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## Article

# A Solution Nonlinear Wilberforce Pendulum

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## Abstract

This work presents solutions to the strongly coupled nonlinear Wilberforce pendulum model. We begin by presenting three specific cases of the equations of motion of the nonlinear Wilberforce Lagrangian. We find solutions governed by Jacobi elliptic functions, Mathieu functions, Weierstrass elliptic functions and obtaining beating behavior of plane waves when their frequencies are close and the phase is proportional to their amplitudes. Similarly, we find analytical solutions showing the power of these methods using the tanh, Riccati, Jacobi, and generalized Jacobi solitary wave solution techniques, without any restrictions.

**Keywords:** Wilberforce pendulum; Jacobi elliptic functions; Mathieu equation; Weierstrass elliptic functions; solitary wave methods

## 1. Introduction

The Wilberforce pendulum is a mechanical device that ingeniously couples the translational and rotational degrees of freedom of a non-point mass attached to a spring [1]. The mass undergoes two motions: the first is vertical, driven by the spring's translational restoring force, and the second is rotational around the axis of translation, also due to the spring's torsional properties [2,3]. Basically, spring stretches and contracts vertically, and twists and untwists around that same translational axis. British physicist Lionel Robert Wilberforce invented it and published his findings in 1896 [1].

Newton's second law, Hamiltonian, or Lagrangian dynamics can describe this device's physical behavior [4–7]. If we work with the latter, we can clearly postulate how the coupling terms appear in the dynamical system and how they impact the laws that describe the device [5–7]. This approach from classical mechanics has two approximations: one linear and the other nonlinear. In scenario one, the linear coupling term is  $\epsilon y\theta$ . Here,  $\epsilon$  represents a phenomenological coupling constant that determines the strength of the nonlinear interaction,  $y$  refers to the translational coordinate, and  $\theta$  describes rotational motion [5–7]. These terms, together with the kinetic and potential energy of each element of the system, generate a pair of coupled linear equations, whose key prediction is the emergence of beat-like behavior between the translational and rotational coordinates. In the second scenario, the reference's nonlinearity is proportional to  $\epsilon(y\theta)^2$  and can be modeled as a perturbative term describing the system's phase-space structure [7].

On the other hand, in recent decades new analytical methods, called solitary wave methods, have emerged to tackle the difficult and very arduous task of obtaining analytical solutions to nonlinear ordinary or partial differential equations. Among these new methods, we can mention the highly important work of W. Malfliet, in which a very insightful strategy is devised to express the solution in terms of solitary waves, offering the advantage that the solution to a given differential equation can be established as a finite sum of hyperbolic tangent terms and their derivatives [8]. Similarly, alternative approaches exist, like those using Riccati equation solutions, which build on Malfliet's idea by representing the desired solution as a finite sum of Riccati solutions [9]. Auxiliary differential equations yield multiple solution families, each as a finite sum satisfying the equation. Solitary wave methods improved by applying Jacobi elliptic functions. When we set combinations of analytical solutions of the Jacobi differential equation [reference], this drives a significant deep insight

into the number of solutions discovered. This improved solitary wave technique now provides advanced solutions for Klein–Gordon equations with quadratic perturbations. This study aims to employ solitary wave methods in analyzing the Wilberforce nonlinear oscillation system. Finally, we present additional results for certain cases given by Jacobi elliptic and Mathieu functions that enlighten how this complex nonlinear oscillator behaves.

2. The Lagrangian Model

A lagrangian that gives into accounts the nonlinear coupling in the system is [7]:

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}k_y y^2 - \frac{1}{2}k_{\theta} \theta^2 - \frac{1}{2}\epsilon(y\theta)^2, \tag{1}$$

The nonlinear equations are.

$$\frac{d}{dt} \frac{dL}{dy} - \frac{dL}{dy} = 0 \rightarrow \frac{d^2y}{dt^2} + \omega_y^2 y + \lambda_y y \theta^2 = 0, \tag{2}$$

$$\frac{d}{dt} \frac{dL}{d\theta} - \frac{dL}{d\theta} = 0 \rightarrow \frac{d^2\theta}{dt^2} + \omega_{\theta}^2 \theta + \lambda_{\theta} \theta y^2 = 0, \tag{3}$$

With,  $\omega_y^2 = \frac{k_y}{m}$   $\omega_{\theta}^2 = \frac{k_{\theta}}{I}$  and  $\lambda_y = \frac{\epsilon}{m}$  and  $\lambda_{\theta} = \frac{\epsilon}{I}$ . Where  $I$  is the moment of inertia,  $m$  the mass,  $k_y$  is the spring constant,  $k_{\theta}$  is the rotational spring constant, and  $\epsilon$  is the term that accounts for the coupling between the rotational and translational degrees of freedom.

3. Approximate Solutions

3.1. Specific Case:  $\lambda_{\theta} = 0, \lambda_y \neq 0$

If we assume that  $\lambda_{\theta} = 0, \lambda_y \neq 0$

$$\frac{d^2\theta}{dt^2} + \omega_{\theta}^2 \theta = 0, \tag{4}$$

This equation becomes a solution of the harmonic oscillator.

$$\theta(t) = B\cos(\omega_{\theta}t + \phi), \tag{5}$$

Substituting into Equation (2)

$$\frac{d^2y}{dt^2} + (\omega_y^2 + \lambda_y B^2 \cos^2(\omega_{\theta}t + \phi))y = 0, \tag{6}$$

Using  $\cos^2(u) = \frac{1+\cos(2u)}{2}$ ,

$$\frac{d^2y}{dt^2} + \left( \omega_y^2 + \lambda_y B^2 \left( \frac{1 + \cos(2(\omega_{\theta}t + \phi))}{2} \right) \right) y = 0, \tag{7}$$

Defining

$$a = \frac{2\omega_y^2 + \lambda_y B^2}{2\omega_{\theta}^2} ; -2q = \frac{\lambda_y B^2}{2\omega_{\theta}^2} ; \tau = (\omega_{\theta}t + \phi), \tag{8}$$

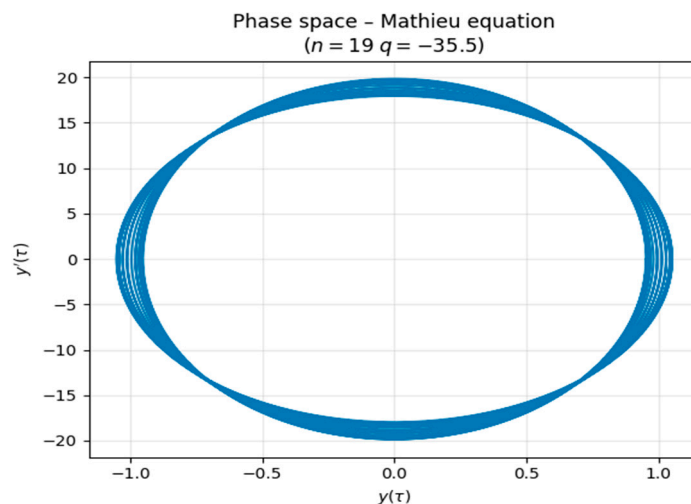
This reduces to a Mathieu equation [8,9]

$$\frac{d^2y}{d\tau^2} + (a - 2q\cos(2\tau))y = 0, \tag{9}$$

The solution is.

$$y(\tau) = Pce_n(\tau, q) + Qse_n(\tau, q), \tag{10}$$

where,  $P$  and  $Q$  are constants determined from initial condition, and  $ce_n$  is the even periodic Mathieu functions and  $se_n$  are the odd Mathieu functions and  $n$  is an integer indexing the characteristic values.



**Figure 1.** Phase space, mass  $m=1$ , Equation (10).

### 3.2. Specific Case: $\omega_\theta = \omega_y, \theta(t) = ky(t)$ .

Also, we have the case where  $\omega_\theta = \omega_y$  equal frequencies, we suppose a proportional solution  $\theta(t) = ky(t)$ , having  $k$  as a constant. So, the system of equations (2) and (3) reduce to

$$\frac{d^2\theta}{dt^2} + \omega_\theta^2\theta + \lambda_\theta\theta y^2 = 0 \rightarrow \frac{d^2ky(t)}{dt^2} + \omega_y^2ky(t) + \lambda_\theta ky(t)y(t)^2 = 0, \quad (12)$$

The system reduces to the Duffing equation [10].

$$\frac{d^2y(t)}{dt^2} + \omega_y^2y(t) + \lambda_\theta y(t)^3 = 0, \quad (13)$$

Doing  $\theta(0) = ky(0)$ , and  $\dot{\theta}(0) = k\dot{y}(0)$ . If we define  $y(0) = A$  and  $\dot{y}(0) = v$  and  $\lambda_\theta > 0$ , and following reference [8,11], we postulate a solution given by

$$y(t) = A \operatorname{cn}[wt, m]$$

where  $A$  is an amplitude and  $\operatorname{cn}$  is the Jacobi elliptic function [8,11], and using the second derivative, we get:

$$y(t) = A \operatorname{cn}\left[\left(\sqrt{\omega^2 + \lambda_\theta A^2}\right)t, \frac{\lambda_\theta A^2}{2(\omega^2 + \lambda_\theta A^2)}\right], \theta(t) = ky(t) \quad (14)$$

### 3.3. Specific Case: Complex Solutions

Also, we postulate solutions [4], given by

$$y(t) = \operatorname{Re}(Ae^{i\omega t + \varphi}), \theta(t) = \operatorname{Re}(Be^{i\omega t + \varphi}), e^\varphi = \frac{1}{\sqrt{AB}} \quad (15)$$

Replacing in Equation (2-3)

$$-m\omega^2 A + k_y A + \epsilon B = 0, \quad (16)$$

$$-I\omega^2 B + k_\theta B + \epsilon A = 0, \quad (17)$$

The frequencies of the normal modes are.

$$\omega_{1,2}^2 = \frac{mk_\theta + Ik_y \pm \sqrt{(mk_\theta + Ik_y)^2 - 4mI(k_\theta k_y - \epsilon^2)}}{2mI}, \quad (18)$$

A general solution  $z(t)$  is:

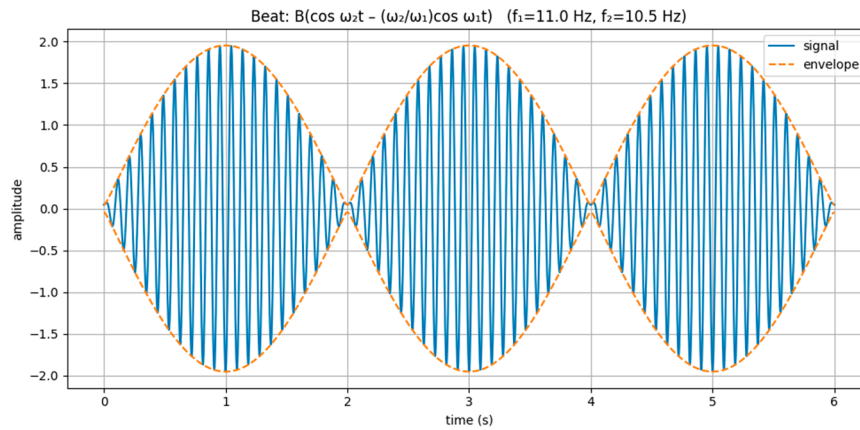
$$z(t) = A_1 e^{i\omega_1 t + \varphi} + A_2 e^{i\omega_2 t + \varphi} \quad (19)$$

Taking initial conditions as  $y(0) = A$  and  $\dot{y}(0) = 0$

$$z(t) = A_1 e^{\varphi} \left( \cos(\omega_1 t) - \frac{\omega_1}{\omega_2} \cos(\omega_2 t) + i \left( \sin(\omega_1 t) - \frac{\omega_1}{\omega_2} \sin(\omega_2 t) \right) \right) \quad (20)$$

And doing  $y(t) = \text{Re}(z(t))$

$$z(t) = A_1 e^{\varphi} \left( \cos(\omega_1 t) - \frac{\omega_1}{\omega_2} \cos(\omega_2 t) \right) \quad (21)$$



**Figure 1.** Beating behavior Equation (21).

3.4. *Specific Case:*  $\theta(t) = ky^2(t)$ ,

In the same way we suppose solution given by  $\theta(t) = ky^2(t)$ , having  $k$  as a constant. Therefore, equations (2) and (3) become:

$$\frac{d^2\theta}{dt^2} + \omega_\theta^2 \theta + \lambda_\theta \theta y^2 = 0 \rightarrow 2 \left( \frac{dy}{dt} \right)^2 + 2y \frac{d^2y}{dt^2} + \omega_\theta^2 y^2 + \lambda_\theta y(t)^4 = 0, \quad (22)$$

Using

$$v = \frac{dy}{dt}; u = v^2$$

Equation (22) is.

$$y \frac{du}{dy} + u + \omega_\theta^2 y^2 + \lambda_\theta y^4 = 0, \quad (23)$$

The integrating factor is  $\mu = y^2$ ,

$$\left( \frac{dy}{dt} \right)^2 = u = -\frac{\omega_\theta^2}{4} y^2 - \frac{\lambda_\theta}{6} y^4 + ky^{-2}, \quad (24)$$

Defining is  $\psi = y^2$ ,

$$\frac{d\psi}{dt} = \frac{d\sqrt{\psi}}{d\psi} \frac{d\psi}{dt} = \frac{1}{2} \frac{1}{\sqrt{\psi}} \frac{d\psi}{dt}, \quad (25)$$

$$\left( \frac{d\psi}{dt} \right)^2 = -\frac{2\lambda_\theta}{3} \psi^3 - \omega_\theta^2 \psi^2 + 4k, \quad (26)$$

Which is the Weierstrass Elliptic differential equation

$$\left( \frac{d\wp}{dt} \right)^2 = 4\wp^3 - g_2\wp - g_3, \quad (27)$$

And  $\wp$  Weierstrass elliptic function. If we want to deplete the quadratic term in equation (26), we define.

$$\psi = az + \beta, \quad (28)$$

where  $a$  and  $\beta$  are parameters. Defining  $\lambda_\theta = -\frac{6}{a}$ , and in equation (23),  $g_2 = -\frac{\omega_\theta^4}{12}$ ,  $g_3 = -\frac{4k}{a}$ . We obtain.

$$\left(\frac{dz}{dt}\right)^2 = 4z^3 - g_2z^2 - g_2, \tag{29}$$

Then, the solution is.

$$y = \sqrt{a\wp + \beta}; \; \theta = ka\wp + \frac{a\omega_\theta^2}{12}, \tag{30}$$

4. Solitary Wave Solutions

4.1. Tanh Solitary Wave Method

We use the Tanh solitary wave method [12], where the solution is:

$$y(t) = \sum_{i=0}^m a_i Y^i, \; \theta(t) = \sum_{i=0}^n b_i Y^i, \; Y = \tanh(\mu t) \tag{22}$$

The derivatives of  $Y^i$ , are.

$$\frac{d}{dt} = \mu(1 - Y^2) \frac{d}{dY}, \tag{23}$$

$$\frac{d^2}{dt^2} = -2Y\mu^2(1 - Y^2) \frac{d}{dY} + \mu^2(1 - Y^2)^2 \frac{d^2}{dY^2}, \tag{24}$$

Replacing in equations (2)-(3):

$$-2Y\mu^2(1 - Y^2) \frac{dy}{dY} + \mu^2(1 - Y^2)^2 \frac{d^2y}{dY^2} + \omega_y^2 y + \lambda_y y\theta^2 = 0 \tag{25}$$

$$-2Y\mu^2(1 - Y^2) \frac{d\theta}{dY} + \mu^2(1 - Y^2)^2 \frac{d^2\theta}{dY^2} + \omega_\theta^2 \theta + \lambda_\theta \theta y^2 = 0, \tag{26}$$

Now, balancing the highest-order linear derivative with the highest order nonlinear terms in equations (25)-(26):

$$Y^4 \frac{d^2y}{dY^2} \rightarrow y\theta^2 \rightarrow m + 2 = m + 2n \rightarrow n = 1, \tag{27}$$

$$Y^4 \frac{d^2\theta}{dY^2} \rightarrow \theta y^2 \rightarrow n + 2 = n + 2m \rightarrow m = 1, \tag{28}$$

Therefore

$$y(t) = a_0 Y^0 + a_1 Y^1, \tag{29}$$

$$\theta(t) = b_0 Y^0 + b_1 Y^1, \tag{30}$$

Their derivatives are.

$$\frac{dy}{dY} = a_1; \; \frac{d^2x}{dY^2} = 0, \tag{31}$$

$$\frac{d\theta}{dY} = b_1; \; \frac{d^2x}{dY^2} = 0, \tag{32}$$

Replacing in equations (25)-(26):

$$-2Y\mu^2(1 - Y^2)(a_1) + \omega_y^2(a_0 + a_1 Y) + \lambda_y(a_0 + a_1 Y)(b_0 + b_1 Y)^2 = 0, \tag{33}$$

$$-2Y\mu^2(1 - Y^2)(b_1) + \omega_\theta^2(b_0 Y^0 + b_1 Y) + \lambda_\theta(a_0 + a_1 Y)^2(b_0 + b_1 Y) = 0, \tag{34}$$

Then, we get a set of equations, order by order in  $Y^i$ . Applying algebra yields:

$$g_1 \rightarrow \{a_0 = 0, b_0 = 0, \omega_y = -\sqrt{2}\mu, \omega_\theta = -\sqrt{2}\mu, \lambda_y = -\frac{2\mu^2}{b_1^2}, \lambda_\theta = -\frac{2\mu^2}{a_1^2}\} \tag{35}$$

$$g_2 \rightarrow \{a_0 = 0, b_0 = 0, \omega_y = -\sqrt{2}\mu, \omega_\theta = \sqrt{2}\mu, \lambda_y = -\frac{2\mu^2}{b_1^2}, \lambda_\theta = -\frac{2\mu^2}{a_1^2}\} \tag{36}$$

$$g_3 \rightarrow \{a_0 = 0, b_0 = 0, \omega_y = \sqrt{2}\mu, \omega_\theta = -\sqrt{2}\mu, \lambda_y = -\frac{2\mu^2}{b_1^2}, \lambda_\theta = -\frac{2\mu^2}{a_1^2}\} \tag{37}$$

$$g_4 \rightarrow \{a_0 = 0, b_0 = 0, \omega_y = \sqrt{2}\mu, \omega_\theta = \sqrt{2}\mu, \lambda_y = -\frac{2\mu^2}{b_1^2}, \lambda_\theta = -\frac{2\mu^2}{a_1^2}\} \tag{38}$$

So, the solutions are.

$$y(t) = a_1 \tanh (\mu t), \tag{39}$$

$$\theta(t) = b_1 \tanh (\mu t), \tag{40}$$

Then, equations (35)-(38) determine  $a_1, b_1$  and  $\mu$ .

4.2. Riccati Solitary Wave Method

Also, we use the Riccati solitary wave method [13]

$$x(t) = \sum_{i=0}^m a_i F^i, \quad y(t) = \sum_{i=0}^n b_i F^i, \tag{41}$$

where  $F$  is the solution of

$$\frac{dF}{dt} = A_1 + C_1 F^2 \rightarrow \frac{d^2F}{dt^2} = 2C_1 F(A_1 + C_1 F^2), \tag{42}$$

Here  $A_1$  and  $C_1$  are constants, Table 1. Again, balancing nonlinear terms in equations (1)-(2), We have  $m=1, n=1$ . Therefore, the solutions and their derivatives are:

$$x = a_0 + a_1 F \rightarrow \frac{dx}{dt} = a_1 \frac{dF}{dt} \rightarrow \frac{d^2x}{dt^2} = (a_1) \frac{d^2F}{dt^2}, \tag{43}$$

$$y = b_0 + b_1 F \rightarrow \frac{dy}{dt} = b_1 \frac{dF}{dt} \rightarrow \frac{d^2y}{dt^2} = (b_1) \frac{d^2F}{dt^2}, \tag{44}$$

**Table 1.** List of solutions to equation (42).

| $A_1$ | $C_1$ | $F$                                                                |
|-------|-------|--------------------------------------------------------------------|
| 1/2   | -1/2  | $\coth(t) \pm \cosh(t), \tanh(t),$<br>$\pm \operatorname{sech}(t)$ |
| 1/2   | 1/2   | $\sec(t) \pm i \tan(t),$                                           |
| -1/2  | -1/2  | $\csc(t) \pm i \cot(t),$                                           |
| 1     | -1    | $\coth(t), \tanh(t),$                                              |
| 1     | 1     | $\tan(t),$                                                         |
| -1    | -1    | $\cot(t),$                                                         |

Replacing in equations (2)-(3)

$$(a_1)2C_1F(A_1 + C_1F^2) + \omega_y^2(a_0 + a_1F) + \lambda_y(a_0 + a_1F)(b_0 + b_1F)^2 = 0, \tag{45}$$

$$(b_1)2C_1F(A_1 + C_1F^2) + \omega_\theta^2(b_0 + b_1F) + \lambda_\theta(a_0 + a_1F)^2(b_0 + b_1F) = 0, \tag{46}$$

We obtain a set of algebraic equations and solving them, we get a set of family’s solutions.



$$f_1 \rightarrow \{a_0 = 0, b_0 = 0, \omega_y = -i\sqrt{2}\sqrt{A_1}\sqrt{C_1}, \omega_\theta = -i\sqrt{2}\sqrt{A_1}\sqrt{C_1}, \lambda_y = -\frac{2C_1^2}{b_1^2}, \lambda_\theta = -\frac{2C_1^2}{a_1^2}\} \quad (47)$$

$$f_2 \rightarrow \{a_0 = 0, b_0 = 0, \omega_y = -i\sqrt{2}\sqrt{A_1}\sqrt{C_1}, \omega_\theta = i\sqrt{2}\sqrt{A_1}\sqrt{C_1}, \lambda_y = -\frac{2C_1^2}{b_1^2}, \lambda_\theta = -\frac{2C_1^2}{a_1^2}\} \quad (48)$$

$$f_3 \rightarrow \{a_0 = 0, b_0 = 0, \omega_y = i\sqrt{2}\sqrt{A_1}\sqrt{C_1}, \omega_\theta = -i\sqrt{2}\sqrt{A_1}\sqrt{C_1}, \lambda_y = -\frac{2C_1^2}{b_1^2}, \lambda_\theta = -\frac{2C_1^2}{a_1^2}\} \quad (49)$$

$$f_4 \rightarrow \{a_0 = 0, b_0 = 0, \omega_y = i\sqrt{2}\sqrt{A_1}\sqrt{C_1}, \omega_\theta = i\sqrt{2}\sqrt{A_1}\sqrt{C_1}, \lambda_y = -\frac{2C_1^2}{b_1^2}, \lambda_\theta = -\frac{2C_1^2}{a_1^2}\} \quad (50)$$

where  $i = \sqrt{-1}$ , represents a complex number. The solutions are:

$$y(t) = a_1 F(t), \quad (51)$$

$$\theta(t) = b_1 F(t), \quad (52)$$

Table 1 lists the solutions for F.

#### 4.3. Soliton Solitary Wave Method

Also, introduce a new solitary wave method, [14]

$$x(t) = \sum_{i=0}^m a_i J^i, \quad y(t) = \sum_{i=0}^m b_i J^i, \quad (53)$$

where  $G$  is the solution of

$$\left(\frac{dJ}{dt}\right)^2 = m_1 J^2 + m_2 J^3 + m_3 J^4, \quad (54)$$

$$\frac{d^2 J}{dt^2} = m_1 J + \frac{3}{2} m_2 J^2 + 2m_3 J^3, \quad (55)$$

And  $m_1$ ,  $m_2$  and  $m_3$  are parameters, and  $J$  the solution. Again, balancing nonlinear terms in equations (2)-(3), we have  $m=1$ ,  $n=1$ . Then, equations (53) are:

$$y(t) = a_0 + a_1 J, \quad \theta(t) = b_0 + b_1 J, \quad (56)$$

$$\frac{d^2 y}{dt^2} = (a_1) \frac{d^2 J}{dt^2} = (a_1) \left( m_1 J + \frac{3}{2} m_2 J^2 + 2m_3 J^3 \right), \quad (57)$$

$$\frac{d^2 \theta}{dt^2} = (b_1) \frac{d^2 J}{dt^2} = (b_1) \left( m_1 J + \frac{3}{2} m_2 J^2 + 2m_3 J^3 \right), \quad (58)$$

Then, equations (54) are:

$$(a_1) \left( m_1 J + \frac{3}{2} m_2 J^2 + 2m_3 J^3 \right) + \omega_y^2 (a_0 + a_1 J) + \lambda_y (a_0 + a_1 J) (b_0 + b_1 J)^2 = 0, \quad (59)$$

$$(b_1) \left( m_1 J + \frac{3}{2} m_2 J^2 + 2m_3 J^3 \right) + \omega_\theta^2 (b_0 + b_1 J) + \lambda_\theta (b_0 + b_1 J) (a_0 + a_1 J)^2 = 0, \quad (60)$$

Again, we get a set of equations, order by order in  $J^i$ , and by doing algebra, we get:



$$\begin{aligned} L_1 \rightarrow \{a_0 = \frac{a_1 m_1}{m_2}, b_0 = \frac{b_1 m_1}{m_2}, \omega_\theta = -\frac{\sqrt{m_1}}{\sqrt{2}}, \omega_y = -\frac{\sqrt{m_1}}{\sqrt{2}}, \lambda_\theta = -\frac{m_2^2}{2a_1^2 m_1}, \lambda_y \\ = -\frac{m_2^2}{2b_1^2 m_1}, m_3 = \frac{m_2^2}{4m_1}\} \end{aligned} \tag{61}$$

$$\begin{aligned} L_2 \rightarrow \{a_0 = \frac{a_1 m_1}{m_2}, b_0 = \frac{b_1 m_1}{m_2}, \omega_\theta = -\frac{\sqrt{m_1}}{\sqrt{2}}, \omega_y = \frac{\sqrt{m_1}}{\sqrt{2}}, \lambda_\theta = -\frac{m_2^2}{2a_1^2 m_1}, \lambda_y \\ = -\frac{m_2^2}{2b_1^2 m_1}, m_3 = \frac{m_2^2}{4m_1}\} \end{aligned} \tag{62}$$

$$\begin{aligned} L_3 \rightarrow \{a_0 = \frac{a_1 m_1}{m_2}, b_0 = \frac{b_1 m_1}{m_2}, \omega_\theta = \frac{\sqrt{m_1}}{\sqrt{2}}, \omega_y = -\frac{\sqrt{m_1}}{\sqrt{2}}, \lambda_\theta = -\frac{m_2^2}{2a_1^2 m_1}, \lambda_y \\ = -\frac{m_2^2}{2b_1^2 m_1}, m_3 = \frac{m_2^2}{4m_1}\} \end{aligned} \tag{63}$$

$$\begin{aligned} L_4 \rightarrow \{a_0 = \frac{a_1 m_1}{m_2}, b_0 = \frac{b_1 m_1}{m_2}, \omega_\theta = \frac{\sqrt{m_1}}{\sqrt{2}}, \omega_y = \frac{\sqrt{m_1}}{\sqrt{2}}, \lambda_\theta = -\frac{m_2^2}{2a_1^2 m_1}, \lambda_y \\ = -\frac{m_2^2}{2b_1^2 m_1}, m_3 = \frac{m_2^2}{4m_1}\} \end{aligned} \tag{64}$$

**Table 1.** List of solutions to equation (55).  $0 < \mu < 1$ .

| $\epsilon$  | $a$         | $b$         | $c$     | $T$                 |
|-------------|-------------|-------------|---------|---------------------|
| -1          | $-\mu^2$    | 1           | 1       | $\text{sn}(t \mu)$  |
| -1          | $\mu^2$     | $1 - \mu^2$ | 1       | $\text{cn}(t \mu),$ |
| -1          | 1           | $\mu^2 - 1$ | 1       | $\text{dn}(t \mu),$ |
| -1          | $-\mu^2$    | 1           | 1       | $\text{cd}(t \mu),$ |
| $\mu^2 - 1$ | $\mu^2$     | 1           | 1       | $\text{sd}(t \mu),$ |
| $1 - \mu^2$ | 1           | -1          | 1       | $\text{nd}(t \mu),$ |
| 1           | 1           | $-\mu^2$    | -1      | $\text{dc}(t \mu),$ |
| 1           | $1 - \mu^2$ | $\mu^2$     | -1      | $\text{nc}(t \mu),$ |
| 1           | $1 - \mu^2$ | 1           | 1       | $\text{sc}(t \mu),$ |
| 1           | 1           | $-\mu^2$    | -1      | $\text{ns}(t \mu),$ |
| 1           | 1           | $\mu^2 - 1$ | $\mu^2$ | $\text{ds}(t \mu)$  |
| 1           | 1           | $1 - \mu^2$ | 1       | $\text{cs}(t \mu),$ |

4.4. Elliptic Solitary Wave Method 1

We employ a Jacobi solitary wave method, [15]

$$x(t) = \sum_{i=0}^r a_i T^i, \quad y(t) = \sum_{i=0}^s b_i T^i, \tag{65}$$

where  $T$  is the solution of

$$\left(\frac{dT}{dt}\right)^2 = (c + \epsilon T^2)(aT^2 + b), \tag{66}$$

$$\frac{d^2T}{dt^2} = (ca + \epsilon b)T + 2\epsilon aT^3, \tag{67}$$

And  $a, b, c$  and  $\epsilon$  are parameters, and  $T$  the solution (referencia Jacobi). Again, balancing nonlinear terms in equations (1)-(2), we have  $r=1, s = 1$ , in summations (46). Then

$$y(t) = a_0 + a_1 T, \quad \theta(t) = b_0 + b_1 T, \tag{68}$$

Therefore

$$\frac{d^2y}{dt^2} = (a_1) \frac{d^2T}{dt^2} = (a_1)((ca + \epsilon b)T + 2\epsilon aT^3), \tag{69}$$

$$\frac{d^2\theta}{dt^2} = (b_1) \frac{d^2T}{dt^2} = (b_1)((ca + \varepsilon b)T + 2\varepsilon aT^3), \quad (70)$$

The movement equations (1)-(2) are:

$$(a_1)((ca + \varepsilon b)T + 2\varepsilon aT^3) + \omega_y^2(a_0 + a_1T) + \lambda_y(a_0 + a_1T)(b_0 + b_1T)^2 = 0, \quad (71)$$

$$(b_1)((ca + \varepsilon b)T + 2\varepsilon aT^3) + \omega_\theta^2(b_0 + b_1T) + \lambda_\theta(b_0 + b_1T)(a_0 + a_1T)^2 = 0, \quad (72)$$

Then, we obtain a group of coupled equations, order by order in  $T^i$ . After performing algebraic operations, the result is as follows:

$$T_1 \rightarrow \{a_0 = b_0 = 0, \omega_\theta = \omega_y = -\frac{\sqrt{-2b\varepsilon^2 + a_1^2c\lambda_\theta}}{\sqrt{2}\sqrt{\varepsilon}}, \lambda_y = \frac{a_1^2\lambda_\theta}{b_1^2}, a = -\frac{a_1^2\lambda_\theta}{2\varepsilon}\} \quad (73)$$

$$T_2 \rightarrow \{a_0 = b_0 = 0, \omega_\theta = -\omega_y = \frac{\sqrt{-2b\varepsilon^2 + a_1^2c\lambda_\theta}}{\sqrt{2}\sqrt{\varepsilon}}, \lambda_y = \frac{a_1^2\lambda_\theta}{b_1^2}, a = -\frac{a_1^2\lambda_\theta}{2\varepsilon}\} \quad (74)$$

$$T_3 \rightarrow \{a_0 = b_0 = 0, \omega_\theta = \omega_y = -\frac{\sqrt{-2b\varepsilon^2 + a_1^2c\lambda_\theta}}{\sqrt{2}\sqrt{\varepsilon}}, \lambda_y = \frac{a_1^2\lambda_\theta}{b_1^2}, a = -\frac{a_1^2\lambda_\theta}{2\varepsilon}\} \quad (75)$$

$$T_4 \rightarrow \{a_0 = b_0 = 0, \omega_\theta = -\omega_y = \frac{\sqrt{-2b\varepsilon^2 + a_1^2c\lambda_\theta}}{\sqrt{2}\sqrt{\varepsilon}}, \lambda_y = \frac{a_1^2\lambda_\theta}{b_1^2}, a = -\frac{a_1^2\lambda_\theta}{2\varepsilon}\} \quad (76)$$

$$T_5 \rightarrow \{a_0 = b_0 = 0, \omega_y = -\omega_\theta, \lambda_y = \frac{a_1^2\lambda_\theta}{b_1^2}, a = -\frac{a_1^2\lambda_\theta}{2\varepsilon}, b = \frac{a_1^2c\lambda_\theta - 2\varepsilon\omega_\theta^2}{2\varepsilon^2}\} \quad (77)$$

$$T_6 \rightarrow \{a_0 = b_0 = 0, \omega_y = \omega_\theta, \lambda_y = \frac{a_1^2\lambda_\theta}{b_1^2}, a = -\frac{a_1^2\lambda_\theta}{2\varepsilon}, b = \frac{a_1^2c\lambda_\theta - 2\varepsilon\omega_\theta^2}{2\varepsilon^2}\} \quad (78)$$

For all the six families of solutions  $T_{1,\dots,6} \rightarrow a_0 = b_0 = 0$ . So, the solutions are:

$$y(t) = a_1T, \quad \theta(t) = b_1T, \quad (79)$$

To show the result, we apply the first row that defines the solution of sn Jacobi elliptic function in table (1), and the conditions exposed in the family of solutions  $T_1$

$$\omega_\theta = \omega_y = -\frac{\sqrt{-2b\varepsilon^2 + a_1^2c\lambda_\theta}}{\sqrt{2}\sqrt{\varepsilon}} \rightarrow \frac{k_y}{m} = \frac{k_\theta}{I} = 1 - \frac{\varepsilon a_1^2}{2I} \quad (80)$$

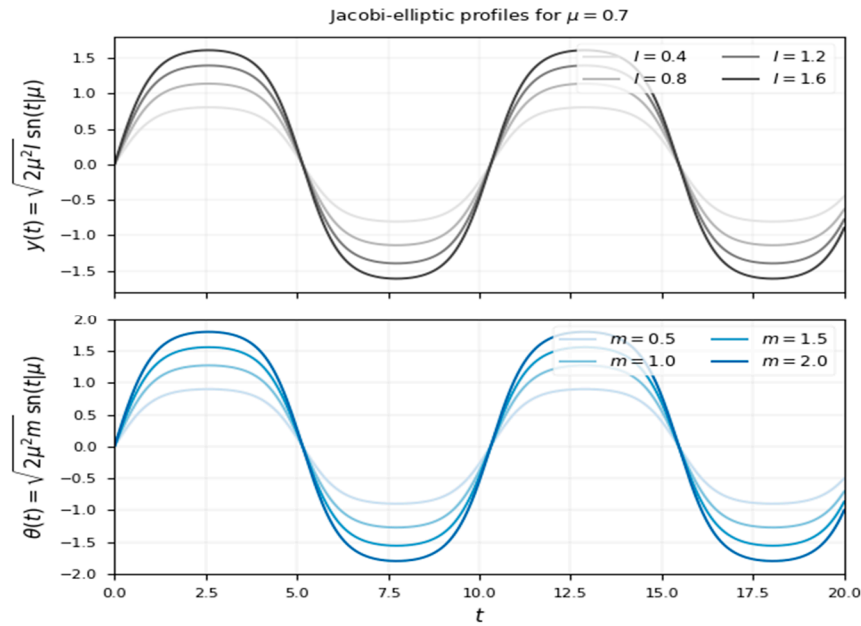
$$\frac{k_\theta}{I} = 1 - \frac{\varepsilon a_1^2}{2I} \rightarrow I > \frac{\varepsilon a_1^2}{2}; \quad \frac{k_y}{m} = 1 - \frac{\varepsilon a_1^2}{2I} \rightarrow I > \frac{\varepsilon a_1^2}{2} \quad (81)$$

And now

$$\lambda_y = \frac{a_1^2\lambda_\theta}{b_1^2} \rightarrow b_1^2 = \frac{a_1^2m}{I} \quad (82)$$

$$a = -\frac{a_1^2\lambda_\theta}{2\varepsilon} \rightarrow a_1^2 = 2\mu^2I \quad (83)$$

$$\begin{aligned} y(t) &= a_1 \operatorname{sn}(t|\mu) = \left(\sqrt{2\mu^2I}\right) \operatorname{sn}(t|\mu), \\ \theta(t) &= b_1 \operatorname{sn}(t|\mu) = \left(\sqrt{2\mu^2m}\right) \operatorname{sn}(t|\mu), \end{aligned} \quad (84)$$



**Figure 2.** Jacobi elliptic function solution sn.

Following with results, we apply the second row that defines the solution of cn Jacobi elliptic function in table (1), and the conditions exposed in the family of solutions  $T_1$

$$\omega_\theta = \omega_y = -\frac{\sqrt{-2b\epsilon^2 + a_1^2 c \lambda_\theta}}{\sqrt{2}\sqrt{\epsilon}} \rightarrow \frac{k_y}{m} = \frac{k_\theta}{I} = 1 - \mu^2 - \frac{a_1^2 \epsilon}{2I} \quad (85)$$

$$\frac{k_\theta}{I} = 1 - \mu^2 - \frac{a_1^2 \epsilon}{2I}; \quad \frac{k_y}{m} = 1 - \mu^2 - \frac{a_1^2 \epsilon}{2I} \quad (86)$$

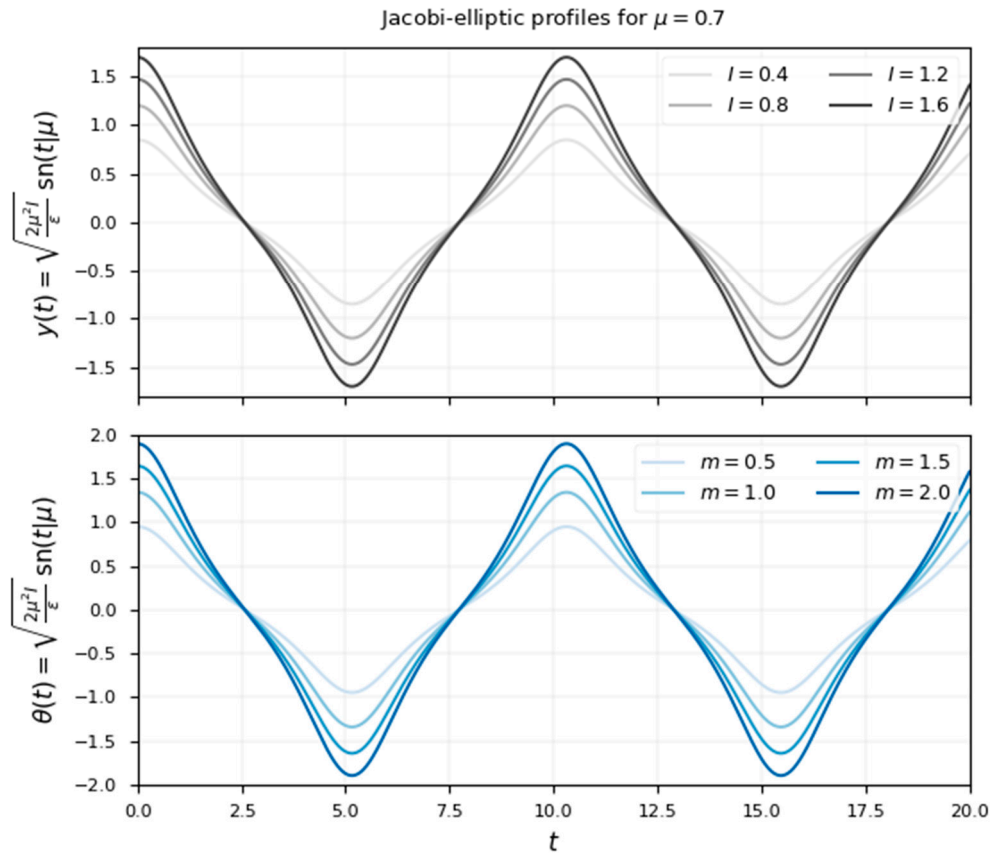
And now

$$\lambda_y = \frac{a_1^2 \lambda_\theta}{b_1^2} \rightarrow b_1^2 = \frac{a_1^2 m}{I} \quad (87)$$

$$a = -\frac{a_1^2 \lambda_\theta}{2\epsilon} \rightarrow \mu^2 = \frac{a_1^2 \epsilon}{2I} \rightarrow a_1^2 = \frac{2\mu^2 I}{\epsilon} \rightarrow b_1^2 = \frac{2\mu^2 m}{\epsilon} \quad (88)$$

$$y(t) = a_1 \text{sn}(t|\mu) = \left( \sqrt{\frac{2\mu^2 I}{\epsilon}} \right) \text{sn}(t|\mu), \quad (89)$$

$$\theta(t) = b_1 \text{sn}(t|\mu) = \left( \sqrt{\frac{2\mu^2 m}{\epsilon}} \right) \text{sn}(t|\mu),$$



**Figure 3.** Jacobi elliptic function sn governing  $y(t)$  and  $\theta(t)$ .

#### 4.5. Elliptic Solitary Wave Method 2

Finally, a more sophisticated solitary wave method [16]

$$x(t) = \sum_{i=0}^m a_i G^i, \quad y(t) = \sum_{i=0}^m b_i G^i, \quad (90)$$

where  $G$  is the solution of

$$\left(\frac{dG}{dt}\right)^2 = r_0 + r_2 G^2 + r_4 G^4 + r_6 G^6, \quad (91)$$

$$\frac{d^2 G}{dt^2} = r_2 G + 2r_4 G^3 + 3r_6 G^5, \quad (92)$$

And  $a$ ,  $b$  and  $\varepsilon$  are parameters, and  $G$  the solution. Again, balancing nonlinear terms in equations (2)-(3), so  $m=1$ ,  $n=1$ . Then, equations (54) are:

$$y(t) = a_0 + a_1 G, \quad \theta(t) = b_0 + b_1 G, \quad (93)$$

$$\frac{d^2 y}{dt^2} = (a_1) \frac{d^2 G}{dt^2} = (a_1)(r_2 G + 2r_4 G^3 + 3r_6 G^5), \quad (94)$$

$$\frac{d^2 \theta}{dt^2} = (b_1) \frac{d^2 G}{dt^2} = (b_1)(r_2 G + 2r_4 G^3 + 3r_6 G^5), \quad (95)$$

Then, equations (54) are:

$$(a_1)(r_2 G + 2r_4 G^3 + 3r_6 G^5) + \omega_y^2(a_0 + a_1 G) + \lambda_y(a_0 + a_1 G)(b_0 + b_1 G)^2 = 0, \quad (96)$$

$$(b_1)(r_2 G + 2r_4 G^3 + 3r_6 G^5) + \omega_\theta^2(b_0 + b_1 G) + \lambda_\theta(b_0 + b_1 G)(a_0 + a_1 G)^2 = 0, \quad (97)$$

Then, we get a set of equations, order by order in  $G^i$ , and by doing some algebra, we get:

$$h_1 \rightarrow \{a_0 = b_0 = 0, \omega_\theta = -i\sqrt{r_2}, \omega_y = -i\sqrt{r_2}, \lambda_\theta = -\frac{2r_4}{a_1^2}, \lambda_y = -\frac{2r_4}{b_1^2}, r_6 = 0\} \quad (98)$$

$$h_2 \rightarrow \{a_0 = b_0 = 0, \omega_\theta = -i\sqrt{r_2}, \omega_y = i\sqrt{r_2}, \lambda_\theta = -\frac{2r_4}{a_1^2}, \lambda_y = -\frac{2r_4}{b_1^2}, r_6 = 0\} \quad (99)$$

$$h_3 \rightarrow \{a_0 = b_0 = 0, \omega_\theta = i\sqrt{r_2}, \omega_y = -i\sqrt{r_2}, \lambda_\theta = -\frac{2r_4}{a_1^2}, \lambda_y = -\frac{2r_4}{b_1^2}, r_6 = 0\} \quad (100)$$

$$h_4 \rightarrow \{a_0 = b_0 = 0, \omega_\theta = i\sqrt{r_2}, \omega_y = i\sqrt{r_2}, \lambda_\theta = -\frac{2r_4}{a_1^2}, \lambda_y = -\frac{2r_4}{b_1^2}, r_6 = 0\} \quad (101)$$

$$h_5 \rightarrow \{a_0 = b_0 = 0, \omega_y = -\omega_\theta, \lambda_y = \frac{a_1^2 \lambda_\theta}{b_1^2}, r_2 = -\omega_\theta^2, r_4 = -\frac{a_1^2 \lambda_\theta}{2}, r_6 = 0\} \quad (102)$$

$$h_6 \rightarrow \{a_0 = b_0 = 0, \omega_y = \omega_\theta, \lambda_y = \frac{a_1^2 \lambda_\theta}{b_1^2}, r_2 = -\omega_\theta^2, r_4 = -\frac{a_1^2 \lambda_\theta}{2}, r_6 = 0\} \quad (103)$$

## 5. Conclusions

Extending the linear Wilberforce pendulum to the nonlinear system poses a complex analytical problem, and to date no analytical solution is known. When we consider special cases—parameter-space restrictions that define the equations of motion—we find approximate equations such as the Duffing equation, the Mathieu equation and Weierstrass elliptic differential equation; similarly, by assuming complex exponential solutions, we observe beating behavior when frequencies are close. Given these perspectives on the complexity of the problem, we turn to novel and up-to-date analytical tools that have proven their worth in various fields of physics and are known as solitary wave solutions. For the problem under consideration, we have applied the tanh method, one of the original methods, Riccati solutions, and two tools based on Jacobi function solutions. We have found a respectable number of solutions demonstrating the method's capabilities. In this same vein, new types of couplings between rotational and translational variables can be designed to reflect internal geometric symmetries in the system's Lagrangian—for example, invariance under coordinate exchange, under multiplication of the degrees of freedom by a complex phase, or under rotations in the internal coordinate space.

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