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Posted Date: 3 September 2024

doi: 10.20944/preprints202409.0181.v1

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On Nilpotent Elements and Armendariz Modules

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Abstract: For a left module ${}_RM$ over non-commutative ring R, the notion for the class of nilpotent elements ($nil_R(M)$) was first introduced and studied by SSevviiri and Groenewald [12]. Moreover, Armendariz and semicommutative modules are generalizations of reduced modules and $nil_R(M) = 0$ in the case of reduced modules. Thus, the nilpotent class plays a vital role in these modules. Motivated by this, we present the concept of nil-Armendariz modules as a generalization of reduced modules and a refinement of Armendariz modules, focusing on the class of nilpotent elements. Further, we demonstrate that the quotient module M/N is nil-Armendariz if and only if N is within the nilpotent class of ${}_RM$. Additionally, we establish that the matrix module $M_n(M)$ is nil-Armendariz over $M_n(R)$ and explore conditions under which nilpotent classes form submodules. Finally, we prove that nil-Armendariz modules remain closed under localization.

Keywords: nilpotent element; armendariz module; armendariz ring, nil-armendariz module

MSC: 16D10, 16S36, 16S50.

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1. Introduction

In this article, R represents an associative ring with identity and ${}_RM$ represents an unital left module over R. Recall that for some $n \in \mathbb{N}$ and $a \in R$, if $a^n = 0$, then a is said be nilpotent element in R. The notation $Nil(R)$ denotes the set of all nilpotent elements in R. If $Nil(R) = \{0\}$, R is called a reduced ring. For a polynomial ring $R[x]$ over R, Armendariz [4] proved a very interesting result that if R is reduced, then the coefficients $u_l m_k = 0$ for each l, k whenever $p(x) = \sum_{l=0}^{n} u_l x^l$ and $m(x) = \sum_{k=0}^{q} u_k x^k$ with coefficients in R satisfy $p(x)m(x) = 0$. Inspired by this result, Rege and Chhawchharia [11] introduced a new class of rings named Armendariz ring as a generalization of reduced rings and provided a sufficient class of rings which are Armendariz but not reduced. A ring R is called Armendariz if $u_l v_k = 0$, whenever $p(x) = \sum_{l=0}^{n} u_l x^l$ and $m(x) = \sum_{k=0}^{q} v_k x^k$ in $R[x]$ satisfies $p(x).m(x) = 0$. R. Antoine [3] introduced nil-Armendariz rings and studied the structure of a nilpotent class in non-commutative rings extensively. A ring R is called nil-Armendariz if $u_l v_k \in Nil(R)$, whenever $p(x) = \sum_{l=0}^{n} u_l x^l$ and $m(x) = \sum_{k=0}^{q} v_k x^k$ in $R[x]$ satisfy $p(x).m(x) \in Nil(R)[x]$. The class of Armendariz and nil-Armendariz rings and their relation with other classes of rings are briefly studied in [3,4,9,10]. In [9], Liu and Zhao introduced weak Armendariz rings to generalize nil-Armendariz rings. A ring R is weak Armendariz if $u_l v_k \in Nil(R)$, whenever $p(x) = \sum_{l=0}^{n} u_l x^l$ and $m(x) = \sum_{k=0}^{q} v_k x^k$ in $R[x]$ satisfy $p(x).m(x) = 0$. Thus, we have the chain: reduced ring $\Longrightarrow$ Armendariz ring $\Longrightarrow$ nil-Armendariz ring $\Longrightarrow$ weak Armendariz, but the converse is not necessarily true. In the field of extensions, Lee and Zhou [8] extended the reduced property to modules. A module ${}_RM$ is said to be reduced If any of the identical conditions listed below hold:

(1) If for $u \in R$ and $v \in M$, we have $u^2 v = 0$, then $uRv = 0$.
(2) Whenever $uv = 0$, then $uM \cap Rv = 0$.

Similarly ${}_RM$ is called rigid if $uv = 0$ holds true whenever $u^2 v = 0$ for $u \in R$ and $v \in M$. A module ${}_RM$ is called Armendariz if $u_l v_k = 0$ whenever $p(x) = \sum_{l=0}^{n} u_l x^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} v_k x^k \in M[x]$ satisfies $p(x).m(x) = 0$. Lee and Zhou recorded many examples of Armendariz modules [8] and

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Rege and Buhphang [5]. They also conducted a comparative study of Armendariz, reduced, and semicommutative modules. A module $_RM$ is called semicommutative if, whenever $uv = 0$ for some $u \in R$ and $v \in M$, it implies $uRv = 0$. Over the past few decades, many algebraists have been generalizing concepts defined for non-commutative rings to modules. In this context, as early as 2014, Ssevvirri and Groenewald [12] proposed the idea of nilpotent elements for modules. An element $v \in {_RM}$ is called nilpotent if either $v = 0$ or $u^t v = 0$ but $uv \neq 0$ for some $u \in R$ and $t \in \mathbb{N}$. The set of all nilpotent elements in $_RM$ is denoted by $nil_R(M)$. In 2019, Ansari and Singh conducted a comparative study of nilpotent elements and established some crucial relationships between nilpotent elements and other classes of modules. They proved that if $_RM$ is reduced, then $_RM$ does not contain non-zero nilpotent elements. Since both nil-Armendariz and weak Armendariz concepts in ring theory are defined through 'nilpotency conditions' on elements, extending these concepts to modules is straightforward. In this direction, Ansari and Singh [1] defined a weak Armendariz module. A module $_RM$ is called weak Armendariz if whenever $p(x) = \sum_{l=0}^{n} u_l x^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} u_k x^k \in M[x]$ satisfy $p(x).m(x) = 0$, then, $u_l v_k \in nil_R(M)$ for each l, k. This new concept further helped study the structure of nilpotent elements and their connection with other subclasses of modules. Recall that an element $v \in {_RM}$ is a torsion element if $uv = 0$ for some non-zero $u \in R$. The set of all torsion elements of $_RM$ is represented by $Tor(M)$. In [12], Ssevvirri and Groenewald raised an important question regarding the conditions under which the set of nilpotent elements forms a submodule of $_RM$. In this article, we noted some conditions on the ring that help make the set of nilpotent elements a submodule.

In the realm of non-commutative rings, extensive studies have been conducted on the generalization of reduced rings, namely the classes of Armendariz and semi commutative rings. However, such developments have yet to be extended to modules due to the absence of various subclass definitions. Therefore, the primary objective of this article is to introduce a new concept called nil-Armendariz modules as an independent category within the class of Armendariz modules, serving as a generalization of reduced modules within the context of the nilpotent class. We examine various properties of this extension and conduct a comparative study between concepts developed in rings and their counterparts in modules.

Among the significant results, we demonstrate the existence of a large class of nil-Armendariz modules but not Armendariz, and vice versa. Additionally, we establish that for a submodule N of $_RM$, the quotient module M/N is nil-Armendariz if and only N is a subset of the nilpotent class of $_RM$. We also prove that for a module $_RM$, matrix module $M_n(M)$ is nil-Armendariz over $M_n(R)$. Furthermore, we explore the structure of the nilpotent class and identify certain conditions under which these classes form a submodule. Additionally, we demonstrate that nil-Armendariz modules maintain closure under localizations.

2. Results on Nil-Armendariz Modules

We begin with the following definition.

Definition 1. *A left R-module M is called nil-Armendariz if whenever* $f(x)m(x) \in nil_R(M)[x]$ *for* $f(x) = \sum_{i=0}^{n} r_i x^i \in R[x]$ *and* $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$*, then* $r_i m_j \in nil_R(M)$*.*

From the above definition, we can easily infer that the nil-Armendariz module class is closed under submodules and that every reduced module is nil-Armendariz. Moreover, all nil-Armendariz modules are weak Armendariz. However, Proposition 2.3 and 2.5, provided later in this article, illustrates that the converse is not true in both cases. Additionally, in the realm of ring theory, it is straightforward to verify that all Armendariz rings are nil-Armendariz. Extending these concepts to module theory may lead one to suspect that all Armendariz modules are nil-Armendariz; however, this assumption does not hold. For instance, consider a module $_RM$. We know that $M_n(M)$ is a module over $M_n(R)$. Any matrix $K = [m_{ij}]_{n\times n} \in M_n(M)$ can be expressed as $K = \sum_{i,j=1}^{n} E_{ij} m_{ij}$, where E_{ij} represents the elementary matrices.

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Lemma 1. *Let M be a left R-module. Then, $nil_R(M_n(M)) = M_n(M)$.*

Proof. Consider any non-zero matrix $[m_{ij}]_{n\times n}$. This implies at least one $m_{ij} \neq 0$ for some $1 \leq i, j \leq n$. Thus, we have two cases as follows:

(a) Suppose $m_{ij} \neq 0$ for $i \neq j$. Then we can take $r = E_{ji}$. Thus, we can easily see that $r^2K = (E_{ji})^2K = 0$, but $rK = e_{ji}K \neq 0$.
(b) Suppose $m_{ij} \neq 0$ for $i = j$. Then we can take $r = E_{li}$ such that $l \neq i$ and $1 \leq l, i \leq n$. Thus, we can easily see that $r^2K = (E_{li})^2K = 0$, but $rK = E_{li}K \neq 0$.

□

Proposition 1. *For a module $_RM$, the matrix module $M_n(M)$ is nil-Armendariz over $M_n(R)$ for $n \geq 2$, but it is not Armendariz.*

Proof. $M_n(M)$ is nil-Armendariz, which easily follows from Lemma 1. Now consider, $p(x) = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1_R \\ 0 & 0 \end{pmatrix} x \in R[x]$ and for any $0 \neq m \in M$, $m(x) = \begin{pmatrix} 0 & 0 \\ 0 & m \end{pmatrix} + \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} x \in M[x]$. Clearly, we see $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \neq 0$, although $p(x)m(x) = 0$. Thus, $M_2(M)$ is not Armendariz over $M_2(R)$. Since $_{M_2(R)}M_2(M)$ is embedded as a submodule in $_{M_n(R)}M_n(M)$ for $n \geq 2$, we can conclude that $_{M_n(R)}M_n(M)$ is not Armendariz. □

Here we have noted an important result regarding the nilpotency of $\mathbb{Z}_{n^2}$ as an $\mathbb{Z}$-module.

Lemma 2. *For any $0 \leq n \in \mathbb{Z}$, $\overline{n} \notin nil_{\mathbb{Z}}(\mathbb{Z}_{n^2})$.*

Proof. Let us suppose that $\overline{n} \in nil_{\mathbb{Z}}(\mathbb{Z}_{n^2})$. Then $\exists\, r \in \mathbb{Z}$ such that $r^2\overline{n} = \overline{0}$ but $r.\overline{n} \neq 0$. This implies $n^2|r^2.n \Rightarrow r = n.l$ for some $l \in \mathbb{Z}$. Thus $n^2|r.n$ which implies $r.\overline{n} = \overline{0}$. Hence a contradiction. □

Proposition 2. *For any $n \in \mathbb{N}$, the $\mathbb{Z}$-module $\mathbb{Z}_{n^2}$ is Armendariz but not nil-Armendariz.*

Proof. Consider $p(x) = 1 + nx \in \mathbb{Z}[x]$ and $m(x) = \overline{1} - \overline{n}x \in \mathbb{Z}n^2[x]$. Then we have $p(x) \cdot m(x) = \overline{1}$. Clearly, $n^2 \cdot \overline{1} = 0$ and $n \cdot \overline{1} \neq 0$. Thus, $\overline{1} \in nil_{\mathbb{Z}}(\mathbb{Z}_{n^2})$ and hence $p(x) \cdot m(x) \in nil_{\mathbb{Z}}(\mathbb{Z}_{n^2})[x]$. However, by Lemma 2, $\overline{n} \notin nil_{\mathbb{Z}}(\mathbb{Z}_{n^2})$. Thus, $\mathbb{Z}_{n^2}$ is not a nil-Armendariz module, but it is an Armendariz module (see Lemma 2.6 in [4]). □

Next, we record some conditions under which the above newly defined concept is equivalent to Armendariz module.

Proposition 3. *For a reduced module $_RM$, the statement given below are equivalent:*

(1) $_RM$ is Armendariz.
(2) $_RM$ is nil-Armendariz.
(3) $_RM$ is weak Armendariz.

Proof. Since the module $_RM$ is reduced, by Corollary 2.11 in [?], we have $nil_R(M) = 0$. Hence, the proof follows straightforwardly. □

Proposition 4. *Let R be a reduced ring. If $_RM$ is torsion free, then the statement given below are equivalent:*

(1) $_RM$ is Armendariz.
(2) $_RM$ is Nil-Armendariz.
(3) $_RM$ is weak Armendariz.

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Proof. The proof follows easily from Proposition 2.7 in [2]. □

Next, for a module ${}_RM$, we provide a large class of submodules of the matrix module ${}_{M_n(R)}M_n(M)$, which are both Armendariz and nil-Armendariz. For this purpose, we denote $T_n(R)$ as the ring of $n \times n$ upper triangular matrices over R. For a left R-module ${}_RM$ and $K = (a_{ij}) \in M_n(R)$, let $KM = \{(a_{ij}m) : m \in M\}$. For elementary matrices E_{ij}, let $U = \sum_{i=0}^{n} E_{i,i+1}$ for $n \geq 2$. We consider $U_n(R) = RI_n + RU + RU^2 + \ldots + RU^{n-1}$ and $U_n(M) = I_nM + UM + U^2M + \ldots + U^{n-1}M$. Then $U_n(R)$ forms a ring, and $U_n(M)$ forms a left module over $U_n(R)$.

There exists a ring isomorphism $\phi : U_n(R) \to \frac{R[x]}{(x^n)}$ defined as $\phi(r_0I_n + r_1U + r_2U^2 + \ldots + r_{n-1}U^{n-1}) = r_0 + r_1x + \ldots + r_{n-1}x^{n-1} + (x^n)$, and an abelian group isomorphism $\theta : U_n(M) \to \dfrac{M[x]}{(M[x](x^n))}$ defined as $\theta(m_0I_n + m_1U + m_2U^2 + \ldots + m_{n-1}U^{n-1}) = m_0 + m_1x + \ldots + m_{n-1}x^{n-1} + M[x](x^n)$, such that $\theta(AW) = \phi(A)\theta(W)$ for all $A \in U_n(R)$ and $W \in U_n(M)$.

In [13], Corollary 3.7, Zhang and Chen proved that ${}_RM$ is a reduced module if and only if $U_n(M)$ is Armendariz over $U_n(R)$. Thus, for a reduced module ${}_RM$, we find a larger class of Armendariz submodules of $T_n(M)$ over $T_n(R)$. We recall the following notations from [7],

Let $k \in \mathbb{N}$ and for $n = 2k \geq 2$, consider

$$A_n^e(M) = \textstyle\sum_{i=1}^{k} \sum_{j=k+i}^{n} E_{i,j}M$$

and for $n = 2k + 1 \geq 3$

$$A_n^o(M) = \textstyle\sum_{i=1}^{k+1} \sum_{j=k+i}^{n} E_{i,j}M.$$

Let

$$A_n(M) = I_nM + UM + \ldots + U^{k-1} + A_n^e(M) \text{ for } n = 2k \geq 2$$

and

$$A_n(M) = I_nM + UM + \ldots + U^{k-1} + A_n^o(M) \text{ for } n = 2k + 1 \geq 3.$$

For example,

$$A_4(M) = \left\{ \begin{pmatrix} v_1 & v_2 & v & w \\ 0 & v_1 & v_2 & z \\ 0 & 0 & v_1 & v_2 \\ 0 & 0 & 0 & v_1 \end{pmatrix} : v_1, v_2, v, w, z \in M \right\}$$

$$A_5(M) = \left\{ \begin{pmatrix} a_1 & a_2 & a & b & c \\ 0 & a_1 & a_2 & d & e \\ 0 & 0 & a_1 & a_2 & f \\ 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_1 \end{pmatrix} : a_1, a_2, a, b, c, d, e, f \in M \right\}.$$

For $A = (a_{ij})$, $B = (b_{ij})$, we write $[A.B]_{ij} = 0$ to mean that $a_{il}b_{lj} = 0$ for $l = 0, , \ldots, n$.

Lemma 3. *([7], Lemma 1.2)For $r(x) = A_0 + A_1x + \ldots + A_px^p \in M_n(R)[x]$ and $m(x) = B_0 + B_1x + \ldots + B_qx^q \in M_n(M)[x]$, let $f_{ij} = a_{ij}^0 + a_{ij}^1x + \ldots + a_{ij}^px^p$ and $g_{ij} = b_{ij}^0 + b_{ij}^1x + \ldots + b_{ij}^qx^q$ where a_{ij}^l are the (i,j)-entries of A_l for $l = 0, 1, \ldots, p$ and b_{ij}^s are the (i,j)-entries of B_s for $s = 0, 1, \ldots, q$. Then $r(x) = (f_{ij}(x)) \in M_n(R[x])$ and $m(x) = (g_{ij}(x)) \in M_n(M[x])$. If ${}_RM$ is Armendariz and $[r(x).m(x)]_{ij} = 0$ for all i, j, then $A_iB_j = 0$ for all i, j.*

The first main result of this paper is the following:

Theorem 1. *Let ${}_RM$ be a reduced module. For $n = 2k + 1 \geq 3$, the following statements are true:*

 doi:10.20944/preprints202409.0181.v1

(1) $_{A_n(R)}A_n(M)$ *is an Armendariz module.*
(2) $_{A_n(R)}A_n(M)$ *is a nil-Armendariz module.*

Proof. (1) Let $r(x) = A_0 + A_1x + \ldots + A_px^p \in A_n(R)[x]$ and $m(x) = B_0 + B_1x + \ldots + B_qx^q \in A_n(M)[x]$ satisfy $r(x).m(x) = 0$. Here we identify $A_n(R)[x]$ with $A_n(R[x])$ and $A_n(M)[x]$ with $A_n(M[x])$ canonically. Then $r(x) = (f_{ij}(x)) \in A_n(R[x])$ and $m(x) = (g_{ij}(x)) \in A_n(M[x])$ where $f_{ij} = a_{ij}^{(0)} + a_{ij}^{(1)}x + \ldots + a_{ij}^{(p)}x^p$ and $g_{ij} = b_{ij}^{(0)} + b_{ij}^{(1)}x + \ldots + b_{ij}^{(q)}x^q$. We will show that $[r(x).m(x)]_{ij} = 0$ for all i, j. Firstly notice that $r(x)$ and $m(x)$ has following properties:

$$
\begin{aligned}
&f_1 := f_{11} = f_{22} = \ldots = f_{nn} && g_1 := g_{11} = g_{22} = \ldots = g_{nn}\\
&f_2 := f_{12} = f_{23} = \ldots = f_{n-1,n} && g_2 := g_{12} = g_{23} = \ldots = g_{n-1,n}\\
&\vdots && \vdots\\
&f_k := f_{1k} = f_{2,k+1} = \ldots = f_{n-k+1,n} && g_k := g_{1k} = g_{2,k+1} = \ldots = g_{n-k+1,n}\\
&f_{i,j} := 0,\ i > j && g_{i,j} := 0,\ i > .
\end{aligned}
$$

Now $r(x).m(x) = 0$ implies

$$\sum_{i+j=t} f_ig_j = 0 \ for \ t = 2, 3, \ldots k+1. \tag{1}$$

We know that $_RM$ is reduced module if and only if $_{R[x]}M[x]$ is reduced([8], Theorem 1.6). Thus from $f_1g_1 = 0$ we get $f_1^2g_1 = 0$ and hence $f_1R[x]g_1 = 0$. Multiplying by f_1 from left side to $f_1g_2 + f_2g_1 = 0$, we get $f_1^2g_2 = 0$ which implies $f_1g_2 = 0$, thus $f_2g_1 = 0$. similarly multiplying by f_1 from left to $f_1g_3 + f_2g_2 + f_3g_1 = 0$, we get $f_1^2g_3 + f_1f_2g_2 + f_1f_3g_1 = 0$, hence $f_1^2g_3$, which implies $f_1g_3 = 0$. Again multiplying f_2 to the same equation we get $f_2^2g_2 + f_2f_3g_1 = 0$, this implies $f_2^2g_2 = 0$ and hence $f_3g_1 = 0$. Similarly Continuing this process, we get

$$f_ig_j = 0 \,\forall\ i + j \le k+1. \tag{2}$$

This implies $[r(x).m(x)]_{ij} = 0$ for all i, j with $(i, j) \notin \Gamma$ where $\Gamma = \{(u, k+u) : u = 1, \ldots, k+1\} \cup \{(u, k+u+1) : u = 1, \ldots, k\} \cup \ldots \cup \{(u, u+n-2) : u = 1, 2\} \cup \{(u, n-1+u) : u = 1\}$.
Again from $r(x).m(x) = 0$, we have

$$
\begin{aligned}
f_1g_{1,k+1} + f_2g_k + f_3g_{k-1} + \ldots + f_kg_2 + f_{1,k+1}g_1 &= 0\\
f_1g_{2,k+2} + f_2g_k + f_3g_{k-1} + \ldots + f_kg_2 + f_{2,k+2}g_1 &= 0\\
&\vdots\\
f_1g_{k+1,2k+1} + f_2g_k + \ldots + f_{k-1}g_3 + f_kg_2 + f_{k+1,2k+1}g_1 &= 0.
\end{aligned}
$$

By applying the same process of left multiplications and using the earlier results obtained in equation (2), we conclude that for $u = 1, 2, \ldots, k+1$

$$f_1g_{u,k+u} = f_{u,k+u}g_1 = 0 \tag{3}$$

and with $i + j = k+2$ for i, j

$$f_ig_j = 0. \tag{4}$$

Thus from equation (3) and (4), we get $[r(x).m(x)]_{u,u+k} = 0$ for $1 \le u \le k+1$.
Now for some $1 \le l \le k$, assume the condition $[r(x).m(x)]_{u,k+u+t} = 0$ holds true for $0 \le t \le l-1$ and $1 \le u \le k-t+1$. Thus it is sufficient to show that for each $u = 1, \ldots, k-t+1$, the equation $[r(x).m(x)]_{u,k+u+l} = 0$ holds true. Again $r(x).m(x) = 0$ gives

$$\sum_{j=1}^{n} f_{u,j}g_{j,k+u+l} = 0 \text{ for } u = 1,\dots,k-l+1.$$

Thus

$$f_1g_{u,k+u+l} + \dots + f_{l+1}g_{u+l,k+u+l} + f_{l+2}g_k + \dots + f_kg_{l+2} + f_{u,k+u}g_{l+1} \\ + \dots + f_{u,k+u+l-1}g_2 + f_{u,k+u+l}g_1 = 0. \quad (5)$$

Again by induction hypothesis and using results obtained in (2), (3) and (4), we obtained the following:

(i) (a) $f_1g_{u,k+u+t} = f_{u,k+u+t}g_1 = 0$, for $1 \le u \le k-t+1; 0 \le t \le l-1$.

(b) $f_2g_{u+1,k+u+t} = f_{u,k+u+t-1}g_2 = 0$, for $1 \le u \le k-t+1; 1 \le t \le l-1$.

$\vdots$

(c) $f_{t+1}g_{u+t,k+u+t} = f_{u,k+u}g_{l+1} = 0$, for $1 \le u \le k-t+1; t = l-1$.

(ii) $f_ig_j = 0$, for $i+j = u+k$, $i,j \ge u$ and $1 \le u =\le l+1$.

Thus (i), (ii) and left multiplication process implies each left side component of equation (5) equals to zero. Hence $[r(x).m(x)]_{u,k+u+t} = 0$ for $1 \le u \le k-l+1$. Hence mathematical induction gives $[r(x).m(x)] = 0\,\forall\,(i,j) \in \Gamma$. Thus ${}_{A_n(R)}A_n(M)$ is Armendariz module.

(2) By using the calculations as in Lemma 1, It is easy to verified that $A_n(M)$ is a nil module over $A_n(R)$. Thus, it is nil-Armendariz.

□

Theorem 2. *Let ${}_RM$ be a reduced module. For $n = 2k \ge 2$, the following statements are true:*

(1) $A_n(M) + E_{1,k}M$ is an Armendariz module over $A_n(R) + E_{1,k}R$.
(2) $A_n(M) + E_{1,k}M$ is nil-Armendariz module over $A_n(R) + E_{1,k}R$.

Proof. The proof of this theorem is almost similar to Theorem 1(1) above. However, for more illustration, we have demonstrated it as follows:

(1) Consider $r(x).m(x) = 0$ for some $r(x) = (f_{ij}) \in A_n(R)$ and $m(x) = (g_{ij}) \in A_n(M)$. Firstly we notice that $r(x)$ and $m(x)$ have following properties:

$$\begin{aligned} &f_1 := f_{11} = f_{22} = \dots = f_{nn} && g_1 := g_{11} = g_{22} = \dots = g_{nn} \\ &f_2 := f_{12} = f_{23} = \dots = f_{n-1,n} && g_2 := g_{12} = g_{23} = \dots = g_{n-1,n} \\ &\vdots && \vdots \\ &f_k := f_{1k} = f_{2,k+1} = \dots = f_{k+1,n} && g_k := g_{1k} = g_{2,k+1} = \dots = g_{k+1,n} \\ &f_0 = f_{1,k} && g_0 = g_{1,k} \\ &f_{i,j} := 0,\ i > j && g_{i,j} := 0,\ i > j. \end{aligned}$$

Now we have

$$\sum_{i+j=t} f_ig_j = 0 \ for\ t = 2,3,\dots k+1 \quad (6)$$

$$f_1g_0 + f_2g_{k-1} + \dots + f_{k-1}g_2 + f_0g_1 = 0. \quad (7)$$

By applying Similar left multiplication with equation (6) and (7), we get

$$f_ig_j = 0\,\forall\,i+j \le k+1. \quad (8)$$

and

$$f_1g_0 = f_0g_1 = 0. \quad (9)$$

This implies $[r(x).m(x)]_{ij} = 0$ for all i, j with $(i, j) \notin \Gamma$ where $\Gamma = \{(u, k+u) : u = 1, \ldots, k+1\} \cup \{(u, k+u+1) : u = 1, \ldots, k\} \cup \ldots \cup \{(u, u+n-2) : u = 1, 2\} \cup \{(u, n-1+u) : u = 1\}$.
Again from $r(x).m(x) = 0$, we have

$$f_1 g_{1,k+1} + f_2 g_k + f_3 g_{k-1} + \ldots + f_{k-1} g_3 + f_0 g_2 + f_{1,k+1} g_1 = 0 \tag{10}$$

and

$$f_1 g_{2,k+2} + f_2 g_k + \ldots + f_k g_2 + f_{2,k+2} g_1 = 0$$
$$\ldots$$
$$f_1 g_{k,2k} + f_2 g_k + \ldots + f_{k-1} g_3 + f_k g_2 + f_{k,2k} g_1 = 0.$$

By applying the same process of left multiplications and using the earlier results obtained in equation (2), we conclude that for $u = 1, 2, \ldots, k+1$

$$f_1 g_{u,k+u} = f_{u,k+u} g_1 = 0 \tag{11}$$

and with $i + j = k + 2$ for i, j

$$f_i g_j = 0 \tag{12}$$

and

$$f_0 g_2 = f_2 g_0 = 0.$$

Thus from equation (11) and (12), we get $[r(x).m(x)]_{u,u+k} = 0$ for $u = 1, 2, \ldots, k$.
Now for some $1 \leq l \leq k$, assume the condition $[r(x).m(x)]_{u,k+u+t} = 0$ holds true for $0 \leq t \leq l$ and $1 \leq u \leq k - t$. Thus it is sufficient to show that for each $u = 1, \ldots, k - l$, the equation $[r(x).m(x)]_{u,k+u+l} = 0$ holds true. For these consider $r(x).m(x) = 0$. This implies

$$\begin{cases} f_1 g_{u,k+u+l} \cdots + f_{l+1} g_{u+l,k+u+l} + f_{l+2} g_k + \ldots + f_k g_{l+2} + f_{u,k+u} g_{l+1} + \\ \ldots + f_{u,k+u+l-1} g_2 + f_{u,k+u} g_{l+1} + \ldots + f_{u,k+u+l-1} g_2 + f_{u,k+u+l} g_1 = \end{cases} \tag{13}$$

and

$$\begin{cases} f_1 g_{1,k+l+l} \cdots + f_{l+1} g_{l+l,k+l+l} + f_{l+2} g_k + \ldots + f_{k-1} g_{l+3} + f_0 g_{l+2} + f_{1,k+1} g_{l+1} + \\ \ldots + f_{1,k+1} g_2 + f_{1,k+l+1} g_1 = 0. \end{cases} \tag{14}$$

Again by induction hypothesis and using results obtained in (8), (9), (11) and (12), we obtained the following:

(i) (a) $f_1 g_{u,k+u+t} = f_{u,k+u+t} g_1 = 0$, for $1 \leq u \leq k - t; 0 \leq t \leq l - 1$.

(b) $f_2 g_{u+1,k+u+t} = f_{u,k+u+t-1} g_2 = 0$, for $1 \leq u \leq k - t; 1 \leq t \leq l - 1$.

$\vdots$

(c) $f_{t+1} g_{u+t,k+u+t} = f_{u,k+u} g_{t+1} = 0$, for $1 \leq u \leq k - t; t = l - 1$.

(ii) $f_i g_j = 0$ for $i + j = u + k$, $i, j \geq u$ for $u 1 \leq u \leq l + 1$.

(iii) $f_0 g_u = 0$ and $1 \leq u \leq l + 1$.

Thus from (i), (ii) and (iii) and left multiplication process helps to obtained that each component of equations (13) and (14) are equals to zero. Hence $[r(x).m(x)]_{u,k+u+t} = 0$ for $u = 1, \ldots, k - l$. Hence mathematical induction gives $[r(x).m(x)] = 0 \,\forall\, (i, j) \in \Gamma$.

(2) By using the calculations as in Lemma 1, It is easy to verified that $A_n(M) + E_{1,k}M$ is a nil module over $A_n(R) + E_{1,k}R$. Thus, it is nil-Armendariz.

□

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Proposition 5. *Let R be a commutative ring. If $nil_R(M) \subsetneq {}_RM$, then the quotient module $M/nil_R(M)$ is rigid.*

Proof. Let $u^2\overline{m} = \overline{0}$ in $M/nil_R(M)$. This implies that $u^2m \in nil_R(M)$. Thus, there exists some $r \in R$ such that $r^2u^2m = 0$ and $ru^2m \neq 0$. Since R is commutative, $r^2u^2m = 0$ implies $(ru)^2m = 0$, but $ru^2m \neq 0$. This implies $um \in nil_R(M)$. Therefore, $u\overline{m} = \overline{0}$ in $M/nil_R(M)$. Hence, $M/nil_R(M)$ is a rigid module. □

Proposition 6. *Let R be a commutative ring and ${}_RM$ be a torsion-free module. If $nil_R(M) \subsetneq {}_RM$, then $M/nil_R(M)$ is torsion-free.*

Proof. Suppose that $\overline{0} \neq \overline{m} \in \mathrm{Tor}(M/nil_R(M))$. Thus, there exists a non-zero $t \in R$ such that $t\overline{m} = \overline{0}$. This means $tm \in nil_R(M)$. Therefore, there exists some $l \in R$ such that $l^2tm = 0$ and $ltm \neq 0$. This implies $(lt)^2m = 0$ but $ltm \neq 0$. Hence, $m \in nil_R(M)$. Therefore, $\overline{m} = \overline{0}$ in $M/nil_R(M)$. □

Proposition 7. *Let R be a commutative ring. If $nil_R(M) \subsetneq {}_RM$, then ${}_RM$ is nil-Armendariz.*

Proof. Recall from [10] that if a module ${}_RM$ is both rigid and semi-commutative, then it is Armendariz. We observe, as per Proposition 5, that $M/\mathrm{nil}_R(M)$ constitutes a rigid module. Since R is commutative, this implies that $M/\mathrm{nil}_R(M)$ is semi-commutative. Thus, $M/\mathrm{nil}_R(M)$ is an Armendariz module. Let's consider $p(x)m(x) \in \mathrm{nil}_RM[x]$. Clearly, $p(x)\overline{m(x)} = \overline{0}$, where $\overline{m(x)}$ signifies the corresponding polynomial in $M/\mathrm{nil}_R(M)[x]$. Consequently, $r\overline{m} = \overline{0}$ for all $r \in \mathrm{coef}(p(x))$ and $\overline{m} \in \mathrm{coef}(\overline{m(x)})$. This suggests that rm is a nilpotent element for all $r \in \mathrm{coef}(p(x))$ and $m \in \mathrm{coef}(m(x))$. □

Proposition 8. *Let N be a submodule of ${}_RM$. If N is a subset of $nil_R(M)$, then ${}_RM$ is nil-Armendariz if and only if M/N is nil-Armendariz over R.*

Proof. Let $f(x) = \sum_{i=0}^{n} r_ix^i \in R[x]$ and $m(x) = \sum_{j=0}^{k} m_jx^j \in M[x]$. We denote $\overline{M} = M/N$. Since N is a nil submodule, then $nil(\overline{M}) = \overline{nil(M)}$. Hence $f(x)m(x) \in nil_R(M)[x]$ if and only if $\overline{f(x)} \cdot \overline{m(x)} \in nil_R(\overline{M})[x]$. Therefore, we conclude that $am \in nil_R(M)$ if and only if $a\overline{m} \in nil(\overline{M})$. Thus, M is nil-Armendariz if and only if $\overline{M}$ is nil-Armendariz. □

For a module ${}_RM$, recall that if R is a commutative domain, then $Tor(M)$ is a submodule and $M/Tor(M)$ is torsion-free. However, the same is not true if R contains a non-zero zero divisor, as illustrated by $M = R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Here, $Tor(\mathbb{Z}_2 \times \mathbb{Z}_2) = (0,0), (1,0), (0,1)$, which is not a submodule. Next, we have identified some conditions for the nil-Armendariz property in the context of the torsion class.

Proposition 9. *Let R be a commutative domain. Then ${}_RM$ is nil-Armendariz if and only if its torsion submodule $Tor(M)$ is nil-Armendariz.*

Proof. Let $p(x) = \sum_{l=0}^{n} u_lx^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} v_kx^k \in M[x]$ satisfy $p(x)m(x) \in nil_R(M)[x]$. Then we have:

$$\begin{aligned} u_0v_0 &\in nil_R(M) \\ u_1v_0 + u_0v_1 &\in nil_R(M) \\ u_2v_0 + u_1v_1 + u_0v_2 &\in nil_R(M) \\ &\vdots \\ u_nv_q &\in nil_R(M). \end{aligned}$$

Since R is a commutative domain, this implies $nil_R(M) \subseteq Tor(M)$. We can assume that $u_0 \neq 0$. Hence, from the first equation, we get $u_0v_0 \in nil_R(M) \Rightarrow l^2u_0v_0 = 0$ for some $l \in R$. Thus, $v_0 \in Tor(M)$. Since

$Tor(M)$ is a submodule of ${}_RM$, this implies $u_1v_0 \in Tor(M)$. Thus, from the second equation, it is clear that $u_0v_1 \in Tor(M)$, which again implies $v_1 \in Tor(M)$. Thus, by repeating the same process finitely many times, we conclude that $m(x) \in Tor(M)[x]$. Therefore, M is a nil-Armendariz module. □

Proposition 10. *Let R be a commutative domain. If ${}_RM$ is nil Armendariz module, then $M/Tor(M)$ is an nil-Armendariz module.*

Proof. We denote the quotient $M/Tor(M)$ by $\overline{M}$. Since $\overline{M}$ is torsion free, so by Proposition 3.2, it is sufficient to show that that $\overline{M}$ is Armendariz. Let $\overline{m(x)} = \sum_{k=0}^{q} \overline{v}_i x^i \in \overline{M}[x]$ and $p(x) = \sum_{l=0}^{n} u_j x^j \in R[x]$ satisfy $p(x)\overline{m}(x) = \overline{0}$ in $\overline{M}[x]$. Then, we have

$$u_0\overline{v}_0 = \overline{0}$$
$$u_0\overline{v}_1 + u_1\overline{v}_0 = \overline{0}$$
$$\dots$$
$$u_n\overline{v}_q = \overline{0}.$$

Now from first equation, we have $u_0v_0 \in Tor(M)$, which further implies $v_0 \in Tor(M)$. Since $Tor(M)$ is a submodule of M, hence $u_1v_0 \in Tor(M)$. Thus, from second equation, we get $u_0v_1 \in Tor(M)$. Thus repeating the same process finitely many times, we conclude that $u_lv_k \in Tor(M)$ for $0 \le l \le n$ and $0 \le k \le q$. Thus, $M/Tor(M)$ is Armendariz module. □

Here we record a "change of rings" result.

Proposition 11. *Let ${}_SM$ be a module and $\phi : R \to S$ be a ring homomorphism. By defining $uv = \phi(u)v$, M can be made R-module. If ϕ is onto then the following are equivalent.*

(1) ${}_RM$ is nil-Armendariz.
(2) ${}_AM$ is nil-Armendariz.

Proof. Firstly, we will show that If $rm \in nil_R(M)$, then $\theta(r)m \in nil_A(M)$. So, let $rm \in nil_R(M)$. Thus, there exists some $k \in R$ such that $k^2rm = 0$ and $krm \neq 0$. Now $0 = k^2rm = \theta(k^2r)m = (\theta(k))^2\theta(r)m$ and $0 \neq krm = \theta(k)\theta(r)m$. Thus, $\theta(r)m \in nil_A(M)$ and vice verse. Thus, the remaining part of the proof easily follows. □

Recall that for a multiplicative closed subset S of the centre C of the ring R, the set $S^{-1}M$ has a left module structure over $S^{-1}R$. In the next proposition, we study localization.

Lemma 4. *For a module ${}_RM$, an element $v \in nil_R(M)$ if and only if $d^{-1}v \in nil_{S^{-1}R}(S^{-1}M)$ for some $d \in S$.*

Proof. Suppose $d^{-1}m \in nil_{S^{-1}R}(S^{-1}M)$ were $d \in S$ and $v \in M$. Thus $\exists\, s^{-1}r \in S^{-1}R$ such that $(s^{-1}r)^2d^{-1}m = 0$ but $s^{-1}rd^{-1}v \neq 0$. This implies $r^2v = 0$ but $rv \neq 0$. Hence $v \in nil_R(M)$. For converse part suppose $v \in nil_R(M)$. Thus, $t^2v = 0$ but $tv \neq 0$ for some $t \in R$. Hence $(r^2/1).v/d = 0$, but $(r/1).v/d \neq 0$. □

Theorem 3. *For a module ${}_RM$, the following conditions are equivalent.*

(1) ${}_RM$ is nil-Armendariz.
(2) $S^{-1}M$ is nil-Armendariz $S^{-1}R$-module for each multiplicatively closed subset S of C.

 doi:10.20944/preprints202409.0181.v1

Proof. (1)⇒(2) Let $f(x) = \sum_{i=0}^{m} \xi_i x^i \in S^{-1}R[x]$ and $m(x) = \sum_{j=0}^{n} \eta_j x^j \in S^{-1}M[x]$ such that $f(x) \cdot m(x) \in nil_{S^{-1}R} S^{-1}M[x]$. Here $\xi_i = s_i^{-1}x^i \in S^{-1}R$ and $\eta_j = t_j^{-1}m_j \in S^{-1}M$. Thus, we have:

$$\begin{cases} \xi_0\eta_0 & \in nil_{S^{-1}R} S^{-1}M \\ \xi_0\eta_1 + \xi_1\eta_0 & \in nil_{S^{-1}R} S^{-1}M \\ \vdots & \\ \xi_m\eta_n & \in nil_{S^{-1}R} S^{-1}M. \end{cases} \tag{15}$$

Let us take $s = (s_0 s_1 \dots s_m)$ and $t = (t_0 t_1 \dots t_n)$ and consider $\overbrace{f(x)} = s.f(x) = \sum_{i=0}^{m} s\xi_i x^i$, $\overbrace{m(x)} = tm(x) = \sum_{j=0}^{n} t\eta_j x^j$. Clearly $\overbrace{f(x)} \in R[x]$ and $\overbrace{m(x)} \in M[x]$ and $\overbrace{f(x)}.\overbrace{m(x)} = s\xi_0 t\eta_0 + (s\xi_0 t\eta_1 + s\xi_1 t\eta_0)x + (s\xi_2 t\eta_0 + s\xi_1 t\eta_1 + s\xi_0 t\eta_2)x^2 + \dots + (s\xi_m t\eta_n)x^{m+n}$. From first equation, we have $\xi_0\eta_0 \in nil_{S^{-1}R} S^{-1}M \Rightarrow (s_0^{-1}a_0)(t_0^{-1}m_0) \in nil_{S^{-1}R}(S^{-1}M) \Rightarrow \exists q^{-1}r \in S^{-1}R$ such that $(q^{-1}r)^2(s_0^{-1}a_0)(t_0^{-1}m_0) = 0$ but $(q^{-1}r)(s_0^{-1}a_0)(t_0^{-1}m_0) \neq 0$. Thus, $r^2 a_0 m_0 = 0$ but $ra_0m_0 \neq 0$, which implies $r^2(s_1 \dots s_m)a_0 (t_1 \dots t_n)m_0 = 0$ and $r(s_1 \dots s_m)a_0(t_1 \dots t_n)m_0 \neq 0$. otherwise, suppose $r(s_1 \dots s_m)a_0(t_1 \dots t_n)m_0 = 0$, then

$$(s_1 \dots s_m)^{-1}(t_1 \dots t_n)^{-1} r(s_1 \dots s_m)a_0(t_1 \dots t_n)m_0 = 0 \Rightarrow ra_0m_0 = 0,$$

which is not possible. Thus, $s\xi_0 t\eta_0 \in nil_R(M)$. Similarly, we can show that $s\xi_m t\eta_n \in nil_R(M)$. Proceeding in similar way, again from first equation $\xi_0\eta_1 + \xi_1\eta_0 \in nil_{S^{-1}R} S^{-1}M$, we have $s_0^{-1}a_0 t_1^{-1}m_1 + s_1^{-1}a_1 t_0^{-1}m_0 \in nil_{S^{-1}R}(S^{-1}M)$, which implies $s_1t_0a_0m_1 + s_0t_1a_1m_0 \in nil_R(M)$. Also, we can see that $s\xi_0 t\eta_1 + s\xi_1 t\eta_0 = (s_1, \dots, s_m)(t_1, \dots, t_n))(s_1a_0t_0m_1 + s_0a_1t_1m_0) \in nil_R(M)$. Thus, similarly, we can show that all the coefficients of x^i in $\overbrace{f(x)}.\overbrace{m(x)}$ are in $nil_R(M)$. Since ${}_RM$ is nil-Armendariz this implies $st\xi_i\eta_j \in nil_R(M) \forall\ i, j)$. Thus by above Lemma 4 $\xi_i\eta_j \in nil_{S^1R}(S^{-1}M)$. (2)⇒(1) Let $f(x).m(x) \in nil_R(M)[x]$, where $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ and $m(x) = \sum_{j=0}^{n} m_j x^j \in M[x]$. Since $f(x) \in S^{-1}R[x]$ and $m(x) \in S^{-1}M[x]$, $a_i m_j \in nil_{S^{-1}M}(S^{-1}M)$, by Lemma 4, $a_i m_j \in nil_R(M)$. □

Theorem 4. *Let R be a commutative domain. Then for a module ${}_RM$, the following are equivalent:*

(1) ${}_RM$ is nil-Armendariz.
(2) ${}_QM$ is nil-Armendariz, were Q is the field of fraction of R.

Proof. The proof of this theorem follows similarly to that of Theorem 3. □

3. Results on Nilpotent Class of Modules

In ring theory, the class of nilpotent elements forms an ideal, provided the ring is commutative, semi-commutative, or even nil-Armendariz. However, the same is not true for the class of nilpotent elements in modules. A finite sum of nilpotent elements of a module ${}_RM$ is not necessarily nilpotent in ${}_RM$, even when ${}_RM$ is defined over a commutative ring R. For example $\overline{1}$ and $\overline{3}$ are nilpotent elements in ${}_{\mathbb{Z}}\mathbb{Z}_8$ since $\overline{2}^3.\overline{1} = 0$ but $\overline{2}.\overline{1} = \overline{2} \neq \overline{0}$ and $\overline{2}^3.\overline{3} = 0$ but $\overline{2}.\overline{3} = \overline{6} \neq \overline{0}$. However, their sum $\overline{4}$ is not nilpotent. Additionally, the class of nilpotent elements is not closed under left multiplication by R, even if R is commutative. For instance, $\overline{2} \in nil_{\mathbb{Z}}\mathbb{Z}_8$, but $2.\overline{2} = 4 \notin nil_{\mathbb{Z}}(\mathbb{Z}_8)$. In [12], SSevviiri and Groenewald posed the question of the conditions under which $nil_R(M)$forms a submodule. Here, we have found some conditions under which $nil_R(M)$ may form a submodule.

Lemma 5. *Let ${}_RM$ be a nil-Armendariz. Then the following are true.*

(1) If $u \in Nil(R)$ and $v \in nil_R(M)$, then $uv \in nil_R(M)$.
(2) If $v, w \in nil_R(M)$, then $v + w \in nil_R(M)$.
(3) If $u, y \in Nil(R)$ and $v \in nil_R(M)$, then $(u + y)v \in nil_R(M)$.

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Proof. (1) Suppose $u \in nil(R)$ and $u^t = 0$. Then

$$(1 + ux + ux^2 + \ldots + u^{t-1}x^{t-1}).(v - uvx) = v \in nil_R(M)[x].$$

Since ${}_RM$ nil-Armendariz, implies $uv \in nil_R(M)$.

(2) Suppose $v, w \in nil_R(M)$.

$$(1 - x).(w + (u + w)x + vx^2) = w + ux - wx^2 - ux^3 \in nil_R(M)[x].$$

Now, since ${}_RM$ is nil-Armendariz, from each polynomial, we can select the suitable coefficients to get $1.(v + w) \in nil_R(M)$.

(3) Suppose $u, y \in Nil(R)$, then $u^k = y^l = 0$. Then

$$(1 + ux + \ldots + u^{k-1}x^{k-1})(1 - ux)(1 - yx)(1 + yx + \ldots + y^{l-1}x^{l-1})v = v.$$

multiplying the intermediate polynomials yields

$$(1 + ux + \ldots + u^{k-1}x^{k-1})(1 - (u + y)x + uyx^2)(1 + yx + \ldots + y^{l-1}x^{l-1})v = v.$$

Now, since ${}_RM$ is nil-Armendariz, and $m \in nil_R(M)[x]$, from each polynomial, we can select the suitable coefficients to get $(u + y)v \in nil_R(M)$.

□

Proposition 12. *Let ${}_RM$ be nil-Armendariz module. If R is nil ring then $nil_R(M)$ is a submodule of ${}_RM$.*

Proof. Since R is a nil ring, it follows directly from (5) that $nil_R(M)$ is submodule of ${}_RM$. □

Proposition 13. *Let ${}_RM$ be a nil-Armendariz module over finitely generated commutative ring R. Then, $nil_R(M)$ is a submodule of ${}_RM$ if every proper ideal of R is nil ideal.*

Proof. Since every proper ideal is nil, it follows by Theorem 2.1 in [6], R that R is a nil ring. Hence, by Lemma 5, $nil_R(M)$ is a submodule of ${}_RM$. □

For a left R-module M, we generally have $Tor(M) \nsubseteq nil_R(M)$ as $\overline{2} \in Tor(\mathbb{Z}_4)$ while $\overline{2} \notin nil_{\mathbb{Z}_4}(\mathbb{Z}_4)$. Considering the definitions of $Tor(M)$ and $nil_R(M)$, one could suspect $nil_R(M)$ to be a subset of $Tor(M)$. However, the example given below will refute this possibility.

Example 1. *Consider the module ${}_{\mathbb{Z}}\mathbb{Z}$. Then, by Lemma 1, The matrix module $M_3(\mathbb{Z})$ is nil module over $M_3(\mathbb{Z})$. On the other hand, consider $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. If possible let us suppose that A is torsion element in $M_3(\mathbb{Z})$. Then, by definition there exits non-zero $L = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ in $M_3(\mathbb{Z})$, satisfying $LA = 0$. But however solving $LA = 0$ implies $a_{ij} = 0 \ \forall\ i, j \Rightarrow L = 0$. Thus, $A \notin Tor(M_3(\mathbb{Z}))$.*

Theorem 5. *If a module ${}_RM$ is torsion free, then $nil_R(M)$ is a submodule of ${}_RM$.*

Proof. If the ring R is reduced, then it is obvious that $nil_R(M) \subseteq Tor(M) = \{0\}$, so $nil_R(M)$ is a submodule. On the other hand, if R is non-reduced, then $\exists$ a non-zero $a \in R$ such that $a^2 = 0$ implying $a^2m = 0$ for every $m \in {}_RM$ and $am \neq 0$ since ${}_RM$ is torsion free. Thus, $nil_R(M)$ is a submodule. □

Author Contributions: All authors have equal contributions.

Funding: This study was carried out with financial support from the Deanship of Scientific Research at Department of Mathematics, College of Science, Taibah University, Saudi Arabia.

Data Availability Statement: Data sharing is not applicable to this article as no data were generated or analyzed during the current study.

Acknowledgments: This work was funded by the Deanship of Scientific Research at Department of Mathematics, College of Science, Taibah University, Saudi Arabia

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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