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The Polynomial $t^2(4x-n)^2 - 2xtn$ Is Always Admitting a Perfect Square

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The Polynomial $t^2(4x-n)^2 - 2xtn$ Is Always Admitting a Perfect Square

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Abstract

In this article, we prove that for every integer $n \geq 2$, there exist positive integers t and x such that the expression $E = t^2(4x-n)^2 - 2xtn$ is always a perfect square.

Keywords: Erdos-Straus conjecture; polynomial; positive integer

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1. Introduction

In his article : Partial Resolution of the Erdős-Straus, Sierpiński, and Generalized Erdős-Straus Conjectures Using New Analytical Formulas, in the page 13, Philemon Urbain MBALLA pose the following question: how can we prove that the polynomial $t^2(4x-n)^2 - 2ntx$ or $4t^2(4x-n)^2 - 8ntx$ always admits a perfect square for all $n \in \mathbb{N}$, with $n \geq 2$? We show in this article that it is indeed the case.

2. Problem

We show that for every integer $n \geq 2$, there exist positive integers t and x such that

$$E = t^2(4x-n)^2 - 2xtn$$

is a perfect square.

Proof. We consider four cases depending on the residue of n modulo 4.

Case 1: $n = 4k$ (so $n \equiv 0 \pmod 4$), with $k \geq 1$ because $n \geq 2$.
Let

$$x = k+1, \qquad t = \frac{k(k+1)}{2}.$$

Then $4x - n = 4(k+1) - 4k = 4$, and

$$E = t^2 \cdot 4^2 - 2 \cdot (k+1) \cdot t \cdot (4k) = 16t^2 - 8k(k+1)t.$$

Substituting $t = \dfrac{k(k+1)}{2}$, we obtain

$$E = 16\left(\frac{k(k+1)}{2}\right)^2 - 8k(k+1) \cdot \frac{k(k+1)}{2} = 4k^2(k+1)^2 - 4k^2(k+1)^2 = 0.$$

Thus $E = 0$, which is a perfect square. Moreover $x = k+1 \geq 2$ and $t = \frac{k(k+1)}{2} \geq 1$ are positive integers.

Case 2: $n = 4k+2$ (so $n \equiv 2 \pmod 4$), with $k \geq 0$ (for $n = 2$, we have $k = 0$).
Let

$$x = k+1, \qquad t = (2k+1)(k+1).$$

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Then $4x - n = 4(k+1) - (4k+2) = 2$, and

$$E = t^2 \cdot 2^2 - 2 \cdot (k+1) \cdot t \cdot (4k+2) = 4t^2 - 2(k+1)t \cdot 2(2k+1) = 4t^2 - 4(2k+1)(k+1)t.$$

Substituting $t = (2k+1)(k+1)$, we get

$$E = 4(2k+1)^2(k+1)^2 - 4(2k+1)(k+1) \cdot (2k+1)(k+1) = 4(2k+1)^2(k+1)^2 - 4(2k+1)^2(k+1)^2 = 0.$$

Hence $E = 0$. For $k = 0$ we have $x = 1, t = 1$; for $k \geq 1$ both are clearly positive.

Case 3: $n = 4k+3$ (so $n \equiv 3 \pmod 4$), with $k \geq 0$ (for $n = 3, k = 0$).
Let

$$x = k+1, \qquad t = 2(4k+3)(k+1).$$

Then $4x - n = 4(k+1) - (4k+3) = 1$, and

$$E = t^2 \cdot 1^2 - 2 \cdot (k+1) \cdot t \cdot (4k+3) = t^2 - 2(k+1)t(4k+3).$$

Substituting $t = 2(4k+3)(k+1)$, we obtain

$$E = 4(4k+3)^2(k+1)^2 - 2(k+1) \cdot 2(4k+3)(k+1) \cdot (4k+3) = 4(4k+3)^2(k+1)^2 - 4(4k+3)^2(k+1)^2 = 0.$$

Thus $E = 0$. For $k = 0$, we have $x = 1, t = 6$, which are positive.

Case 4: $n = 4k+1$ (so $n \equiv 1 \pmod 4$), with $k \geq 1$ because $n \geq 2$ (for $n = 5, k = 1$).
Let

$$x = k, \qquad t = 2k(4k+1).$$

Then $4x - n = 4k - (4k+1) = -1$, so $(4x-n)^2 = 1$, and

$$E = t^2 \cdot 1 - 2 \cdot k \cdot t \cdot (4k+1) = t^2 - 2kt(4k+1).$$

Substituting $t = 2k(4k+1)$, we get

$$E = 4k^2(4k+1)^2 - 2k \cdot 2k(4k+1) \cdot (4k+1) = 4k^2(4k+1)^2 - 4k^2(4k+1)^2 = 0.$$

Hence $E = 0$. For $k = 1$, we have $x = 1, t = 10$, which are positive.
□

3. Conclusions

In every case, we have constructed positive integers t and x such that $t^2(4x-n)^2 - 2xtn = 0$, which is a perfect square. Therefore the statement holds for all $n \geq 2$.

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Reference

1. Philemon Urbain MBALLA, *Partial Resolution of the Erdős-Straus, Sierpiński, and Generalized Erdős-Straus Conjectures Using New Analytical Formulas* ,article.