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Not peer-reviewed version

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Posted Date: 7 August 2025

doi: 10.20944/preprints202508.0469.v1

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Short Note

Banach Equiangularity: Gerzon Universal and van Lint-Seidel Relative Bounds

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Abstract

We introduce the notion of equiangular lines in Banach spaces. We show that Gerzon universal and van Lint-Seidel relative bounds hold.

Keywords: equiangular lines

MSC: Mathematics Subject Classification (2020): 52A40; 05C50

1. Introduction

Let $d \in \mathbb{N}$ and $\gamma \in [0, 1]$. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Recall that a collection $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{K}^d$ is said to be γ -**equiangular lines** [1,2] if

$$|\langle \tau_j, \tau_k \rangle| = \gamma, \quad \forall 1 \leq j, k \leq n, j \neq k.$$

A fundamental problem associated with equiangular lines is the following.

Problem 1.1. *Given $d \in \mathbb{N}$ and $\gamma \in [0, 1]$, what is the upper bound on n such that there exists a collection $\{\tau_j\}_{j=1}^n$ of γ -equiangular lines in $\mathbb{K}^d$?*

Two answers to Problem (1.1) which are driving forces in the study of equiangular lines are the following universal bound of Gerzon [3] and the relative bound of van Lint and Seidel [2,4].

Theorem 1.2 ([3] (Gerzon Universal Bound)). *Let $\{\tau_j\}_{j=1}^n$ be γ -equiangular lines in $\mathbb{K}^d$.*

(i) *If $\mathbb{K} = \mathbb{R}$, then*

$$n \leq \frac{d(d+1)}{2}.$$

(ii) *If $\mathbb{K} = \mathbb{C}$, then*

$$n \leq d^2.$$

Theorem 1.3 ([2,4] (van Lint-Seidel Relative Bound)). *Let $\{\tau_j\}_{j=1}^n$ be γ -equiangular lines in $\mathbb{K}^d$. Then*

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1 - \gamma^2)}{1 - d\gamma^2}.$$

In this note, we introduce the notion of equiangular lines for Banach spaces using L-semi-inner product and derive Theorems 1.2 and 1.3 for these spaces.

2. Banach Equiangular Lines

Our fundamental motivation is to introduce equiangularity in most famous Lebesgue spaces. On the other hand, we need generalization of inner products. Therefore, L-semi-inner products, introduced by Lumer serves our purpose.

Definition 2.1 ([5]). A vector space $\mathcal{X}$ over $\mathbb{K}$ is said to be a L-semi-inner product space if there is a map (called as L-semi-inner product) $[\cdot, \cdot] : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{K}$ satisfying following conditions.

- (i) $[x, x] \geq 0$ for all $x \in \mathcal{X}$. If $x \in \mathcal{X}$ is such that $[x, x] = 0$, then $x = 0$.
- (ii) $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in \mathcal{X}$.
- (iii) $[\lambda x, y] = \lambda[x, y]$ for all $x, y \in \mathcal{X}$, for all $\lambda \in \mathbb{K}$.
- (iv) $|[x, y]|^2 \leq [x, x][y, y]$ for all $x, y \in \mathcal{X}$.

We put forward following notion of equiangular lines in Banach spaces.

Definition 2.2. Let $\gamma \in \mathbb{C}, |\gamma| \leq 1$. Let $\mathcal{X}$ be a L-semi-inner product space. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{X}$ is said to be **Banach γ -equiangular** if following conditions hold.

- (i) $\|\tau_j\| = 1$ for all $1 \leq j \leq n$.
- (ii) $[\tau_j, \tau_k][\tau_k, \tau_j] = \gamma^2$ for all $1 \leq j, k \leq n, j \neq k$.

Theorem 2.3 ((Banach-Gerzon Universal Bound)). Let $\{\tau_j\}_{j=1}^n$ be Banach γ -equiangular lines in a L-semi-inner product space $\mathcal{X}$ of dimension d . If $\gamma \neq 1$, then

$$n \leq d^2.$$

Proof. For $1 \leq j \leq n$, define

$$\tau_j \otimes \tau_j : \mathcal{X} \ni x \mapsto (\tau_j \otimes \tau_j)(x) := [x, \tau_j]\tau_j \in \mathcal{X}.$$

We show that the collection $\{\tau_j \otimes \tau_j\}_{j=1}^n$ is linearly independent. Let $c_1, \dots, c_n \in \mathbb{K}$ be such that

$$\sum_{j=1}^n c_j(\tau_j \otimes \tau_j) = 0.$$

Let $1 \leq k \leq n$ be fixed. Then previous equation gives

$$\sum_{j=1}^n c_j(\tau_j \otimes \tau_j)(\tau_k \otimes \tau_k) = 0.$$

By taking trace we get

$$\begin{aligned} 0 &= \sum_{j=1}^n c_j \text{Tra}((\tau_j \otimes \tau_j)(\tau_k \otimes \tau_k)) = \sum_{j=1}^n c_j [\tau_j, \tau_k][\tau_k, \tau_j] \\ &= \sum_{j=1, j \neq k}^n c_j [\tau_j, \tau_k][\tau_k, \tau_j] + c_k \|\tau_k\|^2 = \sum_{j=1, j \neq k}^n c_j \gamma^2 + c_k \\ &= \gamma^2 \left(\sum_{j=1}^n c_j - c_k \right) + c_k = \gamma^2 \left(\sum_{j=1}^n c_j \right) + (1 - \gamma^2)c_k. \end{aligned}$$

Therefore

$$c_k = \frac{\gamma^2}{\gamma^2 - 1} \sum_{j=1}^n c_j =: c, \quad \forall 1 \leq k \leq n.$$

Now

$$\begin{aligned} 0 &= \text{Tra} \left(\sum_{j=1}^n c_j (\tau_j \otimes \tau_j) \right) = \text{Tra} \left(\sum_{j=1}^n c (\tau_j \otimes \tau_j) \right) \\ &= \sum_{j=1}^n c \text{Tra}(\tau_j \otimes \tau_j) = \sum_{j=1}^n c [\tau_j, \tau_j] = cn. \end{aligned}$$

Hence $c = 0$. Therefore $\{\tau_j \otimes \tau_j\}_{j=1}^n$ is linearly independent. Since the dimension of set of all linear maps from $\mathcal{X}$ to itself is d^2 we must have $n \leq d^2$. $\square$

Now we want to derive van Lint-Seidel relative bound for Banach spaces. We are unable to derive it for exactly Banach equiangular lines. We derive it for a subclass which we define as follows.

Definition 2.4. Let $\{\tau_j\}_{j=1}^n$ be Banach γ -equiangular lines in a L -semi-inner product space $\mathcal{X}$ of dimension d . We say that $\{\tau_j\}_{j=1}^n$ is **diagonalizable** Banach γ -equiangular lines if the operator

$$S_\tau : \mathcal{X} \ni x \mapsto \sum_{j=1}^n [x, \tau_j] \tau_j \in \mathcal{X}$$

is similar (through invertible operator) to a diagonal operator and all its eigenvalues are real.

Note that for Hilbert spaces, equiangular lines are diagonalizable as the operator in Definition 2.4 is positive and hence diagonalizable with all eigenvalues non negative. We cannot ensure this in L -semi-inner product spaces.

Theorem 2.5 ((Banach-van Lint-Seidel Relative Bound)). Let $\{\tau_j\}_{j=1}^n$ be Banach γ -equiangular lines in a L -semi-inner product space $\mathcal{X}$ of dimension d . Then

$$n(1 - d|\gamma|^2) \leq d(1 - |\gamma|^2).$$

In particular, if

$$|\gamma| < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1 - |\gamma|^2)}{1 - d|\gamma|^2}.$$

Proof. We first see that

$$\begin{aligned}\text{Tra}(S_\tau) &= \sum_{j=1}^n [\tau_j, \tau_j] = n, \\ \text{Tra}(S_\tau^2) &= \sum_{j=1}^n \sum_{k=1}^n [\tau_j, \tau_k][\tau_k, \tau_j] = \sum_{j=1}^n \|\tau_j\|^4 + \sum_{j,k=1, j \neq k}^n [\tau_j, \tau_k][\tau_k, \tau_j] = n + \sum_{j,k=1, j \neq k}^n [\tau_j, \tau_k][\tau_k, \tau_j].\end{aligned}$$

Let $\lambda_1, \dots, \lambda_d$ be eigenvalues of S_τ . Since S_τ is diagonalizable and all its eigenvalues are real,

$$\begin{aligned}n^2 &= (\text{Tra}(S_\tau))^2 = \left(\sum_{k=1}^d \lambda_k \right)^2 \leq d \sum_{k=1}^d \lambda_k^2 \\ &= d \text{Tra}(S_\tau^2) = dn + d \sum_{j,k=1, j \neq k}^n [\tau_j, \tau_k][\tau_k, \tau_j] \\ &\leq dn + d \sum_{j,k=1, j \neq k}^n |[\tau_j, \tau_k][\tau_k, \tau_j]| = dn + d(n^2 - n)|\gamma|^2.\end{aligned}$$

Therefore

$$n \leq d + d(n-1)|\gamma|^2 \implies n(1 - d|\gamma|^2) \leq d(1 - |\gamma|^2).$$

□

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