_{i\in I}\Big[\Big[\cdots[[\delta_D(v_{1i}),v_{2i}],v_{3i}]\cdots v_{(n-1)i}\Big],w\Big] \\
&+\sum_{i\in I}(-1)^{|D||v_{1i}|}\Big[\Big[\cdots[[v_{1i},\delta_D(v_{2i})],v_{3i}]\cdots v_{(n-1)i}\Big],w\Big] \\
&+\sum_{i\in I}(-1)^{|D|(|v_{1i}|+|v_{2i}|)}\Big[\Big[\cdots[v_{1i},[v_{2i},\delta_D(v_{3i})]]\cdots v_{(n-1)i}\Big],w\Big] \\
&+\cdots \\
&+\sum_{i\in I}(-1)^{|D||v|}\Big[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots[v_{(n-1)i},\delta_D(w)]\Big]\Bigg).
\end{aligned}
$$

By the arbitrariness of $v=\sum_{i\in I}\Big[\cdots[[v_{1i},v_{2i}],v_{3i}]\cdots v_{(n-1)i}\Big], w, \delta_D \in Der(L)$. □

Lemma 5. *If the base ring R includes $\frac{1}{n-1}$ and L is perfect, then the centralizer of $ad(L)$ in $nDer(L)$ is trivial, i.e., $C_{nDer(L)}(ad(L))=\{0\}$. Consequently, the center of $nDer(L)$ is also trivial.*

Proof. Let $D \in C_{nDer L}(ad(L))$. Thus, for all $v \in L, [D, adv] = 0$. Then, for all $v, u \in hg(L)$, $D([v, u]) - (-1)^{|D||v|}[v, D(u)] = [D, adv](u) = 0$. Therefore, $D([v, u]) = [D(v), u] = (-1)^{|D||v|}[v, D(u)]$. For $v_1, v_2, v_2 \cdots v_{n-1}, v_n \in hg(L)$, we always have that

$$\begin{aligned}
&D([[\cdots[[v_1, v_2], v_3]\cdots v_{n-1}], v_n]) \\
&= (-1)^{|D|(|v_1|+|v_2|+|v_3|+\cdots|v_{n-1}|)}[[\cdots[[v_1, v_2], v_3]\cdots v_{n-1}], D(v_n)] \\
&= (-1)^{|D|(|v_1|+|v_2|+|v_3|+\cdots|v_{n-2}|)}[[\cdots[[v_1, v_2], v_3]\cdots D(v_{n-1})], v_n] \\
&= \cdots \\
&= (-1)^{|D|(|v_1|+|v_2|)}[[\cdots[[v_1, v_2], D(v_3)]\cdots v_{n-1}], v_n] \\
&= (-1)^{|D||v_1|}[[\cdots[[v_1, D(v_2)], v_3]\cdots v_{n-1}], v_n] \\
&= [[\cdots[[D(v_1), v_2], v_3]\cdots v_{n-1}], v_n].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&D([[\cdots[[v_1, v_2], v_3]\cdots v_{n-1}], v_n]) \\
&= [[\cdots[[D(v_1), v_2], v_3]\cdots v_{n-1}], v_n] \\
&+ (-1)^{|D||v_1|}[[\cdots[[v_1, D(v_2)], v_3]\cdots v_{n-1}], v_n] \\
&+ (-1)^{|D|(|v_1|+|v_2|)}[[\cdots[[v_1, v_2], D(v_3)]\cdots v_{n-1}], v_n] \\
&+ \cdots \\
&+ (-1)^{|D|(|v_1|+|v_2|+|v_3|+\cdots|v_{n-2}|)}[[\cdots[[v_1, v_2], v_3]\cdots D(v_{n-1})], v_n] \\
&+ (-1)^{|D|(|v_1|+|v_2|+|v_3|+\cdots|v_{n-1}|)}[[\cdots[[v_1, v_2], v_3]\cdots v_{n-1}], D(v_n)] \\
&= nD([[\cdots[[v_1, v_2], v_3]\cdots v_{n-1}], v_n]) \\
&= 0.
\end{aligned}$$

Hence,

$$(n-1)D([[\cdots[[v_1, v_2], v_3]\cdots v_{n-1}], v_n]) = 0.$$

Because $\frac{1}{n-1} \in R$, the above rebellious gives

$$D([[\cdots[[v_1, v_2], v_3]\cdots v_{n-1}], v_n]) = 0.$$

Since L is perfect, every element of L can be written as a linear combination of elements of the form $[[\cdots[[v_1, v_2], v_3]\cdots v_{n-1}], v_n]$. Therefore, we conclude that $D = 0$, thus completing the proof. □

Lemma 6. *If the base ring R includes* $\frac{1}{n-1}$, *and L is a perfect Lie superalgebra with a trivial center, then* $nDer(L) = Der(L)$.

Proof. Suppose $v \in L, D \in nDer(L)$. By Proposition 1, $[D, adv] = ad\delta_D(v)$. By Lemmas 4 and 1, $ad\delta_D(v) = [\delta_D, adv]$. Hence, $D - \delta_D \in C_{nDer(L)}(ad(L))$. By Lemma 5, $D - \delta_D = 0$, i.e., $D = \delta_D \in Der(L)$. Hence, $nDer(L) \subseteq Der(L)$. The lemma follows from Lemma 4. □

Observe that Lemma 3.7 proves the first first of Theorem 1. Now Next, our aims to prove the second part of Theorem 1.

Lemma 7. *If L is a perfect Lie superalgebra and* $D \in nDer(Der(L))$, *then* $D(ad(L))$ *is contained within* $ad(L)$.

Proof. Since L is perfect, we have

$$\begin{aligned}
D(adv) =& \sum_{i\in I} D([\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{ni}]) \\
=& \sum_{i\in I} D([\cdots[[ad(v_{1i}), v_{2i}], ad(v_{3i})]\cdots ad(v_{ni})]) \\
=& \sum_{i\in I} [\cdots[[D(ad(v_{1i})), ad(v_{2i})], ad(v_{3i})]\cdots ad(v_{ni})] \\
&+ \sum_{i\in I} (-1)^{|D||v_{1i}|} [\cdots[[ad(v_{1i}), D(ad(v_{2i}))], ad(v_{3i})]\cdots ad(v_{ni})] \\
&+ \sum_{i\in I} (-1)^{|D|(|v_{1i}|+|v_{2i}|)} [\cdots[ad(v_{1i}), [ad(v_{2i}), D(ad(v_{3i}))]]\cdots ad(v_{ni})] \\
&+ \cdots + \sum_{i\in I} (-1)^{|D|(|v_1|+|v_2|+|v_3|+\cdots+|v_{N-1}|)} \\
&\Big[\cdots[[ad(v_{1i}), ad(v_{2i})], ad(v_{3i})]\cdots[v_{(n-1)i}, D(ad(v_{ni}))]\Big].
\end{aligned}$$

Hence, $D(adv) \in ad(L)$. The lemma holds thanks to Proposition 1. $\square$

Lemma 8. *Assume that R is the base ring containing $\frac{1}{n-1}$, L is a perfect Lie superalgebra with a trivial center, and $D \in nDer(Der(L))$. If $D(ad(L)) = 0$, then it follows that $D = 0$.*

Proof. For all $d \in Der(L), v \in hg(L)$, since L is perfect,

$$v = \sum_{i\in I}\Big[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}\Big].$$

We have that

$$\begin{aligned}
[adv, D(d)] =& \left[\sum_{i\in I}\Big[\cdots[[ad(v_{1i}), ad(v_{2i})], ad(v_{3i})]\cdots ad(v_{(n-1)i})\Big], D(d)\right] \\
=& \sum_{i\in I}\Big((-1)^{|D||v|} D\Big(\Big[[\cdots[[ad(v_{1i}), ad(v_{2i})], ad(v_{3i})]\cdots ad(v_{(n-1)i})], d\Big]\Big) \\
&-(-1)^{|D||v|}\Big[[\cdots[[D(ad(v_{1i})), ad(v_{2i})], ad(v_{3i})]\cdots ad(v_{(n-1)i})], d\Big] \\
&-(-1)^{|D||v|}(-1)^{|D||v_{i1}|}\Big[[\cdots[[ad(v_{1i}), D(ad(v_{2i}))], ad(v_{3i})]\cdots ad(v_{(n-1)i})], d\Big] \\
&-(-1)^{|D||v|}(-1)^{|D|(|v_{i1}|+|v_{2i}|}\Big[[\cdots[[ad(v_{1i}), ad(v_{2i})], D(ad(v_{3i}))]\cdots ad(v_{(n-1)i})], d\Big] \\
&-\cdots-(-1)^{|D||v|}(-1)^{|D|(|v_{i1}|+|v_{2i}|+|v_{i3}|+\cdots+|v_{i(N-2)}|} \\
&\Big[[\cdots[[ad(v_{1i}), ad(v_{2i})], ad(v_{3i})]\cdots D(ad(v_{(n-1)i}))], d\Big]\Big).
\end{aligned}$$

By Proposition 1, $[adv, d] \in ad(L)$, so $D([adv, d]) = 0$. Hence, $[adv, D(d)] = 0$. Therefore, $D(d) \in C_{\mathrm{nDer}(L)}(ad(L))$. Hence by Lemma 5, $D(d) = 0$. Hence, $D = 0$. The lemma holds. $\square$

Lemma 9. *Let L be a Lie superalgebra over a commutative ring R. Suppose that $\frac{1}{n-1} \in R$, and that L is perfect with a trivial center. If $D \in nDer(Der(L))$, then there exists an element $d \in Der(L)$ such that $D(adv) = ad(d(v))$ for all $v \in L$.*

Proof. For all $D \in nDer(Der(L)), v \in L$, by Lemma 7, $D(adv) \in ad(L)$. Let $u \in L$ and $D(adv) = adu$. Since the center $Z(L)$ is trivial, such u is unique. Clearly, the map $d : L \to L$ given by $d(v) = u$ is an R-module endomorphism of L. Let $v_1, v_2, v_2 \cdots v_{n-1} \in hg(L), v_n \in L$. We have

$$
\begin{aligned}
&ad([\cdots[[v_1, v_2], v_3]\cdots v_n]) = D(ad([\cdots[[v_1, v_2], v_3]\cdots v_n]))\\
=&D([\cdots[[ad(v_1), ad(v_2)], ad(v_3)]\cdots ad(v_n)])\\
=&[\cdots[[D(ad(v_1)), ad(v_2)], ad(v_3)]\cdots ad(v_n)]\\
&+(-1)^{|D||v_{i1}|}[\cdots[[ad(v_1), D(ad(v_2))], ad(v_3)]\cdots ad(v_n)]\\
&+(-1)^{|D|(|v_{i1}|+|v_{2i}|}[\cdots[[ad(v_1), ad(v_2)], D(ad(v_3))]\cdots ad(v_n)]\\
&+\cdots\\
&+(-1)^{|D|(|v_{i1}|+|v_{2i}|+|v_{i3}|+\cdots+|v_{(n-1)i}|)}[\cdots[[ad(v_1), ad(v_2)], ad(v_3)]\cdots D(ad(v_n))]\\
=&ad([\cdots[[d(v_1), v_2], v_3]\cdots v_n] + (-1)^{|D||v_{i1}|}[\cdots[[v_1, d(v_2)], v_3]\cdots v_n]\\
&+(-1)^{|D|(|v_{i1}|+|v_{2i}|}[\cdots[[v_1, v_2], d(v_3)]\cdots v_n]\\
&+\cdots\\
&+(-1)^{|D|(|v_{i1}|+|v_{2i}|+|v_{i3}|+\cdots+|v_{(n-1)i}|)}[\cdots[[v_1, v_2], v_3]\cdots d(v_n)]).
\end{aligned}
$$

Since $Z(L) = \{0\}$,

$$
\begin{aligned}
d([\cdots[[v_1, v_2], v_3]\cdots v_n]) =&[\cdots[[d(v_1), v_2], v_3]\cdots v_n](-1)^{|D||v_{i1}|}[\cdots[[v_1, d(v_2)], v_3]\cdots v_n]\\
&+(-1)^{|D|(|v_{i1}|+|v_{2i}|}[\cdots[[v_1, v_2], d(v_3)]\cdots v_n]\\
&+\cdots\\
&+(-1)^{|D|(|v_{i1}|+|v_{2i}|+|v_{i3}|+\cdots+|v_{(n-1)i}|)}[\cdots[[v_{1i}, v_2], v_3]\cdots d(v_n)].
\end{aligned}
$$

That is, to say, $d \in nDer(L)$. By Lemma 6, $d \in Der(L)$.

Proof of Theorem 1 By Lemma 6, it remains only to prove the second assertion. By Lemma 9, for all $D \in (Der(L)), v \in L$," there exists $d \in Der(L)$ such that for all $v \in L, D(adv) = ad(d(v))$. Using Lemma 1, $ad(d(v)) = [d, adv]$.

Hence,

$$D(adv) = ad(d(v)) = [d, adv] = ad(d)(adv).$$

Thus,

$$(D - ad(d))(adv) = 0.$$

By Lemma 8, $D = ad(d)$. Therefore, $(Der(L)) = ad(Der(L))$. The theorem holds. □

Remark 1. *[8] The condition $\frac{1}{n-1} \in R$ is necessary. For example, if the base ring is field F of characteristic 2 and L is not abelian, then the identity map is a n-derivation but not a derivation.*

Consider the Lie superalgebras L and L' over the commutative ring R. Assume that M is the enveloping Lie superalgebra of $f(L)$ and that f is a n-homomorphism from L to L'. It may be represented as a direct sum of indecomposable ideals and is assumed that L is perfect and M is centerless.

The second main result of this paper the following theorem.

Theorem 2. *Let R be a commutative ring with unity, and assume that 2 is invertible in R. Let L and L' be Lie superalgebras over R, with f being an n-homomorphism from L to L', and let M represent the enveloping Lie superalgebra of $f(L)$. Then, the following hold:*
(a) L is perfect;

(b) M is centerless and can be decomposed into a direct sum of indecomposable ideals. In this case, f is either a homomorphism, an anti-homomorphism, or a direct sum of both a homomorphism and an anti-homomorphism.

Now, we prove the above mentioned result through sequence of lemmas.

Lemma 10. *There exists an even R-linear mapping $\delta_f : L \to L'$ such that for all $v \in hg(L)$ with*

$$v = \sum_{\substack{i\in I \\ \|v_{1i}|+|v_{2i}|+|v_{3i}|\cdots+|v_{(n-1)i}|=|v|}} \left[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}\right], \left(v_{1i}, v_{2i}, v_{3i}\cdots v_{(n-1)i} \in L\right),$$

and

$$\delta_f(v) = \sum_{\substack{i\in I \\ \|v_{1i}|+|v_{2i}|+|v_{3i}|\cdots+|v_{(n-1)i}|=|v|}} \left[\cdots[[f(v_{1i}), f(v_{2i})], f(v_{3i})]\cdots f(v_{(n-1)i})\right]$$

for all $v \in hg(L)$.

Proof. It is sufficient to prove that $\sum[f(v_{i1}), f(v_{i2})]$ is independent of the expression of v. Suppose that $\left[f(v_{1i}), [f(v_{2i}), [f(v_{3i})\cdots[f(v_{i(N-3)})[f(v_{i(N-2)}), f(v_{(n-1)i})]]]]\cdots\right]$.
Assume that

$$v = \sum_{i\in I}\left[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}\right] \in L,$$

and

$$\delta_D(v) = \sum_{i\in I}\left(\left[\cdots[[f(v_{1i}), f(v_{2i})], f(v_{3i})]\cdots f(v_{(n-1)i})\right]\right).$$

In actuality, the definition is unaffected by how v is expressed. To demonstrate it, let

$$\mathbb{S} = \sum_{i\in I}\left(\left[\cdots[[f(v_{1i}), f(v_{2i})], f(v_{3i})]\cdots f(v_{(n-1)i})\right]\right).$$

Let $v = \sum_{i\in I}\left(\left[\cdots[[u_{i1}, u_{i2}], u_{i3}]\cdots u_{i(n-1)}\right]\right)$ and let

$$\mathbb{M} = \sum_{i\in I}\left(\left[\cdots[[f(u_{i1}), f(u_{i2})], f(u_{i3})]\cdots f(u_{i(n-1)})\right]\right).$$

Then, for all $w \in L$, we have

$$\begin{aligned}
[f(w), \mathbb{S} - \mathbb{M}] &= \left[f(w), \sum_{i\in I}\left[\cdots[[f(v_{1i}), f(v_{2i})], f(v_{3i})]\cdots f(v_{(n-1)i})\right]\right. \\
&\quad \left. - \sum_{i\in I}\left[f(w), [\cdots[[f(u_{i1}), f(u_{i2})], f(u_{i3})]\cdots f(u_{i(n-1)})]\right]\right] \\
&= \sum_{i\in I}\left[f(w), [\cdots[[f(v_{1i}), f(v_{2i})], f(v_{3i})]\cdots f(v_{(n-1)i})]\right] \\
&\quad - \sum_{i\in I}\left[f(w), [\cdots[[f(u_{i1}), f(u_{i2})], f(u_{i3})]\cdots f(u_{i(n-1)})]\right] \\
&= f\left(\left[w, \sum_{i\in I}\left[\cdots[[v_{1i}, v_{2i}], v_{3i}]\cdots v_{(n-1)i}\right]\right]\right. \\
&\quad \left. - \left[w, \sum_{i\in I}\left[\cdots[[u_{i1}, u_{i2}], u_{i3}]\cdots u_{i(n-1)}\right]\right]\right) \\
&= f([w, v] - [w, v]) = 0.
\end{aligned}$$

It follows $\mathbb{S} - \mathbb{M} \in Z(M)$ and hence $\mathbb{S} = \mathbb{M}$, since M is centerless. This completes the proof. □

Lemma 11. *Let δ_f be the mapping in Lemma 10. Then, for all $v \in L$, we have that $fadv = ad\delta_f(v)f$.*

Proof. Let $v = \sum_{i \in I}\left[\cdots[[v_{1i}, v_{2i}], v_{3i}] \cdots v_{(n-1)i}\right] \in L$. Then, we have

$$\begin{aligned} fadv(w) = f([v,w]) &= \sum f\left(\left[\left[\cdots[[v_{1i}, v_{2i}], v_{3i}] \cdots v_{(n-1)i}\right], w\right]\right) \\ &= \sum\left[\left[\cdots[[f(v_{1i}), f(v_{2i})], f(v_{3i})] \cdots f(v_{(n-1)i})\right], f(w)\right] \\ &= \left[\sum\left[\cdots[[f(v_{1i}), f(v_{2i})], f(v_{3i})] \cdots f(v_{(n-1)i})\right], f(w)\right] \\ &= \left[\delta_f(v), f(w)\right] = ad\delta_f(v)f(w). \end{aligned}$$

Thus $fadv = ad\delta_f(v)f$ for all $v \in L$ and the lemma holds. □

Lemma 12. *The mapping δ_f is a homomorphism of Lie superalgebras.*

Proof. For all $v, u \in hg(L)$, $w \in L$, it follows from Lemmas 10 and 11 that

$$\begin{aligned} &\left[\delta_f([v,u]) - \left[\delta_f(v), \delta_f(u)\right], f(w)\right] \\ &= \left[\delta_f([v,u]), f(w)\right] - \left[\delta_f(v), \left[\delta_f(u), f(w)\right]\right] + (-1)^{|v||u|}\left[\delta_f(u), \left[\delta_f(v), f(w)\right]\right] \\ &= [[f(v), f(u)], f(w)] - \left[\delta_f(v), ad\delta_f(u)f(w)\right] + (-1)^{|v||u|}\left[\delta_f(u), ad\delta_f(v)f(w)\right] \\ &= [[f(v), f(u)], f(w)] - \left[\delta_f(v)f([y,w])\right] + (-1)^{|v||u|}\left[\delta_f(u), f([v,w])\right] \\ &= [[f(v), f(u)], f(w)] - ad\delta_f(v)f([y,w]) + (-1)^{|v||u|}ad\delta_f(u)f([v,w]) \end{aligned}$$

$$\begin{aligned} &= f([[v,u],w]) - f([v,[y,w]]) + (-1)^{|v||u|}f([y,[v,w]]) \\ &= f\left([[v,u],w] - [v,[y,w]] + (-1)^{|v||u|}[y,[v,w]]\right) \\ &= 0. \end{aligned}$$

By the Jacobi identity, for arbitrary $w \in L$, we have $[\delta_f([v,u]) - [\delta_f(v), \delta_f(u)],\ f(w)] = 0$. Since M is the enveloping Lie superalgebra of $f(L)$, it follows that $\delta_f([v,u]) - [\delta_f(v), \delta_f(u)] \in Z(M)$. Because M is centerless, $\delta_f([v,u]) = [\delta_f(v), \delta_f(u)]$. As $v, u \in L$ were arbitrary, the lemma follows. □

Lemma 13. *Denote $\mathbb{Z}^+ = \mathrm{Im}(f + \delta_f), \mathbb{Z}^- = \mathrm{Im}\, t(f - \delta_f)$. Then, $\mathbb{Z}^+$and $\mathbb{Z}^-$are both ideals of M.*

Proof. It is clear that $\mathbb{Z}^+, \mathbb{Z}^- \subseteq M$. For any $v, u \in L$, by Lemma 11, we have

$$\begin{aligned} \left[f(v) - \delta_f(v), f(u)\right] &= [f(v), f(u)] - [\delta_f(v), f(u)] \\ &= [f(v), f(u)] - ad_{\delta_f(v)}f(u) \\ &= \delta_f([v,u]) - \mathrm{fad}_v(u) \\ &= \delta_f([v,u]) - f([v,u]) \\ &= \left(f - \delta_f\right)([v,u]) \in \mathbb{Z}^-. \end{aligned}$$

Hence, $\mathbb{Z}^-$is an ideal of M. Similarly, $\mathbb{Z}^+$is an ideal of M. □

Lemma 14. *Prove that $[\mathbb{Z}^+, \mathbb{Z}^-] = \{0\}$.*

Proof. Take $v, u, w \in hg(L)$, by Lemmas 10, 11, 12 and 13, we have that

$$\begin{aligned}
&\left[\left[f(v)+\delta_f(v), f(u)-\delta_f(u)\right], f(w)\right] \\
&= [[f(v), f(u)], f(w)] - \left[\left[f(v), \delta_f(u)\right], f(w)\right] \\
&+ \left[\left[\delta_f(v), f(u)\right], f(w)\right] - \left[\left[\delta_f(v), \delta_f(u)\right], f(w)\right] \\
&= f([[v,u],w]) + (-1)^{|v||u|}\left[ad\delta_f(u)f(v), f(w)\right] + \left[ad\delta_f(v)f(u), f(w)\right] \\
&- \left[\delta_f(v), \left[\delta_f(u), f(w)\right]\right] + (-1)^{|v||u|}\left[\delta_f(u), ad\delta_f(v)f(w)\right] \\
&= f([[v,u],w]) + (-1)^{|v||u|}[f([y,v]), f(w)] + [f([v,u]), f(w)] \\
&- \left[\delta_f(v), ad\delta_f(u)f(w)\right] + (-1)^{|v||u|}\left[\delta_f(u), f([v,w])\right] \\
&= f([[v,u],w]) + (-1)^{|v||u|}[f([y,v]), f(w)] + [f([v,u]), f(w)] \\
&- \left[\delta_f(v), f([y,w])\right] - (-1)^{|v||u|}(-1)^{|u|(|v|+|w|}\left[f([v,w]), \delta_f(u)\right] \\
&= f([[v,u],w]) - \left[\delta_f(v), f([y,w])\right] - (-1)^{|u|w}\left[f([v,w]), \delta_f(u)\right] \\
&= f([[v,u],w]) - f([v,[y,w]]) + (-1)^{|u|w}(-1)^{|u|(|v|+|w|)} f([y,[v,w]]) \\
&= f\left([[v,u],w] - [v,[y,w]] + (-1)^{|v||u|}[y,[v,w]]\right) = 0.
\end{aligned}$$

Therefore, $\left[f(v)+\delta_f(v), f(u)-\delta_f(u)\right] \in Z(M)$. Since $Z(M) = \{0\}$, we have that $\left[f(v)+\delta_f(v), f(u)-\delta_f(u)\right] = 0$. The lemma follows. □

Lemma 15. *Prove that* $\mathbb{Z}^+ \cap \mathbb{Z}^- = \{0\}$.

Proof. Let $z \in \mathbb{Z}^+ \cap \mathbb{Z}^-$. By Lemma 14, $[\mathbb{Z}^+, w] = [\mathbb{Z}^-, w] = 0$. Hence, for any $w \in L$,

$$\left[f(v)+\delta_f(v), w\right] = \left[f(v)-\delta_f(v), w\right] = 0$$

Under the assumption $\frac{1}{n-1} \in R$, we have $[f(v), w] = 0$. Since M is the enveloping Lie algebra of $f(L)$ and v is arbitrary, it follows that $w \in Z(M)$. As $Z(M) = \{0\}$, we conclude that $w = 0$. Hence, the lemma follows. □

Lemma 16. *If M cannot be decomposed into a direct sum of two nontrivial ideals, then f is either a homomorphism or an anti-homomorphism of Lie superalgebras.*

Proof. For every $v \in L$, define $\mathbb{Z}^+ = \frac{1}{n-1}(f(v)+\delta_f(v))$ and $\mathbb{Z}^- = \frac{1}{n-1}(f(v)-\delta_f(v))$. It follows that $\mathbb{Z}^+ \in \mathbb{Z}^+$ and $\mathbb{Z}^- \in \mathbb{Z}^-$, and that $f(v) = \mathbb{Z}^+ + \mathbb{Z}^-$. This implies that $f(L) \subseteq \mathbb{Z}^+ + \mathbb{Z}^-$, and consequently, $M \subseteq \mathbb{Z}^+ + \mathbb{Z}^-$. By Lemma 15, we know that $M = \mathbb{Z}^+ \oplus \mathbb{Z}^-$. Since M cannot be written as a direct sum of two nontrivial ideals, exactly one of $\mathbb{Z}^+$ or $\mathbb{Z}^-$ must be trivial. If $\mathbb{Z}^+$ is trivial (that is, $(f+\delta_f)([v,u]) = 0$), then $f([v,u]) = -\delta_f([v,u]) = -[f(v), f(u)] = (-1)^{|v||u|}[f(u), f(v)]$, so f is an anti-homomorphism. Conversely, if $\mathbb{Z}^-$ is trivial (that is, $(f-\delta_f)([v,u]) = 0$), then $f([v,u]) = \delta_f([v,u]) = [f(v), f(u)]$, and thus f is a homomorphism. This completes the proof of the lemma.

Proof of Theorem 2. According to Lemma 16, it is sufficient to prove the theorem when M is decomposable. Given the assumptions, we can express M as the sum

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_{i's}.$$

Each M_i is an indecomposable ideal of M. Since $Z(M) = \{0\}$, so Lemma 10 in [18] implies that every M_i is also centerless. Let $p_i : M \to M_i$ denote the canonical projection. Then, $f = \sum_{i=1}^{s} p_i f$, and each $p_i f : L \to M_i$ is a triple homomorphism, with M_i the enveloping Lie superalgebra of $p_i f(L)$ for $i = 1, 2, \dots, s$. Because each M_i is indecomposable, Lemma 3.7 yields that $p_i f$ is either a homomorphism or an anti-homomorphism from L to M_i. Let $P = \{ i \mid p_i f \text{ is a homomorphism}\}$, and let Q be the complementary set of P within $\{1, 2, \dots, s\}$. Define $M_1 = \sum_{i \in P} M_i$, $M_2 = \sum_{i \in Q} M_i$. Let $f_1 = \sum_{i \in P} p_i f$ and $f_2 = \sum_{i \in Q} p_i f$. By direct verification, we can check that $M = M_1 \oplus M_2$ and that $[M_1, M_2] = 0$. Moreover, $f = f_1 + f_2$, where f_1 is a homomorphism and f_2 is an anti-homomorphism of Lie superalgebras. This completes the proof. □

4. Conclusions

These findings significantly enhance our understanding of n-derivations and n-homomorphisms on perfect Lie superalgebras over a commutative ring R. The results demonstrate that under specific conditions, such as the base ring containing $\frac{1}{n-1}$ and the center of L being zero, n-derivations of L coincide with standard derivations, and every n-derivation of the derivation algebra $Der(L)$ is shown to be inner. Furthermore, the extension of the concept of n-homomorphisms to mappings between Lie superalgebras L and L' reveals that, under certain assumptions, homomorphisms, anti-homomorphisms, and their combinations all qualify as n-homomorphisms. These results contribute to the advancement of the theory of derivations and homomorphisms in non-associative algebras, paving the way for further exploration of their applications in broader algebraic structures.

5. Future Research Problem

The results of this study are established under specific structural and algebraic assumptions-namely, that the Lie superalgebra L is perfect with a trivial center, and that the base ring R contains $\frac{1}{n-1}$. A natural future direction is to extend the analysis of n-derivations to more general classes of Lie superalgebras, such as those that are not perfect or that have a nontrivial center. Additionally, investigating the behavior of n-derivations over rings that do not contain $\frac{1}{n-1}$ could reveal whether this condition is essential or if the main results can be generalized. Another promising direction lies in studying n-homomorphisms between non-isomorphic or structurally different Lie superalgebras, or in exploring their categorical or cohomological interpretations. Expanding the framework to include color Lie superalgebras, graded Lie algebras, or infinite-dimensional cases may also offer deeper insights and broader applicability.

Author Contributions: All authors made equal contributions.

Data Availability Statement: No data were used to support the findings of this study.

Acknowledgments: The authors extend their appreciation to Princess Nourah Bint Abdulrahman University (PNU), Riyadh, Saudi Arabia, for funding this research under Researchers Supporting Project Number (PNURSP2025R231).

Conflicts of Interest: The author declares no conflict of interest in this paper.

References

1. K.I. Beidar and M.A. Chebotar, On Lie derivations of Lie ideals of prime algebras, *Israel J. Math.*, **123** (2001), 131–148.
2. I.N. Herstein and L. Jordan, Structure in simple associative rings, *Bull. Amer. Math. Soc.*, **67** (1961), 517–531.
3. G.A. Swain, Lie derivations of the skew elements of prime rings with involution, *J. Algebra*, **184** (1996), 679–704.
4. D. Müller, Isometries of bi-invariant pseudo-Riemannian metrics on Lie groups, *Geom. Dedicata*, **29** (1989), 65–96.
5. C.R. Miers, Lie n-derivations of von Neumann algebras, *Proc. Amer. Math. Soc.*, **71** (1978), 57–61.
6. P.S. Ji and L. Wang, Lie triple derivations of TUHF algebras, *Linear Algebra Appl.*, **403** (2005), 399–408.

7. J.H. Zhang, B.W. Wu, and H.X. Cao, Lie triple derivations of nest algebras, *Linear Algebra Appl.*, **416** (2006), 559–567.
8. J.H. Zhou, Triple derivations of perfect Lie algebras, *Comm. Algebra*, **41** (2013), 1647–1654.
9. M. Brešar, Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings, *Trans. Amer. Math. Soc.*, **335** (1993), 525–546.
10. N. Jacobson and C.E. Rickart, Jordan homomorphisms of rings, *Trans. Amer. Math. Soc.*, **69** (1950), 479–502.
11. C.R. Miers, Lie homomorphisms of operator algebras, *Pacific J. Math.*, **38** (1971), 717–737.
12. C.R. Miers, Lie*-triple homomorphisms into von Neumann algebras, *Proc. Amer. Math. Soc.*, **58** (1976), 169–172.
13. J.H. Zhou, Triple homomorphisms of perfect Lie algebras, *Comm. Algebra*, **42** (2014), 3724–3730.
14. H. Lian and C. Chen, n-Derivations for finitely generated graded Lie algebras, *Algebra Colloq.*, **23**(2) (2016), 205–212.
15. Y. Li and S. Guo, n-Derivations of Lie color algebras, *arXiv preprint*, arXiv:1903.05827 (2019).
16. J. Zhou, L. Chen, and Y. Ma, Triple derivations and triple homomorphisms of perfect Lie superalgebras, *Indag. Math. (N.S.)*, **28**(2) (2017), 436–445.
17. V.G. Kac, Lie superalgebras, *Adv. Math.*, **26** (1977), 8–96.
18. J.H. Chun and J.S. Lee, On complete Lie superalgebras, *Commun. Korean Math. Soc.*, **11** (1996), 323–334.
19. C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, *Trans. Amer. Math. Soc.*, **63** (1948), 85–124.