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A Study on the Semi-Discrete KP Equation: Bilinear Bäcklund Transformation, Lax Pair and Periodic Solutions

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Abstract

In this paper, we study the integrability and solution structure of the semi-discrete KP equation. We derive the bilinear Bäcklund transformation, Lax pair, and the nonlinear superposition formula of the semi-discrete KP equation. Periodic solutions are constructed by using Hirota's method and Riemann theta functions. The asymptotic behavior of periodic solutions is analyzed and the connection between periodic solutions and soliton solutions is established.

Keywords: Bäcklund transformation; nonlinear superposition formula; Riemann theta function; periodic solution; soliton solution

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1. Introduction

Discrete integrable systems play a fundamental role in modern mathematical physics, serving as integrable discretizations of continuous soliton equations and providing exact lattice models for nonlinear wave phenomena [1–7]. The well-known discrete equation called Hirota-Miwa equation [8,9] reads as

$$a_1(a_2-a_3)T_1(\tau)T_{23}(\tau)+a_2(a_3-a_1)T_2(\tau)T_{31}(\tau)+a_3(a_1-a_2)T_3(\tau)T_{12}(\tau)=0,$$

where τ is a function of discrete variables n_1, n_2 and n_3, T_i is the shift operator defined by $T_i(\tau)=\tau(n_i+1)$, $T_{ij}=T_i(T_j(\tau))$ and a_i are lattice parameters. A semi-discrete KP (sdKP) equation [8] was obtained by taking a continuous limit from the Hirota-Miwa equation, and its bilinear form is

$$(a_1-a_2)(T_1(\tau)T_2(\tau)-T_{12}(\tau)\tau)+a_1a_2D_xT_1(\tau)\cdot T_2(\tau)=0. \tag{1}$$

The above equation can be expressed as the equivalent form

$$(a_1a_2D_x\mathrm{e}^{\frac{D_{n_1}-D_{n_2}}{2}}-(a_1-a_2)\mathrm{e}^{\frac{D_{n_1}+D_{n_2}}{2}}+(a_1-a_2)\mathrm{e}^{\frac{D_{n_1}-D_{n_2}}{2}})\tau\cdot\tau=0, \tag{2}$$

where Hirota's bilinear operators are defined by [11]

$$D_x^mD_t^nf(x,t)\cdot g(x,t)=\frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial s^n}f(x+y,t+s)g(x-y,t-s)|_{s=0,y=0}$$

and

$$\mathrm{e}^{\delta D_n}f(n)\cdot g(n)=f(n+\delta)g(n-\delta).$$

Through the dependent variable transformation $G = -(\ln \tau)_x$, the bilinear sdKP Equation (2) is rewritten in nonlinear form

$$a_1 a_2 (G_1 - G_2)(G_{12} - G_1 - G_2 + G) + (G_{1x} - G_{2x}) = (a_1 - a_2)(G_{12} - G_1 - G_2 + G). \tag{3}$$

Determinant solutions, including soliton solutions of Equation (3) were obtained in [10], while periodic solutions are also an important part of integrable systems. Nakamura presented a direct approach [12,13] to get multi-periodic wave solutions of nonlinear integrable equations, by using Riemann theta functions [14,15]. This method has been applied to some continuous soliton equations including (1+1) and (2+1) dimensions [16–20]. However, its application to discrete equations remains relatively rare [21–23]. The purpose of this paper is to construct periodic solutions of the semi-discrete KP equation and analyze their connection with soliton solutions. This paper is organized as follows. In Section 2, Bäcklund transformations, Lax pair and the nonlinear superposition formula are derived. In Sections 3 and 4, the one-periodic solution and the two-periodic solution are obtained, and the asymptotic properties are discussed. Finally, conclusion and discussions are given in Section 5.

2. Bilinear Bäcklund Transformation and the Nonlinear Superposition Formula

The bilinear sdKP Equation (2) has the following bilinear Bäcklund transformation (BT) [24]

$$D_x \tau \cdot \tilde{\tau} = \theta \tau \cdot \tilde{\tau} + \frac{1}{\lambda}(a_1 - a_2) e^{D_{n_2}} \tau \cdot \tilde{\tau} \tag{4}$$

$$e^{-\frac{D_{n_1}}{2}} \tau \cdot \tilde{\tau} = \mu e^{\frac{D_{n_1}}{2}} \tau \cdot \tilde{\tau} + \frac{1}{\lambda} a_1 a_2 e^{D_{n_2} - \frac{D_{n_1}}{2}} \tau \cdot \tilde{\tau}. \tag{5}$$

where θ, μ are arbitrary constants, and λ is a nonzero constant. If we take $\tau = 1$, $\tilde{\tau} = 1 + e^{\xi}$, $\xi = rx + pn_1 + qn_2 + \xi_0$, and substitute τ, $\tilde{\tau}$ in BT (4)-(5), the relationship of parameters p, q and r can be obtained

$$r = \frac{(a_2 - a_1)(e^p + e^q - e^{p+q} - 1)}{a_1 a_2 (e^p - e^q)}.$$

Therefore the function $\tilde{\tau}$ satisfying the above dispersion relation gives one-soliton solution of the sdKP Equation (3), that is

$$G = -(\ln(1 + e^{\xi}))_x. \tag{6}$$

In addition, a nonlinear superposition formula is derived using the BT as well, and the formula is as follows.

Proposition 1. *Let τ_0 be a solution of Equation (2). Suppose that τ_1, τ_2 are solutions of Equation (2) given by BT (4)-(5) with a starting solution τ_0 and Bäcklund parameters $(\lambda_1, \mu_1, \theta_1)$ and $(\lambda_2, \mu_2, \theta_2)$ respectively, where $(\lambda_1, \mu_1, \theta_1) \neq (\lambda_2, \mu_2, \theta_2)$, $\tau_j \neq 0$, $j = 0, 1, 2$. Then τ_{12} defined by*

$$e^{\frac{D_{n_2}}{2}} \tau_0 \cdot \tau_{12} = C(\lambda_1 e^{-\frac{D_{n_2}}{2}} - \lambda_2 e^{\frac{D_{n_2}}{2}}) \tau_1 \cdot \tau_2 \tag{7}$$

is a new solution of Equation (2), where C is a non-zero constant.

Starting from the bilinear BT (4)-(5), and using the transformation $\varphi = \frac{\tilde{\tau}}{\tau}$, we can also get the Lax pair of the sdKP Equation (3) according to the compatibility condition $(\varphi_x)_{n_1+1} = (\varphi_{n_1+1})_x$. The Lax pair is written as follows

$$\varphi_x = -\theta\varphi - \frac{1}{\lambda}(a_1 - a_2) \frac{\tau_{n_1}^{n_2+1} \tau_{n_1}^{n_2-1}}{(\tau)^2} \varphi_{n_2-1}, \tag{8}$$

$$\varphi_{n_1+1} = \mu\varphi + \frac{1}{\lambda} a_1 a_2 \frac{\tau_{n_1}^{n_2+1} \tau_{n_1+1}^{n_2-1}}{\tau_{n_1+1}^{n_2} \tau} \varphi_{n_1+1}^{n_2-1}. \tag{9}$$

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3. One-Periodic Wave Solution of the sdKP Equation and Its Asymptotic Property

In this part, a one-periodic wave solution of the sdKP Equation (3) is given by Hirota's method and Riemann theta functions. In this case, the theta function takes the form

$$f = \theta(\eta, b) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{2\pi i n\eta - \pi n^2 b} \tag{10}$$

in which $\eta = kx + ln_1 + wn_2 + \eta_0$, $\eta \in C$, and b is a positive real constant.

3.1. One-Periodic Wave Solution

In order to get periodic wave solutions, the sdKP Equation (3) needs to be rewritten as another bilinear form

$$(a_1 a_2 D_x \mathrm{e}^{\frac{D_{n_1} - D_{n_2}}{2}} - (a_1 - a_2)\mathrm{e}^{\frac{D_{n_1} + D_{n_2}}{2}} + (a_1 - a_2)\mathrm{e}^{\frac{D_{n_1} - D_{n_2}}{2}} + c\mathrm{e}^{\frac{D_{n_1} + D_{n_2}}{2}})\tau \cdot \tau = 0, \tag{11}$$

where c is a non-zero constant. If the constant c is zero, the above equation degenerates to the original Equation (2). In what follows, we use the notation $D_i = D_{n_i}, i = 1, 2, 3$, the Equation (11) can be expressed as the equivalent form

$$\begin{aligned}
&\Big[a_1 a_2 D_x \sinh\frac{D_1 - D_2}{2} + (a_1 - a_2)(\cosh\frac{D_1 - D_2}{2} - \cosh\frac{D_1 + D_2}{2}) \\
&\quad + c\cosh\frac{D_1 + D_2}{2})\Big]\tau \cdot \tau = 0
\end{aligned} \tag{12}$$

Now we take the notation

$$\begin{aligned}
F &= F\left(D_x \sinh\frac{D_1 - D_2}{2}, \cosh\frac{D_1 - D_2}{2}, \cosh\frac{D_1 + D_2}{2}, c\right) \\
&= a_1 a_2 D_x \sinh\frac{D_1 - D_2}{2} + (a_1 - a_2)\left(\cosh\frac{D_1 - D_2}{2} - \cosh\frac{D_1 + D_2}{2}\right) \\
&\quad + c\cosh\frac{D_1 + D_2}{2},
\end{aligned} \tag{13}$$

and Equation (12) is in the simple form $F\tau \cdot \tau = 0$. Then substituting the theta function (10) into the LHS of Equation (12), we get

$$Ff \cdot f = \sum_{m=-\infty}^{+\infty} \widetilde{F}(m)\mathrm{e}^{2\pi i m\eta}, \tag{14}$$

where $\widetilde{F}(m)$ has the following definition

$$\begin{aligned}
\widetilde{F}(m) = &\sum_{n=-\infty}^{+\infty} F(2\pi i(2n - m)k \sinh \pi i(2n - m)(l - w), \cosh \pi i(2n - m)(l - w), \\
&\cosh \pi i(2n - m)(l + w), c) \times \mathrm{e}^{\pi i(n^2 + (m-n)^2)b}.
\end{aligned} \tag{15}$$

By shifting the summation index by $n = n' + 1$, we have

$$\begin{aligned}
\widetilde{F}(m) &= \widetilde{F}(m-2)\mathrm{e}^{-2\pi(m-1)b} = \cdots \\
&= \begin{cases} \widetilde{F}(0)\mathrm{e}^{-2\pi n^2 b/2}, & \text{m is even} \\ \widetilde{F}(1)\mathrm{e}^{-2\pi(n^2-1)b/2}, & \text{m is odd,} \end{cases}
\end{aligned}$$

which implies that if the following equations are satisfied,

$$\widetilde{F}(0) = \widetilde{F}(1) = 0 \tag{16}$$

then it shows that $\widetilde{F}(m) = 0$ for all $m \in Z$, and thus the theta function (10) satisfies the bilinear sdKP Equation (12). According to Equations (15) and (16), we have

$$\sum_{n=-\infty}^{+\infty} (4\pi a_1 a_2 ink \sinh 2\pi in(l-w) + (a_1 - a_2)\cosh 2\pi in(l-w) \\ -(a_1 - a_2)\cosh 2\pi in(l+w) + c\cdot\cosh 2\pi in(l+w))\mathrm{e}^{-2\pi n^2 b} = 0, \tag{17}$$

$$\sum_{n=-\infty}^{+\infty} (a_1 a_2(\pi i(2n-1)k)\sinh \pi i(2n-1)(l-w) \\ +(a_1 - a_2)\cosh \pi i(2n-1)(l-w) - (a_1 - a_2)\cosh \pi i(2n-1)(l-w) \\ +c\cdot\cosh \pi i(2n-1)(l+w))\mathrm{e}^{-\pi(2n^2-2n+1)b} = 0 \tag{18}$$

We introduce the notation

$$\begin{aligned}
\phi &= \mathrm{e}^{-\pi b}, \quad a_{11} = \sum_{n=-\infty}^{+\infty} (a_1 a_2 4\pi ink \sinh 2\pi in(l-w))\phi^{2n^2}, \\
a_{12} &= \sum_{n=-\infty}^{+\infty} (\cosh 2\pi in(l+w)\phi^{2n^2}, \\
a_{21} &= \sum_{n=-\infty}^{+\infty} (2a_1 a_2 \pi i(2n-1)\sinh 2\pi i(2n-1)(l-w))\phi^{2n^2-2n+1}, \\
a_{22} &= \sum_{n=-\infty}^{+\infty} (\cosh \pi i(2n-1)(l+w)\phi^{2n^2-2n+1}, \\
d_1 &= \sum_{n=-\infty}^{+\infty} (a_2 - a_1)[\cosh 2\pi in(l-w) - \cosh 2\pi in(l+w)]\phi^{2n^2}, \\
d_2 &= \sum_{n=-\infty}^{+\infty} (a_2 - a_1)[\cosh \pi i(2n-1)(l-w) - \cosh 2\pi i(2n-1)(l+w)]\phi^{2n^2-2n+1},
\end{aligned} \tag{19}$$

Equations (17) and (18) can be written as a system of linear equations about the parameters k and c, i.e.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} k \\ c \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}. \tag{20}$$

Hence we get a one-periodic wave solution of the sdKP Equation (3)

$$G = -(\ln \theta(\eta, b))_x, \tag{21}$$

where the theta function $\theta(\eta, b)$ is given by Equation (10), and the parameters k and c are solved by Equation (20), other parameters l, w, η_0 and b are free, in which l, w and b play a major rule in one periodic wave solution. Here we present the graph of the one-periodic wave solution (21) with the parameter values chosen as $k = 1$, $l = 0.5$, $w = 0.5$ and $n_2 = 1$.

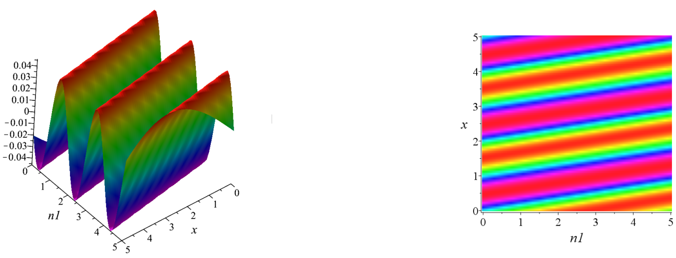

Figure 1. (a) One-periodic wave solution, (b) Density plot of the one-periodic wave solution.

3.2. *Asymptotic Properties of the One-Periodic Wave Solution*

Next, we consider the asymptotic properties of the one-periodic wave solution and establish the relationship of soliton solutions and periodic solutions by taking a limit condition. For this purpose, we give the expansions of the matrix $A = (a_{ij})_{2\times 2}$, the vectors $(d_1, d_2)^T$ and $(k, c)^T$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A_0 + A_1 + A_2\phi^2 + \cdots,$$
$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = D_0 + D_1 + D_2\phi^2 + \cdots,$$
$$\begin{pmatrix} k \\ c \end{pmatrix} = X_0 + X_1\phi + X_2\phi^2 + \cdots,$$

According to the expression of a_{ij}, d_j $(i, j = 1, 2)$ in (19) and expanding these elements to be the sum of ϕ, we have

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & 0 \\ 4a_1a_2\pi i \sinh \pi i(l-w) & 2\cosh \pi i(l+w) \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 8a_1a_2\pi i \sinh 2\pi i(l-w) & 2\cosh 2\pi i(l+w) \\ 0 & 0 \end{pmatrix}, \ A_3 = A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$A_5 = \begin{pmatrix} 0 & 0 \\ 12a_1a_2\pi i \sinh 3\pi i(l-w) & 2\cosh 3\pi i(l+w) \end{pmatrix}, \cdots$$

$$D_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ D_1 = \begin{pmatrix} 0 \\ 2(a_2-a_1)(\cosh \pi i(l-w) - \cosh \pi i(l+w)) \end{pmatrix},$$
$$D_2 = \begin{pmatrix} 2(a_2-a_1)(\cosh 2\pi i(l-w) - \cosh 2\pi i(l+w)) \\ 0 \end{pmatrix}, \ D_3 = D_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$D_5 = \begin{pmatrix} 0 \\ 2(a_2-a_1)(\cosh 3\pi i(l-w) - \cosh 3\pi i(l+w)) \end{pmatrix}, \cdots$$

Therefore, we obtain the following result

$$X_0 = \begin{pmatrix} k_0 \\ 0 \end{pmatrix}, \ X_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ X_2 = \begin{pmatrix} k_2 \\ c_2, \end{pmatrix}, \tag{22}$$

where k_0 and c_2 have the following expression

$$\begin{aligned} k_0 &= \frac{(a_2-a_1)(\cosh \pi i(l-w) - \cosh \pi i(l+w))}{2a_1a_2\pi i \sinh \pi i(l-w)}, \\ c_2 &= 2(a_2-a_1)[\cosh 2\pi i(l-w) - \cosh 2\pi i(l+w) \\ &\quad - \frac{4(\cosh \pi i(l-w) - \cosh \pi i(l+w)) \sinh 2\pi i(l-w)}{\sinh \pi i(l-w)}]. \end{aligned} \tag{23}$$

The relation between the periodic wave solution and the one-soliton solution is given by Theorem 1.

Theorem 1. *If the one-periodic solution of Equation 3 satisfies the following condition*

$$k = \frac{r}{2\pi i}, \ l = \frac{p}{2\pi i}, \ w = \frac{q}{2\pi i}, \ \eta_0 = \frac{\xi_0 + \pi b}{2\pi i}, \tag{24}$$

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where r, p, q and ξ_0 are the same as those in expression (6), we have the following asymptotic properties

$$c \to 0, \quad \eta \to \frac{\xi + \pi b}{2\pi i}, \; \theta(\eta, b) \to 1 + e^{\xi}, \; \phi \to 0.$$

Proof. Firstly, by using the formula in (22), parameters k and c are written as

$$k = k_0 + k_2\phi^2 + \cdots = o(\phi), \quad c = 0 + c_2\phi^2 + \cdots = o(\phi), \tag{25}$$

which implies that $c \to 0$. Now we expand the periodic wave function $\theta(\eta, b)$ in (10) as the form

$$\begin{aligned}\theta(\eta, b) &= 1 + (e^{2\pi i\eta} + e^{-2\pi i\eta})\phi + (e^{4\pi i\eta} + e^{-4\pi i\eta})\phi^4 + (e^{6\pi i\eta} + e^{-6\pi i\eta})\phi^9 + \cdots \\ &= 1 + e^{\eta'} + (e^{-\eta'} + e^{2\eta'})\phi^2 + (e^{-2\eta'} + e^{3\eta'})\phi^6 + \cdots,\end{aligned}$$

and accordingly, η' has the following form

$$\eta' = 2\pi i\eta - \pi b = 2\pi ikx + 2\pi iln_1 + 2\pi iwn_2 + 2\pi i\eta_0 - \pi b = rx + pn_1 + qn_2 + \xi_0.$$

According to the expression of k in (25), we have

$$\eta' \to k_0x + pn_1 + qn_2 + \xi_0, \quad \phi \to 0,$$

where the expression of k_0 is the same as the expression of r in the one-soliton solution (6). Thus we obtain

$$\theta(\eta, b) \to 1 + \mathrm{e}^{\xi}, \quad \phi \to 0.$$

which shows that one periodic wave solution tends to the one soliton solution (6) under the condition $\phi \to 0$. □

4. Two-Periodic Wave Solution of the sdKP Equation and Its Asymptotic Property

In this section, the two-periodic wave solution is considered and in the case $N = 2$, the Riemann theta function is defined as the following series

$$\theta(\eta, B) = \theta(\eta_1, \eta_2, B) = \sum_{s=-\infty}^{+\infty} \mathrm{e}^{2\pi i<\eta,s>-\pi<Bs,s>}, \tag{26}$$

where $s = (s_1, s_2)^T \in Z^2$, $\eta = (\eta_1, \eta_2)^T \in C^2$, $\eta_j = k_jx + l_jn_1 + w_jn_2 + \eta_{0j}, j = 1, 2$, the symbols $<, >$ denote the inner product of vectors, and B is a positive-definite and real-valued symmetric matrix, which can take the form

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \; b_{11} > 0, \; b_{22} > 0, \; b_{11}b_{22} - b_{12}b_{21} > 0.$$

4.1. Two-Periodic Wave Solution

In order to obtain the two-periodic solution, the theta function is substituted into the LHS of Equation (12) and the following result is derived

$$\begin{aligned}
F\theta(\eta,B)\cdot\theta(\eta,B) &= \sum_{s=-\infty}^{+\infty}\sum_{s'=-\infty}^{+\infty} e^{2\pi i<\eta,s'>-\pi<Bs,s'>}\\
&= \sum_{s=-\infty}^{+\infty}\sum_{s'=-\infty}^{+\infty}[a_1a_2 2\pi i<s-s',k>\sinh\pi i<s-s',l-w>\\
&\quad +(a_1-a_2)\cosh\pi i<s-s',l-w>+(c+a_2-a_1)\cosh\pi i<s-s',l+w>]\\
&\quad \times e^{2\pi i<\eta,s+s'>-\pi<BS,s>-\pi<Bs',s'>}\\
&\overset{s'=m-s}{=} \sum_{m=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}[a_1a_2 2\pi i<2s-m,k>\sinh\pi i<2s-m,l-w>\\
&\quad +(a_1-a_2)\cosh\pi i<2s-m,l-w>\\
&\quad +(c+a_2-a_1)\cosh\pi i<2s-m,l+w>]e^{2\pi i<\eta,m>-\pi<BS,s>-\pi<B(m-s),(m-s)>}\\
&= \sum_{m=-\infty}^{+\infty}\bar{F}(m_1,m_2)\times e^{2\pi i<\eta,m>}
\end{aligned} \tag{27}$$

where $s'=(s'_1,s'_2)^T$, $k=(k_1,k_2)^T$, $l=(l_1,l_2)^T$, $w=(w_1,w_2)^T$. In the above formula, the notation $\bar{F}(m_1,m_2)$ is defined by

$$\begin{aligned}
\bar{F}(m_1,m_2) &= \sum_{s=-\infty}^{+\infty}[a_1a_2 2\pi i<2s-m,k>\sinh\pi i<2s-m,l-w>\\
&\quad +(a_1-a_2)\cosh\pi i<2s-m,l-w>\\
&\quad +(c+a_2-a_1)\cosh\pi i<2s-m,l+w>]\\
&\quad \times e^{-\pi<Bs,s>-\pi<B(m-s),(m-s)>}.
\end{aligned} \tag{28}$$

Letting the summation index $s_j=s'_j+\delta_{jl}$ where δ_{jl} is the Kronecker delta, formula (28) then has the form

$$\begin{aligned}
\bar{F}(m_1,m_2) &= \sum_{s=-\infty}^{+\infty}F(2\pi i(\sum_{j=1}^{2}(2s'_j-(m_j-2\delta_{jl})))\sinh(\pi i\sum_{j=1}^{2}(2s'_j-(m_j-2\delta_{jl}))),\\
&\quad \cosh(\pi i\sum_{j=1}^{2}(2s'_j-(m_j-2\delta_{jl}))(l_j-w_j)),\cosh(\pi i\sum_{j=1}^{2}(2s'_j-(m_j-2\delta_{jl}))(l_j+w_j)),c)\\
&\quad \times e^{-\pi<Bs,s>-\pi<B(m-s),(m-s)>}.\\
&= \begin{cases}\bar{F}(m_1-2,m_2)e^{-\pi[2(b_{11}m_1+b_{12}m_2)-2b_{11}]}, & l=1\\ \bar{F}(m_1,m_2-2)e^{-\pi[2(b_{12}m_1+b_{22}m_2)-2b_{22}]}, & l=2\end{cases}.
\end{aligned}$$

The above result indicates that $\bar{F}(m_1,m_2),(m_1,m_2)\in Z^2$ is entirely determined by $\bar{F}(0,0)$, $\bar{F}(0,1)$, $\bar{F}(1,0)$ and $\bar{F}(1,1)$. Hence, if the following four equations are satisfied

$$\bar{F}(0,0)=\bar{F}(0,1)=\bar{F}(1,0)=\bar{F}(1,1)=0, \tag{29}$$

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then $\bar{F}(m_1, m_2) = 0, (m_1, m_2) \in Z^2$, so $F\theta(\eta, B) \cdot \theta(\eta, B) = 0$. That is, $\theta(\eta, B)$ is an exact solution of Equation (12). Next by adopting the following notations

$$
\begin{aligned}
& M = (a_{j1}, a_{j2}, a_{j3}),\ j = 1,2,3,4,\ \ d = (d_1, d_2, d_3, d_4)^T \\
& a_{j1} = 2\pi i a_1 a_2 \sum_{s_1, s_2 \in Z^2} (2s_1 - r_1^j) \sinh \pi i < 2s - r^j, l - w > \epsilon_j(s_1, s_2) \\
& a_{j2} = 2\pi i a_1 a_2 \sum_{s_1, s_2 \in Z^2} (2s_2 - r_2^j) \sinh \pi i < 2s - r^j, l - w > \epsilon_j(s_1, s_2) \\
& a_{j3} = \sum_{s_1, s_2 \in Z^2} (2s_1 - r_1^j) \cosh \pi i < 2s - r^j, l - w > \epsilon_j(s_1, s_2) \\
& d_j = (a_2 - a_1) \sum_{s_1, s_2 \in Z^2} [\cosh \pi i < 2s - r^j, l - w > - \cosh \pi i < 2s - r^j, l + w >] \epsilon_j(s_1, s_2) \\
& \phi_1 = \mathrm{e}^{-\pi b_{11}},\ \phi_2 = \mathrm{e}^{-\pi b_{22}},\ \phi_3 = \mathrm{e}^{-2\pi b_{12}}, \\
& \epsilon_j(s_1, s_2) = \phi_1^{s_1^2 + (s_1 - r_i^j)^2} \phi_2^{s_2^2 + (s_2 - r_2^j)^2} \phi_3^{s_1 s_2 + (s_1 - r_1^j)(s_2 - r_2^j)}, \\
& r^1 = (0,0),\ r^2 = (0,1)\ r^3 = (1,0)\ r^4 = (1,1),\ r^j = (r_1^j, r_2^j),\ j = 1,2,3,4,
\end{aligned}
$$

Equation (29) can be written as the following system

$$
M(k_1, k_2, c)^T = d. \tag{30}
$$

Therefore, we obtain a two-periodic wave solution of Equation (3)

$$
u = -(\ln \theta(\eta, B))_x, \tag{31}
$$

where the parameters $\theta(\eta, B)$, k_1, k_2, c and b_{12} are determined by (26) and (30), while other parameters l_1, l_2, w_1, w_2, b_{11} and b_{22} are free, which play an important role in the two-periodic wave solution.

4.2. Asymptotic Properties of the Two-Periodic Wave Solution

In what follows, we further analyze the asymptotic properties of the two-periodic wave solution (31), and establish the relation between the two-periodic wave and the two-soliton wave [23]. In much the same way as in Theorem 1, we first give the expansions of the matrix M and d in (30)

$$\begin{aligned}
M =& \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 4\pi i a_1 a_2 \sinh \pi i(l_1 - w_1) & 0 & 2\cosh \pi i(l_1 - w_1) \\ 0 & 0 & 0 \end{pmatrix} \phi_1 \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4\pi i a_1 a_2 \sinh \pi i(l_2 - w_2) & 2\cosh \pi i(l_2 - w_2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi_2 \\
&+ \begin{pmatrix} 8\pi i a_1 a_2 \sinh 2\pi i(l_1 - w_1) & 0 & 2\cosh 2\pi i(l_1 - w_1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi_1^2 \\
&+ \begin{pmatrix} 0 & 8\pi i a_1 a_2 \sinh 2\pi i(l_2 - w_2) & 2\cosh 2\pi i(l_2 - w_2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi_2^2 \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ m_1 & m_2 & m_3 \end{pmatrix} \phi_1\phi_2 + o(\phi_1^m \phi_2^j),\ m + j \geq 2,
\end{aligned} \tag{32}$$

in which

$$\begin{aligned}
m_1 &= 4\pi i a_1 a_2 [\phi_3 \sinh \pi i(l_1 - w_1 + l_2 - w_2) + \sinh \pi i(l_1 - w_1 - l_2 + w_2)], \\
m_2 &= 4\pi i a_1 a_2 [\phi_3 \sinh \pi i(l_1 - w_1 + l_2 - w_2) + \sinh \pi i(l_2 - w_2 - l_1 + w_1)], \\
m_3 &= 2[\phi_3 \cosh \pi i(l_1 - w_1 + l_2 - w_2) + \cosh \pi i(l_1 - w_1 - l_2 + w_2)],
\end{aligned}$$

and

$$\begin{aligned}
d =& \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ d_1 \\ 0 \end{pmatrix} \phi_1 + \begin{pmatrix} 0 \\ d_2 \\ 0 \\ 0 \end{pmatrix} \phi_2 \\
&+ \begin{pmatrix} d_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \phi_1^2 + \begin{pmatrix} d_4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \phi_2^2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_5 \end{pmatrix} \phi_1\phi_2 + o(\phi_1^m \phi_2^j),
\end{aligned} \tag{33}$$

$$\begin{aligned}
d_1 &= 2(a_2 - a_1)[\cosh \pi i(l_1 - w_1) - \cosh \pi i(l_1 + w_1)], \\
d_2 &= 2(a_2 - a_1)[\cosh \pi i(l_2 - w_2) - \cosh \pi i(l_2 + w_2)], \\
d_3 &= 2(a_2 - a_1)[\cosh 2\pi i(l_1 - w_1) - \cosh 2\pi i(l_1 + w_1)], \\
d_4 &= 2(a_2 - a_1)[\cosh 2\pi i(l_2 - w_2) - \cosh 2\pi i(l_2 + w_2)], \\
d_5 &= 2(a_2 - a_1)[\phi_3 \cosh \pi i(l_1 - w_1 + l_2 - w_2) + \cosh \pi i(l_1 - w_1 - l_2 + w_2) \\
&\quad - \phi_3 \cosh \pi i(l_1 + w_1 + l_2 + w_2) - \cosh \pi i(l_1 + w_1 - l_2 - w_2)].
\end{aligned}$$

We assume that the solution of system (30) has the following form

$$\begin{aligned}\begin{pmatrix} k_1 \\ k_2 \\ c \end{pmatrix} = \begin{pmatrix} k_{10} \\ k_{20} \\ c_0 \end{pmatrix} + \begin{pmatrix} k_{11} \\ k_{21} \\ c_1 \end{pmatrix} \phi_1 + \begin{pmatrix} k_{12} \\ k_{22} \\ c_2 \end{pmatrix} \phi_2 + \begin{pmatrix} k_{13} \\ k_{23} \\ c_3 \end{pmatrix} \phi_1^2 + \begin{pmatrix} k_{14} \\ k_{24} \\ c_4 \end{pmatrix} \phi_2^2 \\ + \begin{pmatrix} k_{15} \\ k_{25} \\ c_5 \end{pmatrix} \phi_1\phi_2 + o(\phi^m \phi^j),\ m + j \geq 2,\end{aligned} \tag{34}$$

then the relation of the two-periodic solution and the two-soliton solution is given as follows.

Theorem 2. *If the two-periodic solution in the formula (31) satisfies the following condition*

$$k_j = \frac{r_j}{2\pi i},\ l_j = \frac{p_j}{2\pi i},\ w_j = \frac{q_j}{2\pi i},\ \eta_{0j} = \frac{\xi_{0j} + \pi b_{jj}}{2\pi i},\ j = 1, 2, \tag{35}$$

where r_j, p_j, q_j and ξ_{0j} are the same as those in the two-soliton solution, we have the following asymptotic properties

$$c \to 0,\ \ \eta_j \to \frac{\xi_j + \pi b_{jj}}{2\pi i},\ \theta(\eta_1, \eta_2, B) \to 1 + e^{\xi_1} + e^{\xi_2} + A_{12}e^{\xi_1+\xi_2},\ \phi_1, \phi_2 \to 0,$$

where $A_{12} = e^{-2\pi b_{12}}$.

Proof. Substituting formulae (32)-(34) into the system (30), we get the following equalities

$$\begin{aligned}&k_j = k_{j0} + o(\phi_i\phi_2),\ j = 1, 2,\ \ c = c_3\phi_1^2 + c_3\phi_2^2 + o(\phi_1^2\phi_2^2), \\ &\mathrm{e}^{-2\pi b_{12}} = \frac{4\pi a_1 a_2(k_{20} - k_{10}) \sinh \pi(\hat{l}_1 - \hat{l}_2) + 2(a_2 - a_1)(\cosh \pi i(\hat{l}_1 - \hat{l}_2) - \cosh \pi i(\hat{w}_1 - \hat{w}_2)}{4\pi a_1 a_2(k_{20} + k_{10}) \sinh \pi(\hat{l}_1 + \hat{l}_2) + 2(a_2 - a_1)(\cosh \pi i(\hat{l}_1 + \hat{l}_2) - \cosh \pi i(\hat{w}_1 + \hat{w}_2))},\end{aligned}$$

where

$$k_{j0} = \frac{(a_2 - a_1)(\cosh \pi i\hat{l}_j - \cosh \pi i\hat{w}_j)}{2a_1 a_2 \pi i \sinh \pi i\hat{l}_j},\ \ c_3 = \frac{\sinh \pi i\hat{l}_1}{2 \sinh 2\pi i\hat{l}_1},\ \ c_4 = \frac{\sinh \pi i\hat{l}_2}{2 \sinh 2\pi i\hat{l}_2},$$

and $\hat{l}_j = l_j - w_j$, $\hat{w}_j = l_j + w_j$, then we have $c \to 0$ as ϕ_1, $\phi_2 \to 0$. Also, as in the Theorem 1, we expand the theta function $\theta(\eta_1, \eta_2, B)$ in (26) as the form

$$\begin{aligned}\theta(\eta_1, \eta_2, B) &= 1 + (\mathrm{e}^{2\pi i\eta_1} + \mathrm{e}^{-2\pi i\eta_1})\phi_1 + (\mathrm{e}^{2\pi i\eta_2} + \mathrm{e}^{-2\pi i\eta_2})\phi_2 + \cdots \\ &\quad + (\mathrm{e}^{2\pi i(\eta_1+\eta_2)} + \mathrm{e}^{-2\pi i(\eta_1+\eta_2)})\mathrm{e}^{-2\pi b_{12}}\phi_1\phi_2 + \cdots \\ &= 1 + \mathrm{e}^{\hat{\eta}_1} + \mathrm{e}^{\hat{\eta}_2} + \mathrm{e}^{\hat{\eta}_1+\hat{\eta}_2-2\pi b_{12}} + \mathrm{e}^{-\hat{\eta}_1}\phi_1^2 + \mathrm{e}^{-\hat{\eta}_2}\phi_2^2 + \mathrm{e}^{-\hat{\eta}_1-\hat{\eta}_2-2\pi b_{12}}\phi_1^2\phi_2^2 + \cdots,\end{aligned}$$

in which $\hat{\eta}_j = 2\pi i\eta_j - \pi b_{jj}$, $j = 1, 2$. In this case,

$$\hat{\eta}_j = 2\pi i k_j x + 2\pi i l_j n_1 + 2\pi i w_j n_2 + 2\pi i\eta_{0j} - \pi b_{jj} = r_j x + p_j n_1 + q_j n_2 + \xi_{0j}.$$

According to expressions of k_j and ϕ_3 as mentioned above, we obtain

$$\hat{\eta}_j \to \xi_j,\ \ \theta(\eta_1, \eta_2, B) \to 1 + \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + A_{12}\mathrm{e}^{\xi_1+\xi_2},\ \ \phi_1, \phi_2 \to 0,\ \text{where } A_{12} = e^{-2\pi b_{12}}.$$

Therefore, the two-periodic solution tends to a two-soliton solution as ϕ_1, $\phi_2 \to 0$.
□

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5. Conclusions and Discussions

In this paper, we focus on a semi-disceret KP Equation (3) generated by the Hirota-Miwa equation. Its bilinear Bäcklund transformation, Lax pair and nonlinear superposition formula are given. More importantly, using this superposition principle we successfully constructed one-periodic wave solution (21) and two-periodic wave solution (31) which are expressed in terms of Riemann theta functions. Then, one-soliton and two-soliton solutions emerge as natural degenerations by taking appropriate limits of the parameters in the periodic solutions. However, for higher-order periodic solutions, such as the three-periodic wave solution, the associated system of equations for the parameters becomes overdetermined, making it difficult to find exact analytical solutions. Therefore, alternative approaches are needed to obtain such higher-order solutions. For instance, numerical algorithms have been employed in the literature (e.g., in Refs. [19] and [25]) to compute three-periodic solutions for KdV-type and Toda-type equations. In our future work, we will explore the application of numerical methods to simulate three-periodic wave solutions of the semi-discrete KP equation, aiming to gain further insight into its complex nonlinear dynamics. Furthermore, beyond solitons and periodic solutions, integrable systems admit a variety of other important exact solution types including lump solutions and rogue waves. Therefore, another promising direction is to investigate the structure and dynamics of such novel exact solutions for this equation.

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References

1. Bruschi, M.; Manakov, S. V.; Ragnisco, O.; Levi, D. The nonabelian Toda lattice-discrete analogue of the matrix Schrödinger spectral problem. J. Math. Phys. 1980,21(2), 2749–2753.
2. Levi, D.; Ragnisco, O.; Bruschi, M. Extension of the Zakharov-Shabat generalized inverse method to solve differential-difference and difference-difference equations. Nuovo Cimento A 1980,58(1), 56–66.
3. Takahashi, D.; Matsukidaira,J. On discrete soliton equations related to cellular automata, Phys. Lett. A, 1995, 209, 184-188.
4. Hirota, R.; Ohta, Y. Discretization of the new type of soliton equations, J. phys. Soc. Jpn., 2007, 76, 034001.
5. Nagai, H.; Ohta, Y.; Hirota, R. Discrete and ultradiscrete periodic phase soliton equations, 2019, 88, 034001.
6. Hirota, R. Discrete analogue of a generalized Toda equation. J. Phys. Soc. Jpn., 1981, 50, 3785-3791.
7. Amjad, Z.; Haider, B.; Ma, W.X. Integrable discretization and multi-soliton solutions of negative order AKNS equation, 2024, 23(suppl.1), 280.
8. Miwa, T. On Hirota's difference equations. Proc. Japan Acad. A Math. Sci., 1982, 58(1), 9-12.
9. Date, E.; Jimbo, M.; Miwa, T. Method for generating discrete solution equations. J. Phys. Soc. Jpn., 1982, 51, 4116-4124.
10. Li, C.X.; Lafortune, S.; Shen, S.F. A semi-discrete Kadomtsev-Petviashvili equation and its coupled integrable system. J. Math. Phys. 2017, 57, 053503:1-9.
11. Hirota, R. The Direct Method in Soliton Theory. Cambridge University Press, 2004.
12. Nakamura, A. A direct method of calculating periodic wave solutions to nonlinear evolution equations. I. exact two-periodic wave solution. J.Phy.Soc.Jpn, 1979, 47(5), 1701-1705.
13. Nakamura, A. A direct method of calculating periodic-wave solutions to nonlinear evolution equations. II. exact I-periodic and 2-periodic wave solutions of the coupled bilinear equations, J. Phys. Soc. Jpn. 1980, 48, 1365-1370.
14. Whittaker, E.T.; Watson, G.N. Acourse of modern analysis. (Cambridge Univ. Press, Cambride), 1935.
15. Bellman, R. A brief introduction to theta functions. (Rinehart,R; Winston), Newyork, 1961.
16. Hirota, R.; Masaaki, I. A direct approach to multi-periodic wave solutions to nonlinear evolution equations. J. Phys. Soc. Jpn. 1981, 50(1), 338-342.

 doi:10.20944/preprints202510.1487.v1

17. Fan, E.G. Quasi-periodic waves and an asymptotic property for the asymmetrical Nizhnik-Novikov-Veselov equation, J. Phys. A: Math. Theor. 2009, 42, 095206.
18. Ma, W.X.; Zhou, R.G.; Gao, L. Exact one-periodic and two-periodic wave solutions to Hirota bilinear equations in (2+1)dimensions, Modern Phys. Lett. A 2009, 24, 1677-1688.
19. Zhang Y.N.; Hu X.B.; Sun J.Q. A numerical study of the 3-periodic wave solutions to KdV-type equations. J. Comp. Phys. 2018, 355, 566-581.
20. Wei, L. Multiple periodic-soliton solutions to Kadomtsev-Petviashvili equation, Appl. Math. Comp., 2011, 218, 368-375.
21. Hon, Y.C.; Fan E.G.; Qin Z.Y. A kind of explicit quasi-periodic solution and its limit for the Toda lattice equation, Modern Phys. Lett. B, 2008, 22, 547-553.
22. Luo, L. Exact periodic wave solutions for the differntial-difference KP equation, Reports on mathematical physics, 2010, 66(3), 403-417.
23. Wu, G.C.; Xia, T.C. Uniformly constructing exact discrete soliton solutions and periodic solutions to differential-difference equations, Comput. Math. Appl., 2009, 58, 2351-2354.
24. Wan, L.S. Nonlinear superposition formula of a semi-discrete KP equation and its exact solutions (master's thesis), Capital Normal University, Beiing, China, 2024.
25. Zhang Y.N.; Hu X.B.; Sun J.Q. A numerical study of the 3-periodic wave solutions to Toda-type equations, Comm. Comp. Phys. 2019, 26(2), 579-598.