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Article

Global Dynamics of a Predator–Prey System with Variation Multiple Pulse Intervention Effects

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Abstract: A continuous point of a trajectory for an ordinary differential equation can be viewed as a special impulsive point; i.e., the pulsed proportional change rate and the instantaneous increment for the prey and predator populations can be taken as 0. By considering the variation multiple pulse intervention effects (i.e., several indefinite continuous points are regarded as impulsive points), an impulsive predator–prey model for characterizing chemical and biological control processes at different fixed times is first proposed. Our modeling approach can describe all possible realistic situations, and all of the traditional models are some special cases of our model. Due to the complexity of our modelling approach, it is essential to examine the dynamical properties of the periodic solutions using new methods. For example, we investigate the permanence of the system by constructing two uniform lower impulsive comparison systems, indicating the mathematical (or biological) essence of the permanence of our system; furthermore, the existence and global attractiveness of the pest-present periodic solution is analyzed by constructing an impulsive comparison system for a norm $V(t)$, which has not been addressed to date. Based on the implicit function theorem, the bifurcation of the pest-present periodic solution of the system is investigated under certain conditions, which is more rigorous than the corresponding traditional proving method. In addition, by employing the variational method, the eigenvalues of the Jacobian matrix at the fixed point corresponding to the pest-free periodic solution are determined, resulting in a sufficient condition for its local stability, and the threshold condition for the global attractiveness of the pest-free periodic solution is provided in terms of an indicator R_a. Finally, the sensitivity of indicator R_a and bifurcations with respect to several key parameters are determined through numerical simulations, and then, the switch-like transitions among two coexisting attractors show that varying dosages of insecticide applications and the numbers of natural enemies released are crucial.

Keywords: variation multiple pulse intervention effects; positive periodic solution; bifurcation; permanence; global stability

MSC: 34A34; 34K20

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1. Introduction

Pest control has become an increasingly complex issue over the past two decades. Pests in the natural environment not only negatively impact human well-being but also disrupt food production and spread diseases; in the worst scenarios, they even cause death and serious disasters for plants and other living things on Earth [1]. Experimental evidence has demonstrated that these pests reduce the annual production of some agricultural commodities, e.g., 25% shrinkage in rice, 30% shrinkage in pulses, $5-10$% shrinkage in wheat, 20% shrinkage in sugarcane, 35% shrinkage in oil seeds, and 50% shrinkage in cotton [2]. This alarming issue has motivated an urgent demand for controlling the negative impact of pests. The traditional pest control method involves the application of a large variety of single pesticides to crops, which is not favorable for controlling the progress of pest resistance and preserving the quality of the environment. Chemical control by spraying pesticides and the simultaneous deployment of biological control by using natural enemies have been commonly applied in integrated pest management (IPM) systems [3,4].

Various mathematical models have been studied for pest control, including chemical and biological control tactics [5–8]. Researchers have proposed several state-dependent impulsive predator–prey models with integrated pest management to control crop pests and subsequently explored the properties of the Poincaré map and the existence, uniqueness and local or global stability of the order-$k(k \geq 1)$ periodic solutions of the systems by means of the successor function method and the analog of the Poincaré criterion, respectively [9–15]. For example, Cheng *et al.* proposed a state-dependent impulsive predator–prey model with Holling type I and II responses and then investigated the properties of the Poincaré map and the existence, uniqueness and sufficient conditions for the global stability of the order-1 periodic solutions of the system [16]. Al Basir *et al.* formulated an impulsive integrated pest management model for characterizing the application processes of biopesticides and chemical pesticides at fixed times, and the stability of the pest-free periodic solution was investigated using the small amplitude perturbation method, which established a threshold time limit for the impulsive release of various controls as well as some valid theoretical conclusions for effective pest management [17].

The classes of predator–prey systems can be generalized as follows [18,19]:

$$\begin{cases} \frac{dx(t)}{dt} &= ax(t)\left(1 - \frac{x(t)}{K}\right) - \alpha\varphi(x(t))y(t), \\ \frac{dy(t)}{dt} &= -Dy(t) + \beta\varphi(x(t))y(t), \end{cases}$$

where $x(t)$ and $y(t)$ are the densities of the prey and predator populations, respectively, at time t. $a > 0$ denotes the intrinsic growth rate of the prey population, $K > 0$ represents the carrying capacity of the environment, and $\alpha > 0$ is the search efficiency of a predator for prey. $D > 0$ is the mortality rate of the predator population, and $\beta > 0$ is the biomass conversion. $\varphi(x)$ is the generalized functional response.

Since pesticides are harmful to pests and natural enemies, we can introduce the periodic spraying of pesticides and release predators at different instances [20]; then, Liu et al. proposed an impulsive system as follows [21]:

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= ax(t)\left(1 - \frac{x(t)}{K}\right) - \alpha\varphi(x(t))y(t), \\ \frac{dy(t)}{dt} &= -Dy(t) + \beta\varphi(x(t))y(t), \end{aligned}\right\} t \neq (n-1)T + lT, t \neq nT, \\ \left.\begin{aligned} x(t^+) &= (1 - p_P)x(t), \\ y(t^+) &= (1 - p_N)y(t), \end{aligned}\right\} t = (n-1)T + lT, \\ \left.\begin{aligned} x(t^+) &= x(t), \\ y(t^+) &= y(t) + q, \end{aligned}\right\} t = nT, \end{cases} \tag{1}$$

where $0 < p_N < p_P < 1$ represent the constant fractions of the predator and prey populations that die due to pesticides at time $t = (n-1)T + lT(l \in (0,1))$, respectively. $q \geq 0$ represents the amount of natural enemies released at time $t = nT$.

For system (1), the frequencies of pesticide spraying and predator release are the same, which is not realistic. Similar to the modeling approach in [22], system (1) can be modified into the following system:

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= ax(t)\left(1 - \frac{x(t)}{K}\right) - \alpha\varphi(x(t))y(t), \\ \frac{dy(t)}{dt} &= -Dy(t) + \beta\varphi(x(t))y(t), \end{aligned}\right\} t \neq \tau_n, \quad t \neq \lambda_m, \\ \left.\begin{aligned} x(t^+) &= (1 - p_P)x(t), \\ y(t^+) &= (1 - p_N)y(t), \end{aligned}\right\} t = \tau_n, \\ \left.\begin{aligned} x(t^+) &= x(t), \\ y(t^+) &= y(t) + q, \end{aligned}\right\} t = \lambda_m, \end{cases} \tag{2}$$

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where $\tau_n(n = 1, 2, \cdots)$ and $\lambda_m(m = 1, 2, \cdots)$ are impulsive point series at which the chemical and biological control strategies are applied, respectively. Zhao et al. investigated the global attractiveness of the trivial periodic solution only for the following two cases [22]:

Case I: $0 = \lambda_0 < \tau_1 < \tau_2 < \cdots < \tau_{k_p} < \lambda_1 = T_N(> 0)$, where $k_p > 0$;

Case II: $0 = \tau_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{k_N} < \tau_1 = T_p(> 0)$, where $k_N > 0$.

The above two cases are two special cases of real-world implementations of biological and chemical tactics. Thus, a problem arises naturally: is there a more general approach to characterize this kind of impulsive control strategy?

To solve this problem, it should be noted that a continuous point can be viewed as a special impulsive point. Taking system (2) as an example, if $y(t)$ is continuous at $t = t_0$, then it follows that $y(t_0^+) = (1 - p_N)y(t_0)$, where $p_N = 0$; or $y(t_0^+) = y(t_0) + q$, where $q = 0$. Since the Holling type II functional response is common for most of the functional responses mentioned in [18,19], we can take

$$\varphi(x(t)) = \frac{x(t)}{b + x(t)},$$

where $b > 0$ is the half-saturation constant. Then, a novel impulsive system can be formulated as follows [23]:

$$\left\{\begin{array}{lll} \left.\begin{array}{lll} \frac{dy(t)}{dt} & = & -Dy(t) + \frac{\beta x(t)y(t)}{b+x(t)}, \\ \frac{dx(t)}{dt} & = & ax(t)(1 - \frac{x(t)}{K}) - \frac{\alpha x(t)y(t)}{b+x(t)}, \end{array}\right\} & \begin{array}{l} t \notin nT + \{0, \lambda_1, \lambda_2, \cdots, \lambda_m\} \\ t \notin nT + \left\{\tau_1^{(j)}, \tau_2^{(j)}, \cdots, \tau_{m_j}^{(j)}\right\}, 1 \le j \le m+1, \end{array} \\ \left.\begin{array}{lll} y(t^+) & = & \left(1 - p_N^{(i)}\right)y(t), \\ x(t^+) & = & \left(1 - p_P^{(i)}\right)x(t), \end{array}\right\} t = nT + \lambda_i, 1 \le i \le m+1, \\ \left.\begin{array}{lll} y(t^+) & = & \left(1 + p_k^{(j)}\right)y(t) + q_k^{(j)}, \\ x(t^+) & = & x(t), \end{array}\right\} t = nT + \tau_k^{(j)}, 1 \le k \le m_j, 1 \le j \le m+1, \end{array}\right. \tag{3}$$

where $\left(p_N^{(i)}, p_P^{(i)}\right) \in \{(0,0), (p_N, p_P)\}$, $q_k^{(j)} \in \{0, q\}$ and $p_k^{(j)} \in \{0, p_M\}$. $p_M \ge 0$ represents the immigration rate of natural enemies [24–26]. In addition, $m \in Z_+$, and

$$\left\{\begin{array}{l} 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \lambda_{m+1} = T, \\ \lambda_{j-1} < \tau_1^{(j)} < \tau_2^{(j)} < \cdots < \tau_{m_j}^{(j)} < \lambda_j, 1 \le j \le m+1, \end{array}\right.$$

and

$$\left\{\begin{array}{l} \left.\begin{array}{lll} p_N^{(i+n(m+1))} & = & p_N^{(i)}, \\ p_P^{(i+n(m+1))} & = & p_P^{(i)}, \end{array}\right\} \quad 1 \le i \le m+1, \\ \left.\begin{array}{lll} p_k^{(j+n(m+1))} & = & p_k^{(j)}, \\ q_k^{(j+n(m+1))} & = & q_k^{(j)}, \end{array}\right\} \quad 1 \le k \le m_j, 1 \le j \le m+1. \end{array}\right.$$

It should be noted that $\left(p_N^{(i)}, p_P^{(i)}\right) = (0,0)$ indicates that pesticides are not sprayed at time $t = nT + \lambda_i$. Similarly, $q_k^{(j)} = 0$ (or $p_k^{(j)} = 0$) means that there are no natural enemy releases (or immigration) at time $t = nT + \tau_k^{(j)}$. Thus, the modeling approach for system (3) can characterize all of the possible practical cases for the implementation of biological and chemical control tactics.

The goal of our study is mainly to investigate the existence and global stability of pest-present periodic solutions. Unlike the techniques used in [21], the permanence of system (3) can be analyzed by constructing two uniform lower impulsive comparison systems, indicating the mathematical (or biological) essence of the permanence of system (3). To date, the global stability of the pest-present periodic solution of system (1) has not been addressed. Based on the permanence of system (3), we can define a norm with respect to the solution of system (3) and then construct an impulsive comparison system for this norm. Similar to the proof of the contraction mapping principle, we can prove the global stability of the pest-present periodic solution. Due to the complexity of the modeling approach of the

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system (3), the local bifurcation of a pest-present periodic solution cannot be solved by traditional methods [27]. We address this issue by using an implicit function theorem and Taylor's series of two-variable functions. In addition, the local stability of the pest-free periodic solution is investigated by using the variational method, and the global attractiveness of the pest-free periodic solution is also discussed.

The rest of this article is organized as follows. In Section 2, for convenience, we present some definitions and lemmas. In Section 3, the local stability of the pest-free periodic solution is studied using the variational method; and the bifurcation of a pest-present periodic solution of system (3) is studied by using the implicit function theorem and Taylor's series of a two-variable function. Several additional technical computations that were used to establish the results presented in Section 3 are discussed in Appendices $A-D$. Additionally, the global attractiveness of the pest-free periodic solution is discussed in this section. In Section 4, once the threshold condition is satisfied together with a certain other condition, system (3) is permanent; furthermore, the existence and global attractiveness of the pest-present periodic solution are investigated. Numerical simulations that confirm our theoretical findings are discussed in Section 5. Finally, a discussion of the theoretical and numerical results is provided.

2. Preliminaries

In this section, we introduce some definitions and state some preliminary lemmas that will be useful for establishing our results.

Definition 1. *[21] The system (3) is said to be permanent if there are constants $m, M > 0$ (independent of the initial values) and a finite time T_0 such that for all solutions $(y(t), x(t))$ with all initial values $(y(0^+), x(0^+)) > 0$, $m \le y(t), x(t) \le M$ holds for all $T \ge T_0$. Here, T_0 may depend on the initial values $(y(0^+), x(0^+))$.*

Note that $x^* = K$ is a globally stable equilibrium of the system $\frac{dx}{dt} = rx(1-\frac{x}{K})$. Then, we have the following lemma:

Lemma 1. *The positive orthant $\mathbb{R}^2_+$ is an invariant region for the system (3), and for sufficiently large $t > 0$, $x(t) < K$.*

In the following, we assume that $0 < x(t) < K$ for $t \ge 0$.

Consider the following subsystem:

$$\begin{cases} \frac{dy(t)}{dt} = \mu y(t), & t \notin nT + \{0, \lambda_1, \lambda_2, \cdots, \lambda_m\}, \\ & t \notin nT + \left\{\tau_1^{(j)}, \tau_2^{(j)}, \cdots, \tau_{m_j}^{(j)}\right\}, 1 \le j \le m+1, \\ y(t^+) = (1 - p_N^{(i)})y(t), & t = nT + \lambda_i, 1 \le i \le m+1, \\ y(t^+) = (1 + p_k^{(j)})y(t) + q_k^{(j)}, & t \in nT + \tau_k^{(j)}, 1 \le k \le m_j, 1 \le j \le m+1, \end{cases} \tag{4}$$

where $\mu \neq 0$. Then, we have

$$y(t;\mu) = \begin{cases} y(\lambda_{j-1}^+;\mu)e^{\mu(t-\lambda_{j-1})}, & t \in (\lambda_{j-1}, \tau_1^{(j)}], \\ y\left(\left(\tau_{k-1}^{(j)}\right)^+;\mu\right)e^{\mu(t-\tau_{k-1}^{(j)})}, & t \in \left(\tau_{k-1}^{(j)}, \tau_k^{(j)}\right], \\ & k = 2, 3, \cdots, m_j, \\ y\left(\left(\tau_{m_j}^{(j)}\right)^+;\mu\right)e^{\mu(t-\tau_{m_j}^{(j)})}, & t \in \left(\tau_{m_j}^{(j)}, \lambda_j\right], \end{cases} \tag{5}$$

for $j = 1, 2, \cdots, m+1$, where $y(t;\mu)$ is a solution of (4).

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It follows from (4) and (5) that

$$\begin{cases} y\left(\left(\tau_1^{(j)}\right)^+;\mu\right) &= (1+p_1^{(j)})e^{\mu(\tau_1^{(j)}-\lambda_{j-1})}y(\lambda_{j-1}^+;\mu)+q_1^{(j)}, \\ y\left(\left(\tau_k^{(j)}\right)^+;\mu\right) &= (1+p_k^{(j)})e^{\mu(\tau_k^{(j)}-\tau_{k-1}^{(j)})}y\left(\left(\tau_{k-1}^{(j)}\right)^+;\mu\right)+q_k^{(j)}, \\ & \quad k=2,3,\cdots,m_j. \end{cases} \tag{6}$$

Then, it can be obtained by mathematical induction that

$$\begin{aligned} y(\lambda_j^+;\mu) &= \left((1-p_N^{(j)})\prod_{l=1}^{m_j}(1+p_l^{(j)})\right)e^{\mu(\lambda_j-\lambda_{j-1})}y(\lambda_{j-1}^+;\mu)+C_j, \\ & \quad j=1,2,\cdots,m,m+1, \end{aligned} \tag{7}$$

where $\prod\limits_{k+1}^{k}=1$ and

$$C_j = \sum_{k=1}^{m_j}\left((1-p_N^{(j)})\prod_{l=k+1}^{m_j}(1+p_l^{(j)})\right)e^{\mu(\lambda_j-\tau_k^{(j)})}q_k^{(j)}, \tag{8}$$

and, furthermore,

$$\begin{aligned} y(T^+;\mu) &= \prod_{h=1}^{m+1}\left((1-p_N^{(h)})\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\mu T}y(0^+;\mu) \\ &+\sum_{s=1}^{m+1}\left[\prod_{h=s+1}^{m+1}\left((1-p_N^{(h)})\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\mu(T-\lambda_s)}C_s\right]. \end{aligned} \tag{9}$$

Letting $y(T^+;\mu)=y(0^+;\mu)$ yields

$$\begin{aligned} y^*(T^+;\mu) &= y^*(0^+;\mu) \\ &= \frac{\sum\limits_{s=1}^{m+1}\left[\prod\limits_{h=s+1}^{m+1}\left((1-p_N^{(h)})\prod\limits_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\mu(T-\lambda_s)}C_s\right]}{1-\prod\limits_{h=1}^{m+1}\left((1-p_N^{(h)})\prod\limits_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\mu T}}. \end{aligned}$$

Then, if (10) is valid, system (4) has a unique positive periodic solution as follows:

$$y^*(t;\mu)=\begin{cases} y^*(\lambda_{j-1}^+;\mu)e^{\mu(t-\lambda_{j-1})}, & t\in(\lambda_{j-1},\tau_1^{(j)}], \\ y^*\left(\left(\tau_{k-1}^{(j)}\right)^+;\mu\right)e^{\mu(t-\tau_{k-1}^{(j)})}, & t\in\left(\tau_{k-1}^{(j)},\tau_k^{(j)}\right], \\ & k=2,3,\cdots,m_j, \\ y^*\left(\left(\tau_{m_j}^{(j)}\right)^+;\mu\right)e^{\mu(t-\tau_{m_j}^{(j)})}, & t\in\left(\tau_{m_j}^{(j)},\lambda_j\right], \end{cases}$$

where $j=1,2,\cdots,m,m+1$.

Hence, we reach the following conclusions:

Lemma 2. *System (4) has a unique positive solution $y^*(t;\mu)$ if and only if*

$$\prod_{h=1}^{m+1}\left((1-p_N^{(h)})\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\mu T}<1, \tag{10}$$

and for every solution $y(t;\mu)$ of (4), it follows that $\lim\limits_{t\to+\infty}(y(t;\mu)-y^*(t;\mu))=0$ (see Figure 1).

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Proof. From (4), we have

$$\begin{cases} \frac{dz(t)}{dt} = \mu z(t), & t \notin nT + \{0, \lambda_1, \lambda_2, \cdots, \lambda_m\}, \\ & t \notin nT + \left\{\tau_1^{(j)}, \tau_2^{(j)}, \cdots, \tau_{m_j}^{(j)}\right\}, 1 \le j \le m+1, \\ z(t^+) = (1 - p_N^{(i)}) z(t), & t = nT + \lambda_i, 1 \le i \le m+1, \\ z(t^+) = (1 + p_k^{(j)}) z(t), & t \in nT + \tau_k^{(j)}, 1 \le k \le m_j, 1 \le j \le m+1, \end{cases}$$

where $z(t) = y(t;\mu) - y^*(t;\mu)$. Similar to (9) and (8), it follows that

$$z(nT^+) = \prod_{h=1}^{m+1} \left((1 - p_N^{(h)}) \prod_{l=1}^{m_h} (1 + p_l^{(h)}) \right) e^{\mu T} z((n-1)T^+). \tag{11}$$

(11), (10), (8), (7), (6) and (5) indicate that $\lim_{t \to +\infty} z(t) = 0$.

This completes the proof. □

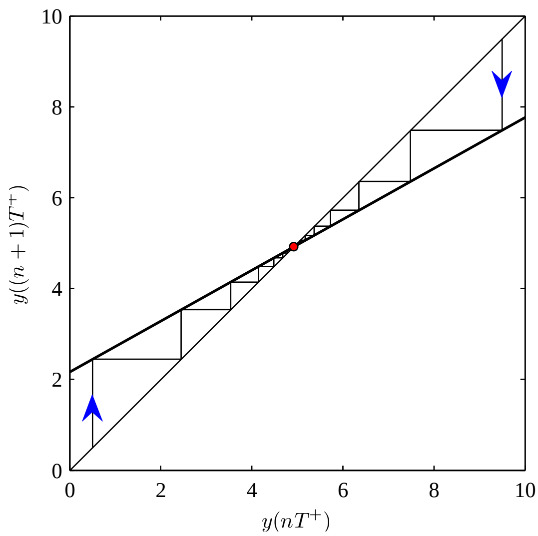

Figure 1. Dynamic cobweb method to show the relationship of $y(nT^+)$ and $y((n+1)T^+)$ corresponding to the special case demonstrated in system (52). The baseline parameter values are fixed as follows: $D = 0.08$, $p_N = 0.45$, $p_M = 0.05$, $q = 2$, $T = 1$ and $m = 2$.

In the following, we always assume that

$$\prod_{h=1}^{m+1} \left((1 - p_N^{(h)}) \prod_{l=1}^{m_h} (1 + p_l^{(h)}) \right) e^{-DT} < 1, \tag{12}$$

and denote

$$y^*(t) = y^*(t; -D).$$

3. Bifurcation of the Pest-Present Periodic Solution

Denote by $\Phi(t; t_0, X^0)$ the solution of the first two equations of (3) for the initial data $t = t_0$ and $X^0 = (y^0, x^0)$; also,

$$\Phi(t; t_0, X^0) = (y(t; t_0, X^0), x(t; t_0, X^0)).$$

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Additionally, we define some relevant mappings by

$$\begin{cases} P^{(i)}(y,x) &= \begin{pmatrix} 1-p_N^{(i)} & 0 \\ 0 & 1-p_P^{(i)} \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix}, \\ & \quad i=1,\cdots,m,m+1, \\ N_k^{(j)}(y,x) &= \begin{pmatrix} 1+p_k^{(j)} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} q_k^{(j)} \\ 0 \end{pmatrix}, \\ & \quad 1 \le k \le m_j, 1 \le j \le m+1, \end{cases} \tag{13}$$

and

$$\begin{cases} I_1^{(1)}(X^0) &= N_1^{(1)}(\Phi(\tau_1^{(1)};X^0)) \\ &\triangleq (I_{1_1}^{(1)}(X^0), I_{1_2}^{(1)}(X^0))^T, \\ I_k^{(1)}(X^0) &= N_k^{(1)}(\Phi(\tau_k^{(1)}-\tau_{k-1}^{(1)};I_{k-1}^{(1)}(X^0))) \\ &\triangleq (I_{k_1}^{(1)}(X^0), I_{k_2}^{(1)}(X^0))^T, \\ & \quad 2 \le k \le m_1, \\ \Psi^{(1)}(X^0) &= P^{(1)}(\Phi(\lambda_1-\tau_{m_1}^{(1)};I_{m_1}^{(1)}(X^0))) \\ &\triangleq (\Psi_1^{(1)}(X^0), \Psi_2^{(1)}(X^0)), \\ I_1^{(j)}(X^0) &= N_1^{(j)}(\Phi(\tau_1^{(j)}-\lambda_{j-1};\Psi^{(j-1)}(X^0))) \\ &\triangleq (I_{1_1}^{(j)}(X^0), I_{1_2}^{(j)}(X^0))^T, \\ I_k^{(j)}(X^0) &= N_k^{(j)}(\Phi(\tau_k^{(j)}-\tau_{k-1}^{(j)};I_{k-1}^{(j)}(X^0))) \\ &\triangleq (I_{k_1}^{(j)}(X^0), I_{k_2}^{(j)}(X^0))^T, \\ & \quad 2 \le k \le m_j, \\ \Psi^{(j)}(X^0) &= P^{(j)}(\Phi(\lambda_j-\tau_{m_j}^{(j)};I_{m_j}^{(j)}(X^0))) \\ &\triangleq (\Psi_1^{(j)}(X^0), \Psi_2^{(j)}(X^0)), \\ & \quad 2 \le j \le m+1. \end{cases} \tag{14}$$

Clearly, it follows that

$$\begin{cases} \begin{pmatrix} I_{1_1}^{(1)}(X^0) \\ I_{1_2}^{(1)}(X^0) \end{pmatrix} &= \begin{pmatrix} (1+p_1^{(1)})y(\tau_1^{(1)};X^0)+q_1^{(1)} \\ x(\tau_1^{(1)};X^0) \end{pmatrix}, \\ \begin{pmatrix} I_{k_1}^{(1)}(X^0) \\ I_{k_2}^{(1)}(X^0) \end{pmatrix} &= \begin{pmatrix} (1+p_k^{(1)})y(\tau_k^{(1)}-\tau_{k-1}^{(1)};(I_{(k-1)_1}^{(1)}(X^0), I_{(k-1)_2}^{(1)}(X^0)))+q_k^{(1)} \\ x(\tau_k^{(1)}-\tau_{k-1}^{(1)};(I_{(k-1)_1}^{(1)}(X^0), I_{(k-1)_2}^{(1)}(X^0))) \end{pmatrix}, \\ & \quad k=2,3,\cdots,m_1, \\ \begin{pmatrix} \Psi_1^{(1)}(X^0) \\ \Psi_2^{(1)}(X^0) \end{pmatrix} &= \begin{pmatrix} (1-p_N^{(1)})y(\lambda_1-\tau_{m_1}^{(1)};(I_{(m_1)_1}^{(1)}(X^0), I_{(m_1)_2}^{(1)}(X^0))) \\ (1-p_P^{(1)})x(\lambda_1-\tau_{m_1}^{(1)};(I_{(m_1)_1}^{(1)}(X^0), I_{(m_1)_2}^{(1)}(X^0))) \end{pmatrix}, \\ \begin{pmatrix} I_{1_1}^{(j)}(X^0) \\ I_{1_2}^{(j)}(X^0) \end{pmatrix} &= \begin{pmatrix} (1+p_1^{(j)})y(\tau_1^{(j)}-\lambda_{j-1};(\Psi_1^{(j-1)}(X^0), \Psi_2^{(j-1)}(X^0)))+q_1^{(j)} \\ x(\tau_1^{(j)}-\lambda_{j-1};(\Psi_1^{(j-1)}(X^0), \Psi_2^{(j-1)}(X^0))) \end{pmatrix}, \\ \begin{pmatrix} I_{k_1}^{(j)}(X^0) \\ I_{k_2}^{(j)}(X^0) \end{pmatrix} &= \begin{pmatrix} (1+p_k^{(j)})y(\tau_k^{(j)}-\tau_{k-1}^{(j)};(I_{(k-1)_1}^{(j)}(X^0), I_{(k-1)_2}^{(j)}(X^0)))+q_k^{(j)} \\ x(\tau_k^{(j)}-\tau_{k-1}^{(j)};(I_{(k-1)_1}^{(j)}(X^0), I_{(k-1)_2}^{j)}(X^0))) \end{pmatrix}, \\ & \quad k=2,3,\cdots,m_j, \\ \begin{pmatrix} \Psi_1^{(j)}(X^0) \\ \Psi_2^{(j)}(X^0) \end{pmatrix} &= \begin{pmatrix} (1-p_N^{(j)})y(\lambda_j-\tau_{m_j}^{(j)};(I_{(m_j)_1}^{(j)}(X^0), I_{(m_j)_2}^{(j)}(X^0))) \\ (1-p_P^{(j)})x(\lambda_j-\tau_{m_j}^{(j)};(I_{(m_j)_1}^{(j)}(X^0), I_{(m_j)_2}^{(j)}(X^0))) \end{pmatrix}, \\ & \quad j=2,3,\cdots,m,m+1. \end{cases} \tag{15}$$

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For $j = 1, 2, \cdots, m, m+1$, it follows from (A7) that

$$c^{(j)}(X_0^*) = (1 - p_P^{(j)}) e^{\int_{\lambda_{j-1}^+}^{\lambda_j} (a - \frac{\alpha y^*(t)}{b}) dt}, \tag{16}$$

where $X_0^* = (y^*(0^+), 0)^T$. Then, from (A8), (12) and (16), we see that the two eigenvalues of the Jacobian matrix of the map $\Psi(X^0)$ at the point X_0^* are

$$\begin{cases} \mu_1 = \prod_{h=1}^{m+1} \left((1 - p_N^{(h)}) \prod_{l=1}^{m_h} (1 + p_l^{(h)}) \right) e^{-DT} < 1, \\ \mu_2 = \left(\prod_{j=1}^{m+1} (1 - p_P^{(j)}) \right) e^{\int_{0^+}^{T} (a - \frac{\alpha y^*(t)}{b}) dt}, \end{cases} \tag{17}$$

which results in the following theorem [28]:

Theorem 1. *The pest-free periodic solution $(y^*(t), 0)$ of system (3) is locally asymptotically stable provided that*

$$R_s \triangleq \left(\prod_{j=1}^{m+1} (1 - p_P^{(j)}) \right) e^{\int_{0^+}^{T} (a - \frac{\alpha y^*(t)}{b}) dt} < 1.$$

Using the comparison theorem, we find that [29,30]

Theorem 2. *The pest-free periodic solution $(y^*(t), 0)$ of system (3) is globally attractive provided that*

$$R_a \triangleq \left(\prod_{i=1}^{m+1} (1 - p_P^{(i)}) \right) e^{\int_{0^+}^{T} \left(a - \frac{\alpha y^*(t)}{b+K} \right) dt} < 1. \tag{18}$$

In fact, we can prove that a necessary condition for the bifurcation of a pest-present periodic solution is [31]

$$\mu_2 = \left(\prod_{j=1}^{m+1} (1 - p_P^{(j)}) \right) e^{\int_{0^+}^{T} (a - \frac{\alpha y^*(t)}{b}) dt} = 1. \tag{19}$$

According to Theorem 3, under a certain condition, (19) is sufficient as well.

To investigate the bifurcation of the pest-present periodic solution of system (3), we can consider the system as follows:

$$\begin{cases} \left. \begin{aligned} \frac{dy(t)}{dt} &= -Dy(t) + \frac{\beta x(t) y(t)}{b + x(t)}, \\ \frac{dx(t)}{dt} &= ax(t)(1 - \frac{x(t)}{K}) - \frac{\alpha x(t) y(t)}{b + x(t)}, \end{aligned} \right\} \begin{aligned} & t \notin nT + \{0, L_1 T, L_2 T, \cdots, L_m T\} \\ & t \notin nT + \left\{ l_1^{(j)} T, l_2^{(j)} T, \cdots, l_{m_j}^{(j)} T \right\}, 1 \le j \le m+1, \end{aligned} \\ \left. \begin{aligned} y(t^+) &= \left(1 - p_N^{(i)}\right) y(t), \\ x(t^+) &= \left(1 - p_P^{(i)}\right) x(t), \end{aligned} \right\} t = nT + L_i T, 1 \le i \le m+1, \\ \left. \begin{aligned} y(t^+) &= \left(1 + p_k^{(j)}\right) y(t) + q_k^{(j)}, \\ x(t^+) &= x(t), \end{aligned} \right\} t \in nT + l_k^{(j)} T, 1 \le k \le m_j, 1 \le j \le m+1, \end{cases} \tag{20}$$

where

$$\begin{cases} 0 = L_0 < L_1 < L_2 < \cdots < L_m < L_{m+1} = 1, \\ L_{j-1} = l_0^{(j)} < l_1^{(j)} < l_2^{(j)} < \cdots < l_{m_j}^{(j)} < l_{m_j+1}^{(j)} = L_j, 1 \le j \le m+1. \end{cases}$$

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Theorem 3. *Suppose that*

$$\begin{cases} \left(\prod_{j=1}^{m+1}(1-p_P^{(j)})\right)e^{\int_{0^+}^{T}(a-\frac{\alpha y^*(t)}{b})dt} & = & 1, \\ \frac{ab}{K\alpha} & \geq & \frac{\max\limits_{1\leq j\leq m+1}\max\limits_{1\leq k\leq m_j} y^*(l_k^{(j)}T^+)}{b}, \end{cases} \tag{21}$$

holds, then the pest-free periodic solution $(y^*(t), 0)$ *of system* (20) *bifurcates into a pest-present periodic solution.*

Proof. Assume that the period T involved in (21) is rewritten as T_b, i.e.,

$$\begin{cases} \left(\prod_{j=1}^{m+1}(1-p_P^{(j)})\right)e^{\int_{0^+}^{T_b}(a-\frac{\alpha y^*(t)}{b})dt} & = & 1, \\ \frac{ab}{K\alpha} & \geq & \frac{\max\limits_{1\leq j\leq m+1}\max\limits_{1\leq k\leq m_j} y^*(l_k^{(j)}T_b^+)}{b}. \end{cases} \tag{22}$$

Denote

$$\begin{aligned} G(T, X^0) & = \Psi^{(m+1)}(X^0) - X^0 \\ & \triangleq (g_1(T, X^0), g_2(T, X^0))^T. \end{aligned} \tag{23}$$

Then, it follows from (A2), (17), (A4) and (22) that

$$\begin{aligned} D_{X^0}G(X_0^*) & = D_{X^0}\Psi^{(m+1)}(X_0^*) - E_2 \\ & = \begin{pmatrix} a_0^* & b_0^* \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{24}$$

where

$$\begin{cases} a_0^* & = & \prod_{h=1}^{m+1}\left((1-p_N^{(h)})\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{-DT_b} - 1 < 0, \\ b_0^* & > & 0. \end{cases}$$

On the linear space $\mathbb{R}^2$ defined in the field $\mathbb{R}$, we can define a linear transformation $\mathscr{A}$ as follows:

$$\mathscr{A}(y, x) = D_{X^0}G(X_0^*)\begin{pmatrix} y \\ x \end{pmatrix}, \tag{25}$$

where $X = (y, x)^T \in \mathbb{R}^2$. Then, for $(\alpha_0, \beta_0)^T \in \mathscr{A}^{-1}(0)$, it follows that $\mathscr{A}(\alpha_0, \beta_0) = 0$; i.e.,

$$\begin{pmatrix} a_0^* & b_0^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = 0,$$

which implies that

$$\begin{cases} \dim(\mathscr{A}(\mathbb{R}^2)) & = & \mathrm{rank}(D_{X^0}G(X_0^*)) = 1, \\ \dim(\mathscr{A}^{-1}(0)) & = & 2 - \dim(\mathscr{A}(\mathbb{R}^2)) = 1, \end{cases}$$

and a basis in $\mathscr{A}^{-1}(0)$ is $V_p = (\frac{b_0^*}{-a_0^*}, 1)^T$.

On the other hand, it follows from (25) and (24) that $\mathscr{A}(\mathbb{R}^2) = \{(y, 0)^T | y \in \mathbb{R}\}$, which possesses a basis $V_f' = (a_0^*, 0)^T$. Let $\mathscr{A}(\alpha_1, \beta_1) = V_f'$, i.e.,

$$\begin{pmatrix} a_0^* & b_0^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} a_0^* \\ 0 \end{pmatrix}.$$

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Then, we can take $(\alpha_1, \beta_1)^T = (1,0)^T \triangleq V_f$. Then, a group of bases of the linear space $\mathbb{R}^2$ can be taken as

$$\begin{cases} V_p &= (\frac{b_0^*}{-a_0^*}, 1)^T, \\ V_f &= (1,0)^T. \end{cases}$$

Let $X^0 = c_p V_p + c_f V_f$, where $c_p, c_f \in \mathbb{R}$, and

$$\begin{cases} f_1(T, c_p, c_f) &= g_1(T, c_p V_p + c_f V_f), \\ f_2(T, c_p, c_f) &= g_2(T, c_p V_p + c_f V_f). \end{cases} \tag{26}$$

It follows from (26), (23) and (24) that

$$\begin{cases} f_1(T_b, 0, y^*(0^+)) &= 0, \\ \frac{\partial f_1}{\partial c_f}(T_b, 0, y^*(0^+)) &= \frac{\partial g_1}{\partial y^0}(T_b, (y^*(0^+), 0)) \\ &= a_0^* < 0. \end{cases}$$

According to the implicit function theorem, there exists a unique function $c_f = c_f(T, c_p)$ in some neighborhood of $(T_b, 0, y^*(0^+))$ such that

$$\begin{cases} c_f(T_b, 0) &= y^*(0^+), \\ f_1(T, c_p, c_f(T, c_p)) &= 0. \end{cases} \tag{27}$$

Let us denote

$$f(T, c_p) = f_2(T, c_p, c_f(T, c_p)). \tag{28}$$

Thus, from (28), (26), (23), (27), (24) and (20), we see that

$$\begin{cases} f(T_b, 0) &= g_2(T_b, X_0^*) \\ &= 0, \\ \frac{\partial f}{\partial c_p}(T_b, 0) &= \frac{\partial g_2}{\partial y^0}(T, c_p V_p + c_f(T, c_p) V_f)\Big|_{(T, c_p)=(T_b, 0)} \left(\frac{b_0^*}{-a_0^*} + \frac{\partial c_f(T_b, 0)}{\partial c_p} \right) \\ &\quad + \frac{\partial g_2}{\partial x^0}(T, c_p V_p + c_f V_f)\Big|_{(T, c_p)=(T_b, 0)} \\ &= 0, \\ \frac{\partial^n f}{\partial T^n}(T, 0) &= \frac{\partial^n}{\partial T^n}(g_2(T, c_f(T, 0) V_f)) \\ &= 0, \quad n = 1, 2, \cdots. \end{cases} \tag{29}$$

From (29), we can determine the Taylor series of f at point $(T_b, 0)$ as follows:

$$f(T, c_p) = c_p \hat{f}(T, c_p),$$

where

$$\begin{aligned} \hat{f}(T, c_p) \quad = \quad & \frac{1}{2!}\left(2(T - T_b)\frac{\partial^2 f}{\partial T \partial c_p}(T_b, 0) + c_p \frac{\partial^2 f}{\partial c_p^2}(T_b, 0) \right) \\ & + \cdots \\ & + \frac{1}{n!}\Bigg(C_n^1 (T - T_b)^{n-1} \frac{\partial^n f}{\partial T^{n-1} \partial c_p}(T_b, 0) + C_n^2 (T - T_b)^{n-2} c_p \frac{\partial^n f}{\partial T^{n-2} \partial c_p^2}(T_b, 0) \\ & \qquad + \cdots + c_p^{n-1} \frac{\partial^n f}{\partial c_p^n}(T_b, 0) \Bigg) \\ & + \cdots. \end{aligned}$$

From (28), (26), (23) and (14), we see that

$$f(T, c_p) = \Psi_2^{(m+1)}(c_p V_p + c_f(T, c_p) V_f) - c_p, \tag{30}$$

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and thus, it follows from (30), (A27) and (A33) that

$$\begin{cases} \frac{\partial^2 f(T_b,0)}{\partial c_p^2} & = & \frac{\partial^2}{\partial c_p^2}\left(\Psi_2^{(m+1)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ & < & 0, \\ \frac{\partial^2 f(T_b,0)}{\partial c_p \partial T} & = & \frac{\partial^2}{\partial c_p \partial T}\left(\Psi_2^{(m+1)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ & > & 0. \end{cases} \tag{31}$$

Note that

$$\begin{aligned} \frac{\partial \hat{f}}{\partial c_p}(T,c_p) \quad = \quad & \frac{1}{2!}\frac{\partial^2 f}{\partial c_p^2}(T_b,0) \\ & + \cdots \\ & + \frac{1}{n!}\left(C_n^2(T-T_b)^{n-2}\frac{\partial^n f}{\partial T^{n-2}\partial c_p^2}(T_b,0) + \cdots + (n-1)c_p^{n-2}\frac{\partial^n f}{\partial c_p^n}(T_b,0)\right) \\ & + \cdots, \end{aligned}$$

and furthermore,

$$\frac{\partial \hat{f}}{\partial c_p}(T_b,0) = \frac{1}{2!}\frac{\partial^2 f}{\partial c_p^2}(T_b,0) < 0 \tag{32}$$

based on (31). In addition, $\hat{f}(T_b,0) = 0$. Then, there exists a unique function $c_p = c_p(T)$ in some neighborhood of $(T_b,0)$ such that

$$\begin{cases} c_p(T_b) & = & 0, \\ \hat{f}(T,c_p(T)) & = & 0. \end{cases} \tag{33}$$

It follows from (33) that

$$\frac{\partial \hat{f}}{\partial T}(T_b,0) + \frac{\partial \hat{f}}{\partial c_p}(T_b,0)\frac{dc_p(T_b)}{dT} = 0. \tag{34}$$

Then, from (33), (34), (31) and (32), we have

$$\begin{aligned} \lim_{T\to T_b^+}\frac{c_p(T)}{T-T_b} & = \lim_{T\to T_b^+}\frac{c_p(T)-c_p(T_b)}{T-T_b} \\ & = \frac{\frac{\partial \hat{f}}{\partial T}(T_b,0)}{-\frac{\partial \hat{f}}{\partial c_p}(T_b,0)} \\ & = \frac{\frac{\partial^2 f}{\partial c_p \partial T}(T_b,0)}{-\frac{\partial \hat{f}}{\partial c_p}(T_b,0)} \\ & > 0. \end{aligned}$$

Thus, there exists a $\Delta T > 0$ such that when $T \in (T_b, T_b + \Delta T)$, it follows that $c_p(T) > 0$. Then, a pest-present periodic solution emerges.

This completes the proof. □

4. Existence and Global Attractiveness of the Pest-Present Periodic Solution of System (3)

In this section, we provide some sufficient conditions ensuring the permanence and the existence and global attractiveness of the pest-present periodic solution of system (3).

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Theorem 4. *Assume that*

$$\begin{cases} \prod_{h=1}^{m+1}\left((1-p_N^{(h)})\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{(-D+\frac{\beta K}{b+K})T} & < & 1, \\ R_s = \left(\prod_{j=1}^{m+1}(1-p_P^{(j)})\right)e^{\int_{0^+}^{T}(a-\frac{\alpha y^*(t)}{b})dt} & > & 1, \end{cases} \tag{35}$$

We draw the following conclusions:

(i) *the system (3) is permanent;*

(ii) *if*

$$\prod_{h=1}^{m+1}\left(\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\hat{C}_1 T} < 1 \tag{36}$$

holds, where

$$\hat{C}_1 \triangleq \max\left\{-D+\frac{\beta K}{b+K}+\frac{\alpha}{b+m_P}, m_P\left(\frac{(b\beta+\alpha)M_N}{(b+m_P)^2}-\frac{a}{K}\right)\right\},$$

then there exists a globally attractive pest-present periodic solution for system (3).

Proof. (i) According to Lemmas 2, (3) and (35), we may assume that $m_N \le y(t) \le M_N$ for $t \ge 0$, where

$$\begin{cases} m_N &=& \min_{1\le j\le m+1}\min_{1\le k\le m_j} y^*(l_k^{(j)}T)-\varepsilon_1, \\ M_N &=& \max_{1\le j\le m+1}\max_{t\in[\lambda_{j-1}^+,\tau_1^{(j)}]\cup\cdots\cup[(\tau_{m_j}^{(j)})^+,\lambda_j]} y^*\left(t;-D+\frac{\beta K}{b+K}\right)+\varepsilon_2, \end{cases}$$

with $\varepsilon_2 > 0$ and $\varepsilon_1 \in \left(0, \min_{1\le j\le m+1}\min_{1\le k\le m_j} y^*(l_k^{(j)}T)\right)$. According to the definition of the permanence of system (3), we only need to find $m_P > 0$ such that $x(t) \ge m_P$ for sufficiently large t.

In view of (12) and (35), we can choose $m_0 > 0$ and $\varepsilon_3 > 0$ such that

$$\begin{cases} \prod_{h=1}^{m+1}\left((1-p_N^{(h)})\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{(-D+\frac{\beta m_0}{b+m_0})T} & < & 1, \\ \zeta & > & 1, \end{cases}$$

where

$$\zeta \triangleq \left(\prod_{i=1}^{m+1}(1-p_P^{(i)})\right)e^{\int_{0^+}^{T}\left[a-\frac{am_0}{K}-\frac{\alpha}{b}\left(y^*\left(t;-D+\frac{\beta m_0}{b+m_0}\right)+\varepsilon_3\right)\right]dt}.$$

Consider the following system:

$$\begin{cases} \frac{dy(t)}{dt} &=& (-D+\frac{\beta m_0}{b+m_0})y(t), & t\notin nT+\{0,\lambda_1,\lambda_1,\cdots,\lambda_m\}, \\ & & & t\notin nT+\left\{\tau_1^{(j)},\tau_2^{(j)},\cdots,\tau_{m_j}^{(j)}\right\}, \\ & & & 1\le j\le m+1, \\ y(t^+) &=& (1-p_N^{(i)})y(t), & t=nT+\lambda_i, 1\le i\le m+1, \\ y(t^+) &=& (1+p_k^{(j)})y(t)+q_k^{(j)}, & t=nT+\tau_k^{(j)}, 1\le k\le m_j, 1\le j\le m+1. \end{cases}$$

According to Lemma 2, we see that $\lim_{t\to+\infty}(y(t;-D+\frac{\beta m_0}{b+m_0})-y^*(t;-D+\frac{\beta m_0}{b+m_0})) = 0$. Then, there exists a $t_3 > 0$ such that

$$y(t;-D+\frac{\beta m_0}{b+m_0}) < y^*(t;-D+\frac{\beta m_0}{b+m_0})+\varepsilon_3, \quad \text{for} \quad t > t_3. \tag{37}$$

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Similar to (45), we can prove that there exists a $t_4 > t_3$ such that $x(t_4) \geq m_0$. Without loss of generality, assume that there exist some $t > t_4$ such that $x(t) < m_0$. Set $t^* = \inf\{t|t > t_4, x(t) < m_0\}$; then, $t^* \geq t_4 > t_3$. Then, $x(t) \geq m_0$ holds for $t \in (t_4, t^*]$, and assume that $t^* \in (n_1T, (n_1+1)T]$, where $n_1 \in Z_+$.

Consider the following system:

$$\begin{cases} \begin{aligned} \frac{dx(t)}{dt} &= x(t)\left(\underbrace{a - \frac{am_0}{K} - \max\left\{\frac{\alpha M_N}{b}, a\right\}}_{\triangleq \eta}\right) \\ &\triangleq x(t)\eta, \end{aligned} & \begin{array}{l} t \notin nT + \{0, \lambda_1, \lambda_1, \cdots, \lambda_m\}, \\ t \notin nT + \left\{\tau_1^{(j)}, \tau_2^{(j)}, \cdots, \tau_{m_j}^{(j)}\right\}, \\ 1 \leq j \leq m+1, \end{array} \\ x(t^+) = (1 - p_P^{(i)}))x(t), & t = nT + \lambda_i, 1 \leq i \leq m+1, \\ x(t^+) = x(t), & t = nT + \tau_k^{(j)}, 1 \leq k \leq m_j, 1 \leq j \leq m+1, \\ x(n_1T^+) = \frac{m_0}{2}, & \end{cases} \tag{38}$$

where $\eta = a - \frac{am_0}{K} - \max\left\{\frac{\alpha M_N}{b}, a\right\} < 0$. Then, it follows that

$$\begin{aligned} \hat{x}(t^*) &< \hat{x}(n_1T^+) \\ &= \frac{m_0}{2} \\ &< x(t^*), \end{aligned} \tag{39}$$

where $\hat{x}(t)$ is a solution of system (38).

We claim that $x(t) \geq \hat{x}(t)$ holds within the interval $(t^*, +\infty)$. By contrast, assume that there exists a $t_5 > t^*$ such that $x(t_5) < \hat{x}(t_5)$. Set

$$t^{**} = \inf\{t|t > t^*, x(t) < \hat{x}(t)\}.$$

From (39), $x(t^{*+}) > \hat{x}(t^{*+})$. Then, $t^{**} > t^*$, and it follows for the interval $(t^*, t^{**}]$ that $x(t) \geq \hat{x}(t)$. Obviously, it is impossible that $x(t^{**}) > \hat{x}(t^{**})$. Therefore, assume that $x(t^{**}) = \hat{x}(t^{**}) < \hat{x}(n_1T^+) = \frac{m_0}{2}$. Then, $x(t^{**+}) = \hat{x}(t^{**+}) < m_0$, and furthermore, there exists a $\delta > 0$ such that when $t \in (t^{**}, t^{**} + \delta)$, it follows that $x(t) < m_0$. Moreover, it follows for the interval $(t^{**}, t^{**} + \delta)$ that

$$\begin{cases} \begin{aligned} \frac{dx(t)}{dt} &\geq x(t)\left(\underbrace{a - \frac{am_0}{K} - \max\left\{\frac{\alpha M_N}{b}, a\right\}}_{\triangleq \eta}\right) \\ &\triangleq x(t)\eta, \end{aligned} & \begin{array}{l} t \notin nT + \{0, \lambda_1, \lambda_1, \cdots, \lambda_m\}, \\ t \notin nT + \left\{\tau_1^{(j)}, \tau_2^{(j)}, \cdots, \tau_{m_j}^{(j)}\right\}, \\ 1 \leq j \leq m+1, \end{array} \\ x(t^+) = (1 - p_P^{(i)})x(t), & t = nT + \lambda_i, 1 \leq i \leq m+1, \\ x(t^+) = x(t), & t = nT + \tau_k^{(j)}, 1 \leq k \leq m_j, 1 \leq j \leq m+1, \end{cases} \tag{40}$$

Thus, $x(t) \geq \hat{x}(t)$ is valid in the interval $(t^{**}, t^{**} + \delta)$, contradicting the definition of $t^{**} = \inf\{t|t > t^*, x(t) < \hat{x}(t)\}$.

Remark: In fact, system (38) can be replaced by the following system:

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$$\begin{cases} \frac{dx(t)}{dt} = x(t)\Bigg(\underbrace{\min\Big\{\min_{(y,x)\in[m_N,M_N]\times[0,K]}\Big\{a-\frac{ax}{K}-\frac{\alpha y}{b+x}\Big\},0\Big\}-\epsilon}_{\triangleq a_-}\Bigg) & t \notin nT+\{0,\lambda_1,\lambda_1,\cdots,\lambda_m\}, \\ \qquad\triangleq x(t)a_-, & t\notin nT+\Big\{\tau_1^{(j)},\tau_2^{(j)},\cdots,\tau_{m_j}^{(j)}\Big\}, \\ & 1\le j\le m+1, \\ & t=nT+\lambda_i, 1\le i\le m+1, \\ x(t^+) = (1-p_P^{(i)}))x(t), & t=nT+\tau_k^{(j)}, 1\le k\le m_j, 1\le j\le m+1. \\ x(t^+) = x(t), & \\ x(n_1T^+) = \frac{m_0}{2}, & \end{cases} \tag{41}$$

where $\epsilon > 0$ and $a_- = \min\Big\{\min_{(y,x)\in[m_N,M_N]\times[0,K]}\Big\{a-\frac{ax}{K}-\frac{\alpha y}{b+x}\Big\},0\Big\}-\epsilon < 0$. Similarly, we can also prove that $x(t) > \hat{x}(t)$ holds in the interval $(t^*,+\infty)$, where $\hat{x}(t)$ is a solution of system (41).

Consider the following system:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)\Big[a-\frac{am_0}{K}-\frac{\alpha}{b}\Big(y^*\Big(t;-D+\frac{\beta m_0}{b+m_0}\Big)+\varepsilon_3\Big)\Big], & t\notin nT+\{0,\lambda_1,\lambda_1,\cdots,\lambda_m\}, \\ & t\notin nT+\Big\{\tau_1^{(j)},\tau_2^{(j)},\cdots,\tau_{m_j}^{(j)}\Big\}, \\ & 1\le j\le m+1, \\ x(t^+) = (1-p_P^{(i)}))x(t), & t=nT+\lambda_i, 1\le i\le m+1, \\ x(t^+) = x(t), & t=nT+\tau_k^{(j)}, 1\le k\le m_j, 1\le j\le m+1, \\ x((n_1+1)T) = \hat{x}((n_1+1)T), & \end{cases} \tag{42}$$

We have

$$\widetilde{x}((n+1)T) = \widetilde{x}(nT)\zeta,$$

where $\widetilde{x}(t)$ is a solution of system (42). Thus, it follows that

$$\begin{aligned} \widetilde{x}((n_1+1)T+n_2T) &= \widetilde{x}((n_1+1)T)\zeta^{n_2} \\ &= \hat{x}((n_1+1)T)\zeta^{n_2} \\ &> m_0, \end{aligned} \tag{43}$$

where

$$n_2 = \left[\frac{\ln\frac{m_0}{\hat{x}((n_1+1)T)}}{\ln\zeta}\right]+1.$$

Without loss of generality, assume that for all $t\in(t^*,(n_1+1)T]$, $x(t) < m_0$. We claim that there exists a $t_6 \in ((n_1+1)T,(n_1+1)T+n_2T)$ such that $x(t_6) > m_0$. By contrast, assume that $x(t)\le m_0$ for all $t\in((n_1+1)T,(n_1+1)T+n_2T]$. Then, when $t\in((n_1+1)T,(n_1+1)T+n_2T]$, it follows from (3) and (37) that

$$\begin{cases} \frac{dx(t)}{dt} \ge x(t)\Big[a-\frac{am_0}{K}-\frac{\alpha}{b}\Big(y^*\Big(t;-D+\frac{\beta m_0}{b+m_0}\Big)+\varepsilon_3\Big)\Big], & t\notin nT+\{0,\lambda_1,\lambda_1,\cdots,\lambda_m\}, \\ & t\notin nT+\Big\{\tau_1^{(j)},\tau_2^{(j)},\cdots,\tau_{m_j}^{(j)}\Big\}, \\ & 1\le j\le m+1, \\ x(t^+) = (1-p_P^{(i)}))x(t), & t=nT+\lambda_i, 1\le i\le m+1, \\ x(t^+) = x(t), & t=nT+\tau_k^{(j)}, 1\le k\le m_j, 1\le j\le m+1. \end{cases} \tag{44}$$

Note that $x((n_1+1)T^+;t^*,x(t^*)) \ge \hat{x}((n_1+1)T^+)$. It follows from (44), (42) and (43) that

$$\begin{aligned} x((n_1+1)T+n_2T) &\ge \widetilde{x}((n_1+1)T+n_2T) \\ &> m_0, \end{aligned} \tag{45}$$

which is a contradiction.

For the interval (t^*,t_6),

$$\begin{aligned} x(t) &\ge \hat{x}(t) \\ &> \hat{x}(t_6) \\ &> \hat{x}((n_1+1)T+n_2T) \triangleq m_P. \end{aligned}$$

For $t > t_6$, the same arguments can be continued because $x(t_6) > m_0$.

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(ii) To investigate the existence and global attractiveness of the pest-present periodic solution of system (3), we only need to carry it out it in the set S, where

$$S \triangleq \left\{ (y(t), x(t))^T \in \mathbb{R}_+^2 \middle| m_N \le y(t) \le M_N, m_P \le x(t) \le K \right\}.$$

(3) shows that when $t \notin nT + \{0, \lambda_1, \lambda_2, \cdots, \lambda_m\}$ and $t \notin nT + \left\{\tau_1^{(j)}, \tau_2^{(j)}, \cdots, \tau_{m_j}^{(j)}\right\}$, where $1 \le j \le m+1$, we have the following:

$$\begin{cases} \frac{dy(t)}{dt} &= -Dy(t) + \frac{\beta x(t)y(t)}{b+x(t)}, \\ \frac{d\ln x(t)}{dt} &= a(1 - \frac{x(t)}{K}) - \frac{\alpha y(t)}{b+x(t)}. \end{cases} \tag{46}$$

Let us assume that $(y_i(t), x_i(t)) \in S$ and $i = 1, 2$ are two solutions of system (3). Then, when

$$\begin{cases} t \notin nT + \{0, \lambda_1, \lambda_2, \cdots, \lambda_m\}, \\ t \notin nT + \left\{\tau_1^{(j)}, \tau_2^{(j)}, \cdots, \tau_{m_j}^{(j)}\right\}, \qquad 1 \le j \le m+1, \\ (y_2(t) - y_1(t))(x_2(t), x_1(t)) \neq 0, \end{cases}$$

it follows from (46) that

$$\begin{aligned} \frac{d|y_2(t)-y_1(t)|}{dt} &= \operatorname{sgn}(y_2(t) - y_1(t))\left(\frac{dy_2(t)}{dt} - \frac{dy_1(t)}{dt}\right) \\ &= -D|y_2(t) - y_1(t)| + \operatorname{sgn}(y_2(t) - y_1(t))\left[\frac{\beta x_2(t)y_2(t)}{b+x_2(t)} - \frac{\beta x_1(t)y_1(t)}{b+x_1(t)}\right] \\ &= -D|y_2(t) - y_1(t)| \\ &\quad +\operatorname{sgn}(y_2(t) - y_1(t))\left[\frac{\beta\eta_1}{b+\eta_1}(y_2(t) - y_1(t)) + \frac{b\beta\xi_1}{(b+\eta_1)^2}(x_2(t) - x_1(t))\right] \\ &\le (-D + \frac{\beta\eta_1}{b+\eta_1})|y_2(t) - y_1(t)| + \frac{b\beta\xi_1}{(b+\eta_1)^2}|x_2(t) - x_1(t)|, \end{aligned} \tag{47}$$

where $(\xi_1, \eta_1) = (y_1(t), x_1(t)) + \theta_1((y_2(t), x_2(t)) - (y_1(t), x_1(t)))$, $\theta_1 \in (0, 1)$, and

$$\begin{aligned} \frac{d|\ln x_2(t)-\ln x_1(t)|}{dt} &= \operatorname{sgn}(x_2(t) - x_1(t))\left(\frac{d\ln x_2(t)}{dt} - \frac{d\ln x_1(t)}{dt}\right) \\ &= -\frac{a}{K}|x_2(t) - x_1(t)| \\ &\quad -\operatorname{sgn}(x_2(t) - x_1(t))\left[\frac{\alpha y_2(t)}{b+x_2(t)} - \frac{\alpha y_1(t)}{b+x_1(t)}\right] \\ &= -\frac{a}{K}|x_2(t) - x_1(t)| \\ &\quad -\operatorname{sgn}(x_2(t) - x_1(t))\frac{\alpha}{(b+x_2(t))(b+x_1(t))} \\ &\qquad \times[(b + x_1(t))(y_2(t) - y_1(t)) - y_1(t)(x_2(t) - x_1(t))] \\ &\le \frac{\alpha}{b+x_2(t)}|y_2(t) - y_1(t)| \\ &\quad +\left(\frac{\alpha y_1(t)}{(b+x_2(t))(b+x_1(t))} - \frac{a}{K}\right)|x_2(t) - x_1(t)|. \end{aligned} \tag{48}$$

Note that

$$|\ln x_2(t) - \ln x_1(t)| \le \frac{1}{m_P}|x_2(t) - x_1(t)|. \tag{49}$$

Define

$$V(t) \triangleq |y_2(t) - y_1(t)| + |\ln x_2(t) - \ln x_1(t)|.$$

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Then, it follows from (47), (48), (36) and (49) that

$$\begin{aligned}
\frac{dV(t)}{dt} &\le (-D+\frac{\beta\eta_1}{b+\eta_1}+\frac{\alpha}{b+x_2(t)})|y_2(t)-y_1(t)| \\
&\quad +\left(\frac{b\beta\xi_1}{(b+\eta_1)^2}+\frac{\alpha y_1(t)}{(b+x_2(t))(b+x_1(t))}-\frac{a}{K}\right)|x_2(t)-x_1(t)| \\
&\le (-D+\frac{\beta K}{b+K}+\frac{\alpha}{b+m_P})|y_2(t)-y_1(t)| \\
&\quad +m_P\left(\frac{b\beta M_N}{(b+m_P)^2}+\frac{\alpha M_N}{(b+m_P)^2}-\frac{a}{K}\right)\frac{1}{m_P}|x_2(t)-x_1(t)| \\
&\le (-D+\frac{\beta K}{b+K}+\frac{\alpha}{b+m_P})|y_2(t)-y_1(t)| \\
&\quad +m_P\left(\frac{(b\beta+\alpha)M_N}{(b+m_P)^2}-\frac{a}{K}\right)|\ln x_2(t)-\ln x_1(t)| \\
&\le \hat{C}_1 V(t).
\end{aligned}$$

However, it is easy to verify that

$$\begin{cases}
V(t^+) &= |y_2(t^+)-y_1(t^+)|+|\ln x_2(t^+)-\ln x_1(t^+)| \\
&= (1-p_N^{(i)})|y_2(t)-y_1(t)|+|\ln x_2(t)-\ln x_1(t)| \\
&\le V(t), \qquad t=nT+\lambda_i, 1\le i\le m+1, \\
V(t^+) &= |y_2(t^+)-y_1(t^+)|+|\ln x_2(t^+)-\ln x_1(t^+)| \\
&= (1+p_k^{(j)})|y_2(t)-y_1(t)|+|\ln x_2(t)-\ln x_1(t)| \\
&\le (1+p_k^{(j)})V(t), \qquad t=nT+\tau_k^{(j)}, 1\le k\le m_j, 1\le j\le m+1.
\end{cases}$$

Similar to (11), we obtain

$$V((n+1)T^+) \le \prod_{h=1}^{m+1}\left(\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\hat{C}_1 T}V(nT^+). \tag{50}$$

For any solution $(y(t),x(t))^T\in S$ of system (3), it follows from (50) and (36) that

$$\begin{aligned}
\rho_n &\triangleq |y((n+1)T^+)-y(nT^+)|+|\ln x((n+1)T^+)-\ln x(nT^+)| \\
&= |y(nT^+;(y(T^+),x(T^+)))-y(nT^+)| \\
&\quad +|\ln x(nT^+;(y(T^+),x(T^+)))-\ln x(nT^+)| \\
&\le \prod_{h=1}^{m+1}\left(\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\hat{C}_1 T} \\
&\quad \times(|y((n-1)T^+;(y(T^+),x(T^+)))-y((n-1)T^+)| \\
&\qquad +|\ln x((n-1)T^+;(y(T^+),x(T^+)))-\ln x((n-1)T^+)|) \\
&= \prod_{h=1}^{m+1}\left(\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\hat{C}_1 T} \\
&\quad \times(|y(nT^+)-y((n-1)T^+)|+|\ln x(nT^+)-\ln x((n-1)T^+)|) \\
&= \prod_{h=1}^{m+1}\left(\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\hat{C}_1 T}\rho_{n-1} \\
&\triangleq \hat{C}_2\rho_{n-1},
\end{aligned}$$

and furthermore,

$$\rho_n \le \hat{C}_2\rho_{n-1}\le\cdots\le\hat{C}_2^n\rho_0, \tag{51}$$

where

$$\begin{cases}
\rho_n &= |y((n+1)T^+)-y(nT^+)|+|\ln x((n+1)T^+)-\ln x(nT^+)|, \\
\hat{C}_2 &= \prod_{h=1}^{m+1}\left(\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{\hat{C}_1 T}.
\end{cases}$$

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Then, for arbitrary $p > 0$, it follows from (51) that

$$\begin{aligned}
& |y((n+p)T^+) - y(nT^+)| + |\ln x((n+p)T^+) - \ln x(nT^+)| \\
\leq\ & |y((n+p)T^+) - y((n+p-1)T^+)| \\
& \quad +|\ln x((n+p)T^+) - \ln x((n+p-1)T^+)| \\
& +|y((n+p-1)T^+) - y((n+p-2)T^+)| \\
& \quad +|\ln x((n+p-1)T^+) - \ln x((n+p-2)T^+)| \\
& +\cdots \\
& +|y((n+1)T^+) - y(nT^+)| + |\ln x((n+1)T^+) - \ln x(nT^+)| \\
=\ & \rho_{n+p-1} + \rho_{n+p-2} + \cdots + \rho_n \\
\leq\ & (\hat{C}_2^{n+p-1} + \hat{C}_2^{n+p-2} + \cdots + \hat{C}_2^n)\rho_0 \\
=\ & \frac{\hat{C}_2^n(1-\hat{C}_2^p)}{1-\hat{C}_2}\rho_0 \\
<\ & \frac{\hat{C}_2^n}{1-\hat{C}_2}\rho_0,
\end{aligned}$$

which implies that $\{y(nT^+)\}$ and $\{\ln x(nT^+)\}$ are both Cauchy sequences. Thus, both $\lim_{n\to\infty} y(nT^+)$ and $\lim_{n\to\infty} \ln x(nT^+)$ exist.

Denote

$$\begin{cases} \lim_{n\to\infty} y(nT^+) = y', \\ \lim_{n\to\infty} \ln x(nT^+) = \ln x', \end{cases}$$

where $y' \in [m_N, M_N]$, and

$$x' = \lim_{n\to\infty} x(nT^+) \in [m_P, K].$$

Then, it follows that

$$\begin{cases}
& \lim_{n\to\infty} y(nT^+) \\
= & \lim_{n\to\infty} y(T^+; (y((n-1)T^+), x((n-1)T^+))) \\
= & y(T^+; (\lim_{n\to\infty} y((n-1)T^+), \lim_{n\to\infty} x((n-1)T^+))) \\
= & y(T^+; (y', x')), \\
& \lim_{n\to\infty} \ln x(nT^+) \\
= & \ln \lim_{n\to\infty} x(T^+; (y((n-1)T^+), x((n-1)T^+))) \\
= & \ln x(T^+; (\lim_{n\to\infty} y((n-1)T^+), \lim_{n\to\infty} x((n-1)T^+))) \\
= & \ln x(T^+; (y', x')).
\end{cases}$$

Thus, we have

$$\begin{cases} y(T^+; (y', x')) = y', \\ x(T^+; (y', x')) = x'. \end{cases}$$

That is, $(y', x')^T \in S$ is a fixed point of the Poincaré map of system (3).

If $(y'', x'')^T \in S$ is a fixed point of the Poincaré map of system (3), then

$$\begin{cases} y(T^+; (y'', x'')) = y'', \\ x(T^+; (y'', x'')) = x''. \end{cases}$$

Since

$$\begin{aligned}
& |y'' - y'| + |\ln x'' - \ln x'| \\
=\ & |y(T^+; (y'', x'')) - y(T^+; (y', x'))| \\
& +|\ln x(T^+; (y'', x'')) - \ln x(T^+; (y', x'))| \\
\leq\ & \hat{C}_2(|y(0^+; (y'', x'')) - y(0^+; (y', x'))| \\
& \quad +|\ln x(0^+; (y'', x'')) - \ln x(0^+; (y', x'))|) \\
=\ & \hat{C}_2(|y'' - y'| + |\ln x'' - \ln x'|),
\end{aligned}$$

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we have $y'' = y'$ and $x'' = x'$ because $\hat{C}_2 \in (0,1)$. Therefore, $(y', x')^T \in S$ is a unique fixed point of the Poincaré map of system (3).

Therefore, $(y', x')^T \in S$ is a globally attractive fixed point of the Poincaré map of system (3).

This completes the proof. □

We note that the first inequality of (35) can be inferred from (36), and

$$\begin{cases} \lim\limits_{m_0 \to 0^+} \left(-D + \frac{\beta K}{b+K} + \frac{\alpha}{b+m_P}\right) & = & -D + \frac{\beta K}{b+K} + \frac{\alpha}{b}, \\ \lim\limits_{(m_0,\varepsilon_2) \to (0^+,0^+)} \left(\frac{(b\beta+\alpha)M_N}{(b+m_P)^2} - \frac{a}{K}\right) & = & \frac{b\beta+\alpha}{b^2} \max\limits_{1\le j\le m+1} \max\limits_{t\in[\lambda_{j-1}^+, \tau_1^{(j)}]\cup\cdots\cup[(\tau_{m_j}^{(j)})^+, \lambda_j]} y^*(t; -D + \frac{\beta K}{b+K}) - \frac{a}{K}. \end{cases}$$

Then, the following claim is valid:

Corollary 1. *Let us assume that $p_M = 0$, the second inequality of (35) and*

$$\begin{cases} -D + \frac{\beta K}{b+K} + \frac{\alpha}{b} & < & 0, \\ \frac{b\beta+\alpha}{b^2} \max\limits_{1\le j\le m+1} \max\limits_{1\le k\le m_j} y^*(l_k T^+; -D + \frac{\beta K}{b+K}) - \frac{a}{K} & < & 0, \end{cases}$$

hold; then, there exists a globally attractive pest-present periodic solution for the system (3).

5. Numerical analysis

In the following, we perform numerical simulations for the special case of system (3) as follows:

$$\begin{cases} \left.\begin{array}{lcl} \frac{dy(t)}{dt} & = & -Dy(t) + \frac{\beta x(t) y(t)}{b+x(t)}, \\ \frac{dx(t)}{dt} & = & ax(t)\left(1 - \frac{x(t)}{K}\right) - \frac{\alpha x(t) y(t)}{b+x(t)}, \end{array}\right\} t \neq nT, t \in nT + \{\tau_1, \tau_2, \cdots, \tau_m\}, \\ \left.\begin{array}{lcl} y(t^+) & = & (1-p_N)y(t), \\ x(t^+) & = & (1-p_P)x(t), \end{array}\right\} t = nT, \\ \left.\begin{array}{lcl} y(t^+) & = & (1+p_M)y(t) + q, \\ x(t^+) & = & x(t), \end{array}\right\} t \in nT + \{\tau_1, \tau_2, \cdots, \tau_m\}, \end{cases} \tag{52}$$

where $0 < \tau_1 < \tau_2 < \cdots < \tau_m < T$.

We are interested in how the key factors affect the threshold value R_a defined in (18). Figures $2(A) - (C)$ show that when p_N decreases to 0 or p_M and q increase (to the threshold value $p_M^* = 0.1064$), R_a monotonically decreases, revealing that fewer natural enemies should be killed, or more natural enemies should migrate or be released. Then, the optimal control strategy is achieved when p_N is sufficiently small or p_M and q are appropriately large. Similarly, Figures $2(D) - (E)$ show that R_a monotonically decreases to 0 as T decreases to the threshold value $T^* = 1.7324$ or m increases. Thus, more frequent integrated pest management and release of natural enemies are beneficial for pest control.

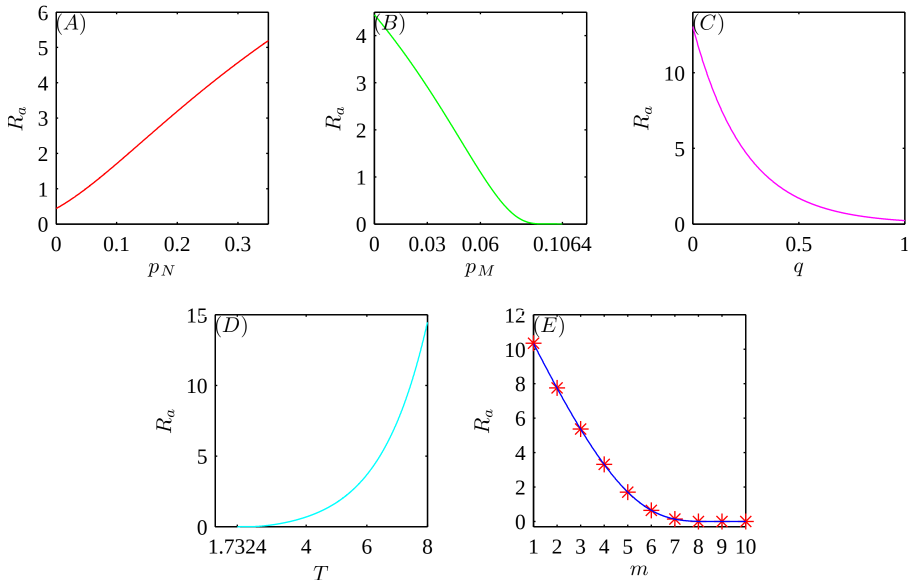

Figure 2. Simulations of the effects of p_N, p_M, q, T and m on R_a. The baseline parameter values are fixed as follows: $D = 0.08$, $b = 1.5$, $a = 0.6$, $K = 10$, $\alpha = 0.5$, $p_N = 0.1$, $p_P = 0.35$, $p_M = 0.05$, $q = 0.5$, $T = 5$ and $m = 5$.

According to Theorems 1 and 2, when $R_s < 1$, $R_a < 1$ and $R_s > 1$, the pest-free periodic solution $(y^*(t), 0)$ is locally asymptotically stable, globally attractive, and unstable, respectively (see Figure 3(A)).

Figure 3$(B) - (C)$ shows that when $T < T_a = 10.3459$, the global attractiveness of the pest-free periodic solution $(y^*(t), 0)$ can be validated (see Figure 4$(A) - (B)$); however, when $T > T_b = 16.5495$, the emergence of a pest-present solution leads to the loss of the local stability of the pest-free periodic solution, i.e., system (52) is permanent (see Figure 5$(A) - (C)$). Similar numerical analysis is suitable for the bifurcation of a pest-present periodic solution with respect to q (see Figures 3$(D) - (E)$ and Figures 4$(C) - (D)$). In addition, Figures 5$(D) - (F)$ confirm that when Corollary 1 holds, the global attractive pest-present periodic solution emerges.

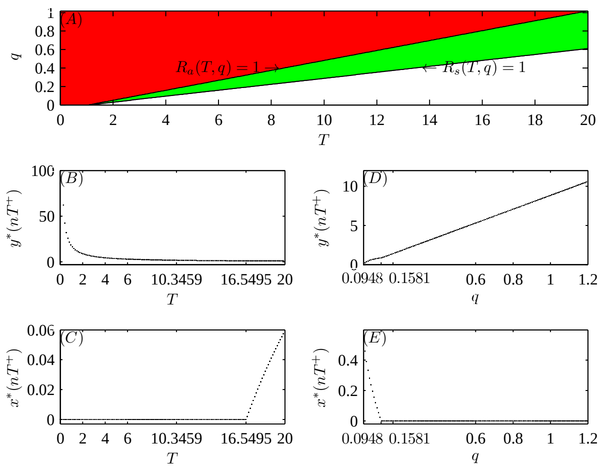

Figure 3. Bifurcation diagrams of system (52) with respect to T and q, where the other parameter values are fixed as follows: $D = 0.08$, $\beta = 0.19$, $b = 1.5$, $a = 0.4$, $K = 1$, $\alpha = 0.5$, $p_N = 0.08$, $p_P = 0.35$, $p_M = 0.028$ and $m = 3$. (A): two-dimensional bifurcation diagram with respect to T and q, where the green, red and white regions denotes the locally stable, globally stable and unstable regions of the pest-free periodic solution $(y^*(t), 0)$, respectively; $(B) - (C)$: one-dimensional diagrams with respect to T, where $q = 0.5$ and the initial values are $(0, 10^6)$ and $(1.0277, 10^{-10})$; $(D) - (E)$: one-dimensional diagrams with respect to q, where $T = 4$ and the initial values are $(0, 10^6)$ and $(0.8369, 10^{-10})$.

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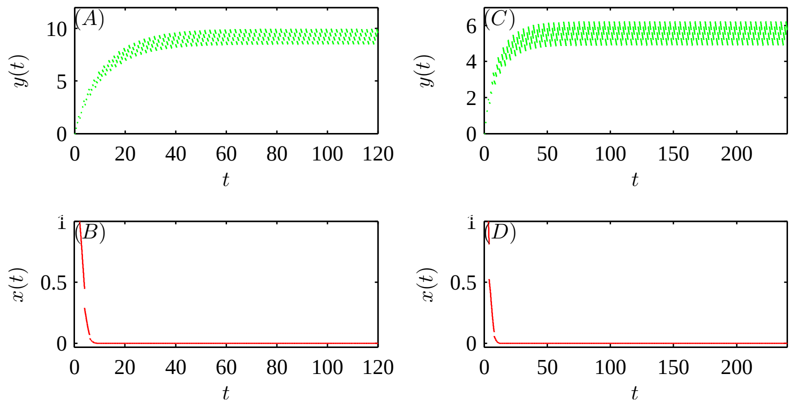

Figure 4. Dynamic behaviors of system (52) with integrated pest management. $(A)-(B)$: $q=0.5$, $T=2$; $(C)-(D)$: $q=0.6$, $T=4$. The other parameters are identical to those in Figure 3, and the initial values are $(0,10^6)$.

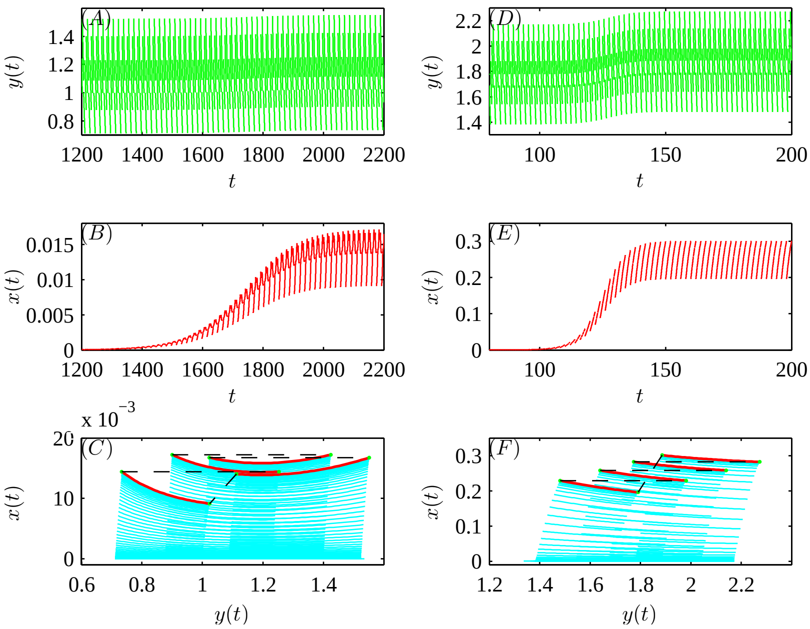

Figure 5. Dynamic behaviors of system (52) with integrated pest management. $(A)-(C)$: $q=0.5$, $T=17$, and the other parameters are identical to those in Figure 3; $(D)-(F)$: The parameter values are fixed as follows: $D=0.4$, $\beta=0.16$, $b=1.5$, $a=1$, $K=1$, $\alpha=0.5$, $p_N=0.05$, $p_P=0.35$, $p_M=0$, $q=0.5$, $T=2$ and $m=3$. The initial values are: $(A)-(C)$ $(1.0277,10^{-10})$; $(D)-(F)$ $(10^{-12},10^{-12})$.

Furthermore, Figures 6 and 7 show that model (52) exhibits more complex and interesting dynamic behaviors, including periodic doubling bifurcation, chaotic solutions, periodic adding\reducing, periodic windows, periodic halving bifurcation, and chaos crisis with increasing implementation period T and amount of released natural enemies q, respectively [32].

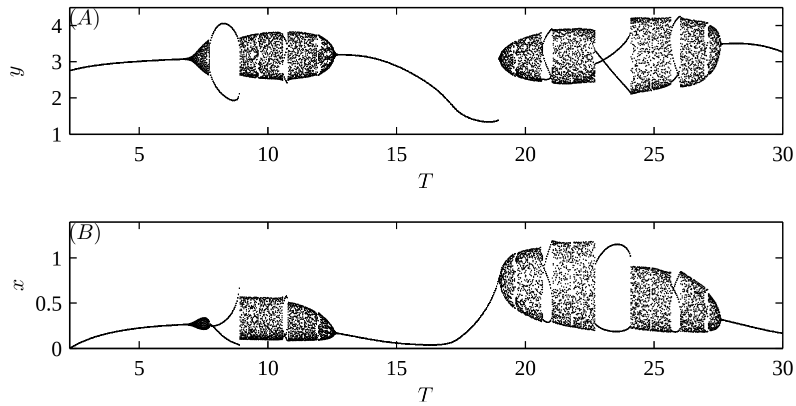

Figure 6. Bifurcation diagrams of system (52) with respect to parameter T. The other parameter values are fixed as follows: $D = 0.2$, $\beta = 0.91$, $b = 2.04$, $a = 1.5$, $K = 3.5$, $\alpha = 1$, $p_N = 0.06$, $p_P = 0.2$, $p_M = 0.1$, $q = 0.5$, $m = 2$, and the initial value is $(2, 1)$.

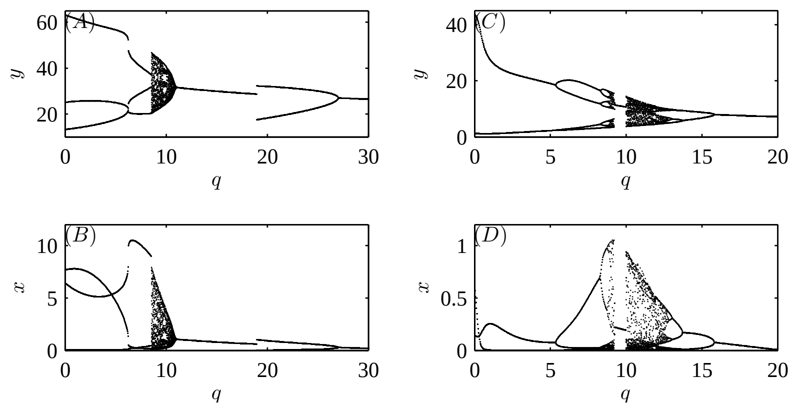

Figure 7. Bifurcation diagrams of system (52) with respect to parameter q. $(A) - (B)$: $D = 0.2$, $\beta = 1.17$, $b = 27.5$, $a = 1.5$, $K = 55$, $\alpha = 1.3$, $p_N = 0.06$, $p_P = 0.3$, $p_M = 0.1$, $T = 12$, $m = 2$, and the initial value is $(2, 1)$; $(C) - (D)$: $D = 0.34$, $\beta = 1.17$, $b = 10$, $a = 1.5$, $K = 44$, $\alpha = 1.3$, $p_N = 0.06$, $p_P = 0.7$, $p_M = 0.1$, $T = 11.25$, $m = 2$, and the initial values is $(5, 5)$.

Moreover, random perturbations due to variations in the dosages applied or releases (migration) of natural enemies can be also taken into account with parameters p_N, p_P, p_M and q. For example, when small random perturbations are introduced in one of the parameters p_N, p_M and q, as shown in Figures 8, 9$(A) - (B)$ and 10, the stable attractor with initial value $(9, 6)$ can switch to another attractor with a smaller amplitude at a random time. Conversely, if a small random perturbation is introduced in parameter p_P, Figures 9$(C) - (D)$ show that the stable attractor with initial value $(2, 1)$ can switch to another attractor with a larger amplitude at a random time. Thus, different doses of pesticide application and natural enemy release (or migration) can influence the dynamics of the system (3).

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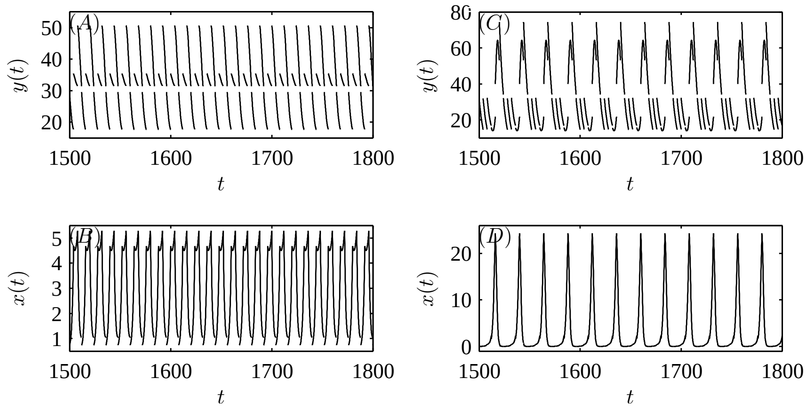

Figure 8. Two coexisting attractors of system (52) with $q = 16$ and the other parameters identical to those in Figures 7$(A) - (B)$. The initial values are: $(A) - (B)(2,1)$; $(C) - (D)(9,6)$.

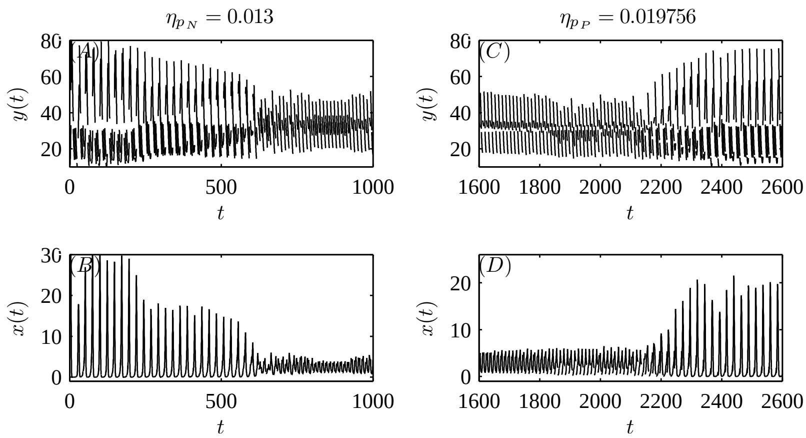

Figure 9. Attractors' switch-like behaviors of system (52) with small random perturbations on parameters p_N and p_P, i.e., $\eta_{p_N} = 0.013$ and $\eta_{p_P} = 0.019756$. The baseline parameters are identical to those in Figure 8, and the initial values are: $(A) - (B)(9,6)$; $(C) - (D)(2,1)$.

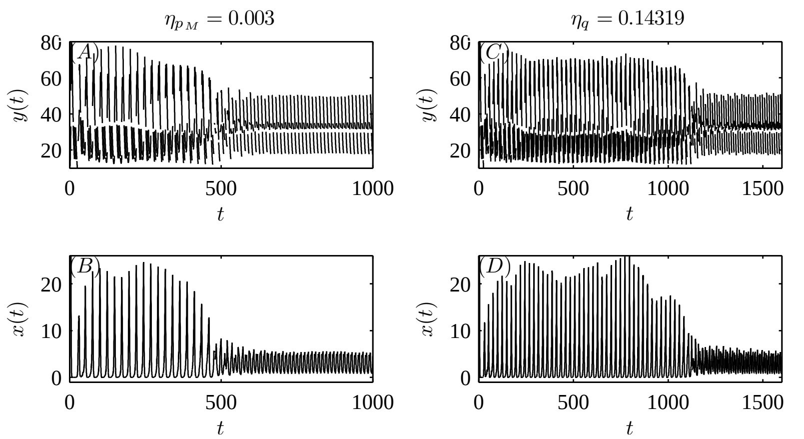

Figure 10. Attractors' switch-like behaviors of system (52) with small random perturbations on parameters p_M and q, i.e., $\eta_{p_M} = 0.003$ and $\eta_q = 0.14319$. The baseline parameters are identical to those in Figure 8, and the initial values are $(9,6)$.

Moreover, Figure 11(A) shows that the pest population behaves in a "positive" fashion with pesticide spraying alone. As natural enemy release (or migration) is introduced to control pests, the pest population also oscillates in a periodic cycle but with a much smaller amplitude, revealing that

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the tendency of pests to become extinct becomes increasingly significant as m increases gradually (see Figures 11$(B) - (C)$). In particular, when m is sufficiently large, the pest population is eventually eradicated with integrated pest management (see Figure 11(D)). Thus, we demonstrate that integrated pest management is more effective than pesticide spraying alone.

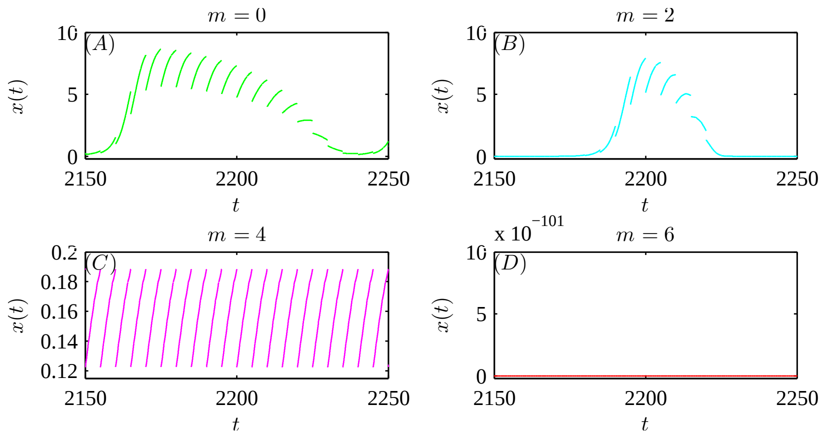

Figure 11. Dynamic behaviors of system (52) with integrated pest management. (A): $m = 0$, $p_M = 0$, $q = 0$; (B): $m = 2$; (C): $m = 4$; (D): $m = 6$. The other parameter values are fixed as follows: $D = 0.08$, $\beta = 0.16$, $b = 1.5$, $a = 0.6$, $K = 10$, $\alpha = 0.5$, $p_N = 0.1$, $p_P = 0.35$, $p_M = 0.05$, $q = 0.1$, $T = 5$, and the initial value is $(2, 1)$.

6. Discussion

In this study, we developed a predator–prey model with variation multiple pulse intervention effects to determine how chemical and biological control tactics affect pest control dynamics.

By employing the variational method, we determine the eigenvalues of the Jacobian matrix at the fixed point corresponding to the pest-free periodic solution. This fact was used to obtain the local stability threshold condition. A sufficient condition for the global attractiveness of the pest-free periodic solution was given as in (18). Moreover, the loss of local stability of the pest-free periodic solution resulted in the emergence of a pest-present periodic solution in the local neighborhood of the bifurcation point; furthermore, the permanence of system (3) was valid under certain conditions. Based on the techniques used in [33–42], the existence and global attractiveness of the pest-present periodic solution were analyzed by using Cauchy's convergence principle under certain conditions, and a special case was obtained.

Our results demonstrate that the effectiveness of integrated pest management is crucial for pest prevention and resurgence. The numerical results presented in Section 5 indicate that R_a is sensitive to small changes in several key parameters, such as p_N, p_M, q, T, and m. Furthermore, we perform two-parameter bifurcation analyses on the threshold values R_s and R_a, which involve the local stability and global attractiveness of the pest-free periodic solution. Figure 3 shows the impacts of the period T and the amount q of natural enemies released on the behaviors of system (52).

Figures 6 and 7 show that model (52) has several interesting dynamic behaviors, such as periodic doubling bifurcations, chaos, periodic adding\reducing, periodic windows, and periodic halving bifurcation. Moreover, the switch-like transitions between the two attractors revealed that both varying dosages of insecticide and the numbers of natural enemies released (or migrated) are crucial for pest control. In addition, further numerical simulations indicated that integrated pest management is more effective than pesticide spraying alone.

Compared with other relevant studies, the highlights of our study are as follows:
(i) A continuous point can be viewed as a special impulsive point, i.e., the corresponding pulsed killing rate for the prey and predator populations, and the pulsed increment or migration rate of the natural enemies can be regarded as 0. Then, we propose model (3) to characterize all the possible situations of the implementation of the IPM strategy in reality. The traditional modeling approaches illustrated in [21,22,43,44] are special cases of our model.

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(ii) Due to the complexity of our modeling approach, it is difficult for us to prove the permanence of system (3) by using the traditional proof method demonstrated in [21,45]. Therefore, we prove the permanence of system (3) by constructing two uniform lower impulsive comparison systems on the interval $(t_3, +\infty)$, where (37) holds. If $\hat{x}(t)$ is viewed as the second prey population, (40) and (39) indicate that the change rate of $x(t)$ is greater than that of $\hat{x}(t)$; thus, $x(t) \geq \hat{x}(t)$ holds within the interval $[t^{*+}, +\infty)$. However, if we view $\widetilde{x}(t)$ as the third prey population, (44) indicates that the change rate of $x(t)$ is greater than that of $\widetilde{x}(t)$. Thus, when $\widetilde{x}(t)$ jumps above the threshold line $x = m_0$ at time $t = (n_1+1)T + n_2T$, $x(t)$ also jumps above the threshold line $x = m_0$ at some $t_6 \in ((n_1+1)T, (n_1+1)T + n_2T)$. Since the change rate of $\hat{x}(t)$ is negative or the density of $\hat{x}(t)$ decreases as $t \to +\infty$, it follows on the interval (t^*, t_6) that $x(t) > \hat{x}((n_1+1)T + n_2T)$, which only depends on m_0.

In addition, the fact that $t = t^*$ and $t = t^{**}$ are always viewed as impulsive times facilitates our proof. $\forall t \in ((n_1+1)T, t_6)$, it may hold that $x(t) < m_0$, $x(t) = m_0$ or $x(t) > m_0$.

(iii) To date, the global attractiveness of the pest-present periodic solution of system (3) has not been addressed. Then, based on the permanence of system (3), we define a norm $V(t)$ with respect to the solution of system (3) and then construct an impulsive comparison system for $V(t)$. Similar to the proof of the contraction mapping principle, we prove the existence and global attractiveness of the pest-present periodic solution of system (3).

(iv) According to the conclusion in [27], the bifurcation of the nontrivial periodic solution in [45–49] can be determined under certain conditions. However, the bifurcation of the pest-present periodic solution of system (3) cannot be handled in the same manner. Based on the implicit function theorem, we address this problem successfully. Compared to the following equation mentioned in [46,47]

$$2B\widetilde{\tau} + Ca + \frac{1}{a}o(\widetilde{\tau}, a)(\widetilde{\tau}^2 + a^2) = 0,$$

the existence of the pest-present periodic solution is inferred from $\hat{f}(T, c_p) = 0$ more rigorously.

As T gradually increases in interval $[T_b, T_b + \bar{\delta})$, the IPM strategy is implemented less frequently so as not to control the pests. Then, a pest-present periodic solution emerges. Biologically speaking, during the evolution of T, the pest-free periodic solution becomes unstable; intuitively, to maintain the ecological/systematic balance, both the prey and predator populations should eventually stabilize around a pest-present periodic solution. Given the numerical results in Figure 3, we hypothesize that the pest-present periodic solution obtained above is locally stable. We plan to investigate the stability of the pest-free solution in our future research.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest: The authors declare no conflicts of interest.

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Appendix A. First-Order Partial Derivatives of $y(t)$ and $x(t)$ with Initial Values

(1) Let

$$\begin{cases} y = y(t; t_0, X^0), \\ x = x(t; t_0, X^0), \end{cases}$$

be a solution of the first two equations of system (3), where $X^0 = (y^0, x^0)^T \in \mathbb{R}^2$. Then, we have

$$\begin{cases} \frac{dy(t;t_0,X^0)}{dt} &= -Dy(t;t_0,X^0) + \frac{\beta x(t;t_0,X^0) y(t;t_0,X^0)}{b + x(t;t_0,X^0)}, \\ \frac{dx(t;t_0,X^0)}{dt} &= ax(t;t_0,X^0)\left(1 - \frac{x(t;t_0,X^0)}{K}\right) - \frac{\alpha x(t;t_0,X^0) y(t;t_0,X^0)}{b + x(t;t_0,X^0)}, \end{cases} \quad \text{(A1)}$$

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and

$$x(t; t_0, X_0) = 0, \quad \forall t \geq t_0, \tag{A2}$$

where $X_0 = (y^0, 0)^T \in \mathbb{R}^2$. Moreover, it holds that

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial y(t;t_0,X_0)}{\partial y^0}\right) &= -D\frac{\partial y(t;t_0,X_0)}{\partial y^0}, \\ \frac{d}{dt}\left(\frac{\partial y(t;t_0,X_0)}{\partial x^0}\right) &= -D\frac{\partial y(t;t_0,X_0)}{\partial x^0} + \frac{\beta y(t;t_0,X_0)}{b}\frac{\partial x(t;t_0,X_0)}{\partial x^0}, \\ \frac{d}{dt}\left(\frac{\partial x(t;t_0,X_0)}{\partial x^0}\right) &= \left(a - \frac{\alpha y(t;t_0,X_0)}{b}\right)\frac{\partial x(t;t_0,X_0)}{\partial x^0}, \\ \frac{\partial y(t_0;t_0,X_0)}{\partial y^0} &= 1, \\ \frac{\partial y(t_0;t_0,X_0)}{\partial x^0} &= 0, \\ \frac{\partial x(t_0;t_0,X_0)}{\partial x^0} &= 1. \end{cases} \tag{A3}$$

Solving (A3) yields

$$\begin{cases} \frac{\partial y(t;t_0,X_0)}{\partial y^0} &= e^{-D(t-t_0)} > 0, \\ \frac{\partial y(t;t_0,X_0)}{\partial x^0} &= * > 0, \\ \frac{\partial x(t;t_0,X_0)}{\partial x^0} &= e^{\int_{t_0}^{t}(a - \frac{\alpha y(s;t_0,X_0)}{b})ds} > 0. \end{cases} \tag{A4}$$

(2) In the following, assume that $j = 1, 2, 3, \cdots, m, m+1$, and define

$$\begin{cases} I_0^{(j)}(X^0) &= \Psi^{(j-1)}(X^0), \\ \Psi^{(0)}(X^0) &= X^0. \end{cases} \tag{A5}$$

From (14), (13), (15), (A4) and (A2), we have

$$\begin{aligned} & D_{X^0}(\Psi^{(j)}(X_0)) \\ = & \begin{pmatrix} 1 - p_N^{(j)} & 0 \\ 0 & 1 - p_P^{(j)} \end{pmatrix} \begin{pmatrix} e^{-D(\lambda_j - \tau_{m_j}^{(j)})} & \frac{\partial y(\lambda_j - \tau_{m_j}^{(j)}; I_{m_j}^{(j)}(X_0))}{\partial x^0} \\ 0 & e^{\int_{0^+}^{\lambda_j - \tau_{m_j}^{(j)}}(a - \frac{\alpha y(t; I_{m_j}^{(j)}(X_0))}{b})dt} \end{pmatrix} \\ & \times \begin{pmatrix} 1 + p_{m_j}^{(j)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-D(\tau_{m_j}^{(j)} - \tau_{m_j-1}^{(j)})} & \frac{\partial y(\tau_{m_j}^{(j)} - \tau_{m_j-1}^{(j)}; I_{m_j-1}^{(j)}(X_0))}{\partial x^0} \\ 0 & e^{\int_{0^+}^{\tau_{m_j}^{(j)} - \tau_{m_j-1}^{(j)}}(a - \frac{\alpha y(t; I_{m_j-1}^{(j)}(X_0))}{b})dt} \end{pmatrix} \\ & \times \begin{pmatrix} 1 + p_{m_j-1}^{(j)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-D(\tau_{m_j-1}^{(j)} - \tau_{m_j-2}^{(j)})} & \frac{\partial y(\tau_{m_j-1}^{(j)} - \tau_{m_j-2}^{(j)}; I_{m_j-2}^{(j)}(X_0))}{\partial x^0} \\ 0 & e^{\int_{0^+}^{\tau_{m_j-1}^{(j)} - \tau_{m_j-2}^{(j)}}(a - \frac{\alpha y(t; I_{m_j-2}^{(j)}(X_0))}{b})dt} \end{pmatrix} \\ & \times \cdots \\ & \times \begin{pmatrix} 1 + p_2^{(j)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-D(\tau_2^{(j)} - \tau_1^{(j)})} & \frac{\partial y(\tau_2^{(j)} - \tau_1^{(j)}; I_1^{(j)}(X_0))}{\partial x^0} \\ 0 & e^{\int_{0^+}^{\tau_2^{(j)} - \tau_1^{(j)}}(a - \frac{\alpha y(t; I_1^{(j)}(X_0))}{b})dt} \end{pmatrix} \\ & \times \begin{pmatrix} 1 + p_1^{(j)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-D(\tau_1^{(j)} - \lambda_{j-1})} & \frac{\partial y(\tau_1^{(j)} - \lambda_{j-1}; \Psi^{(j-1)}(X_0))}{\partial x^0} \\ 0 & e^{\int_{0^+}^{\tau_1^{(j)} - \lambda_{j-1}}(a - \frac{\alpha y(t; \Psi^{(j-1)}(X_0))}{b})dt} \end{pmatrix} \\ & \times D_{X^0}(\Psi^{(j-1)}(X_0)), \end{aligned}$$

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i.e.,

$$
\begin{aligned}
&D_{X^0}(\Psi^{(j)}(X_0)) \\
= &\begin{pmatrix} \left((1-p_N^{(j)})\prod_{l=1}^{m_j}(1+p_l^{(j)})\right)e^{-D(\lambda_j-\lambda_{j-1})} & * \\ 0 & c^{(j)}(X_0) \end{pmatrix} D_{X^0}(\Psi^{(j-1)}(X_0)),
\end{aligned} \tag{A6}
$$

where

$$
\begin{aligned}
c^{(j)}(X_0) \quad = \quad & (1-p_P^{(j)})e^{\int_{0^+}^{\lambda_j-\tau_{m_j}^{(j)}}(a-\frac{\alpha y(t;I_{m_j}^{(j)}(X_0))}{b})dt} \\
& \times e^{\int_{0^+}^{\tau_{m_j}^{(j)}-\tau_{m_j-1}^{(j)}}(a-\frac{\alpha y(t;I_{m_j-1}^{(j)}(X_0))}{b})dt} \\
& \times e^{\int_{0^+}^{\tau_{m_j-1}^{(j)}-\tau_{m_j-2}^{(j)}}(a-\frac{\alpha y(t;I_{m_j-2}^{(j)}(X_0))}{b})dt} \\
& \times \cdots \\
& \times e^{\int_{0^+}^{\tau_2^{(j)}-\tau_1^{(j)}}(a-\frac{\alpha y(t;I_1^{(j)}(X_0))}{b})dt} \\
& \times e^{\int_{0^+}^{\tau_1^{(j)}-\lambda_{j-1}}(a-\frac{\alpha y(t;\Psi^{(j-1)}(X_0))}{b})dt}.
\end{aligned} \tag{A7}
$$

Similarly, from (A6), it can be obtained by mathematical induction that

$$
\begin{aligned}
&D_{X^0}(\Psi^{(m+1)}(X_0)) \\
= &\begin{pmatrix} \prod_{h=1}^{m+1}\left((1-p_N^{(h)})\prod_{l=1}^{m_h}(1+p_l^{(h)})\right)e^{-DT} & * \\ 0 & \prod_{h=1}^{m+1}c^{(h)}(X_0) \end{pmatrix}.
\end{aligned} \tag{A8}
$$

Appendix B. Several Propositions for Determining the Signs of $\frac{\partial^2\Psi_2^{(m+1)}(c_pV_p+c_f(T,c_p)V_f)}{\partial c_p^2}$ and $\frac{\partial^2\Psi_2^{(m+1)}(c_pV_p+c_f(T,c_p)V_f)}{\partial c_p\partial T}$ at Point $(T_b,0)$

Similar to (A5), assume that $j=1,2,3,\cdots,m,m+1$, and define

$$
\begin{cases}
I_0^{(j)}(c_pV_p+c_f(T,c_p)V_f) &= \Psi^{(j-1)}(c_pV_p+c_f(T,c_p)V_f), \\
\Psi^{(j)}(c_pV_p+c_f(T,c_p)V_f) &= \begin{pmatrix} 1-p_N^{(j)} & 0 \\ 0 & 1-p_P^{(j)} \end{pmatrix} I_{m_j+1}^{(j)}(c_pV_p+c_f(T,c_p)V_f),
\end{cases}
$$

where

$$
\begin{cases}
\Psi^{(0)}(c_pV_p+c_f(T,c_p)V_f) &= c_pV_p+c_f(T,c_p)V_f, \\
I_{m_j+1}^{(j)}(c_pV_p+c_f(T,c_p))V_f &= \Phi(\lambda_j-\tau_{m_j}^{(j)};I_{m_j}^{(j)}(c_pV_p+c_f(T,c_p)V_f)) \\
&= \begin{pmatrix} y((L_j-l_{m_j}^{(j)})T;I_{m_j}^{(j)}(c_pV_p+c_f(T,c_p)V_f)) \\ x((L_j-l_{m_j}^{(j)})T;I_{m_j}^{(j)}(c_pV_p+c_f(T,c_p)V_f)) \end{pmatrix}.
\end{cases} \tag{A9}
$$

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Proposition A1. *It holds that*

$$\begin{cases} \left.\begin{array}{rcl} \frac{\partial}{\partial T}\left(\Psi_1^{(0)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} & = & \frac{\partial c_f(T_b,0)}{\partial T} < 0, \\ \frac{\partial}{\partial T}\left(I_{k_1}^{(1)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} & < & 0, \\ & & k = 1,2,3,\cdots,m_1, \end{array}\right\} j = 1, \\ \left.\begin{array}{rcl} \frac{\partial}{\partial T}\left(\Psi_1^{(j-1)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} & < & 0, \\ \frac{\partial}{\partial T}\left(I_{k_1}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} & < & 0, \\ & & k = 1,2,3,\cdots,m_j, \end{array}\right\} j = 2,3,\cdots,m,m+1. \end{cases}$$

Proof. From (15), (A9), (A4) and (20), we see that

$$\begin{aligned} & \frac{\partial}{\partial T}\left(I_{k_1}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = \ & (1+p_k^{(j)})e^{-D(l_k^{(j)}-l_{k-1}^{(j)})T_b}\frac{\partial}{\partial T}\left(I_{(k-1)_1}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ & +(1+p_k^{(j)})(l_k^{(j)} - l_{k-1}^{(j)})\frac{\partial y((l_k^{(j)}-l_{k-1}^{(j)})T_b; I_{k-1}^{(j)}(X_0^*))}{\partial t} \\ \triangleq \ & (1+p_k^{(j)})e^{-D(l_k^{(j)}-l_{k-1}^{(j)})T_b}\frac{\partial}{\partial T}\left(I_{(k-1)_1}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ & +\overline{R}_k^{(j)}, \\ & k = 1,2,3,\cdots,m_j,m_j+1, \end{aligned}$$

where $p_{m_j+1} = 0$, and

$$\begin{aligned} \overline{R}_k^{(j)} & = & (1+p_k^{(j)})(l_k^{(j)} - l_{k-1}^{(j)})\underbrace{\frac{\partial y((l_k^{(j)} - l_{k-1}^{(j)})T_b; I_{k-1}^{(j)}(X_0^*))}{\partial t}}_{<0} \\ & < & 0, \\ & & k = 1,2,3,\cdots,m_j,m_j+1. \end{aligned} \tag{A10}$$

Then, it can be obtained by mathematical induction that

$$\begin{aligned} & \frac{\partial}{\partial T}\left(\Psi_1^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & \left((1-p_N^{(j)})\prod_{h=1}^{m_j}(1+p_h^{(j)})\right)e^{-D(L_j-L_{j-1})T_b}\frac{\partial}{\partial T}\left(\Psi_1^{(j-1)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ & + \sum_{s=1}^{m_j+1}\left((1-p_N^{(j)})\prod_{h=s+1}^{m_j}(1+p_h^{(j)})\right)e^{-D(L_j-l_s^{(j)})T_b}\overline{R}_s^{(j)}. \end{aligned} \tag{A11}$$

Additionally, from (A11), it can be obtained by mathematical induction that

$$\begin{aligned} & \frac{\partial}{\partial T}\left(\Psi_1^{(m+1)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & \underbrace{\prod_{i=1}^{m+1}\left((1-p_N^{(i)})\prod_{h=1}^{m_i}(1+p_h^{(i)})\right)e^{-D(\overbrace{L_{m+1} - L_0}^{=1})T_b}}_{<1}\frac{\partial c_f(T_b,0)}{\partial T} \\ & + \prod_{i=2}^{m+1}\left((1-p_N^{(i)})\prod_{h=1}^{m_i}(1+p_h^{(i)})\right)\sum_{s=1}^{m_1+1}\left((1-p_N^{(1)})\prod_{h=s+1}^{m_1}(1+p_h^{(1)})\right)e^{-D(L_{m+1}-l_s^{(1)})T_b}\overline{R}_s^{(1)} \\ & + \prod_{i=3}^{m+1}\left((1-p_N^{(i)})\prod_{h=1}^{m_i}(1+p_h^{(i)})\right)\sum_{s=1}^{m_2+1}\left((1-p_N^{(2)})\prod_{h=s+1}^{m_2}(1+p_h^{(2)})\right)e^{-D(L_{m+1}-l_s^{(2)})T_b}\overline{R}_s^{(2)} \\ & + \cdots \\ & + \left((1-p_N^{(m+1)})\prod_{h=1}^{m_{m+1}}(1+p_h^{(m+1)})\right)\sum_{s=1}^{m_m+1}\left((1-p_N^{(m)})\prod_{h=s+1}^{m_m}(1+p_h^{(m)})\right)e^{-D(L_{m+1}-l_s^{(m)})T_b}\overline{R}_s^{(m)} \\ & + \sum_{s=1}^{m_{m+1}+1}\left((1-p_N^{(m+1)})\prod_{h=s+1}^{m_{m+1}}(1+p_h^{(m+1)})\right)e^{-D(L_{m+1}-l_s^{(m+1)})T_b}\overline{R}_s^{(m+1)}. \end{aligned} \tag{A12}$$

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On the other hand, from (27), (26), (23) and (14), we see that

$$\Psi_1^{(m+1)}(c_pV_p + c_f(T, c_p)) = \frac{b_0^*}{-a_0^*}c_p + c_f(T, c_p),$$

and thus,

$$\frac{\partial}{\partial T}\left(\Psi_1^{(m+1)}(c_pV_p + c_f(T, c_p))\right)\Big|_{(T,c_p)=(T_b,0)} = \frac{\partial c_f(T_b, 0)}{\partial T}. \tag{A13}$$

Therefore, it holds from (A12), (A13), (12) and (A10) that

$$\frac{\partial c_f(T_b, 0)}{\partial T} < 0.$$

This completes the proof. □

Proposition A2. *It is considered that*

$$\begin{cases} \left.\begin{aligned} \frac{\partial}{\partial cp}\left(\Psi_2^{(0)}(c_pV_p + c_f(T,cp)V_f)\right)\Big|_{(T,cp)=(T_b,0)} &= 1, \\ \frac{\partial}{\partial cp}\left(I_{k_2}^{(1)}(c_pV_p + c_f(T,cp)V_f)\right)\Big|_{(T,cp)=(T_b,0)} &= e^{\int_{0^+}^{l_k^{(1)}T_b}(a-\frac{\alpha y^*(t)}{b})dt}, \\ & \quad k = 1,2,\cdots,m_1, \end{aligned}\right\} j = 1, \\ \left.\begin{aligned} \frac{\partial}{\partial cp}\left(\Psi_2^{(j-1)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} &= \left(\prod_{i=1}^{j-1}(1-p_P^{(i)})\right)e^{\int_{0^+}^{L_{j-1}T_b}(a-\frac{\alpha y^*(t)}{b})dt}, \\ \frac{\partial}{\partial cp}\left(I_{k_2}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} &= \left(\prod_{i=1}^{j-1}(1-p_P^{(i)})\right)e^{\int_{0^+}^{l_k^{(j)}T_b}(a-\frac{\alpha y^*(t)}{b})dt}, \\ & \quad k = 1,2,\cdots,m_j. \end{aligned}\right\} j = 2,3,\cdots,m,m+1. \end{cases} \tag{A14}$$

Proof. From (15), (20) and (A4), we see that

$$\begin{aligned} & \frac{\partial}{\partial cp}\left(I_{k_2}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & \underbrace{\frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_pV_p + c_f(T,c_p)V_f))}{\partial y^0}\Big|_{(T,c_p)=(T_b,0)}}_{=0} \frac{\partial}{\partial cp}\left(I_{(k-1)_1}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ & + \frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_pV_p + c_f(T,c_p)V_f))}{\partial x^0}\Big|_{(T,c_p)=(T_b,0)} \frac{\partial}{\partial cp}\left(I_{(k-1)_2}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & \frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_pV_p + c_f(T,c_p)V_f))}{\partial x^0}\Big|_{(T,c_p)=(T_b,0)} \frac{\partial}{\partial cp}\left(I_{(k-1)_2}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & e^{\int_{0^+}^{(l_k^{(j)} - l_{k-1}^{(j)})T_b}(a - \frac{\alpha y(t; I_{k-1}^{(j)}(X_0^*))}{b})dt} \frac{\partial}{\partial cp}\left(I_{(k-1)_2}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & e^{\int_{0^+}^{(l_k^{(j)} - l_{k-1}^{(j)})T_b}(a - \frac{\alpha y^*(t + l_{k-1}^{(j)}T_b)}{b})dt} \frac{\partial}{\partial cp}\left(I_{(k-1)_2}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & e^{\int_{l_{k-1}^{(j)}T_b^+}^{l_k^{(j)}T_b}(a - \frac{\alpha y^*(t)}{b})dt} \frac{\partial}{\partial cp}\left(I_{(k-1)_2}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}, \\ & k = 1,2,3,\cdots,m_j,m_j+1, \end{aligned}$$

i.e.,

$$\begin{aligned} & \frac{\partial}{\partial c_p}\left(I_{k_2}^{(j)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & e^{\int_{L_{j-1}T_b^+}^{l_k^{(j)}T_b}(a - \frac{\alpha y^*(t)}{b})dt} \frac{\partial}{\partial c_p}\left(\Psi_2^{(j-1)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}, \\ & k = 1,2,3,\cdots,m_j,m_j+1. \end{aligned} \tag{A15}$$

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Then, it follows from (A15) that

$$\begin{aligned}&\frac{\partial}{\partial c_p}\left(\Psi_2^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}\\ =\ &(1-p_P^{(j)})e^{\int_{L_{j-1}T_b^+}^{L_jT_b}(a-\frac{\alpha y^*(t)}{b})dt}\frac{\partial}{\partial c_p}\left(\Psi_2^{(j-1)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}.\end{aligned} \tag{A16}$$

Since

$$\frac{\partial}{\partial c_p}\left(\Psi_2^{(0)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}=1,$$

it follows from (A16) that

$$\frac{\partial}{\partial c_p}\left(\Psi_2^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} = \left(\prod_{i=1}^{j}(1-p_P^{(i)})\right)e^{\int_{0^+}^{L_jT_b}(a-\frac{\alpha y^*(t)}{b})dt}, \quad j=1,2,3,\cdots,m. \tag{A17}$$

Furthermore, from (A17) and (A15), (A14) is valid.

This completes the proof. □

Proposition A3. *It is considered that*

$$\begin{cases}\left.\begin{aligned}&\frac{\partial}{\partial c_p}\left(\Psi_1^{(0)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} = \underbrace{\frac{b_0^*}{-a_0^*}}_{>0}+\underbrace{\frac{\partial c_f(T_b,0)}{\partial c_p}}_{=0}>0,\\ &\frac{\partial}{\partial c_p}\left(I_{k_1}^{(1)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} > 0,\\ &\qquad k=1,2,\cdots,m_1,\end{aligned}\right\}j=1,\\ \left.\begin{aligned}&\frac{\partial}{\partial c_p}\left(\Psi_1^{(j-1)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} > 0,\\ &\frac{\partial}{\partial c_p}\left(I_{k_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} > 0,\\ &\qquad k=1,2,\cdots,m_j,\end{aligned}\right\}j=2,3,\cdots,m,m+1.\end{cases}$$

Proof. From (15), (A4) and Proposition A2, we see that

$$\begin{aligned}&\frac{1}{1+p_k^{(j)}}\frac{\partial}{\partial c_p}\left(I_{k_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}\\ =\ &\underbrace{\frac{\partial y((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\Big|_{(T,c_p)=(T_b,0)}}_{>0}\frac{\partial}{\partial c_p}\left(I_{(k-1)_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}\\ &+\underbrace{\frac{\partial y((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\Big|_{(T,c_p)=(T_b,0)}}_{>0}\underbrace{\frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}}_{>0},\\ &k=1,2,3,\cdots,m_j,m_j+1,\end{aligned}$$

where $p_{m_j+1}^{(j)}=0$.

On the other hand, it follows from (27), (26), (23) and (24) that

$$\frac{\partial g_1(T_b,X_0^*)}{\partial y^0}\left(\frac{b_0^*}{-a_0^*}+\frac{\partial c_f(T_b,X_0^*)}{\partial c_p}\right)+\frac{\partial g_1(T_b,X_0^*)}{\partial x^0}=0,$$

or,

$$a_0^*\left(\frac{b_0^*}{-a_0^*}+\frac{\partial c_f(T_b,0)}{\partial c_p}\right)+b_0^*=0,$$

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and hence we obtain that
$$\frac{\partial c_f(T_b,0)}{\partial c_p} = 0.$$
Then, we have
$$\frac{\partial}{\partial c_p}\left(\Psi_1^{(0)}(c_pV_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} = \underbrace{\frac{b_0^*}{-a_0^*}}_{>0} + \underbrace{\frac{\partial c_f(T_b,0)}{\partial c_p}}_{=0} > 0.$$

This completes the proof. □

Proposition A4. *(i)* *If the following condition holds:*
$$\frac{ab}{K\alpha} \geq \frac{\max\limits_{1\leq j\leq m+1}\max\limits_{1\leq k\leq m_j} y^*(l_k^{(j)}T_b^+)}{b}, \tag{A18}$$
then the following inqualities
$$\begin{cases} \frac{\partial^2 x((l_1^{(j)}-L_{j-1})T;\Psi^{(j-1)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^{0\,2}}\Big|_{(T,c_p)=(T_b,0)} & < \quad 0, \\ \frac{\partial^2 x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^{0\,2}}\Big|_{(T,c_p)=(T_b,0)} & < \quad 0, \\ & \quad\ k=2,3,\cdots,m_j,m_j+1, \end{cases} \tag{A19}$$
hold for $j = 1,2,3,\cdots,m,m+1$.

(ii) *It is considered that*
$$\frac{\partial^2 x(t;t_0,X_0)}{\partial y^0\partial x^0} < 0$$
where $X_0 = (y^0,0)^T \in \mathbb{R}^2$ and $t_0 \geq 0$.

Proof. (i) It follows from (A1) that
$$\frac{d}{dt}\left(\frac{\partial x}{\partial x^0}(t;t_0,X^0)\right) = \frac{\partial F_2(\Phi(t;t_0,X^0))}{\partial y}\frac{\partial y}{\partial x^0}(t;t_0,X^0) + \frac{\partial F_2(\Phi(t;t_0,X^0))}{\partial x}\frac{\partial x}{\partial x^0}(t;t_0,X^0),$$
where
$$F_2(y,x) = ax\left(1-\frac{x}{K}\right) - \frac{\alpha xy}{b+x}.$$
Then, we have
$$\begin{aligned}&\frac{d}{dt}\left(\frac{\partial^2 x}{\partial x^{0\,2}}(t;t_0,X^0)\right)\\ =\ &\frac{\partial F_2(\Phi(t;t_0,X^0))}{\partial x}\frac{\partial^2 x}{\partial x^{0\,2}}(t;t_0,X^0) + h_1(t,t_0,X^0),\end{aligned} \tag{A20}$$
where
$$\begin{aligned}&h_1(t,t_0,X^0)\\ =\ &\frac{\partial}{\partial x^0}\left(\frac{\partial F_2(\Phi(t;t_0,X^0))}{\partial y}\right)\frac{\partial y}{\partial x^0}(t;t_0,X^0) + \frac{\partial}{\partial x^0}\left(\frac{\partial F_2(\Phi(t;t_0,X^0))}{\partial x}\right)\frac{\partial x}{\partial x^0}(t;t_0,X^0)\\ &+\frac{\partial F_2(\Phi(t;t_0,X^0))}{\partial y}\frac{\partial^2 y}{\partial x^{0\,2}}(t;t_0,X^0).\end{aligned}$$

Note that
$$\frac{\partial^2 x}{\partial x^{0\,2}}(t_0;t_0,X^0) = 0. \tag{A21}$$

Since

$$
\begin{aligned}
& h_1(t,t_0,X^0) \\
= & -\tfrac{2\alpha b}{(b+x(t;t_0,X^0))^2}\tfrac{\partial y}{\partial x^0}(t;t_0,X^0)\tfrac{\partial x}{\partial x^0}(t;t_0,X^0) \\
& +\left(\tfrac{2\alpha b y(t;t_0,X^0)}{(b+x(t;t_0,X^0))^3}-\tfrac{2a}{K}\right)\left(\tfrac{\partial x}{\partial x^0}(t;t_0,X^0)\right)^2 \\
& +\left(-\tfrac{\alpha x(t;t_0,X^0)}{b+x(t;t_0,X^0)}\right)\tfrac{\partial^2 y}{\partial x^{0 2}}(t;t_0,X^0),
\end{aligned}
$$

We have

$$
\begin{aligned}
& h_1(t,t_0,X_0) \\
= & -\tfrac{2\alpha}{b}\tfrac{\partial x}{\partial x^0}(t;t_0,X_0)\left[\tfrac{\partial y}{\partial x^0}(t;t_0,X_0)+\left(\tfrac{ab}{K\alpha}-\tfrac{y(t;t_0,X_0)}{b}\right)\tfrac{\partial x}{\partial x^0}(t;t_0,X_0)\right].
\end{aligned} \tag{A22}
$$

Then, according to the comparison theorem, and (A20), (A21), (A22) and (A18), we know that (A19) is valid.

(ii) Similar to (A20), we have

$$
\frac{d}{dt}\left(\frac{\partial^2 x}{\partial y^0\partial x^0}(t;t_0,X^0)\right) = \frac{\partial F_2(\Phi(t;t_0,X^0))}{\partial x}\frac{\partial^2 x}{\partial y^0\partial x^0}(t;t_0,X^0)+h_2(t,t_0,X^0),
$$

where

$$
\begin{aligned}
& h_2(t,t_0,X^0) \\
= & \tfrac{\partial}{\partial y^0}\left(\tfrac{\partial F_2(\Phi(t;t_0,X^0))}{\partial y}\right)\tfrac{\partial y}{\partial x^0}(t;t_0,X^0)+\tfrac{\partial}{\partial y^0}\left(\tfrac{\partial F_2(\Phi(t;t_0,X^0))}{\partial x}\right)\tfrac{\partial x}{\partial x^0}(t;t_0,X^0) \\
& +\tfrac{\partial F_2(\Phi(t;t_0,X^0))}{\partial y}\tfrac{\partial^2 y}{\partial y^0\partial x^0}(t;t_0,X^0).
\end{aligned}
$$

Note that

$$
\frac{\partial^2 x}{\partial y^0\partial x^0}(t_0;t_0,X^0)=0.
$$

Since

$$
\begin{aligned}
& h_2(t,t_0,X^0) \\
= & -\tfrac{\alpha b}{(b+x(t;t_0,X^0))^2}\tfrac{\partial x(t;t_0,X^0)}{\partial y^0}\tfrac{\partial y}{\partial x^0}(t;t_0,X^0) \\
& +\left(\left(\tfrac{2\alpha b y(t;t_0,X^0)}{(b+x(t;t_0,X^0))^3}-\tfrac{2a}{K}\right)\tfrac{\partial x(t;t_0,X^0)}{\partial y^0}-\tfrac{\alpha b}{(b+x(t;t_0,X^0))^2}\tfrac{\partial y(t;t_0,X^0)}{\partial y^0}\right)\tfrac{\partial x}{\partial x^0}(t;t_0,X^0) \\
& +\left(-\tfrac{\alpha x(t;t_0,X^0)}{b+x(t;t_0,X^0)}\right)\tfrac{\partial^2 y}{\partial y^0\partial x^0}(t;t_0,X^0),
\end{aligned}
$$

We have

$$
h_2(t,t_0,X_0)=-\frac{\alpha}{b}\frac{\partial y(t;t_0,X_0)}{\partial y^0}\frac{\partial x}{\partial x^0}(t;t_0,X_0)<0,
$$

which implies that

$$
\frac{\partial^2 x}{\partial y^0\partial x^0}(t;t_0,X_0)<0.
$$

This completes the proof. □

Proposition A5. *For $j=1,2,3,\cdots,m,m+1$, it is true that*

$$
\begin{cases}
\frac{\partial}{\partial c_p}\left(\frac{\partial x((l_1^{(j)}-L_{j-1})T;\Psi^{(j-1)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\right)\Bigg|_{(T,c_p)=(T_b,0)} < 0, \\
\frac{\partial}{\partial c_p}\left(\frac{\partial x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j-1)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\right)\Bigg|_{(T,c_p)=(T_b,0)} < 0, \\
\qquad\qquad k=2,3,\cdots,m_j, \\
\frac{\partial}{\partial c_p}\left(\frac{\partial x((L_j-l_{m_j}^{(j)})T;I_{m_j}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\right)\Bigg|_{(T,c_p)=(T_b,0)} < 0,
\end{cases}
$$

and

$$\begin{cases} \frac{\partial}{\partial c_p}\left(\frac{\partial x((l_1^{(j)}-L_{j-1})T;\Psi^{(j-1)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\right)\Big|_{(T,c_p)=(T_b,0)} & < & 0, \\ \frac{\partial}{\partial c_p}\left(\frac{\partial x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j-1)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\right)\Big|_{(T,c_p)=(T_b,0)} & < & 0, \\ & & k=2,3,\cdots,m_j, \\ \frac{\partial}{\partial c_p}\left(\frac{\partial x((L_j-l_{m_j}^{(j)})T;I_{m_j}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\right)\Big|_{(T,c_p)=(T_b,0)} & < & 0. \end{cases}$$

Proof. (1) From (20) and Propositions A4 and A2, we obtain that

$$\begin{aligned} & \frac{\partial}{\partial c_p}\left(\frac{\partial x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & \underbrace{\frac{\partial^2 x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial {y^0}^2}\Big|_{(T,c_p)=(T_b,0)}}_{=0} \frac{\partial}{\partial c_p}\left(I_{(k-1)_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ & + \frac{\partial^2 x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0\partial y^0}\Big|_{(T,c_p)=(T_b,0)} \frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & \underbrace{\frac{\partial^2 x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0\partial y^0}\Big|_{(T,c_p)=(T_b,0)}}_{<0} \underbrace{\frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}}_{>0} \\ < & 0, \\ & k=1,2,3,\cdots,m_j,m_j+1. \end{aligned}$$

(2) Similarly, we obtain that

$$\begin{aligned} & \frac{\partial}{\partial c_p}\left(\frac{\partial x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\right)\Big|_{(T,c_p)=(T_b,0)} \\ = & \underbrace{\frac{\partial^2 x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0\partial x^0}\Big|_{(T,c_p)=(T_b,0)}}_{<0} \underbrace{\frac{\partial}{\partial c_p}\left(I_{(k-1)_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}}_{>0} \\ & + \underbrace{\frac{\partial^2 x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial {x^0}^2}\Big|_{(T,c_p)=(T_b,0)}}_{<0} \underbrace{\frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}}_{>0} \\ < & 0, \\ & k=1,2,3,\cdots,m_j,m_j+1, \end{aligned}$$

This completes the proof. □

Appendix C. The Determination of the Sign of $\frac{\partial^2\Psi_2^{(m+1)}(c_pV_p+c_f(T,c_p)V_f)}{\partial c_p^2}$ at Point $(T_b,0)$

For $j=1,2,3,\cdots,m,m+1$, according to (15),

$$\begin{aligned} & \frac{\partial}{\partial c_p}\left(I_{k_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right) \\ = & \frac{\partial}{\partial c_p}\left(x(\tau_k^{(j)}-\tau_{k-1}^{(j)};I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))\right) \\ = & \frac{\partial}{\partial c_p}\left(x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))\right) \\ = & \frac{\partial x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\frac{\partial}{\partial c_p}\left(I_{(k-1)_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right) \\ & + \frac{\partial x((l_k^{(j)}-l_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right), \\ & k=1,2,\cdots,m_j,m_j+1, \end{aligned} \tag{A23}$$

and then,

$$
\begin{aligned}
&\frac{\partial^2}{\partial c_p^2}\left(I_{k_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right) \\
= \;& \frac{\partial}{\partial c_p}\left(\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\right)\frac{\partial}{\partial c_p}\left(I_{(k-1)_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right) \\
&+\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\frac{\partial^2}{\partial c_p^2}\left(I_{(k-1)_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right) \\
&+\frac{\partial}{\partial c_p}\left(\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\right)\frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right) \\
&+\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\frac{\partial^2}{\partial c_p^2}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right), \\
&k=1,2,\cdots,m_j,m_j+1.
\end{aligned}
\tag{A24}
$$

Hence, from (A24), (A4), (A2), and Propositions A5, A3 and A2, we have

$$
\begin{aligned}
&\frac{\partial^2}{\partial c_p^2}\left(I_{k_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
=\;& \underbrace{\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\Big|_{(T,c_p)=(T_b,0)}}_{>0}\frac{\partial^2}{\partial c_p^2}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
&+\hat{R}_k^{(j)}, \\
&k=1,2,\cdots,m_j,m_j+1,
\end{aligned}
\tag{A25}
$$

where

$$
\begin{aligned}
&\hat{R}_k^{(j)} \\
=\;& \frac{\partial}{\partial c_p}\left(\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\right)\Big|_{(T,c_p)=(T_b,0)}\frac{\partial}{\partial c_p}\left(I_{(k-1)_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
&+\underbrace{\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\Big|_{(T,c_p)=(T_b,0)}}_{=0}\frac{\partial^2}{\partial c_p^2}\left(I_{(k-1)_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
&+\frac{\partial}{\partial c_p}\left(\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\right)\Big|_{(T,c_p)=(T_b,0)}\frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
=\;& \underbrace{\frac{\partial}{\partial c_p}\left(\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial y^0}\right)\Big|_{(T,c_p)=(T_b,0)}}_{<0}\underbrace{\frac{\partial}{\partial c_p}\left(I_{(k-1)_1}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}}_{>0} \\
&+\underbrace{\frac{\partial}{\partial c_p}\left(\frac{\partial x((I_k^{(j)}-I_{k-1}^{(j)})T;I_{k-1}^{(j)}(c_pV_p+c_f(T,c_p)V_f))}{\partial x^0}\right)\Big|_{(T,c_p)=(T_b,0)}}_{<0}\underbrace{\frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}}_{>0} \\
<\;& 0, \\
&k=1,2,\cdots,m_j,m_j+1.
\end{aligned}
\tag{A26}
$$

Since

$$
\begin{cases}
\frac{\partial^2}{\partial c_p^2}\left(\Psi_2^{(0)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} = 0, \\
\frac{\partial^2}{\partial c_p^2}\left(\Psi_2^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} = (1-p_P^{(j)})\frac{\partial^2}{\partial c_p^2}\left(I_{(m_j+1)_2}^{(j)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}, \\
\qquad j=1,2,\cdots,m,m+1,
\end{cases}
$$

it follows from (A25) and (A26) that

$$
\frac{\partial^2}{\partial c_p^2}\left(\Psi_2^{(m+1)}(c_pV_p+c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} < 0. \tag{A27}
$$

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Appendix D. Determination of the Sign of $\frac{\partial^2 \Psi_2^{(m+1)}(c_p V_p + c_f(T,c_p)V_f)}{\partial c_p \partial T}$ at Point $(T_b, 0)$

Similar to (A23), we see that

$$\begin{aligned}
& \frac{\partial}{\partial T}\left(I_{k_2}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right) \\
= & \frac{\partial}{\partial T}\left(x(\tau_k^{(j)} - \tau_{k-1}^{(j)}; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))\right) \\
= & \frac{\partial}{\partial T}\left(x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))\right) \\
= & (l_k^{(j)} - l_{k-1}^{(j)})\frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))}{\partial t} \\
& + \frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))}{\partial y^0}\frac{\partial}{\partial T}\left(I_{(k-1)_1}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right) \\
& + \frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))}{\partial x^0}\frac{\partial}{\partial T}\left(I_{(k-1)_2}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right), \\
& k = 1, 2, \cdots, m_j, m_j + 1,
\end{aligned}$$

and thus, it follows from (20) and Propositions A5 and A1 that

$$\begin{aligned}
& \frac{\partial^2}{\partial c_p \partial T}\left(I_{k_2}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
= & (l_k^{(j)} - l_{k-1}^{(j)})\frac{\partial}{\partial c_p}\left(\frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))}{\partial t}\right)\Bigg|_{(T,c_p)=(T_b,0)} \\
& + \frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))}{\partial x^0}\Bigg|_{(T,c_p)=(T_b,0)}\frac{\partial^2}{\partial c_p \partial T}\left(I_{(k-1)_2}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
& + \widetilde{R}_k^{(j)}, \\
& k = 1, 2, \cdots, m_j, m_j + 1,
\end{aligned} \tag{A28}$$

where

$$\begin{aligned}
& \widetilde{R}_k^{(j)} \\
= & \underbrace{\frac{\partial}{\partial c_p}\left(\frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))}{\partial y^0}\right)\Bigg|_{(T,c_p)=(T_b,0)}}_{<0}\underbrace{\frac{\partial}{\partial T}\left(I_{(k-1)_1}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)}}_{<0} \\
> & 0, \\
& k = 1, 2, \cdots, m_j, m_j + 1.
\end{aligned} \tag{A29}$$

In addition, from (20), (A3), (A4) and Proposition A2, we see that

$$\begin{aligned}
& \frac{\partial}{\partial c_p}\left(\frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))}{\partial t}\right)\Bigg|_{(T,c_p)=(T_b,0)} \\
= & \frac{\partial^2 x((l_k^{(j)} - l_{k-1}^{(j)})T; I_{k-1}^{(j)}(c_p V_p + c_f(T,c_p)V_f))}{\partial x^0 \partial t}\Bigg|_{(T,c_p)=(T_b,0)}\frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
= & \left(a - \frac{\alpha y((l_k^{(j)} - l_{k-1}^{(j)})T_b; I_{k-1}^{(j)}(X_0^*))}{b}\right)\frac{\partial x((l_k^{(j)} - l_{k-1}^{(j)})T_b; I_{k-1}^{(j)}(X_0^*))}{\partial x^0} \\
& \times \frac{\partial}{\partial c_p}\left(I_{(k-1)_2}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
= & \left(a - \frac{\alpha y((l_k^{(j)} - l_{k-1}^{(j)})T_b; I_{k-1}^{(j)}(X_0^*))}{b}\right)e^{\int_{0^+}^{(l_k^{(j)} - l_{k-1}^{(j)})T_b}(a - \frac{\alpha y(t; I_{k-1}^{(j)}(X_0^*))}{b})dt} \\
& \times \left(\prod_{i=1}^{j-1}(1 - p_P^{(i)})\right)e^{\int_{0^+}^{l_{k-1}^{(j)}}(a - \frac{\alpha y^*(t)}{b})dt} \\
= & \left(\prod_{i=1}^{j-1}(1 - p_P^{(i)})\right)\left(a - \frac{\alpha y^*(l_k^{(j)} T_b)}{b}\right)e^{\int_{0^+}^{l_k^{(j)} T_b}(a - \frac{\alpha y^*(t)}{b})dt}, \\
& k = 1, 2, \cdots, m_j, m_j + 1.
\end{aligned} \tag{A30}$$

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From (A28), (A30) and (A4), we have

$$
\begin{aligned}
&\frac{\partial^2}{\partial c_p \partial T}\left(I_{k_2}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
=\;& (l_k^{(j)} - l_{k-1}^{(j)})\left[\left(\prod_{i=1}^{j-1}(1-p_P^{(i)})\right)\left(a - \frac{\alpha y^*(l_k^{(j)}T_b)}{b}\right)e^{\int_{0^+}^{l_k^{(j)}T_b}(a-\frac{\alpha y^*(t)}{b})dt}\right] \\
&+ e^{\int_{0^+}^{(l_k^{(j)}-l_{k-1}^{(j)})T_b}(a-\frac{\alpha y(t;I_{k-1}^{(j)}(X_0^*))}{b})dt}\frac{\partial^2}{\partial c_p \partial T}\left(I_{(k-1)_2}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
&+ \widetilde{R}_k^{(j)} \\
=\;& \left(\prod_{i=1}^{j-1}(1-p_P^{(i)})\right)\left((l_k^{(j)}-l_{k-1}^{(j)})a - (l_k^{(j)}-l_{k-1}^{(j)})\frac{\alpha y^*(l_k^{(j)}T_b)}{b}\right)e^{\int_{0^+}^{l_k^{(j)}T_b}(a-\frac{\alpha y^*(t)}{b})dt} \\
&+ e^{\int_{l_{k-1}^{(j)}T_b^+}^{l_k^{(j)}T_b}(a-\frac{\alpha y^*(t)}{b})dt}\frac{\partial^2}{\partial c_p \partial T}\left(I_{(k-1)_2}^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
&+ \widetilde{R}_k^{(j)}, \\
&k = 1,2,\cdots,m_j,m_j+1.
\end{aligned}
\tag{A31}
$$

Then, from (A31), it can be obtained by mathematical induction that

$$
\begin{aligned}
&\frac{\partial^2}{\partial c_p \partial T}\left(\Psi_2^{(j)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
=\;& \left(\prod_{i=1}^{j}(1-p_P^{(i)})\right)e^{\int_{0^+}^{L_jT_b}(a-\frac{\alpha y^*(t)}{b})dt} \\
&\times\left[(L_j - L_{j-1})a - \left((L_j - l_{m_j}^{(j)})\frac{\alpha y^*(L_jT_b)}{b} + \cdots + (l_2^{(j)}-l_1^{(j)})\frac{\alpha y^*(l_2^{(j)}T_b)}{b} + (l_1^{(j)}-L_{j-1})\frac{\alpha y^*(l_1^{(j)}T_b)}{b}\right)\right] \\
&+ (1-p_P^{(j)})e^{\int_{L_{j-1}T_b^+}^{L_jT_b}(a-\frac{\alpha y^*(t)}{b})dt}\frac{\partial^2}{\partial c_p \partial T}\left(\Psi_2^{(j-1)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
&+ (1-p_P^{(j)})\left(\sum_{s=1}^{m_j+1} e^{\int_{l_s^{(j)}T_b^+}^{L_jT_b}(a-\frac{\alpha y^*(t)}{b})dt}\widetilde{R}_s^{(j)}\right).
\end{aligned}
$$

Furthermore, it can also be obtained by mathematical induction that

$$
\begin{aligned}
&\frac{\partial^2}{\partial c_p \partial T}\left(\Psi_2^{(m+1)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
=\;& \left(\prod_{i=1}^{m+1}(1-p_P^{(i)})\right)e^{\int_{0^+}^{L_{m+1}T_b}(a-\frac{\alpha y^*(t)}{b})dt} \\
&\times\Big[(L_{m+1}-L_0)a \\
&\quad -\Big((L_{m+1}-l_{m_{m+1}}^{(m+1)})\frac{\alpha y^*(L_{m+1}T_b)}{b} + \cdots + (l_2^{(m+1)}-l_1^{(m+1)})\frac{\alpha y^*(l_2^{(m+1)}T_b)}{b} + (l_1^{(m+1)}-L_m)\frac{\alpha y^*(l_1^{(m+1)}T_b)}{b} \\
&\qquad\qquad \vdots \\
&\qquad + (L_2 - l_{m_2}^{(2)})\frac{\alpha y^*(L_2T_b)}{b} + \cdots + (l_2^{(2)}-l_1^{(2)})\frac{\alpha y^*(l_2^{(2)}T_b)}{b} + (l_1^{(2)}-L_1)\frac{\alpha y^*(l_1^{(2)}T_b)}{b} \\
&\qquad + (L_1 - l_{m_1}^{(1)})\frac{\alpha y^*(L_1T_b)}{b} + \cdots + (l_2^{(1)}-l_1^{(1)})\frac{\alpha y^*(l_2^{(1)}T_b)}{b} + (l_1^{(1)}-L_0)\frac{\alpha y^*(l_1^{(1)}T_b)}{b}\Big)\Big] \\
&+ \sum_{h=1}^{m+1}\left(\prod_{i=h}^{m+1}(1-p_P^{(i)})\right)\left(\sum_{s=1}^{m_h+1} e^{\int_{l_s^{(h)}T_b^+}^{L_{m+1}T_b}(a-\frac{\alpha y^*(t)}{b})dt}\widetilde{R}_s^{(h)}\right).
\end{aligned}
\tag{A32}
$$

Then, from (A32), (22), (20) and (A29), we obtain that

$$
\begin{aligned}
&\frac{\partial^2}{\partial c_p \partial T}\left(\Psi_2^{(m+1)}(c_p V_p + c_f(T,c_p)V_f)\right)\Big|_{(T,c_p)=(T_b,0)} \\
=\;& \frac{\left(\prod_{i=1}^{m+1}(1-p_p^{(i)})\right) e^{\int_{0^+}^{L_{m+1}T_b}(a-\frac{\alpha y^*(t)}{b})dt}}{T_b} \\
&\times \Bigg[\underbrace{(L_{m+1}-L_0)}_{=1} a T_b \\
&\quad - \Bigg((\underbrace{L_{m+1}}_{=1} - l_{m_{m+1}}^{(m+1)})T_b \frac{\alpha y^*(\overbrace{L_{m+1}T_b}^{=1})}{b} + \cdots + (l_2^{(m+1)} - l_1^{(m+1)})T_b \frac{\alpha y^*(l_2^{(m+1)}T_b)}{b} + (l_1^{(m+1)} - L_m)T_b \frac{\alpha y^*(l_1^{(m+1)}T_b)}{b} \\
&\quad\quad \vdots \\
&\quad + (L_2 - l_{m_2}^{(2)})T_b \frac{\alpha y^*(L_2T_b)}{b} + \cdots + (l_2^{(2)} - l_1^{(2)})T_b \frac{\alpha y^*(l_2^{(2)}T_b)}{b} + (l_1^{(2)} - L_1)T_b \frac{\alpha y^*(l_1^{(2)}T_b)}{b} \\
&\quad + (L_1 - l_{m_1}^{(1)})T_b \frac{\alpha y^*(L_1T_b)}{b} + \cdots + (l_2^{(1)} - l_1^{(1)})T_b \frac{\alpha y^*(l_2^{(1)}T_b)}{b} + (l_1^{(1)} - \underbrace{L_0}_{=0})T_b \frac{\alpha y^*(l_1^{(1)}T_b)}{b} \Bigg) \Bigg] \\
&+ \sum_{h=1}^{m+1}\left(\prod_{i=h}^{m+1}(1-p_p^{(i)})\right)\left(\sum_{s=1}^{m_h+1} e^{\int_{l_s^{(h)}T_b^+}^{L_{m+1}T_b}(a-\frac{\alpha y^*(t)}{b})dt} \bar{R}_s^{(h)} \right) \\
=\;& \frac{\left(\prod_{i=1}^{m+1}(1-p_p^{(i)})\right) e^{\int_{0^+}^{L_{m+1}T_b}(a-\frac{\alpha y^*(t)}{b})dt}}{T_b} \\
&\times \Bigg[\underbrace{\Bigg(\ln \frac{1}{\prod_{j=1}^{m+1}(1-p_p^{(j)})} + \int_{0^+}^{T_b} \frac{\alpha y^*(t)}{b} dt \Bigg) - \Bigg(\int_{l_{m_{m+1}}^{(m+1)}T_b^+}^{T_b} \frac{\alpha y^*(T_b)}{b} dt + \cdots + \int_{l_1^{(m+1)}T_b^+}^{l_2^{(m+1)}T_b} \frac{\alpha y^*(l_2^{(m+1)}T_b)}{b} dt + \int_{L_mT_b^+}^{l_1^{(m+1)}T_b} \frac{\alpha y^*(l_1^{(m+1)}T_b)}{b} dt \\
&\quad\quad \vdots \\
&\quad + \int_{l_{m_2}^{(2)}T_b^+}^{L_2T_b} \frac{\alpha y^*(L_2T_b)}{b} dt + \cdots + \int_{l_1^{(2)}T_b^+}^{l_2^{(2)}T_b} \frac{\alpha y^*(l_2^{(2)}T_b)}{b} dt + \int_{L_1T_b^+}^{l_1^{(2)}T_b} \frac{\alpha y^*(l_1^{(2)}T_b)}{b} dt \\
&\quad + \int_{l_{m_1}^{(1)}T_b^+}^{L_1T_b} \frac{\alpha y^*(L_1T_b)}{b} dt + \cdots + \int_{l_1^{(1)}T_b^+}^{l_2^{(1)}T_b} \frac{\alpha y^*(l_2^{(1)}T_b)}{b} dt + \int_{0^+}^{l_1^{(1)}T_b} \frac{\alpha y^*(l_1^{(1)}T_b)}{b} dt \Bigg)}_{>0} \Bigg] \\
&+ \sum_{h=1}^{m+1}\left(\prod_{i=h}^{m+1}(1-p_p^{(i)})\right)\left(\sum_{s=1}^{m_h+1} e^{\int_{l_s^{(h)}T_b^+}^{L_{m+1}T_b}(a-\frac{\alpha y^*(t)}{b})dt} \underbrace{\bar{R}_s^{(h)}}_{>0} \right) \\
>\;& 0.
\end{aligned}
\tag{A33}
$$

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