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Article

Gauge Theories in Rainbow Space-Time

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Abstract

We construct Maxwell and Yang-Mills theories in the rainbow space-time. We show that the time-dependent Aharonov-Bohm phases for both Abelian and non-Abelian gauge fields are non-zero in the rainbow space-time, unlike in Minkowski space-time, where they are zero. Starting from the Poisson brackets between coordinates and velocities of a massive particle moving in rainbow space-time and interacting with Abelian/non-Abelian gauge fields, we derive Maxwell's/Yang-Mills equations in rainbow space-time using a variant of Feynman's approach. In this approach, one replaces kinetic momentum with conjugate momentum and the gauge field to calculate the Poisson brackets. Finding the time-derivative of these Poisson brackets and exploiting their anti-symmetry property, a second rank anti-symmetric tensor is introduced which is the field strength associated with Maxwell/Yang-Mills theory, respectively. We show the gauge invariance/covariance of these field strengths explicitly and by demanding the gauge invariance, construct Maxwell's/Yang-Mills Lagrangians. We derive the Lorentz force equation for the particle moving in rainbow space-time in the presence of Abelian/non-Abelian gauge fields. Using the gauge field strength in rainbow space-time, expressed in terms of field strength in the Minkowski space-time and rainbow parameters, we explicitly evaluate the Aharonov-Bohm phases for different choices of rainbow functions and also for different possible gauge field configurations. We show that the time-dependent Aharonov-Bohm phases for Abelian as well as non-Abelian fields are, in general, non-zero in the rainbow space-time. The non-vanishing of time-dependent Aharonov-Bohm phase, which distinguishes rainbow space-time from Minkowski space-time, is a clear signal to be measured to validate the rainbow space-time.

Keywords: rainbow space-time; abelian and non-abelian gauge theory; aharonov-bohm phase

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1. Introduction

Reconciliation of quantum theory and general relativity is a mainstay of physics for many decades now [1]. Efforts in this direction have led to the development of various approaches for the construction of microscopic gravity. These studies have shown that a consistent quantum theory of gravity certainly requires quantisation of space-time. This requires the introduction of a dimension-full parameter and this fundamental invariant quantity to be a characteristic of quantum gravity. The existence of this fundamental constant with dimensions (of length or energy) necessitates the revamping of the principle of relativity. This led to the development of doubly special relativity (DSR), where the transformations between inertial observers are energy scale dependent [2–5].

Further, the possibility that the global Lorentz invariance is only an approximate symmetry, seen only for the low-energy effective theory of quantum gravity [6], also motivated the construction of models where this symmetry is seen only at the leading order in $l_P E_o$, where l_P is the Planck length and E_o is the energy scale of observation. One of these approaches is the DSR, where the energy scale, presumably Planck energy E_P, acts as a separation between classical and quantum regimes of gravity, and the value of E_P is observer independent. Generalisation of this to include gravitation introduces a parameter-dependent space-time metric known as the rainbow gravity (RG) model. Here the rainbow functions (f and g) are functions of $\frac{E_o}{E_P}$ where E_o is the energy scale of observation. In the limit of these

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functions going to unity, i.e., $f, g \to 1$, one recovers classical, that is, Minkowski space-time. In this framework, based on the energy scale of the observations, different observers may assign different metrics to the same region of space-time at a given instant of time; the same observer may also assign different metrics to particles having different energies at the same instant in the same neighbourhood.

As the notions of distances and time intervals become energy dependent, this paradigm also allows the possibility of energy-dependent velocity of light. This approach also necessitates a non-linear energy-momentum dispersion relation. In the rainbow space-time (RST), coordinates as well as 4-momenta transform under non-linear realisation of the Lorentz group.

Rainbow space-time (RST) and physical models constructed on this space-time reduce to the corresponding classical ones in the limit where the rainbow functions go to identity. RST also accommodate a generalised equivalence principle: All freely falling observers making measurements at the energy scale E_o, such that $\frac{1}{R} << E_o << E_P$ (i.e., $\frac{1}{R} << \frac{hc}{\lambda} << \frac{hc}{\lambda_P}$) agree on their results up to first order in $\frac{1}{R}$. Here R is the radius of curvature of the space-time [6].

Modified Einstein equations and their solutions were constructed and analysed in the rainbow framework. Scalar field theory in RST was introduced in [7,8]. Insisting that plane waves be a solution to the free field theory in the rainbow framework forces transformation between inertial coordinates and the space-time metric to be energy dependent [8].

In [8] Klein-Gordon equations corresponding to different possible choices of the map U (see eq.(6)) were introduced. Investigations of Schrödinger equation, Klein-Gordon equation, Dirac equation, and Duffin-Kemmer-Petiau equation, in different space-times modified by rainbow gravity functions (see eq.(5) below) were carried out in [9–17]. These studies were motivated to see the implications of gravity at high energy scales on particles whose dynamics are described by the above-mentioned equations.

The choices of rainbow functions, inspired by physical considerations and studied in [7,18–20] are

$$f(E_o) = g(E_o) = \frac{1}{1 - \lambda\left(\frac{E_o}{E_P}\right)}, \tag{1}$$

$$f(E_o) = \frac{e^{\frac{\beta E_o}{E_P}} - 1}{\beta\left(\frac{E_o}{E_P}\right)}, \quad g(E_o) = 1, \tag{2}$$

$$f(E_o) = 1,\ g(E_o) = \sqrt{1 - \eta\left(\frac{E_o}{E_P}\right)^n}. \tag{3}$$

Here λ, β, and η are the rainbow parameters corresponding to each choice of the rainbow functions. As the maximum energy of a particle cannot exceed E_P, we see $\frac{E_o}{E_P} < 1$.

Construction and analysis of gauge theories on RST is not much explored. Here, we try to bridge this gap by constructing Abelian and non-Abelian gauge theories on RST. We also analyse these theories and show the existence of signals that distinguish rainbow effects.

In this paper, we construct $U(1)$ and $SU(N)$ gauge invariant Lagrangians in the RST. We employ a variant of Feynman's approach in deriving the gauge field equations [21–24]. We start with the definition of kinetic momentum $\hat{p}_\mu = m\frac{d\hat{x}_\mu}{d\tau}$ in the RST, where hat-variables are those defined in the RST and τ is the time parameter. Role of τ has been studied in [25,26], and in particular the reparametrisation invariance of the derivation of Maxwell's theory in [22] has been examined in [26] and shown that the Maxwell's theory emerges in a specific limit. Two other ingredients used are the Poisson brackets between the coordinates $\hat{x}_\mu$ and four-velocities $\frac{d\hat{x}_\mu}{d\tau}$ (implied by the Poisson bracket between $\hat{x}_\mu$ and $\hat{p}_\mu$) and covariant form of Newton's law. For deriving Maxwell's electrodynamics in RST, we start with the above relations for a charged particle moving in an electromagnetic field. This fact is thus used to introduce canonical momentum $\hat{\pi}_\mu$ for this particle by minimal prescription, i.e.,

$\hat{\pi}_\mu = \hat{p}_\mu + e\hat{A}_\mu = m\dot{\hat{x}}_\mu + e\hat{A}_\mu$ where over-dot represents derivative with respect to τ, and $\hat{A}_\mu = \hat{A}_\mu(\hat{x})$ is the gauge field in RST.

We then take the derivatives of the Poisson bracket between the coordinate and velocity of RST with respect to τ. The resulting relation has two terms, one involving acceleration $\ddot{\hat{x}}_\mu$ and the second is anti-symmetric in indices. We re-express the first using Newton's law, $\hat{F}_\mu = m\ddot{\hat{x}}_\mu$, and identify the second term with an arbitrary second rank anti-symmetric tensor $\hat{F}_{\mu\nu}$. Explicit evaluation of this second term, after rewriting velocities in terms of conjugate momentum and $U(1)$ gauge field, shows that this second rank tensor is the 4-curl of $\hat{A}_\mu$ in RST, that is, it is Maxwell's field strength tensor in RST. Next, we use the Jacobi identity of Poisson brackets involving three momenta, which are rewritten in terms of $\hat{\pi}_\mu$ and $\hat{A}_\mu$, leading to homogeneous Maxwell's equations in RST. The remaining two Maxwell's equations are then derived starting from the nested Poisson brackets of four momentum, i.e., $\{\hat{p}_\nu, \{\hat{p}_\mu, \{\hat{p}^\mu, \hat{p}^\nu\}\}\}$. Here again, we substitute kinetic momentum in terms of conjugate momenta and Abelian gauge field, use the definition of $\hat{F}_{\mu\nu}$ in terms of Poisson bracket between momenta, before explicitly evaluating these Poisson brackets. This allows us to introduce a conserved four-vector $\hat{j}^\mu$, which is related to the divergence of Maxwell's field strength, thus allowing us to identify it with the conserved current associated with Maxwell's theory in RST.

We show that the field strength tensor $\hat{F}_{\mu\nu}$ is invariant under $U(1)$ transformation of the four-vector field in RST. Using this invariant field strength, we write down the Maxwell Lagrangian in RST using gauge invariance and being quadratic in derivatives as guiding principles. We show that the Maxwell equations constructed are exactly the same as the Euler-Lagrange equations following from this Lagrangian. Next, using the relation between the Poisson bracket between Newton's force $\hat{F}_\mu$, $\hat{x}_\mu$, and the field strength tensor, we derive the Lorentz force equation in RST. For different choices of rainbow functions (see eq.(1), eq.(2), and eq.(3)), we obtain the field strength, Lagrangian, equations of motion, and Lorentz force, explicitly. All our results, in the limit $f, g \to 1$, reduce to the well-known results in the Minkowski space-time.

We next consider an isospin carrying particle of mass m, moving in the presence of the non-Abelian gauge field and derive the Yang-Mills equations in RST, following similar steps as used in deriving Maxwell's equations. Here, after deriving the Yang-Mills field strength tensor $\hat{F}^a_{\mu\nu}$ and two homogeneous equations, we show the covariance of $\hat{F}^a_{\mu\nu}$ under $SU(N)$ transformation of the non-Abelian gauge field living in RST. This allows us to construct a $SU(N)$ invariant Lagrangian in RST from which the remaining Yang-Mills equations follow as Euler-Lagrange equations. Then, starting from the definition of the non-Abelian field strength tensor and using Newton's force equation for a charged particle under $SU(N)$ gauge group, we derive the corresponding Lorentz force equation. In deriving this Lorentz force equation, one extra input is needed, in comparison to the derivation of the corresponding equation in the electromagnetic case, which is Wong's equation: an equation describing the time evolution of $su(N)$ generators. Using the condition that the Wong's equation in RST should reduce to that in the Minkowski space-time in the appropriate limit, we generalise the Wong's equation to RST. Using this, we calculate the Poisson bracket between velocities in RST and $su(N)$ Lie algebra valued functions in RST. This allows us to define all the terms appearing in the non-Abelian Lorentz equation. Here, again, we obtain expressions of non-Abelian field strength, Lagrangian, Yang-Mills equations, and non-Abelian Lorentz force for specific choices of the rainbow functions. As in the Abelian case, here too, all our results correctly reproduce the Minkowski results in the limit $f, g \to 1$.

The field strength tensor for both Abelian and non-Abelian theories are modified, and in general, the "electric" and "magnetic" components are modified differently. These differences show up vividly in the Aharonov-Bohm (A-B) phase associated with Abelian and non-Abelian gauge fields in RST. Since the time and space coordinates of RST are related to the Minkowski ones through scaling by different factors (see eq.(11)), this also contributes to the extra terms in the A-B phases (compared to those in the Minkowski space-time). This difference is more drastic in the case of the A-B effect for the time-dependent gauge field configurations. Here, we show that the time-dependent A-B phase factor is non-zero in RST, unlike in Minkowski space-time [27–30]. This can be a possible signal, amenable

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to experimental verification of RST. Note that the A-B phase for the time-independent case also gets modified, both for Abelian and non-Abelian gauge fields. We explicitly evaluate these A-B phases for different choices of gauge field configurations, for both Abelian and non-Abelian theories, and also for different choices of rainbow functions f and g.

This paper is organised as follows: In section 2, we give a summary of the RST. This is essential for our later calculations. In section 3, we present the construction of Maxwell's electrodynamics in RST. Here, using the minimal coupling procedure described above, we derive homogeneous Maxwell's equations first and then the inhomogeneous equations. This is followed by the construction of the Lagrangian for Abelian gauge theory in RST. We then show the derivation of the Lorentz force equation in RST. We end this section with explicit calculations, for three different choices of rainbow functions, of the field strength tensor, Maxwell's equations, Lagrangian, and Lorentz force. Section 4 presents our construction of Yang-Mills theory in RST. As for Abelian theory, we use the minimal coupling method, but now with a non-Abelian gauge field. Here, we show the derivation of the Yang-Mills field strength and two of the field equations. Next, we construct the $SU(N)$ invariant Lagrangian and obtain the remaining equations as Euler-Lagrange equations. This is followed by the construction of the force equation for an isospin-carrying particle in the presence of a non-Abelian gauge field. We further show explicit forms of field strength, Yang-Mills equations, Lagrangian, and Lorentz force equations for three choices of the rainbow functions. In section 5, we investigate the time-dependent A-B phase in RST and show that it is non-zero, unlike in the Minkowski case. First, we calculate the A-B phase for the Abelian gauge field in RST. For different time-dependent Abelian potentials, we find the A-B phases for different choices of rainbow functions. We repeat this for the non-Abelian gauge field also and find the expression for the time-dependent A-B phase. Here, different time-dependent potentials are considered, and we obtain the corresponding A-B phase for three choices of rainbow functions. For both Abelian and non-Abelian gauge fields, the time-dependent A-B phase is, in general, non-zero and thus a possibility of experimental verification of RST exists. We present the concluding remarks in section 6.

2. Rainbow Space-Time

In the RST framework, one modifies the energy-momentum dispersion relation by introducing functions that depend on the ratio of the energy of the observer/probe (E_o) and Planck energy (E_P). The modified dispersion relation of the test particle in RST is given by [6,7]

$$E_o^2 f^2\left(\frac{E_o}{E_P}\right) - p^2 g^2\left(\frac{E_o}{E_P}\right) = m^2, \tag{4}$$

with $c = 1$. Here $f(\frac{E_o}{E_P})$ and $g(\frac{E_o}{E_P})$ are the rainbow functions, parametrised by the ratio of observer's/probe particle's energy E_o to the Planck energy E_P, m is the mass, and p is the 3-momentum of the test particle. The fact that the usual energy-momentum dispersion relation should be recovered in the Minkowski limit implies that the rainbow functions satisfy,

$$\lim_{\frac{E_o}{E_P}\to 0} f\left(\frac{E_o}{E_P}\right) = 1, \quad \lim_{\frac{E_o}{E_P}\to 0} g\left(\frac{E_o}{E_P}\right) = 1. \tag{5}$$

The modified dispersion relation is invariant under the action of a Lorentz group through a non-linear representation in the momentum space, where the boost generators are now given by $K^i = U^{-1}L_0^i U$ while the rotation generators remain unaffected. Here, U, which maps the generator of Lorentz boost onto the non-linear representation of momentum space, is given by[6,7]

$$U_\mu(p) \equiv U_\mu(E_o, p_i) = (U_0, U_i) = \Big(E_o f(\frac{E_o}{E_P}), p_i g(\frac{E_o}{E_P})\Big). \tag{6}$$

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Dual of this non-linear momentum space is set up by demanding that $U_\mu(p)$ must contract with $U^\mu(x)$ [8], i.e.,

$$U_\mu(p)U^\mu(x) = p_\mu x^\mu, \tag{7}$$

where $\mu = 0, 1, 2, 3$. The non-linear map $U^\mu(x)$ is given by [8]

$$U^\mu(x) = (U^0, U^i) = \Big(\frac{t}{f(\frac{E_o}{E_P})}, \frac{x^i}{g(\frac{E_o}{E_P})}\Big). \tag{8}$$

The line element that is invariant under the corresponding, modified Lorentz transformation is [8]

$$s^2 = \eta_{\mu\nu}U^\mu(x)U^\nu(x) = \frac{t^2}{f^2(\frac{E_o}{E_P})} - \frac{(x^i)^2}{g^2(\frac{E_o}{E_P})}, \tag{9}$$

where $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$, which is the same as that in the Minkowski space-time [1], but note the coordinates are $\hat{x}_\mu$ rather than Minkowski coordinates x_μ. The coordinates in RST are related to the Minkowski coordinates through

$$\hat{x}^0 = \frac{x^0}{f(\frac{E_o}{E_P})}, \quad \hat{x}^i = \frac{x^i}{g(\frac{E_o}{E_P})}. \tag{11}$$

Note that in the limit where $f(\frac{E_o}{E_P})$ =1= $g(\frac{E_o}{E_P})$, the coordinates of the RST reduce to the Minkowski space-time coordinates. This implies that an arbitrary function $\hat{\phi}(\hat{x})$ in RST, in the limit where the rainbow functions (f and g) go to 1, will reduce to $\phi(x)$, the corresponding function in Minkowski space-time.

3. Maxwell's Electrodynamics in Rainbow Space-Time

For constructing Maxwell's electrodynamics in RST, we consider the Poisson brackets between coordinates and velocities of RST. By considering the anti-symmetry property of the Poisson brackets between velocities in RST, we introduce an anti-symmetric tensor $\hat{F}_{\mu\nu}$. By applying minimal coupling prescription, we introduce the Abelian gauge field $\hat{A}_\mu(\hat{x})$ and replace the kinetic momentum ($\hat{p}_\mu = m\dot{\hat{x}}_\mu$) with $\hat{\pi}_\mu - e\hat{A}_\mu$, where $\hat{\pi}_\mu$ is the canonically conjugate momentum in RST. Note that $\hat{A}_\mu(\hat{x})$ is function of the RST coordinates. By calculating the Poisson brackets explicitly and equating them with the anti-symmetric tensor $\hat{F}_{\mu\nu}$ introduced, we obtain the explicit form of this anti-symmetric tensor in RST. Using the explicit form of this anti-symmetric tensor, we obtain Maxwell's equations in RST. The Lagrangian describing Maxwell's field in RST is constructed using this field strength tensor, and the corresponding Euler-Lagrange equations are derived. Using Newton's force equation in covariant form in the Poisson brackets, we also derive the Lorentz force equation in RST.

3.1. Maxwell's Equations

We assume that the RST coordinates given in eq.(11) and the corresponding velocities satisfy

$$m\{\hat{x}^\mu, \dot{\hat{x}}^\nu\} = -\hat{\eta}^{\mu\nu} \tag{12}$$

where $\hat{\eta}^{\mu\nu} = (+1, -1, -1, -1)$ is the same as $\eta_{\mu\nu}$ given in eq.(9). Note that in the limit $f \to 1, g \to 1$, the above equation reduces to the corresponding relation in the Minkowski space-time. From this, it follows

$$\{\hat{x}^\mu, \hat{\phi}(\hat{x})\} = 0, \quad m\{\dot{\hat{x}}^\mu, \hat{\phi}(\hat{x})\} = m\{\dot{\hat{x}}^\mu, \hat{x}^\nu\}\frac{\partial\hat{\phi}(\hat{x})}{\partial\hat{x}^\nu} = \frac{\partial\hat{\phi}(\hat{x})}{\partial\hat{x}_\mu} \tag{13}$$

[1] If in eq.(9), we express the RHS in terms of Minkowski coordinates, x_μ, then the metric will be [8],

$$g_{\mu\nu} = diag\Big(\frac{1}{f^2(\frac{E_o}{E_P})}, -\frac{1}{g^2(\frac{E_o}{E_P})} - \frac{1}{g^2(\frac{E_o}{E_P})} - \frac{1}{g^2(\frac{E_o}{E_P})}\Big). \tag{10}$$

where $\hat{\phi}(\hat{x})$ is a function of rainbow coordinates. Here, the over-dot stands for derivative with respect to τ. We use minimal coupling procedure to introduce interaction between an electrically charged particle and the electromagnetic field. Following this, we define conjugate momentum corresponding to the coordinate $\hat{x}_\mu$ as

$$\begin{aligned}\hat{\pi}_0 &= m\dot{\hat{x}}_0 + e\hat{A}_0 = \frac{m\dot{x}_0}{f(E_o)} + e\hat{A}_0, \\ \hat{\pi}_i &= m\dot{\hat{x}}_i + e\hat{A}_i = \frac{m\dot{x}_i}{g(E_o)} + e\hat{A}_i. \end{aligned} \tag{14}$$

The kinetic momentum is $\hat{p}_\mu = m\dot{\hat{x}}_\mu$ and e is the coupling strength. Here $\hat{A}_\mu = (\hat{A}_0(\hat{x}), \hat{A}_i(\hat{x}))$ and in the limit $f, g \to 1$, $\hat{x} \to x$ and $\hat{A}_\mu(\hat{x}) \to A_\mu(x)$. Using the above equation and the definition of the Poisson bracket, we find

$$m\{\hat{x}_\mu, \dot{\hat{x}}_\nu\} = \{\hat{x}_\mu, \hat{\pi}_\nu - e\hat{A}_\nu\} = \{\hat{x}_\mu, \hat{\pi}_\nu\} = -\hat{\eta}_{\mu\nu}. \tag{15}$$

Here we have used the fact that as we are working with Abelian theory and hence $\{\hat{A}_\mu, \hat{A}_\nu\} = 0$ and $\hat{A}_\mu$ is function only of $\hat{x}$ where $\hat{x}$ is the position coordinates of RST. We next start with the relation

$$m\{\hat{x}_0, \dot{\hat{x}}_i\} = 0 \tag{16}$$

and differentiating it with respect to τ gives [22–25],

$$\{\hat{x}_0, \hat{F}_i\} = -m\{\dot{\hat{x}}_0, \dot{\hat{x}}_i\} \tag{17}$$

where $\hat{F}_i = m\ddot{\hat{x}}_i$. By using the anti-symmetric property of the RHS of the above equation, we introduce an anti-symmetric tensor as

$$\frac{e}{m}\hat{F}_{0i} = -m\{\dot{\hat{x}}_0, \dot{\hat{x}}_i\}. \tag{18}$$

Re-expressing velocities on the RHS using eq.(14), we find

$$-\frac{1}{m}\{m\dot{\hat{x}}_0, m\dot{\hat{x}}_i\} = -\frac{1}{m}\{\hat{\pi}_0 - e\hat{A}_0, \hat{\pi}_i - e\hat{A}_i\} = \frac{e}{m}\left(\hat{\partial}_0\hat{A}_i - \hat{\partial}_i\hat{A}_0\right) = \frac{e}{m}\hat{F}_{0i} \tag{19}$$

In the above $\hat{\partial}_\mu = \frac{\partial}{\partial\hat{x}^\mu}$. In obtaining the above equation, we have used $\{\hat{A}_0, \hat{A}_i\} = 0$, as we are considering Abelian field, $\hat{A}_0(\hat{x})$, $\hat{A}_i(\hat{x})$ and they are functions only of $\hat{x}_\mu$ and $\{\hat{\pi}_0, \hat{\pi}_i\} = 0$, where $\hat{\pi}_\mu$ is the canonical momentum in RST. Using $\hat{\partial}_0 = f\frac{\partial}{\partial x^0} = f\partial_0$ and $\hat{\partial}_i = g\frac{\partial}{\partial x^i} = g\partial_i$ where ∂_μ is the derivative with respect to the Minkowski space-time coordinates x^μ, we re-express $\hat{F}_{0i}$ as

$$\hat{F}_{0i} = f\partial_0\hat{A}_i - g\partial_i\hat{A}_0, \tag{20}$$

where f and g are the rainbow functions (see the choices of these functions given in eq.(1), eq.(2), and eq.(3)). Next, we start with,

$$m\{\hat{x}_i, \dot{\hat{x}}_j\} = -\hat{\eta}_{ij} \tag{21}$$

after differentiating with respect to τ, we get

$$\{\hat{x}_i, \hat{F}_j\} = -m\{\dot{\hat{x}}_i, \dot{\hat{x}}_j\}, \tag{22}$$

where $\hat{F}_j = m\ddot{\hat{x}}_j$. As the RHS of the above equation is anti-symmetric as earlier, we introduce the anti-symmetric tensor,

$$\frac{e}{m}\hat{F}_{ij} = -m\{\dot{\hat{x}}_i, \dot{\hat{x}}_j\}. \tag{23}$$

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Following similar steps used in deriving eq.(19), we find the explicit form of $\hat{F}_{ij}$, i.e.,

$$\begin{aligned} -\frac{1}{m}\{m\dot{\hat{x}}_i, m\dot{\hat{x}}_j\} &= -\frac{1}{m}\{\hat{\pi}_i - e\hat{A}_i, \hat{\pi}_j - e\hat{A}_j\} \\ &= \frac{e}{m}\left(\frac{\partial \hat{A}_j}{\partial \hat{x}^i} - \frac{\partial \hat{A}_i}{\partial \hat{x}^j}\right) = \frac{e}{m} g\left(\partial_i \hat{A}_j - \partial_j \hat{A}_i\right) = \frac{e}{m}\hat{F}_{ij}. \end{aligned} \tag{24}$$

This gives the explicit form of spatial components of the second rank anti-symmetric tensor which is

$$\hat{F}_{ij} = g\left(\partial_i \hat{A}_j - \partial_j \hat{A}_i\right). \tag{25}$$

Note that in the limit when the rainbow functions f and g go to 1, the field strength $\hat{F}_{\mu\nu}$ obtained above (see eq.(19), eq.(24) and equivalently, eq.(20) and eq.(25)) reduce to the Maxwell's field strength $F_{\mu\nu}$ in the Minkowski space-time.

The Jacobi identity between the components of the kinetic momentum $\hat{p}_\mu = m\dot{\hat{x}}_\mu$ in the RST is

$$\epsilon^{\alpha\mu\nu\sigma}\{\hat{p}_\mu, \{\hat{p}_\nu, \hat{p}_\sigma\}\} = 0. \tag{26}$$

For $\alpha = 0$, after using eq.(23), this gives,

$$-e\epsilon^{0ijk}\{\hat{p}_i, \hat{F}_{jk}\} = 0$$

Rewriting $\hat{p}_i$ using eq.(14), in the above expression gives,

$$\epsilon^{0ijk}\{\hat{\pi}_i - e\hat{A}_i, \hat{F}_{jk}\} = 0. \tag{27}$$

We next substitute $m^2\{\dot{\hat{x}}_\mu, \dot{\hat{x}}_\nu\} = \{\hat{p}_\mu, \hat{p}_\nu\} = -e\hat{F}_{\mu\nu}$ (see eq.(18) and eq.(23)) in the first term on the LHS of the Jacobi identity $\{\hat{x}_\mu, \{\hat{p}_\nu, \hat{p}_\rho\}\} + \{\hat{p}_\nu, \{\hat{p}_\rho, \hat{x}_\mu\}\} + \{\hat{p}_\rho, \{\hat{x}_\mu, \hat{p}_\nu\}\} = 0$ and noting that the second and third terms are identically zero (after using eq.(15)), we find $\{\hat{x}_\mu, \hat{F}_{\nu\rho}\} = 0$. This shows that $\hat{F}_{\mu\nu}$ is function of $\hat{x}$ alone and hence $\{\hat{A}_i, \hat{F}_{jk}\} = 0$. Thus the eq.(27) reduces to

$$\hat{\partial}_i \hat{F}_{jk} + \hat{\partial}_j \hat{F}_{ki} + \hat{\partial}_k \hat{F}_{ij} = 0, \tag{28}$$

which is one of the Maxwell's equations in RST. Since $\frac{\partial}{\partial \hat{x}^i} = g\frac{\partial}{\partial x^i} = g\partial_i$, the above equation can be rewritten as,

$$g\left(\partial_i \hat{F}_{jk} + \partial_j \hat{F}_{ki} + \partial_k \hat{F}_{ij}\right) = 0. \tag{29}$$

In the limit when $g \to 1$, $\hat{F}_{ij} \to F_{ij}$ and the above equation reduces to the Maxwell's equation in Minkowski space-time. The second homogeneous Maxwell's equation in RST is obtained by setting $\alpha = i$ in the eq.(26), which gives,

$$\{\hat{p}_0, \{\hat{p}_j, \hat{p}_k\}\} + \{\hat{p}_j, \{\hat{p}_k, \hat{p}_0\}\} + \{\hat{p}_k, \{\hat{p}_0, \hat{p}_j\}\} = 0. \tag{30}$$

Using eq.(18) and eq.(23), this becomes

$$\{\hat{p}_0, \hat{F}_{jk}\} + \{\hat{p}_j, \hat{F}_{k0}\} + \{\hat{p}_k, \hat{F}_{0j}\} = 0. \tag{31}$$

Replacing $\hat{p}_0$ with $\hat{\pi}_0 - e\hat{A}_0$ and $\hat{p}_i$ with $\hat{\pi}_i - e\hat{A}_i$ in the above expression and after simplifying, we get

$$\hat{\partial}_0 \hat{F}_{jk} + 2\hat{\partial}_j \hat{F}_{k0} = 0, \tag{32}$$

which is the second homogeneous Maxwell's equation in RST. This can also be written as,

$$f\partial_0 \hat{F}_{jk} + 2g\partial_j \hat{F}_{k0} = 0 \tag{33}$$

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where $\frac{\partial}{\partial \hat{x}^0}$ is replaced with $f\frac{\partial}{\partial x^0} = f\partial_0$ and $\frac{\partial}{\partial \hat{x}^i}$ is replaced with $g\frac{\partial}{\partial x^i} = g\partial_i$. Here also, Maxwell's equation in Minkowski space-time is obtained in the limit where f and g, the rainbow functions, go to 1. Note that the homogeneous Maxwell's equations in RST derived in eq.(28) and eq.(32) are exact.

Now, for deriving the inhomogeneous Maxwell's equation in RST, consider

$$\begin{aligned} \{\hat{p}_\nu, \{\hat{p}_\mu, \{\hat{p}^\mu, \hat{p}^\nu\}\}\} &= -e\{\hat{p}_\nu, \{\hat{p}_\mu, \hat{F}^{\mu\nu}\}\} \\ &= -e\{\hat{\pi}_\nu - e\hat{A}_\nu, \{\hat{\pi}_\mu - e\hat{A}_\mu, \hat{F}^{\mu\nu}\}\} \\ &= -e\{\hat{\pi}_\nu - e\hat{A}_\nu, \{\hat{\pi}_\mu, \hat{F}^{\mu\nu}\}\} = -e\{\hat{\pi}_\nu, \{\hat{\pi}_\mu, \hat{F}^{\mu\nu}\}\} = 0 \end{aligned} \tag{34}$$

where we have used the minimal coupling rule, the relations $\{\hat{A}_\mu, \hat{F}^{\mu\nu}\} = 0$ and $\{\hat{A}_\nu, \{\hat{\pi}_\mu, \hat{F}^{\mu\nu}\}\} = \{\hat{A}_\nu, \frac{\partial \hat{F}^{\mu\nu}}{\partial \hat{x}^\mu}\} = 0$. Also by using the anti-symmetry property of $\hat{F}^{\mu\nu}$, we get $\{\hat{\pi}_\nu, \{\hat{\pi}_\mu, \hat{F}^{\mu\nu}\}\} = \hat{\partial}_\nu\hat{\partial}_\mu\hat{F}^{\mu\nu} = 0$. From these results, we define

$$\{\hat{p}_\mu, \{\hat{p}^\mu, \hat{p}^\nu\}\} = -e\hat{j}^\nu \tag{35}$$

where $\hat{j}^\nu$ satisfy $\{\hat{p}_\nu, \hat{j}_\nu\} = \hat{\partial}_\nu\hat{j}_\nu = 0$. In the above equation, by taking $\nu = 0$, we get

$$\{\hat{p}_j, \{\hat{p}^j, \hat{p}^0\}\} = -e\hat{j}^0 \tag{36}$$

The LHS of the above equation can be written using eq.(18) as,

$$\{\hat{p}_j, \{\hat{p}^j, \hat{p}^0\}\} = -e\{\hat{p}_j, \hat{F}^{j0}\} \tag{37}$$

Substituting for the kinetic momentum $\hat{p}_j = \hat{\pi}_j - e\hat{A}_j$ in the LHS of the above equation gives,

$$\{\hat{p}_j, \{\hat{p}^j, \hat{p}^0\}\} = -e\{\hat{\pi}_j - e\hat{A}_j, \hat{F}^{j0}\} = -e\{\hat{\pi}_j, \hat{F}^{j0}\} = -e\frac{\partial \hat{F}^{j0}}{\partial \hat{x}^j}. \tag{38}$$

Comparing eq.(36) and eq.(38) gives,

$$\frac{\partial \hat{F}^{j0}}{\partial \hat{x}^j} = \hat{j}^0 \tag{39}$$

which can be written as

$$g\partial_j\hat{F}^{j0} = \hat{j}^0. \tag{40}$$

In eq.(35), taking $\nu = i$ we get,

$$\{\hat{p}_0, \{\hat{p}^0, \hat{p}^i\}\} + \{\hat{p}_j, \{\hat{p}^j, \hat{p}^i\}\} = -e\hat{j}^i \tag{41}$$

Here, using the definition of the field strength tensor for Maxwell's field in RST (see eq.(18) and eq.(23)) and the minimal coupling rule given in eq.(14), we get the second inhomogeneous Maxwell's equation as

$$\hat{\partial}_0\hat{F}^{0i} + \hat{\partial}_j\hat{F}^{ji} = \hat{j}^i \tag{42}$$

which can be rewritten as

$$f\partial_0\hat{F}^{0i} + g\partial_j\hat{F}^{ji} = \hat{j}^i. \tag{43}$$

These Maxwell's equations, eq.(39) and eq.(42), are also exact. The above equations (eq.(39), eq.(40), eq.(42), and eq.(43)) reduce to the corresponding Minkowski ones when the value of the rainbow functions f and g becomes 1, as required.

Under the transformation

$$\hat{A}_0 \rightarrow \hat{A}_0 + \frac{\partial \alpha}{\partial \hat{x}^0} = \hat{A}_0 + f\partial_0\alpha, \tag{44}$$

and

$$\hat{A}_i \rightarrow \hat{A}_i + \frac{\partial \alpha}{\partial \hat{x}^i} = \hat{A}_i + g\partial_i\alpha, \tag{45}$$

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we see that the Abelian field strength in RST, that is, $\hat{F}_{0i} = f\partial_0\hat{A}_i - g\partial_i\hat{A}_0$ and $\hat{F}_{ij} = g\left(\partial_i\hat{A}_j - \partial_j\hat{A}_i\right)$ derived in eq.(20) and eq.(25) respectively, are invariant. Note here that f and g are rainbow functions and not gauge couplings. Note that in the limit $f, g \to 1$, the above gauge transformation becomes exactly the same as the $U(1)$ gauge transformation, which is the symmetry of Maxwell's theory in the Minkowski space-time.

Next, using the explicit form of the gauge invariant field strength in RST, we construct Maxwell's Lagrangian in the RST. We demanded that the Lagrangian is (i) invariant under the $U(1)$ transformation given in eq.(44) and eq.(45), (ii) quadratic in derivatives, and these requirements uniquely fix the Lagrangian to be

$$\hat{L} = -\frac{1}{2}\hat{F}_{0i}\hat{F}^{0i} - \frac{1}{4}\hat{F}_{ij}\hat{F}^{ij}, \tag{46}$$

which is clearly invariant under the $U(1)$ transformation of $\hat{A}_0$ and $\hat{A}_i$ given in eq.(44) and eq.(45). The Euler-Lagrange equations of motion following from eq.(46) are

$$\hat{\partial}_i\hat{F}^{0i} = 0 \quad \text{or} \quad g\partial_i\hat{F}^{0i} = 0 \tag{47}$$

$$\hat{\partial}_0\hat{F}^{0i} + \hat{\partial}_j\hat{F}^{ji} = 0 \quad \text{or} \quad f\partial_0\hat{F}^{0i} + g\partial_j\hat{F}^{ji} = 0 \tag{48}$$

Eq.(47) and eq.(48) reproduce eq.(39) (eq.(40)) and eq.(42) (eq.(43)) for the source free region. The Maxwell's equations in RST, which we obtained above as the Euler-Lagrange equations and also the corresponding Lagrangian (eq.(46)), go to the Maxwell's equations and the Lagrangian in Minkowski space-time, when the rainbow functions f and g go to one.

3.2. Lorentz Equations in RST

Next, we derive the Lorentz force equation in the RST: For this, we explicitly evaluate the LHS of $\{\hat{x}_0, \hat{F}_i\} = \frac{e}{m}\hat{F}_{0i}$

$$-\hat{\eta}_{\rho\sigma}\left(\frac{\partial\hat{x}_0}{\partial\hat{x}_\rho}\frac{\partial\hat{F}_i}{\partial\hat{\pi}_\sigma} - \frac{\partial\hat{F}_i}{\partial\hat{x}_\rho}\frac{\partial\hat{x}_0}{\partial\hat{\pi}_\sigma}\right) = -\frac{\partial\hat{F}_i}{\partial\hat{\pi}^0} \tag{49}$$

Noting $\hat{F}_{0i} = \hat{F}_{0i}(\hat{x}_\mu)$, and all the modifications coming from RST are functions of $\frac{E_o}{E_P}$ alone, we can integrate the above equation with respect to $\hat{\pi}^0$ to get (note that $\hat{F}_{0i}$ is a function only of $\hat{x}_\mu$, see discussion after eq.(27))

$$\hat{F}_i = \frac{e}{m}\hat{F}_{i0}\hat{\pi}^0 + \tilde{G}_i(\hat{x}) \tag{50}$$

In the above expression, $\hat{\pi}_0 = m\dot{\hat{x}}_0 + e\hat{A}_0$. Taking $\tilde{G}_i(\hat{x}) = \hat{G}_i(\hat{x}) - \frac{e^2}{m}\hat{F}_{i0}\hat{A}^0$ allows us to re-express RHS of the above equation as

$$\hat{F}_i = \frac{e}{m}\hat{F}_{i0}(\hat{\pi}^0 - e\hat{A}^0) + \hat{G}_i(\hat{x}) = e\hat{F}_{i0}\dot{\hat{x}}^0 + \hat{G}_i(\hat{x}). \tag{51}$$

Similarly, $\{\hat{x}_i, \hat{F}_j\} = \frac{e}{m}\hat{F}_{ij}$ gives,

$$\frac{\partial\hat{F}_j}{\partial\hat{\pi}^i} = -\frac{e}{m}\hat{F}_{ij} \tag{52}$$

which on integrating with respect to $\hat{\pi}^i$ gives (note $\hat{F}_{ij}$ is a function only of $\hat{x}_\mu$),

$$\hat{F}_i = \frac{e}{m}\hat{F}_{ij}\hat{\pi}^j + \tilde{G}_i(\hat{x}). \tag{53}$$

Substituting $\tilde{G}_i(\hat{x}) = \hat{G}_i(\hat{x}) - \frac{e^2}{m}\hat{F}_{ij}\hat{A}^j$ in the above gives,

$$\hat{F}_i = \frac{e}{m}\hat{F}_{ij}(\hat{\pi}^j - e\hat{A}^j) + \hat{G}_i(\hat{x}) = e\hat{F}_{ij}\dot{\hat{x}}^j + \hat{G}_i(\hat{x}), \tag{54}$$

which is similar in form of Lorentz force equation involving velocities in the Minkowski space-time.

By combining the eq.(51) and eq.(54), we get the i^{th} component of the Lorentz force experienced by a charged particle moving in Maxwell's field in RST as

$$\hat{F}_i = e\hat{F}_{i0}\dot{\hat{x}}^0 + e\hat{F}_{ij}\dot{\hat{x}}^j + \hat{G}_i(\hat{x}), \tag{55}$$

which can also be written as,

$$\hat{F}_i = \frac{e}{f}\hat{F}_{i0}\dot{x}^0 + \frac{e}{g}\hat{F}_{ij}\dot{x}^j + \hat{G}_i(\hat{x}). \tag{56}$$

Note that in the limit f and g go to one (and $\hat{F}_{\mu\nu} \to F_{\mu\nu}$), the expression for the spatial component of the Lorentz force equation given above reduces to the corresponding expression in Minkowski space-time. For finding the zeroth component of the Lorentz force experienced by the charged particle in the presence of Maxwell's field in RST, consider

$$\{\hat{x}_i, \hat{F}_0\} = \frac{e}{m}\hat{F}_{i0}. \tag{57}$$

Using the definition of Poisson bracket in the LHS of the above expression gives,

$$\frac{\partial \hat{F}_0}{\partial \hat{\pi}^i} = -\frac{e}{m}\hat{F}_{i0}. \tag{58}$$

On integrating the above expression with respect to $\hat{\pi}^i$, we get

$$\hat{F}_0 = \frac{e}{m}\hat{F}_{0i}\hat{\pi}^i + \tilde{G}_0(\hat{x}). \tag{59}$$

As earlier substituting $\tilde{G}_0(\hat{x}) = \hat{G}_0(\hat{x}) - \frac{e^2}{m}\hat{F}_{0i}\hat{A}^i$ gives,

$$\hat{F}_0 = \frac{e}{m}\hat{F}_{0i}(\hat{\pi}^i - e\hat{A}^i) + \hat{G}_0(\hat{x}) = e\hat{F}_{0i}\dot{\hat{x}}^i + \hat{G}_0(\hat{x}) \tag{60}$$

which can also be written as

$$\hat{F}_0 = \frac{e}{g}\hat{F}_{0i}\dot{x}^i + \hat{G}_0(\hat{x}). \tag{61}$$

This is the zeroth component of the Lorentz force acting on a charged particle moving in Maxwell's field in RST. Note that $\hat{F}_i$ and $\hat{F}_0$ in the eq.(55), eq.(60) (equivalently in eq.(56) and eq.(61)) reduce to the familiar expressions in the Minkowski limit obtained by sending $f \to 1$ and $g \to 1$. Note that the Lorentz force equations in RST derived in eq.(55) and eq.(60) are exact. Expressing $\hat{G}_0(\hat{x})$ and $\hat{G}_i(\hat{x})$ by inverting eq.(55) and eq.(60), we find

$$\begin{aligned} m\{\dot{\hat{x}}_0, \hat{G}_i(\hat{x})\} - m\{\dot{\hat{x}}_i, \hat{G}_0(\hat{x})\} = 0, \\ m\{\dot{\hat{x}}_i, \hat{G}_j(\hat{x})\} - m\{\dot{\hat{x}}_j, \hat{G}_i(\hat{x})\} = 0, \end{aligned} \tag{62}$$

where we have used the definition of field strength tensor given in eq.(18) and eq.(23) and Maxwell's equations that we have obtained earlier. This implies,

$$\begin{aligned} \hat{\partial}_0\hat{G}_i(\hat{x}) - \hat{\partial}_i\hat{G}_0(\hat{x}) = 0, \\ \hat{\partial}_i\hat{G}_j(\hat{x}) - \hat{\partial}_j\hat{G}_i(\hat{x}) = 0. \end{aligned} \tag{63}$$

3.3. Choices of Rainbow Functions

We have constructed the field strength tensor in RST (eq.(20) and eq.(25)), the Maxwell's equations (eq.(29), eq.(33), eq.(47) and eq.(48)), corresponding Lagrangian in RST (eq.(46)) and, Lorentz force equation of a particle moving in the presence of this Maxwell's field in RST (eq.(55) and eq.(60)). Here, all these quantities are written in terms of rainbow functions, $f(\frac{E_o}{E_P})$ and $g(\frac{E_o}{E_P})$. We now obtain these explicitly for 3-different choices of rainbow functions given in eq.(1), eq.(2), and eq.(3), respectively, keeping terms up to first order in the rainbow parameters.

Choice 1: For the first choice of rainbow function given in eq.(1), using Taylor expansion, we see that a general function of RST coordinates will take the form, $\hat{\phi}_\mu(\hat{x}) = \phi_\mu(x) - \frac{\lambda E_o}{E_P}(x_\nu \partial^\nu \phi_\mu) + O(\lambda^2)$. Using this we expand $\hat{A}_0$ and $\hat{A}_i$ in $\hat{F}_{0i} = f\partial_0 \hat{A}_i - g\partial_i \hat{A}_0$ and $\hat{F}_{ij} = g\left(\partial_i \hat{A}_j - \partial_j \hat{A}_i\right)$ and keeping terms up to first order in λ, we find, the components of field strength tensor in RST as,

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{\lambda E_o}{E_P}(x_\alpha \partial^\alpha F_{\mu\nu}), \tag{64}$$

which is same as the Taylor expansion of $\hat{F}_{\mu\nu}(\hat{x})$ to first order in λ, and thus the Lagrangian valid up to first order in λ is

$$\hat{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda E_o}{2E_P}F_{\mu\nu}(x_\alpha \partial^\alpha F^{\mu\nu}). \tag{65}$$

The corresponding Maxwell's equations in RST are

$$\epsilon^{\alpha\mu\nu\rho}\left(1 - \frac{\lambda E_o}{E_P}x_\sigma \partial^\sigma\right)\partial_\mu F_{\nu\rho} = 0; \quad \left(1 - \frac{\lambda E_o}{E_P}x_\nu \partial^\nu\right)\partial_\mu F^{\alpha\mu} = 0. \tag{66}$$

The Lorentz force equation for the choice of rainbow function given in eq.(1) is,

$$\hat{F}_\mu = F_\mu - \frac{\lambda E_o}{E_P}\left(F_\mu + e(x_\nu \partial^\nu F_{\mu\alpha})\dot{x}^\alpha\right) + \hat{G}_\mu(\hat{x}). \tag{67}$$

Here $F_i = e(F_{i0}\dot{x}^0 + F_{ij}\dot{x}^j)$ and $F_0 = eF_{0i}\dot{x}^i$. In the limit when the rainbow parameter $\lambda \to 0$, $\hat{F}_{\mu\nu}$, $\hat{L}$, Maxwell's equations and $\hat{F}_\mu$ reduce to the Minkowski expressions as expected.

Choice 2: For the second choice of rainbow function given in eq.(2), applying Taylor expansion of a general function, $\hat{\phi}_\mu(\hat{x}) = \phi_\mu(x) - \frac{\beta E_o}{2E_P}(x_0 \partial^0 \phi_\mu) + O(\beta^2)$ to $\hat{A}_\mu$ appearing in $\hat{F}_{\mu\nu}$ (see eq.(20) and eq.(25)) we get the field strength tensor in RST as

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{\beta E_o}{2E_P}(x_0 \partial^0 F_{\mu\nu}) \tag{68}$$

up to the first order in β. The Lagrangian to the first order in β is

$$\hat{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\beta E_o}{4E_P}F_{\mu\nu}(x_0 \partial^0 F^{\mu\nu}). \tag{69}$$

Maxwell's equations are now given by

$$\epsilon^{\alpha\mu\nu\rho}\left(1 - \frac{\beta E_o}{2E_P}x_0 \partial^0\right)\partial_\mu F_{\nu\rho} = 0; \quad \left(1 - \frac{\beta E_o}{2E_P}x_0 \partial^0\right)\partial_\mu F^{\alpha\mu} = 0. \tag{70}$$

And the components of the Lorentz force are

$$\hat{F}_0 = F_0 - \frac{\beta E_o}{2E_P}e\left(x_0 \partial^0 F_{0i}\dot{x}^i\right) + \hat{G}_0(\hat{x}) \;\; ; \;\; \hat{F}_i = F_i - \frac{\beta E_o}{2E_P}e\left(F_{i0}\dot{x}^0 + x_0 \partial^0 F_{i\alpha}\dot{x}^\alpha\right) + \hat{G}_i(\hat{x}) \tag{71}$$

Note that comparing to $\hat{F}_0$, here $\hat{F}_i$ have extra terms because when $g(E_o) = 1$, the rainbow parameter dependent term in eq.(61) is coming only from $\hat{F}_{0i}$ part whereas $\hat{F}_i$ has $f(E_o)$ dependence which contribute extra rainbow parameter dependent terms. Note that $\beta \to 0$ is the Minkowski limit, and we get back the corresponding expressions in Minkowski space-time in the limit $f, g \to 1$ from the above equations.

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Choice 3: For the third choice of rainbow function (eq.(3)), Taylor series expansion of a general function of RST $\hat{\phi}_\mu(\hat{x}) = \phi_\mu(x) + \frac{\eta E_o}{2E_P}\left(x_m \partial^m \phi_\mu\right) + O(\eta^2)$, applied to eq.(20) and eq.(25) gives field strength tensor to be

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{\eta E_o}{2E_P}(x_m \partial^m F_{\mu\nu}), \tag{72}$$

up to the first order in η. Thus, the Lagrangian is

$$\hat{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\eta E_o}{4E_P}F_{\mu\nu}(x_m \partial^m F^{\mu\nu}) \tag{73}$$

up to the first order in η. The corresponding Maxwell's equations in RST are

$$\epsilon^{\alpha\mu\nu\rho}\left(1 + \frac{\eta E_o}{2E_P}x_m \partial^m\right)\partial_\mu F_{\nu\rho} = 0; \quad \left(1 + \frac{\eta E_o}{2E_P}x_m \partial^m\right)\partial_\mu F^{\alpha\mu} = 0. \tag{74}$$

The components of Lorentz force experienced by a particle moving in the presence of Maxwell's field in RST for the choice of rainbow function given in eq.(3) are,

$$\hat{F}_\mu = F_\mu + \frac{\eta E_o}{2E_P}e\left(F_{\mu j}\dot{x}^j + x_m \partial^m F_{\mu\alpha}\dot{x}^\alpha\right) + \hat{G}_\mu(\hat{x}). \tag{75}$$

Note that in the Minkowski limit, i.e., when the rainbow parameter η goes to zero, all the above expressions in RST reduce to the corresponding expressions in Minkowski space-time.

4. Yang-Mills theory in Rainbow Space-Time

For constructing the modified Yang-Mills field theory in RST, we follow the same procedure used in constructing Maxwell's electrodynamics in RST. Here the non-Abelian gauge field is introduced through minimal coupling prescription, by replacing the kinetic momentum $\hat{p}_\mu$ $(m\dot{\hat{x}}_\mu)$ with $\hat{\pi}_\mu - e\hat{A}_\mu$, where $\hat{\pi}_\mu$ is the canonical momentum in RST and $\hat{A}_\mu(\hat{x}) = \hat{A}^a_\mu(\hat{x})I^a$ is the non-Abelian gauge field in RST.

4.1. Yang-Mills Equations

We start the derivation of the Yang-Mills equations in RST with the following assumptions,

1. the canonical momentum $(\hat{\pi}_\mu)$ obtained using the minimal coupling rule is

$$\hat{\pi}_0 = m\dot{\hat{x}}_0 + e\hat{A}^a_0 I^a, \quad \hat{\pi}_i = m\dot{\hat{x}}_i + e\hat{A}^a_i I^a \tag{76}$$

   where $m\dot{\hat{x}}_\mu = \hat{p}_\mu$ is the kinetic momentum of the particle, $\hat{A}^a_0 I^a$ and $\hat{A}^a_i I^a$ are the components of the $SU(N)$ gauge field in RST, e is the gauge coupling strength.
2. the generators of $su(N)$ algebra, I^a, $a = 1, 2, ..N^2 - 1$ satisfy the relation

$$\{I^a, I^b\} = f^{abc}I^c. \tag{77}$$

As in section 3, we start with

$$m\{\hat{x}_0, \dot{\hat{x}}_i\} = \{\hat{x}_0, \hat{\pi}_i - e\hat{A}_i\} = \{\hat{x}_0, \hat{\pi}_i\} = 0. \tag{78}$$

where the non-Abelian gauge field $\hat{A}_\mu$ depends only on the space-time coordinates in RST, that is $\hat{A}_\mu = \hat{A}^a_\mu(\hat{x})I^a$. Differentiating the LHS of the above equation with respect to τ, we find

$$m\{\dot{\hat{x}}_0, \dot{\hat{x}}_i\} + m\{\hat{x}_0, \ddot{\hat{x}}_i\} = 0 \tag{79}$$

where the 'over-dot' stands for derivative with respect to τ. As for Maxwell's theory, here also, we define an anti-symmetric tensor as

$$m\{\dot{\hat{x}}_0, \dot{\hat{x}}_i\} = -\frac{e}{m}\hat{F}_{0i}. \tag{80}$$

where now $\hat{F}_{0i} = \hat{F}^a_{0i} I^a$. To derive the explicit form of $\hat{F}_{0i}$, we evaluate the LHS of the above equation using the minimal coupling prescription given in eq.(76),

$$\begin{aligned} -m\{\dot{\hat{x}}_0, \dot{\hat{x}}_i\} &= -\frac{1}{m}\{\hat{\pi}_0 - e\hat{A}_0, \hat{\pi}_i - e\hat{A}_i\} \\ &= \frac{e}{m}\left(\hat{\partial}_0 \hat{A}^a_i - \hat{\partial}_i \hat{A}^a_0 - e f^{abc} \hat{A}^b_0 \hat{A}^c_i\right) I^a \end{aligned} \tag{81}$$

Here we have used eq.(77), the fact that $\{\hat{x}_\mu, I^a\} = 0$ and $\{\hat{\pi}_\mu, I^a\} = 0$ [22]. Using this in eq.(80), we find

$$\hat{F}^a_{0i} = \hat{\partial}_0 \hat{A}^a_i - \hat{\partial}_i \hat{A}^a_0 - e f^{abc} \hat{A}^b_0 \hat{A}^c_i \tag{82}$$

We can also write $\hat{F}^a_{0i}$ as,

$$\hat{F}^a_{0i} = f\partial_0 \hat{A}^a_i - g\partial_i \hat{A}^a_0 - e f^{abc} \hat{A}^b_0 \hat{A}^c_i. \tag{83}$$

Here f and g are the rainbow functions and e is the gauge coupling strength. Similarly, for deriving the explicit form of $\hat{F}^a_{ij}$, consider,

$$m\{\hat{x}_i, \dot{\hat{x}}_j\} = \{\hat{x}_i, \hat{\pi}_j - e\hat{A}_j\} = -\hat{\eta}_{ij} \tag{84}$$

where we used $\{\hat{x}_i, \hat{A}_j\} = 0$. Differentiating above equation with respect to the parameter τ leads to $\{\dot{\hat{x}}_i, \hat{F}_j\} + m\{\dot{\hat{x}}_i, \dot{\hat{x}}_j\} = 0$. Now we introduce an anti-symmetric tensor through

$$-m\{\dot{\hat{x}}_i, \dot{\hat{x}}_j\} = \frac{e}{m}\hat{F}_{ij}, \tag{85}$$

where $\hat{F}_{ij} = \hat{F}^a_{ij} I^a$. From the above equation, we get

$$-m\{\dot{\hat{x}}_i, \dot{\hat{x}}_j\} = -\frac{1}{m}\{\hat{\pi}_i - e\hat{A}_i, \hat{\pi}_j - e\hat{A}_j\} = \frac{e}{m}\left(\hat{\partial}_i \hat{A}^a_j - \hat{\partial}_j \hat{A}^a_i - e f^{abc} \hat{A}^b_i \hat{A}^c_j\right) I^a = \frac{e}{m}\hat{F}_{ij}. \tag{86}$$

As for $\hat{F}^a_{0i}$, we rewrite $\hat{F}^a_{ij}$ as

$$\hat{F}^a_{ij} = g(\partial_i \hat{A}^a_j - \partial_j \hat{A}^a_i) - e f^{abc} \hat{A}^b_i \hat{A}^c_j. \tag{87}$$

In the above, g is the rainbow function and e is the gauge coupling strength. Note that both $\hat{F}_{0i}$ and $\hat{F}_{ij}$ reduce to the corresponding expressions in Minkowski space-time when the rainbow functions f and g go to one. Now we use this explicit form of $\hat{F}_{0i}$ and $\hat{F}_{ij}$ given in eq.(82) and eq.(86), respectively, for deriving the Yang-Mills field equations in RST. The Yang-Mills field strength tensor derived here in eq.(82) and eq.(86) is exact.

We now show that the components of $\hat{F}_{\mu\nu}$ obtained in eq.(82) and eq.(86) obey the Yang-Mills field equations in RST. Using the explicit forms of $\hat{F}^a_{0i}$ and $\hat{F}^a_{ij}$ obtained above, we obtain one of the homogeneous Yang-Mills field equations

$$\frac{\partial \hat{F}^a_{ij}}{\partial \hat{x}^0} + \frac{\partial \hat{F}^a_{j0}}{\partial \hat{x}^i} + \frac{\partial \hat{F}^a_{0i}}{\partial \hat{x}^j} = -e f^{abc}\left(f\partial_0(\hat{A}^b_i \hat{A}^c_j) + g\partial_i(\hat{A}^b_j \hat{A}^c_0) + g\partial_j(\hat{A}^b_0 \hat{A}^c_i)\right) \tag{88}$$

which becomes

$$\left(\hat{\partial}_0 \hat{F}_{ij} + e\hat{\partial}_0\{\hat{A}_i, \hat{A}_j\}\right) + \left(\hat{\partial}_i \hat{F}_{j0} + e\hat{\partial}_i\{\hat{A}_j, \hat{A}_0\}\right) + \left(\hat{\partial}_j \hat{F}_{0i} + e\hat{\partial}_j\{\hat{A}_0, \hat{A}_i\}\right) = 0. \tag{89}$$

After simplification, this can be written as,

$$\left(\hat{\partial}_0\hat{F}_{ij}+e\{\hat{F}_{ij},\hat{A}_0\}\right)+\left(\hat{\partial}_i\hat{F}_{j0}+e\{\hat{F}_{j0},\hat{A}_i\}\right)+\left(\hat{\partial}_j\hat{F}_{0i}+e\{\hat{F}_{0i},\hat{A}_j\}\right)=0. \tag{90}$$

which suggests that the time and space components of the covariant derivative in RST are of the form,

$$(\hat{D}_0)^{ac}=\delta^{ac}\hat{\partial}_0-ef^{abc}\hat{A}_0^b \tag{91}$$

and

$$(\hat{D}_i)^{ac}=\delta^{ac}\hat{\partial}_i-ef^{abc}\hat{A}_i^b. \tag{92}$$

Here $\hat{A}_0^b$ and $\hat{A}_i^b$ are the components of the Yang-Mills field in RST. Using eq.(91) and eq.(92), we re-express the Yang-Mills equation in eq.(90) as,

$$(\hat{D}_0\hat{F}_{ij})^a+(\hat{D}_i\hat{F}_{j0})^a+(\hat{D}_j\hat{F}_{0i})^a=0. \tag{93}$$

Using the explicit form of the field strength tensor in RST obtained in eq.(86), we find

$$\frac{\partial\hat{F}^a_{jk}}{\partial\hat{x}^i}+\frac{\partial\hat{F}^a_{ki}}{\partial\hat{x}^j}+\frac{\partial\hat{F}^a_{ij}}{\partial\hat{x}^k}=-ef^{abc}\left(\hat{\partial}_i(\hat{A}^b_j\hat{A}^c_k)+\hat{\partial}_j(\hat{A}^b_k\hat{A}^c_i)+\hat{\partial}_k(\hat{A}^b_i\hat{A}^c_j)\right), \tag{94}$$

which on simplification gives,

$$(\hat{\partial}_i\hat{F}^a_{jk}-ef^{abc}\hat{A}^b_i\hat{F}^c_{jk})+(\hat{\partial}_j\hat{F}^a_{ki}-ef^{abc}\hat{A}^b_j\hat{F}^c_{ki})+(\hat{\partial}_k\hat{F}^a_{ij}-ef^{abc}\hat{A}^b_k\hat{F}^c_{ij})=0. \tag{95}$$

Using the eq.(92), we rewrite eq.(95) in a familiar form as

$$(\hat{D}_i\hat{F}_{jk})^a+(\hat{D}_j\hat{F}_{ki})^a+(\hat{D}_k\hat{F}_{ij})^a=0. \tag{96}$$

Using $\hat{\partial}_0=f\partial_0$ and $\hat{\partial}_i=g\partial_i$, we re-express eq.(93) and eq.(96) as

$$(f\delta^{ac}\partial_0-ef^{abc}\hat{A}^b_0)\hat{F}^c_{ij}+(g\delta^{ac}\partial_i-ef^{abc}\hat{A}^b_i)\hat{F}^c_{j0}+(g\delta^{ac}\partial_j-ef^{abc}\hat{A}^b_j)\hat{F}^c_{0i}=0, \tag{97}$$

and

$$(g\delta^{ac}\partial_i-ef^{abc}\hat{A}^b_i)\hat{F}^c_{jk}+(g\delta^{ac}\partial_j-ef^{abc}\hat{A}^b_j)\hat{F}^c_{ki}+(g\delta^{ac}\partial_k-ef^{abc}\hat{A}^b_k)\hat{F}^c_{ij}=0, \tag{98}$$

respectively. The Yang-Mills equations in RST derived in (eq.(93) and eq.(96) (equivalently in eq.(97) and eq.(98))) are exact. Note that in [22], where the non-Abelian gauge theory in Minkowski space-time is constructed using Feynman's approach, one shows the second rank field strength tensor satisfies homogeneous Yang-Mills equations and then derives its explicit form using Wongs equation [31]. Here, first we derive the explicit form of the field strength $\hat{F}_{\mu\nu}$ and then we use this explicit form to show that the field strength satisfies homogeneous Yang-Mills equations. Also note that the form of the field strength tensor obtained in eq.(82) and eq.(86) is compatible with the gauge covariant derivative given in eq.(91) and eq.(92), that is,

$$(\{\hat{D}_0,\hat{D}_i\}\phi)^a=-ef^{abc}\hat{F}^b_{0i}\phi^c;\quad(\{\hat{D}_i,\hat{D}_j\}\phi)^a=-ef^{abc}\hat{F}^b_{ij}\phi^c. \tag{99}$$

Next, we take up the construction of the Yang-Mills Lagrangian in RST. Under the transformations

$$\hat{A}^a_0\to\hat{A}^{a'}_0=\hat{A}^a_0+(\hat{D}_0\alpha)^a=\hat{A}^a_0+(f\delta^{ac}\partial_0\alpha^c-ef^{abc}\hat{A}^b_0\alpha^c), \tag{100}$$

$$\hat{A}^a_i\to\hat{A}^{a'}_i=\hat{A}^a_i+(\hat{D}_i\alpha)^a=\hat{A}^a_i+(g\delta^{ac}\partial_i\alpha^c-ef^{abc}\hat{A}^b_i\alpha^c), \tag{101}$$

The components of the field strength tensor in the RST background given in eq.(82) and eq.(86) are covariant. The above $SU(N)$ gauge transformations in RST reduce to the well-known symmetry transformations of $SU(N)$ theory in the Minkowski space-time, in the limit $f, g \to 1$, as expected.

Using the above gauge covariant field strength in RST, we construct the Lagrangian corresponding to the Yang-Mills field in RST. Here, in constructing the Lagrangian, we insist that the Lagrangian is (i) invariant under the $SU(N)$ transformations given in eq.(100) and eq.(101); (ii) quadratic in derivatives. The Lagrangian satisfying these requirements is

$$L = -\frac{1}{2}\hat{F}^a_{0i}\hat{F}^{a0i} - \frac{1}{4}\hat{F}^a_{ij}\hat{F}^{aij}. \tag{102}$$

Here $\hat{F}^a_{0i}$ and $\hat{F}^a_{ij}$ are the components of the Yang-Mills field strength tensor in RST given in eq.(82) and eq.(86). Using this Lagrangian of Yang-Mills field theory in RST, we find the remaining two Yang-Mills field equations as the Euler-Lagrangian equations following from eq.(102), and they are

$$(\hat{D}_i\hat{F}^{i0})^a = 0; \quad (\hat{D}_0\hat{F}^{0i})^a + (\hat{D}_j\hat{F}^{ji})^a = 0. \tag{103}$$

Here $\hat{D}_0$ and $\hat{D}_i$ are the components of the gauge covariant derivative in RST, which we have obtained in eq.(91) and eq.(92). In the limit when the rainbow functions f and g become 1, eq.(102) and eq.(103) reduce to the corresponding equations in the Minkowski space-time.

4.2. Force Equation in Presence of Non-Abelian Gauge Field in RST

We now derive the force experienced by an isospin-carrying particle in the presence of a Yang-Mills field in RST. To obtain the expression for the spatial component $\hat{F}_i$ of this force, we start from

$$m\{\hat{x}_0, \ddot{\hat{x}}_i\} = \frac{e}{m}\hat{F}_{0i}. \tag{104}$$

Since $\hat{F}_i = m\ddot{\hat{x}}_i$, the LHS of the above equation can be written as,

$$\{\hat{x}_0, \hat{F}_i\} = -\frac{\partial \hat{F}_i}{\partial \hat{\pi}_0}. \tag{105}$$

Comparing this with the RHS of eq.(104), gives $\frac{\partial \hat{F}_i}{\partial \hat{\pi}_0} = -\frac{e}{m}\hat{F}_{0i}$. Since $\hat{F}_{0i}$ is function only of $\hat{x}^\mu$, we integrate this equation with respect to $\hat{\pi}_0$ and obtain

$$\hat{F}_i = \frac{e}{m}\hat{F}_{i0}\hat{\pi}^0 + \tilde{G}_i(\hat{x}), \tag{106}$$

where $\tilde{G}_i$ is a function of $\hat{x}$ alone. Now for writing the Lorentz force in terms of the velocity $\dot{\hat{x}}_0$, we take $\tilde{G}_i(\hat{x}) = \hat{G}_i(\hat{x}) - \frac{e^2}{m}\hat{F}_{i0}\hat{A}^0$, and eq.(106) becomes,

$$\hat{F}_i = \frac{e}{m}\hat{F}_{i0}(\hat{\pi}^0 - e\hat{A}^0) + \hat{G}_i(\hat{x}) = e\hat{F}_{i0}\dot{\hat{x}}_0 + \hat{G}_i(\hat{x}). \tag{107}$$

Next consider

$$\{\hat{x}_i, \hat{F}_j\} = -\frac{\partial \hat{F}_j}{\partial \hat{\pi}^i} = \frac{e}{m}\hat{F}_{ij}, \tag{108}$$

which, on integration, gives

$$\hat{F}_i = \frac{e}{m}\hat{F}_{ij}\hat{\pi}^j + \tilde{G}_i(\hat{x}), \tag{109}$$

where $\tilde{G}_i(\hat{x})$ is a function of $\hat{x}$. Setting $\tilde{G}_i(\hat{x}) = \hat{G}_i(\hat{x}) - \frac{e^2}{m}\hat{F}_{ij}\hat{A}^j$, for expressing the Lorentz force in terms of velocity, the above equation becomes,

$$\hat{F}_i = e\hat{F}_{ij}\dot{\hat{x}}^j + \hat{G}_i(\hat{x}). \tag{110}$$

Combining eq.(107) and eq.(110) gives,

$$\hat{F}_i = e\hat{F}_{i0}\dot{\hat{x}}^0 + e\hat{F}_{ij}\dot{\hat{x}}^j + \hat{G}_i(\hat{x}). \tag{111}$$

This gives the i^{th}-component of the Lorentz force acting on a particle carrying isospin in the RST background. To get the zeroth component of the Lorentz force in the RST, we start with

$$\{\hat{x}_i, \hat{F}_0\} = -\frac{\partial \hat{F}_0}{\partial \hat{\pi}^i} = \frac{e}{m}\hat{F}_{i0}.$$

which leads to

$$\hat{F}_0 = \frac{e}{m}\hat{F}_{0i}\hat{\pi}^i + \tilde{G}_0(\hat{x}). \tag{112}$$

On substituting $\tilde{G}_0(\hat{x}) = \hat{G}_0(\hat{x}) - \frac{e^2}{m}\hat{F}_{0i}\hat{A}^i$, the zeroth component of Lorentz force becomes,

$$\hat{F}_0 = e\hat{F}_{0i}\dot{\hat{x}}^i + \hat{G}_0(\hat{x}). \tag{113}$$

Note that in eq.(111) and eq.(113), $\hat{G}_\mu(\hat{x}) = \hat{G}^a_\mu I^a$. Further $\hat{F}_0$ and $\hat{F}_i$ derived here are exact.

Next we obtain the condition of $\hat{G}_0(\hat{x})$ and $\hat{G}_i(\hat{x})$. To this end, we start with the Wong's equation in RST, which is given by

$$\dot{I}^a = ef^{abc}\dot{\hat{x}}^0\hat{A}^b_0 I^c + ef^{abc}\dot{\hat{x}}^i\hat{A}^b_i I^c \tag{114}$$

which reduces to the corresponding equation in the Minkowski space-time, in the limit $f, g \to 1$ [31]. Considering an arbitrary function $\hat{\phi}(\hat{x})$ in the RST, we find,

$$m\{\dot{\hat{x}}_0, \hat{\phi}^a(\hat{x})I^a\} = m\{\dot{\hat{x}}_0, \hat{\phi}^a(\hat{x})\}I^a + m\hat{\phi}^a(\hat{x})\{\dot{\hat{x}}_0, I^a\}, \tag{115}$$

Assuming [22]

$$\{\hat{x}_0, I^a\} = 0,$$

and differentiating this equation with respect to τ will introduce a term containing $\dot{I}^a$. Rewriting this $\dot{I}^a$ using the Wong's equation in RST given in eq.(114) gives

$$\begin{aligned} m\{\dot{\hat{x}}_0, I^a\} &= -m\{\hat{x}_0, \dot{I}^a\} \\ &= -m\{\hat{x}_0, ef^{abc}\dot{\hat{x}}^0\hat{A}^b_0 I^c + ef^{abc}\dot{\hat{x}}^i\hat{A}^b_i I^c\} \\ &= -ef^{abc}\{\hat{x}_0, m\dot{\hat{x}}^0\}\hat{A}^b_0 I^c = -ef^{abc}\{\hat{x}_0, \hat{\pi}^0 - e\hat{A}^0\}\hat{A}^b_0 I^c = ef^{abc}\hat{A}^b_0 I^c. \end{aligned} \tag{116}$$

Using eq.(116) in eq.(115) gives,

$$\begin{aligned} m\{\dot{\hat{x}}_0, \hat{\phi}^a(\hat{x})I^a\} &= \{\hat{\pi}_0 - e\hat{A}_0, \hat{\phi}^a(\hat{x})\}I^a + ef^{abc}\hat{\phi}^a(\hat{x})\hat{A}^b_0 I^c \\ &= \delta^{ac}\frac{\partial \hat{\phi}^c}{\partial \hat{x}^0}I^a - ef^{abc}\hat{A}^b_0\hat{\phi}^c I^a = (\delta^{ac}\hat{\partial}_0 - ef^{abc}\hat{A}^b_0)\hat{\phi}^c I^a = (\hat{D}_0\hat{\phi})^a I^a. \end{aligned} \tag{117}$$

Note that the expression for the zeroth component of the covariant derivative $\hat{D}^{ac}_0$ obtained above using Wong's equation in RST matches with one in eq.(91), which was derived by using the explicit form of the Yang-Mills field strength tensor in RST. Similarly, we consider,

$$m\{\dot{\hat{x}}_i, \hat{\phi}^a(\hat{x})I^a\} = m\{\dot{\hat{x}}_i, \hat{\phi}^a(\hat{x})\}I^a + m\hat{\phi}^a(\hat{x})\{\dot{\hat{x}}_i, I^a\}. \tag{118}$$

We start with

$$\{\hat{x}_i, I^a\} = 0. \tag{119}$$

and differentiating this equation with respect to τ and using Wong's equation (eq.(114)), we calculate the second term on the RHS of eq.(118), i.e.,

$$\begin{aligned} m\{\dot{\hat{x}}_i, I^a\} &= -m\{\hat{x}_i, \dot{I}^a\} \\ &= -m\{\hat{x}_i, ef^{abc}\dot{\hat{x}}^0 \hat{A}_0^b I^c + ef^{abc}\dot{\hat{x}}^j \hat{A}_j^b I^c\} \\ &= -ef^{abc}\{\hat{x}_i, m\dot{\hat{x}}_j\}\hat{A}^{jb} I^c = -ef^{abc}\{\hat{x}_i, \hat{\pi}_j - e\hat{A}_j\}\hat{A}^{jb} I^c = ef^{abc}\hat{A}_i^b I^c. \end{aligned} \tag{120}$$

Using eq.(120) in eq.(118) gives,

$$\begin{aligned} m\{\dot{\hat{x}}_i, \hat{\phi}^a(\hat{x}) I^a\} &= \{\hat{\pi}_i - e\hat{A}_i, \hat{\phi}^a(\hat{x})\} I^a + ef^{abc}\hat{\phi}^a(\hat{x})\hat{A}_i^b I^c \\ &= (\delta^{ac}\hat{\partial}_i - ef^{abc}\hat{A}_i^b)\hat{\phi}^c I^a = (\hat{D}_i\hat{\phi})^a I^a \end{aligned} \tag{121}$$

Here $\hat{D}_i^{ac}$ is the i^{th} component of the covariant derivative in the RST background, which matches with the expression obtained in eq.(92). In particular, taking the function $\hat{\phi}$ to be $\hat{G}_\mu = \hat{G}_\mu^a I^a$, we find

$$\begin{aligned} m\{\dot{\hat{x}}_0, \hat{G}_i^a(\hat{x}) I^a\} &= (\hat{D}_0\hat{G}_i)^a I^a, \\ m\{\dot{\hat{x}}_i, \hat{G}_j^a(\hat{x}) I^a\} &= (\hat{D}_i\hat{G}_j)^a I^a \end{aligned} \tag{122}$$

Using eq.(122) and the results from the Lorentz force equation shows that,

$$\begin{aligned} \hat{D}_0\hat{G}_i - \hat{D}_i\hat{G}_0 &= 0, \\ \hat{D}_i\hat{G}_j - \hat{D}_j\hat{G}_i &= 0. \end{aligned} \tag{123}$$

4.3. Choices of Rainbow Functions

In the previous section, we have obtained the general expressions for components of Yang-Mills field strength tensor in RST (eq.(82) and eq.(86)), the Yang-Mills equations (eq.(93), eq.(96) and eq.(103)), corresponding Lagrangian (eq.102) and also the expression for components of the Lorentz force experienced by a $SU(N)$ charged particle moving in the Yang-Mills field in rainbow framework (eq.(111) and eq.(113)). These expressions are written in terms of rainbow functions $f(\frac{E_o}{E_P})$, $g(\frac{E_o}{E_P})$ and the fields $\hat{A}_\mu(\hat{x})$. Next, we construct these explicitly for the three choices of rainbow functions, keeping terms up to first order in the rainbow parameters.

For choice 1: We find the non-Abelian field strength to be

$$\hat{F}_{\mu\nu}^a = F_{\mu\nu}^a - \frac{\lambda E_o}{E_P}(x_\alpha \partial^\alpha F_{\mu\nu}^a), \tag{124}$$

and the Lagrangian to be

$$\hat{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{\lambda E_o}{2E_P}F_{\mu\nu}^a(x_\alpha\partial^\alpha F^{a\mu\nu}). \tag{125}$$

The corresponding Yang-Mills equations are

$$\epsilon^{\alpha\mu\nu\rho}\left(1 - \frac{\lambda E_o}{E_P}x_\sigma\partial^\sigma\right)(D_\mu F_{\nu\rho})^a = 0 \;\; ; \;\; \left(1 - \frac{\lambda E_o}{E_P}x_\nu\partial^\nu\right)(D_\mu F^{\alpha\mu})^a = 0. \tag{126}$$

The components of the Lorentz force equation for the choice of rainbow function given in eq.(1) are,

$$\hat{F}_\alpha = F_\alpha - \frac{\lambda E_o}{E_P}(F_\alpha + e\left(x_\nu\partial^\nu F_{\alpha\beta})\dot{x}^\beta\right) + \hat{G}_\alpha(\hat{x}). \tag{127}$$

Here $F_i = e(F_{i0}\dot{x}^0 + F_{ij}\dot{x}^j)$ and $F_0 = eF_{0i}\dot{x}^i$, where $F_{\mu\nu} = F_{\mu\nu}^a I^a$. In the limit $\lambda \to 0$, all the above equations reduce to the corresponding equation in the Minkowski space-time.

For choice 2: For choice given in eq.(2), we have

$$\hat{F}^a_{\mu\nu} = F^a_{\mu\nu} - \frac{\beta E_o}{2E_P}(x_0\partial^0 F^a_{\mu\nu}), \tag{128}$$

and the Lagrangian is

$$\hat{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\beta E_o}{4E_P}F^a_{\mu\nu}(x_0\partial^0 F^{a\mu\nu}). \tag{129}$$

The Yang-Mills equations following from the above Lagrangian are

$$\epsilon^{\alpha\mu\nu\rho}\left(1 - \frac{\beta E_o}{2E_P}x_0\partial^0\right)(D_\mu F_{\nu\rho})^a = 0; \ \left(1 - \frac{\beta E_o}{2E_P}x_0\partial^0\right)(D_\mu F^{\alpha\mu})^a = 0. \tag{130}$$

The components of the Lorentz force equation for the choice of rainbow function given in eq.(2) are

$$\hat{F}_0 = F_0 - \frac{\beta E_o}{2E_P}e\left(x_0\partial^0 F_{0i}\dot{x}^i\right) + \hat{G}_0(\hat{x}); \ \hat{F}_i = F_i - \frac{\beta E_o}{2E_P}e\left(F_{i0}\dot{x}^0 + x_0\partial^0 F_{i\alpha}\dot{x}^\alpha\right) + \hat{G}_i(\hat{x}). \tag{131}$$

The corresponding expressions in Minkowski space-time are obtained in the limit $\beta \to 0$, as expected.

For choice 3:

$$\hat{F}^a_{\mu\nu} = F^a_{\mu\nu} + \frac{\eta E_o}{2E_P}(x_m\partial^m F^a_{\mu\nu}), \tag{132}$$

We have the Lagrangian is

$$\hat{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{\eta E_o}{4E_P}F^a_{\mu\nu}(x_m\partial^m F^{a\mu\nu}). \tag{133}$$

The corresponding Yang-Mills equations are

$$\epsilon^{\alpha\mu\nu\rho}\left(1 + \frac{\eta E_o}{2E_P}x_m\partial^m\right)(D_\mu F_{\nu\rho})^a = 0; \ \left(1 + \frac{\eta E_o}{2E_P}x_m\partial^m\right)(D_\mu F^{\alpha\mu})^a = 0. \tag{134}$$

The components of the Lorentz force equation for this choice of rainbow function are

$$\hat{F}_\mu = F_\mu + \frac{\eta E_o}{2E_P}e\left(F_{\mu j}\dot{x}^j + x_m\partial^m F_{\mu\alpha}\dot{x}^\alpha\right) + \hat{G}_\mu(\hat{x}). \tag{135}$$

As the rainbow parameter η goes to zero, all the above expressions go to the corresponding expressions in Minkowski space-time.

5. Aharonov-Bohm Effect in Rainbow Space-Time

In this section, we investigate the Aharonov-Bohm (A-B) phase for a particle moving in an energy-dependent space-time, the RST. We show that the time-dependent A-B phase is, in general, non-zero in RST, unlike in Minkowski space-time, where it is zero identically. This gives a possible, experimentally measurable signal to validate the RST proposal. Here, we first present a brief description of the Aharonov-Bohm phase in Minkowski space-time and then generalise this to RST. In [27], it was shown that the wave function of a quantum mechanical particle interacting with a magnetic potential would pick up an extra phase factor. This phase factor depends only on the vector potential and not on the magnetic field. Thus, the wave function of a charged particle moving through a region where the magnetic field $\vec{B}(= \vec{\nabla} \times \vec{A})$ is absent but the vector potential $\vec{A}$ is present, picks up this extra phase. For a charged particle interacting with a gauge field, the wave function is given by,

$$\psi = e^{\frac{ie}{\hbar}\oint A_\mu dx^\mu}\psi_0 \tag{136}$$

where ψ_0 is the wave function when the interaction is absent. i.e, comparing to the case when the interaction is absent, the wave function has an extra phase factor.

Since a long solenoid carrying a steady current generates a magnetic field only in its inner region, and no magnetic field is present outside the solenoid, while the vector potential $\vec{A}$ is non-zero even outside provides the possibility to verify this prediction of extra phase due to the interaction of a charged particle with vector potential, experimentally. This effect was experimentally confirmed in [30]. Here, the vector potential was time-independent. A time-dependent potential generates an electric field ($\vec{E} = -\partial_t \vec{A}$), which leads to the wave function picking up another phase factor. The sign of this phase factor is such that it cancels the phase factor generated by the magnetic flux present in the region, thus leading to the absence of net A-B phase [30].

In a region where $A_0 = 0$, the A-B phase in the presence of an Abelian field is given by

$$\oint A_i dx^i = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \int \vec{B} \cdot d\vec{S} \tag{137}$$

For studying the time-dependent A-B effect, one uses the covariant generalisation of A-B phase, i.e., $\oint A_\mu dx^\mu = \frac{1}{2} \int F_{\mu\nu} dx^\mu \wedge dx^\nu$. The cancellation of the A-B phase for the time-dependent vector potential has been experimentally verified in [32].

We first consider the Abelian case and construct the general expression for the time-dependent A-B phase in RST by replacing the quantities in Minkowski space-time with the corresponding ones in RST. Here, the explicit form of the components of the field strength tensor, which are obtained in the previous section, is used. Then, by considering various time-dependent potentials, we explicitly calculate the corresponding A-B phases. We repeat this for the non-Abelian case also. For each of these potentials, we consider three choices of rainbow functions and find the corresponding A-B phase.

5.1. Abelian Aharonov-Bohm phase in RST

The expression for covariant A-B phase in Minkowski space-time is given by [27,29]

$$\alpha_{AB} = \frac{e}{\hbar} \oint dA = \frac{e}{2\hbar} \int F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{138}$$

where $dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$. Here $F_{\mu\nu}$ is the anti-symmetric field strength, dx^μ and dx^ν are the differential four vectors, and $\wedge$ is the wedge product, which is anti-symmetric. $dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ in terms of the components is

$$\begin{aligned} dA &= F_{10} dx \wedge dt + F_{20} dy \wedge dt + F_{30} dz \wedge dt + F_{12} dx \wedge dy + F_{23} dy \wedge dz + F_{31} dz \wedge dx \\ &= (E_x dx + E_y dy + E_z dz) \wedge dt + (B_z dx \wedge dy + B_x dy \wedge dz + B_y dz \wedge dx), \end{aligned} \tag{139}$$

where $E_i, i = 1, 2, 3$ are the components of electric field and B_i are the components of magnetic field. Note that our metric is $\eta_{\mu\nu} = (+1, -1, -1, -1)$.

Using eq.(139) in eq.(138), we see that the A-B phase is identically zero when $\vec{E} = -\partial_t \vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$, ($dx \wedge dy = dS\hat{k}$, $dy \wedge dz = dS\hat{\imath}$, $dz \wedge dx = dS\hat{\jmath}$) i.e.,

$$\begin{aligned} \alpha_{AB} &= \frac{e}{\hbar} \Big(\int (\vec{E} \cdot d\vec{l}) dt + \int \vec{B} \cdot d\vec{S} \Big) \\ &= \frac{e}{\hbar} \Big(- \int (\partial_t \vec{A} \cdot d\vec{l}) dt + \int \vec{B} \cdot d\vec{S} \Big) \\ &= \frac{e}{\hbar} \Big(- \int \vec{A} \cdot d\vec{l} + \int \vec{B} \cdot d\vec{S} \Big) = \frac{e}{\hbar} \Big(- \int \vec{B} \cdot d\vec{S} + \int \vec{B} \cdot d\vec{S} \Big) = 0. \end{aligned} \tag{140}$$

In this section, we derive the expression for the A-B phase of a charged particle moving in an energy-dependent space-time, i.e., in the RST. As mentioned earlier, the RST coordinates are given in eq.(11). Note that the RST metric $\hat{\eta}_{\mu\nu}$ is same as the Minkowski space-time metric $\eta_{\mu\nu}$, which together with the anti-commutation relation of Dirac matrices $\hat{\gamma}^\mu$ given by

$$[\hat{\gamma}^\mu, \hat{\gamma}^\nu]_+ = 2\hat{\eta}^{\mu\nu} = 2\eta^{\mu\nu} \tag{141}$$

implies that the Dirac matrices are not modified in RST. Thus,

$$(i\gamma^{\mu}\hat{\partial}_{\mu} - e\gamma^{\mu}\hat{A}_{\mu} - m)\hat{\psi} = 0, \tag{142}$$

is the Dirac equation of a spin $\frac{1}{2}$ particle in RST, where γ^{μ} is the Dirac matrices, which are not modified. Note that here, $\hat{\partial}_{\mu} = (\hat{\partial}_0, \hat{\partial}_i) = (f\partial_0, g\partial_i)$ are the partial derivatives and $\hat{A}_{\mu} = \hat{A}_{\mu}(\hat{x})$ is the gauge potential in RST. The Schrödinger equation in RST is given by

$$i\hbar\frac{\partial\hat{\psi}}{\partial\hat{t}} = \left(\frac{1}{2m}(-i\hbar\hat{\partial}_i - e\hat{A}_i)^2 + e\hat{A}_0\right)\hat{\psi} = \hat{E}\hat{\psi}. \tag{143}$$

The solution of the above equation is $\hat{\psi} = e^{\frac{ie}{\hbar}\oint \hat{A}_{\mu}d\hat{x}^{\mu}}\hat{\psi}_0$, where $\hat{\psi}_0$ is the wave function, when the interaction is absent. Note that here $d\hat{x}^{\mu} = (\frac{dx^0}{f(E_o)}, \frac{dx^i}{g(E_o)})$. This implies that, when the charged, spin $\frac{1}{2}$ particle is interacting with a gauge field in RST, it picks up a phase, and this A-B phase is given by

$$\hat{\alpha}_{AB} = \oint \hat{A}_{\mu}d\hat{x}^{\mu} = \frac{e}{2\hbar}\int \hat{F}_{\mu\nu}d\hat{x}^{\mu}\wedge d\hat{x}^{\nu} = \frac{e}{\hbar}\left(\int \hat{F}_{i0}d\hat{x}^i\wedge d\hat{x}^0 + \int \frac{1}{2}\hat{F}_{ij}d\hat{x}^i\wedge d\hat{x}^j\right) \tag{144}$$

where $\hat{F}_{\mu\nu}$ is the Maxwell field strength in RST and $d\hat{x}^{\mu} = (\frac{dx^0}{f(E_o)}, \frac{dx^i}{g(E_o)})$.

Using Maxwell's field strength in RST obtained (see eq.(20) and eq.(25)), we now evaluate $\hat{\alpha}_{AB}$, valid up to first order in the rainbow parameters. Note that $F_{i0} = E_i$ and $F_{ij} = \epsilon_{ijk}B_k$. This $\hat{F}_{\mu\nu}$ is written in terms of $\hat{A}_{\mu}(\hat{x})$ and the rainbow functions ($f(\frac{E_o}{E_P})$ and $g(\frac{E_o}{E_P})$). By using the Taylor expansion of $\hat{A}_{\mu}(\hat{x})$ for different choices of the rainbow functions, we derive the expression for the field strength tensor $\hat{F}_{\mu\nu}$ in RST. This $\hat{F}_{\mu\nu}$ is written in terms of the field strength $F_{\mu\nu}$ in Minkowski space-time and the rainbow parameters. The choice of rainbow functions and the corresponding expression for the field strength tensor in RST is given below,

$$f(E_o) = g(E_o) = 1 + \frac{\lambda E_o}{E_P} + O(\lambda^2); \quad \hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{\lambda E_o}{E_P}x_{\alpha}\partial^{\alpha}F_{\mu\nu} + O(\lambda^2) \tag{145}$$

$$f(E_o) = 1 + \frac{\beta E_o}{2E_P} + O(\beta^2), \quad g(E_o) = 1; \qquad \hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{\beta E_o}{2E_P}x_0\partial^0 F_{\mu\nu} + O(\beta^2) \tag{146}$$

$$f(E_o) = 1, \ g(E_o) = 1 - \frac{\eta E_o}{2E_P} + O(\eta^2); \quad \hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{\eta E_o}{2E_P}x_m\partial^m F_{\mu\nu} + O(\eta^2). \tag{147}$$

Note that in the above expressions λ, β, and η are the rainbow parameters, and throughout this study, we are considering terms only up to first order in the rainbow parameters. From the above expressions, it is clear that the general form of Maxwell's field strength tensor in RST is of the form,

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + kx_{\alpha}\partial^{\alpha}F_{\mu\nu} \tag{148}$$

where k and α take values $-\frac{\lambda E_o}{E_P}$, $(0,1,2,3)$, $-\frac{\beta E_o}{2E_P}$, (0) and $\frac{\eta E_o}{2E_P}$, $(1,2,3)$ for the choices given in eq.(145), eq.(146) and eq.(147), respectively.

Using eq.(148) in eq.(144), we now evaluate the A-B phase for the above choices of rainbow functions, keeping terms up to first order in the rainbow parameters. For $A_0 = 0$, $\vec{E} = -\partial_t\vec{A}$ and $\vec{B} = \vec{\nabla}\times\vec{A} \neq 0$, we find

$$\begin{aligned} \frac{1}{2}\hat{F}_{\mu\nu}d\hat{x}^{\mu}\wedge d\hat{x}^{\nu} &= (F_{i0} + kx_{\alpha}\partial^{\alpha}F_{i0})\frac{dx^i\wedge dx^0}{fg} + \frac{1}{2}(F_{ij} + kx_{\alpha}\partial^{\alpha}F_{ij})\frac{dx^i\wedge dx^j}{g^2} \\ &= F_{i0}\frac{dx^i\wedge dx^0}{fg} + F_{ij}\frac{dx^i\wedge dx^j}{2g^2} + k\Big(x_{\alpha}\partial^{\alpha}F_{i0}\frac{dx^i\wedge dx^0}{fg} \\ &+ x_{\alpha}\partial^{\alpha}F_{ij}\frac{dx^i\wedge dx^j}{2g^2}\Big) \end{aligned} \tag{149}$$

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As earlier, we use $F_{i0} = E_i$, $F_{ij} = \epsilon_{ijk} B_k$, $\epsilon_{ijk} \frac{dx^i \wedge dx^j}{2} = dS\hat{k}$ and obtain the corresponding A-B phase as

$$\hat{\alpha}_{AB} = \frac{e}{\hbar}\Bigg(\int \frac{\vec{E}\cdot\vec{dl}dt}{fg} + \int \frac{\vec{B}\cdot\vec{dS}}{g^2} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot\frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot\frac{\vec{dS}}{g^2}\Big)\Bigg) \tag{150}$$

which gives

$$\begin{aligned}
\hat{\alpha}_{AB} &= \frac{e}{\hbar}\Bigg(\int -\frac{\partial_t\vec{A}\cdot\vec{dl}dt}{fg} + \int \frac{\vec{B}\cdot\vec{dS}}{g^2} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot\frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot\frac{\vec{dS}}{g^2}\Big)\Bigg) \\
&= \frac{e}{\hbar}\Bigg(-\int \frac{\vec{\nabla}\times\vec{A}\cdot\vec{dS}}{fg} + \int \frac{\vec{B}\cdot\vec{dS}}{g^2} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot\frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot\frac{\vec{dS}}{g^2}\Big)\Bigg) \\
&= \frac{e}{\hbar}\Bigg(\frac{1}{g}\Big(\frac{1}{g}-\frac{1}{f}\Big)\int \vec{B}\cdot\vec{dS} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot\frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot\frac{\vec{dS}}{g^2}\Big)\Bigg)
\end{aligned} \tag{151}$$

This is the general expression for the A-B phase due to the interaction with a time-dependent Abelian potential in RST when $A_0 = 0$. We now explicitly calculate the above A-B phase in RST for the time-dependent potential

$$A_0 = 0, \quad \vec{A} = \Big(\frac{\mathcal{K}I_0 cos(\omega_0 t)}{r} + \frac{\alpha I r}{2}\Big)\hat{\phi}. \tag{152}$$

Here I_0 and I are steady currents, $\mathcal{K}$ and α are constants. Note that here $\omega_0 = \frac{2\pi}{T}$, T is the time period. The corresponding electric field is

$$\vec{E} = -\partial_t\vec{A} = \frac{\mathcal{K}\omega_0 I_0 sin(\omega_0 t)}{r}\hat{\phi} \tag{153}$$

and the magnetic field is

$$\vec{B} = \alpha I\hat{z}. \tag{154}$$

Using the electric field and magnetic field given in eq.(153) and eq.(154) in eq.(151), we get

for $f(E_o) = g(E_o) = 1 + \frac{\lambda E_o}{E_P}$

$$\hat{\alpha}_{AB} = 0, \tag{155}$$

for $f(E_o) = 1 + \frac{\beta E_o}{2E_P}$, $\quad g(E_o) = 1$

$$\hat{\alpha}_{AB} = \frac{e}{\hbar}\frac{\beta E_o}{2E_P}\int \vec{B}\cdot\vec{dS} = \frac{e}{\hbar}\frac{\beta E_o}{2E_P}\Phi, \tag{156}$$

and for $f(E_o) = 1$, $\quad g(E_o) = 1 - \frac{\eta E_o}{2E_P}$

$$\hat{\alpha}_{AB} = \frac{e}{\hbar}\frac{\eta E_o}{2E_P}\int \vec{B}\cdot\vec{dS} = \frac{e}{\hbar}\frac{\eta E_o}{2E_P}\Phi. \tag{157}$$

In the above, $\Phi = \int \vec{B}\cdot\vec{dS}$ is the magnetic flux.

Note that in obtaining the above results, we have used the fact that the k-dependent terms in eq.(151) go to zero as $\vec{B}$ is constant (see eq.(154)) and the integration in the first k-dependent term of eq.(151) is carried over the period T. It is clear that for the choice of potential given in eq.(152), the A-B phase is zero for the first choice of rainbow function (see eq.(155)), whereas for the other two choices of rainbow functions, we get a non-zero value for the A-B phase (see eq.(156) and eq.(157)), both of which are modified due to RST. Here in the limit where the rainbow parameters go to zero, the A-B phases become zero, as expected.

Next consider

$$A_0 = A_3 = 0, \;\; A_1 = \mathcal{K}(t - \frac{y}{2}), \;\; A_2 = \mathcal{K}(t + \frac{x}{2}) \tag{158}$$

Here we get a non-zero electric field and magnetic field, $\vec{E} = -\mathcal{K}(\hat{\imath} + \hat{\jmath})$ and $\vec{B} = \mathcal{K}\hat{k}$. Using this in eq.(151), we find

$$\hat{\alpha}_{AB} = \frac{e}{\hbar}\frac{1}{g}(\frac{1}{g} - \frac{1}{f})\int \vec{B}\cdot d\vec{S} \tag{159}$$

which is zero for the choice where $f(E_o) = g(E_o) = 1 + \frac{\lambda E_o}{E_P}$, whereas for the other two choices, we get $\frac{e}{\hbar}\frac{\beta E_o}{2E_P}\Phi$ and $\frac{e}{\hbar}\frac{\eta E_o}{2E_P}\Phi$, respectively. Here, the A-B phases have rainbow parameter dependence, and it is non-zero for two choices of rainbow functions.

For the case when the time-dependent Abelian potential is given by [33]

$$\begin{aligned} A^\mu &= (0, 0, \frac{B_0 tr}{2}\hat{\phi}, 0), r < R \\ &= (0, 0, \frac{B_0 t R^2}{2r}\hat{\phi}, 0), r \geq R, \end{aligned} \tag{160}$$

the electric field in the outside region ($r \geq R$) is $\vec{E} = -\frac{B_0 R^2}{2r}\hat{\phi}$ and magnetic field is $\vec{B} = 0$. For finding the corresponding A-B phase, we use (see eq.(150))

$$\hat{\alpha}_{AB} = \frac{e}{\hbar}\Bigg(\int \frac{\vec{E}\cdot d\vec{l}dt}{fg} + \int \frac{\vec{B}\cdot d\vec{S}}{g^2} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot\frac{d\vec{l}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot\frac{d\vec{S}}{g^2}\Big)\Bigg).$$

For the potential given in eq.(160), as $\vec{B} = 0$, the second and fourth terms of the above equation go to zero. For the first choice of rainbow function, the A-B phase is

$$\hat{\alpha}_{AB} = \frac{e}{\hbar}\Big((1 - \frac{2\lambda E_o}{E_P})\int -\frac{B_0 R^2}{2r}\hat{\phi}\cdot d\vec{l}dt - \frac{\lambda E_o}{E_P}\int \frac{B_0 R^2}{2r}\hat{\phi}\cdot d\vec{l}dt\Big) = -\frac{e}{\hbar}(1 - \frac{\lambda E_o}{E_P})B_0 R^2 \pi t \tag{161}$$

where we have used $k = -\frac{\lambda E_o}{E_P}$, $x_\alpha \partial^\alpha \vec{E} = \frac{B_0 R^2}{2r}\hat{\phi}$. While for the second choice, (as $x_0 \partial^0 \vec{E} = 0$), the only non-zero term of eq.(150) is the first term and

$$\hat{\alpha}_{AB} = \frac{e}{\hbar}\Big((1 - \frac{\beta E_o}{2E_P})\int -\frac{B_0 R^2}{2r}\hat{\phi}\cdot d\vec{l}dt = -\frac{e}{\hbar}(1 - \frac{\beta E_o}{2E_P})B_0 R^2 \pi t. \tag{162}$$

For the third choice of rainbow function (see eq.(3)), we find

$$\hat{\alpha}_{AB} = \frac{e}{\hbar}\Big((1 + \frac{\eta E_o}{2E_P})\int -\frac{B_0 R^2}{2r}\hat{\phi}\cdot d\vec{l}dt + \frac{\eta E_o}{2E_P}\int \frac{B_0 R^2}{2r}\hat{\phi}\cdot d\vec{l}dt\Big) = -\frac{e}{\hbar}B_0 R^2 \pi t. \tag{163}$$

Note that, in the above equations (eq.(161), eq.(162) and eq.(163)), the dt integration is over the range 0 to a finite t because, normally, the potential is generated in the solenoid by applying current for a finite time. Note that here the A-B phase is non-zero for all three cases, but for the third choice of rainbow functions, the A-B phase is the same as that in the Minkowski space-time, without any modification coming from RST. This is due to the fact that the contribution to the A-B phase due to RST coming from the electric field and magnetic field (first and third terms of eq.(150)) exactly cancel each other.

5.2. Non-Abelian Aharonov-Bohm phase in RST

In this section, we consider the interaction of a charged particle with the non-Abelian gauge field and analyse the A-B phase acquired by its wave function. Note that in the presence of a non-Abelian field, the wave function of the particle is $\hat{\psi} = e^{\frac{ie}{\hbar}\oint \hat{A}^a_\mu I^a d\hat{x}^\mu}\hat{\psi}_0$[34,35]. The expression for the Yang-Mills field strength in Minkowski space-time is given by

$$F_{\mu\nu} = F^a_{\mu\nu} I^a = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e f^{abc} A^b_\mu A^c_\nu) I^a \tag{164}$$

where the magnetic field is

$$\vec{B} = \vec{B}^a I^a = \left((\vec{\nabla} \times \vec{A})^a - \frac{e}{2} f^{abc} A^b \times A^c \right) I^a. \tag{165}$$

Assuming $A^a_0 = 0$ and $\vec{E}^a = -\partial_t \vec{A}^a$, the A-B phase in non-Abelian case becomes [36],

$$\oint A_\mu dx^\mu = \int (\frac{1}{2} F_{\mu\nu} + e f^{abc} A^b_\mu A^c_\nu I^a) dx^\mu \wedge dx^\nu \tag{166}$$

Using $\int \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = -\int \partial_t \vec{A} \cdot \vec{dl} dt + \int \vec{B} \cdot \vec{dS}$ and substituting eq.(165) for the second term gives, $\int \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = -\frac{e}{2} f^{abc} \int (A^b \times A^c) I^a \cdot \vec{dS}$. On substituting this, the RHS of eq.(166) reduces to zero. That is, in the Minkowski space-time,

$$\oint A_\mu dx^\mu = 0, \tag{167}$$

which implies that the A-B phase due to a time-dependent, non-Abelian gauge field vanishes in the Minkowski space-time.

Now we consider the particle moving in the energy-dependent space-time, the RST. The Aharonov-Bohm phase for a non-Abelian field in RST is thus given by

$$\oint \hat{A}_\mu d\hat{x}^\mu = \int \frac{1}{2} \hat{F}_{\mu\nu} d\hat{x}^\mu \wedge d\hat{x}^\nu + e f^{abc} \int \hat{A}^b_\mu \hat{A}^c_\nu I^a d\hat{x}^\mu \wedge d\hat{x}^\nu, \tag{168}$$

where $\hat{F}_{\mu\nu} = \hat{F}^a_{\mu\nu} I^a = \left(\hat{\partial}_\mu \hat{A}^a_\nu - \hat{\partial}_\nu \hat{A}^a_\mu - e f^{abc} \hat{A}^b_\mu \hat{A}^c_\nu \right) I^a$ is the Yang-Mills field strength in RST and $d\hat{x}^\mu = (\frac{dx^0}{f(E_o)}, \frac{dx^i}{g(E_o)})$. As in the Abelian case, here also we consider different choices for the rainbow functions, given in eq.(1), eq.(2), and eq.(3). Using minimal coupling prescription and by considering the anti-symmetric property of the Poisson bracket between velocities, the expression for the components of the Yang-Mills field strength in RST is obtained in eq.(82) and eq.(86). Using the Taylor series expansion of field $\hat{A}^a_\mu(\hat{x})$ for different choices of rainbow functions ($f(\frac{E_o}{E_P})$ and $g(\frac{E_o}{E_P})$), we write the Yang-Mills field strength ($\hat{F}^a_{\mu\nu}$) in terms of the non-Abelian field strength $F^a_{\mu\nu}$ in the Minkowski space-time and the rainbow parameters. For the choices of rainbow functions given in eq.(1), eq.(2), and eq.(3), the non-Abelian field strength valid up to first order in the rainbow parameter is given, respectively,

$$\hat{F}_{\mu\nu} = \hat{F}^a_{\mu\nu} I^a = (F^a_{\mu\nu} - \frac{\lambda E_o}{E_P} x_\alpha \partial^\alpha F^a_{\mu\nu}) I^a, \tag{169}$$

$$\hat{F}_{\mu\nu} = \hat{F}^a_{\mu\nu} I^a = (F^a_{\mu\nu} - \frac{\beta E_o}{2E_P} x_0 \partial^0 F^a_{\mu\nu}) I^a, \tag{170}$$

and

$$\hat{F}_{\mu\nu} = \hat{F}^a_{\mu\nu} I^a = (F^a_{\mu\nu} + \frac{\eta E_o}{2E_P} x_m \partial^m F^a_{\mu\nu}) I^a. \tag{171}$$

Here λ, β, and η are the rainbow parameters, and we are considering terms valid only up to the first order in these rainbow parameters. From the above equations, the general form of the non-Abelian strength tensor in RST up to the first order in the rainbow parameters can be written as,

$$\hat{F}_{\mu\nu} = \hat{F}^a_{\mu\nu} I^a = (F^a_{\mu\nu} + k x_\alpha \partial^\alpha F^a_{\mu\nu}) I^a, \tag{172}$$

where k takes values $-\frac{\lambda E_o}{E_P}$, $-\frac{\beta E_o}{2E_P}$ and $\frac{\eta E_o}{2E_P}$ and α takes values $(0,1,2,3)$, (0) and $(1,2,3)$ for the choices given in eq.(169), eq.(170) and eq.(171), respectively.

Let us take the case where $A_0^a = 0$, $\vec{E}^a = -\partial_t \vec{A}^a$ and $\vec{B}^a \neq 0$. Then,

$$\begin{aligned}
\frac{1}{2}\hat{F}_{\mu\nu} d\hat{x}^\mu \wedge d\hat{x}^\nu &= \hat{F}_{i0} d\hat{x}^i \wedge d\hat{x}^0 + \frac{1}{2}\hat{F}_{ij} d\hat{x}^i \wedge d\hat{x}^j \\
&= (F_{i0} + kx_\alpha \partial^\alpha F_{i0})\frac{dx^i \wedge dx^0}{fg} + \frac{1}{2}(F_{ij} + kx_\alpha \partial^\alpha F_{ij})\frac{dx^i \wedge dx^j}{g^2} \\
&= F_{i0}\frac{dx^i \wedge dx^0}{fg} + F_{ij}\frac{dx^i \wedge dx^j}{2g^2} + k\Big(x_\alpha \partial^\alpha F_{i0}\frac{dx^i \wedge dx^0}{fg} \\
&+ x_\alpha \partial^\alpha F_{ij}\frac{dx^i \wedge dx^j}{2g^2}\Big)
\end{aligned} \tag{173}$$

and thus, with $F_{i0} = E_i$, $F_{ij} = \epsilon_{ijk} B_k$, $\epsilon_{ijk}\frac{dx^i \wedge dx^j}{2} = dS\hat{k}$, we find

$$\begin{aligned}
\int \frac{1}{2}\hat{F}_{\mu\nu} d\hat{x}^\mu \wedge d\hat{x}^\nu &= \int \frac{\vec{E}\cdot \vec{dl}dt}{fg} + \int \frac{\vec{B}\cdot \vec{dS}}{g^2} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot \frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot \frac{\vec{dS}}{g^2}\Big) \\
&= \int -\frac{\partial_t \vec{A}\cdot \vec{dl}dt}{fg} + \int \frac{\vec{B}\cdot \vec{dS}}{g^2} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot \frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot \frac{\vec{dS}}{g^2}\Big) \\
&= -\int \frac{\vec{A}\cdot \vec{dl}}{fg} + \int \frac{\vec{B}\cdot \vec{dS}}{g^2} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot \frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot \frac{\vec{dS}}{g^2}\Big).
\end{aligned} \tag{174}$$

Using eq.(165) in the second term of the RHS, we rewrite the above equation as,

$$\begin{aligned}
\int \frac{1}{2}\hat{F}_{\mu\nu} d\hat{x}^\mu \wedge d\hat{x}^\nu = &- \frac{1}{fg}\int \vec{A}\cdot \vec{dl} + \frac{1}{g^2}\Big(\int \vec{A}\cdot \vec{dl} - \frac{e}{2} f^{abc}\int A^b \times A^c I^a \cdot \vec{dS}\Big) \\
&+ k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot \frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot \frac{\vec{dS}}{g^2}\Big) \\
&= \frac{1}{g}\Big(\frac{1}{g} - \frac{1}{f}\Big)\int \vec{A}\cdot \vec{dl} - \frac{e}{2g^2} f^{abc}\int A^b \times A^c I^a \cdot \vec{dS} \\
&+ k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot \frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot \frac{\vec{dS}}{g^2}\Big).
\end{aligned} \tag{175}$$

Using the above expression, we get,

$$\begin{aligned}
\oint \hat{A}_\mu d\hat{x}^\mu &= \int \frac{1}{2}\hat{F}_{\mu\nu} d\hat{x}^\mu \wedge d\hat{x}^\nu + e f^{abc}\int \hat{A}_\mu^b \hat{A}_\nu^c I^a d\hat{x}^\mu \wedge d\hat{x}^\nu \\
&= \frac{1}{g}\Big(\frac{1}{g} - \frac{1}{f}\Big)\int \vec{A}\cdot \vec{dl} - \frac{e}{2g^2} f^{abc}\int A^b \times A^c I^a \cdot \vec{dS} \\
&+ k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot \frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot \frac{\vec{dS}}{g^2}\Big) + \frac{e}{g^2} f^{abc}\int \hat{A}_j^b \hat{A}_k^c I^a dx^j \wedge dx^k
\end{aligned} \tag{176}$$

where in writing the last term, we have used $A_0^a = 0$ and eq.(11). Using the Taylor expansion of $\hat{A}_i(\hat{x})$ in RST, i.e, $\hat{A}_i = A_i + kx_\alpha \partial^\alpha A_i$, where A_i is the vector potential in Minkowski space-time, $k = -\frac{\lambda E_o}{E_P}$, $\alpha = 0, 1, 2, 3$; $k = -\frac{\beta E_o}{2E_P}$, $\alpha = 0$ and $k = \frac{\eta E_o}{2E_P}$, $\alpha = 1, 2, 3$, respectively for the three choices of rainbow functions, we get,

$$\begin{aligned}
\frac{e}{g^2} f^{abc}\hat{A}_j^b \hat{A}_k^c I^a dx^j \wedge dx^k &= \frac{e}{g^2} f^{abc}(A_j^b + kx_\alpha \partial^\alpha A_j^b)(A_k^c + kx_\alpha \partial^\alpha A_k^c) I^a dx^j \wedge dx^k \\
&= \frac{e}{g^2} f^{abc}\Big(A_j^b A_k^c + kx_\alpha \partial^\alpha (A_j^b A_k^c)\Big) I^a dx^j \wedge dx^k + O(k^2)
\end{aligned} \tag{177}$$

Substituting this in eq.(176) and keeping terms up to first order in k gives,

$$\oint \hat{A}_\mu d\hat{x}^\mu = \frac{1}{g}\Big(\frac{1}{g} - \frac{1}{f}\Big)\int \vec{A}\cdot \vec{dl} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot \frac{\vec{dl}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot \frac{\vec{dS}}{g^2} + \frac{e f^{abc}}{2g^2}\int x_\alpha \partial^\alpha (A^b \times A^c) I^a \cdot \vec{dS}\Big) \tag{178}$$

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This is the general expression for the non-Abelian A-B phase factor in RST, valid up to first order in the rainbow parameter. Now we use this result for different choices of the time-dependent potentials and find the corresponding time-dependent, non-Abelian A-B phase.

For example, consider the time-dependent non-Abelian potential of the form

$$A_0^a = 0; \;\; \vec{A}^a = \Big(\frac{\mathcal{K}^a I_0 cos(\omega_0 t)}{r} + \frac{\alpha^a I r}{2}\Big)\hat{\phi}, \tag{179}$$

where $\omega_0 = \frac{2\pi}{T}$, T is the time period, I_0 and I are constant currents. The corresponding electric and magnetic fields are

$$\vec{E}^a = -\partial_t \vec{A}^a = \frac{\mathcal{K}^a \omega_0 I_0 sin(\omega_0 t)}{r}\hat{\phi}; \quad \vec{B}^a = \alpha^a I \hat{z}. \tag{180}$$

Using this in eq.(178), the corresponding A-B phase is obtained as,

$$\begin{aligned}\oint \hat{A}_\mu d\hat{x}^\mu &= \frac{1}{g}\Big(\frac{1}{g} - \frac{1}{f}\Big)\int \vec{B}\cdot d\vec{S} + k\Big(\int x_\alpha \partial^\alpha \vec{E}\cdot \frac{d\vec{l}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot \frac{d\vec{S}}{g^2}\Big)\\ &= \frac{1}{g}\Big(\frac{1}{g} - \frac{1}{f}\Big)\int \vec{B}\cdot d\vec{S}\end{aligned} \tag{181}$$

Here, as the potential given in eq.(179) has only one non-zero spatial component, the last term of eq.(178) goes to zero and also $(\vec{\nabla}\times\vec{A})^a = \vec{B}^a$. Here, the first k-dependent term of eq.(181) vanishes because the integration range is over the period T and the second k-dependent term becomes zero as $\vec{B}$ is constant. Now we consider different choices for the rainbow functions given in eq.(1), eq.(2), and eq.(3), and the corresponding A-B phases obtained from eq.(181) explicitly as

$$\oint \hat{A}_\mu d\hat{x}^\mu = 0, \tag{182}$$

$$\oint \hat{A}_\mu d\hat{x}^\mu = \frac{\beta E_o}{2E_P}\Phi, \tag{183}$$

$$\oint \hat{A}_\mu d\hat{x}^\mu = \frac{\eta E_o}{2E_P}\Phi, \tag{184}$$

where $\int \vec{B}\cdot d\vec{S} = \Phi$ is the magnetic flux. Thus, it is clear that for the non-Abelian potential given in eq.(179), the time-dependent A-B phase is non-zero for two choices of rainbow functions.

Consider another potential,

$$A_0^a = A_3^a = 0, \;\; A_1^a = K^a\big(t - \frac{y}{2}\big), \;\; A_2^a = K^a\big(t + \frac{x}{2}\big) \tag{185}$$

Here, $\vec{E} = -K^a(\hat{i}+\hat{j})$ and $\vec{B} = K^a\hat{k}$, but note that $x_\alpha\partial^\alpha\vec{E} = 0 = x_\alpha\partial^\alpha\vec{B}$, and the anti-symmetric property of f^{abc} makes the last term of eq.(178) zero. Thus, only the first term in the RHS of eq.(178) is non-zero. Therefore, we see that the A-B phase is zero for the first choice of rainbow function, while it is $\frac{\beta E_o}{2E_P}\Phi$ and $\frac{\eta E_o}{2E_P}\Phi$ for the other two choices of rainbow functions, respectively.

Next, consider a time-dependent potential that produces a non-zero electric field but zero magnetic field. i.e., for the potential [33],

$$A^{\mu a} = \Big(0, 0, \frac{B_0^a t r}{2}\hat{\phi}, 0\Big), r < R, \tag{186}$$

$$= \Big(0, 0, \frac{B_0^a t R^2}{2r}\hat{\phi}, 0\Big), r \geq R, \tag{187}$$

for the region $r \geq R$, the electric field is $\vec{E}^a = -\frac{B_0^a R^2}{2r}\hat{\phi}$ and magnetic field is $\vec{B}^a = 0$. For finding the corresponding A-B phase, consider,

$$\oint \hat{A}_\mu d\hat{x}^\mu = \int \frac{1}{2}\hat{F}_{\mu\nu} d\hat{x}^\mu \wedge d\hat{x}^\nu + ef^{abc}\int \hat{A}^b_\mu \hat{A}^c_\nu I^a d\hat{x}^\mu \wedge d\hat{x}^\nu \tag{188}$$

Note that the last term on the RHS is identically zero for the above choice of potential as there is only one non-zero component. Thus we find

$$\begin{aligned}\oint \hat{A}_\mu d\hat{x}^\mu &= \int \frac{1}{2}\hat{F}_{\mu\nu} d\hat{x}^\mu \wedge d\hat{x}^\nu = \int \frac{\vec{E}\cdot d\vec{l}dt}{fg} + \int \frac{\vec{B}\cdot d\vec{S}}{g^2} + k\Big(\int (x_\alpha \partial^\alpha \vec{E})\cdot \frac{d\vec{l}dt}{fg} + \int x_\alpha \partial^\alpha \vec{B}\cdot \frac{d\vec{S}}{g^2}\Big) \\ &= \int \frac{\vec{E}\cdot d\vec{l}dt}{fg} + k\int (x_\alpha \partial^\alpha \vec{E})\cdot \frac{d\vec{l}dt}{fg}.\end{aligned} \tag{189}$$

For the first choice of rainbow function, the A-B phase is

$$\begin{aligned}\oint \hat{A}_\mu d\hat{x}^\mu &= (1 - \frac{2\lambda E_o}{E_P})\int -\frac{B_0^a R^2}{2r}\hat{\phi}\cdot d\vec{l}dt - \frac{\lambda E_o}{E_P}\int \frac{B_0^a R^2}{2r}\hat{\phi}\cdot d\vec{l}dt \\ &= -(1 - \frac{\lambda E_o}{E_P}) B_0^a R^2 \pi t,\end{aligned} \tag{190}$$

where we have used $k = -\frac{\lambda E_o}{E_P}$, $x_\alpha \partial^\alpha \vec{E} = \frac{B_0^a R^2}{2r}\hat{\phi}$. For the second choice of the rainbow function,

$$\oint \hat{A}_\mu d\hat{x}^\mu = -(1 - \frac{\beta E_o}{2E_P}) B_0^a R^2 \pi t \tag{191}$$

and for the third choice,

$$\oint \hat{A}_\mu d\hat{x}^\mu = -B_0^a R^2 \pi t. \tag{192}$$

Here, the dt integration is over the range 0 to a finite t. It is interesting to see that for all three choices the A-B phase is non-zero, but there is no rainbow parameter dependent modification for the third choice (see eq.(192)), as the η dependent terms coming from the two terms of eq.(189) cancel each other.

6. Conclusion

We have constructed both Maxwell's theory and Yang-Mills theory in an energy-dependent space-time, the RST. Here, the space-time coordinates are modified by rainbow functions, which are functions parametrized by the ratio of the energy of the observer/probe (E_o) to Planck energy (E_P). Using the expression for the field-strength tensors derived for Maxwell's and Yang-Mills theory, we have studied the Aharonov-Bohm phase in RST, for both Abelian and non-Abelian fields.

Here, we have followed the minimal coupling prescription to construct the Abelian and non-Abelian field theories in RST. We start with a massive, charged particle interacting with an Abelian gauge field. Considering the Poisson bracket between coordinate and velocity in RST, we differentiate it with respect to the time parameter τ and obtain the Poisson bracket between velocities [2]. By considering the anti-symmetric property of the Poisson bracket between velocities, we introduce

[2] In [22], it has been pointed out that the Poisson bracket in eq.(12) breaks the reparametrisation invariance and hence τ cannot be interpreted as proper-time. In [26], the role of τ in Feynman's approach was investigated, and it was shown that τ has to be taken as an extra coordinate, thereby increasing the space-time dimension by one. The extra degrees of freedom introduced by this enlargement of the space-time dimension to $4+1$ is compensated by the field G_μ, $\mu = 0,1,2,3$ appearing in the Lorentz force equation (see for example, eq.(55) and eq.(61)). It was shown that the Poisson bracket in eq.(12) led to a Lagrangian describing the dynamics of the massive particle interacting with external gauge fields in $4+1$ dimensional space-time. The Euler-Lagrange equation following from this Lagrangian is exactly the same as the Lorentz force equation one derives using Feynman's approach in $3+1$ dimensional space-time [22]. The dynamics of these external fields, when restricted to $3+1$ dimensions, is described by the gauge field equations-Maxwell's/Yang-Mills equations-derived using Feynman's approach in [22]. Thus, even though the parameter τ used in the Feynman approach is not the proper-time, the gauge field equations derived describes Maxwell's/Yang-Mills theory in $3+1$ dimensions.

an anti-symmetric tensor in RST. We have then shown that this anti-symmetric tensor is the field strength tensor associated with Maxwell's electrodynamics. We have evaluated the Poisson brackets between velocities explicitly by re-expressing $m\dot{\hat{x}}_\mu = \hat{p}_\mu = \hat{\pi}_\mu - e\hat{A}_\mu$, where we have used the minimal coupling prescription to introduce the Abelian gauge field. We then equated this with the definition of a second-rank anti-symmetric tensor, mentioned earlier, and obtained the explicit form of the Abelian field strength tensor in RST. Using the definition of the field strength tensor, we have shown that it satisfies two of Maxwell's equations in RST. The remaining two equations are obtained using nested Poisson brackets. The Maxwell field strength tensor in RST is shown to be invariant under the $U(1)$ transformations in RST, explicitly. By demanding the Lagrangian to be $U(1)$ invariant and quadratic in derivatives, we constructed Maxwell's Lagrangian in RST. This Lagrangian is invariant under the gauge transformations $\hat{A}_0 \to \hat{A}_0 + \hat{\partial}_0\alpha$ and $\hat{A}_i \to \hat{A}_i + \hat{\partial}_i\alpha$, where $\hat{\partial}_0 = f\partial_0$ and $\hat{\partial}_i = g\partial_i$. The Euler-Lagrange equations that follow are exactly the same as the ones constructed by the minimal coupling variant of Feynman's approach. This shows that the gauge-invariant Lagrangian constructed is consistent with Maxwell's theory derived using Feynman's approach.

The Maxwell's equations in RST for the three different choices of rainbow functions valid up to first order in the rainbow parameters are obtained as

$$\epsilon^{\alpha\mu\nu\rho}\Big(1 - \frac{\lambda E_o}{E_P}x_\sigma\partial^\sigma\Big)\partial_\mu F_{\nu\rho} = 0; \quad \Big(1 - \frac{\lambda E_o}{E_P}x_\nu\partial^\nu\Big)\partial_\mu F^{\alpha\mu} = 0, \tag{193}$$

$$\epsilon^{\alpha\mu\nu\rho}\Big(1 - \frac{\beta E_o}{2E_P}x_0\partial^0\Big)\partial_\mu F_{\nu\rho} = 0; \quad \Big(1 - \frac{\beta E_o}{2E_P}x_0\partial^0\Big)\partial_\mu F^{\alpha\mu} = 0. \tag{194}$$

$$\epsilon^{\alpha\mu\nu\rho}\Big(1 + \frac{\eta E_o}{2E_P}x_m\partial^m\Big)\partial_\mu F_{\nu\rho} = 0; \quad \Big(1 + \frac{\eta E_o}{2E_P}x_m\partial^m\Big)\partial_\mu F^{\alpha\mu} = 0, \tag{195}$$

respectively. Note that all these equations reduce to the corresponding Maxwell's equations in Minkowski space-time when the rainbow parameters go to zero. From the form of the above equations, which are written in terms of Maxwell's equations in Minkowski space-time and the rainbow parameters, it is clear that the solutions to Maxwell's equations in Minkowski space-time will also be solutions in RST.

The components of the Lorentz force acting on a particle interacting with the Maxwell field in RST are derived. The components of field strength tensor (eq.(20) and eq.(25)), Maxwell's equations (eq.(28), eq.(32), eq.(47) and eq.(48)), Lagrangian (eq.(46)), and the Lorentz force equations (eq.(56) and eq.(61)) that have obtained are written in terms of rainbow functions $f(\frac{E_o}{E_P})$ and $g(\frac{E_o}{E_P})$ and $\hat{A}_\mu(\hat{x})$. Here, we have considered three different choices of rainbow functions $f(E_o)$ and $g(E_o)$, and using the Taylor expansion of $\hat{A}_\mu(\hat{x})$, we have expressed these quantities in terms of the corresponding quantities in Minkowski space-time and the rainbow parameters. Note that throughout our study, we have considered terms which are valid only up to first order in the rainbow parameters (λ, β, η). In the limit when these rainbow parameters go to zero, or equivalently when the rainbow functions (f, g) go to one, all the results in RST reduce to the corresponding ones in Minkowski space-time.

Following a similar procedure, we then constructed the Yang-Mills theory in RST using minimal coupling. Here we have followed the same steps as in the construction of Maxwell's equation in RST, but the kinetic momentum $(\hat{p}_\mu)$ is now replaced by $\hat{\pi}_\mu - e\hat{A}_\mu$, where $\hat{A}_\mu = \hat{A}^a_\mu I^a$ is the non-Abelian gauge potential with I^a representing the generators of $su(N)$ algebra. We have obtained the expressions for the Yang-Mills field strength $\hat{F}^a_{\mu\nu}$ (see eq.(82) and eq.(86)) and, using its explicit form, derived the Yang-Mills field equations (eq.(93), eq.(96) and eq.(103)) in RST. We have explicitly shown the covariance of field strength tensor $\hat{F}^a_{\mu\nu}$ under the gauge transformations $\hat{A}^a_0 \to \hat{A}^a_0 + (\hat{D}_0\alpha)^a$ and $\hat{A}^a_i \to \hat{A}^a_i + (\hat{D}_i\alpha)^a$, where $\hat{D}_\mu$ is the covariant derivative in RST. We have calculated the Poisson brackets between different components of the covariant derivative $\hat{D}_\mu$ in RST and showed that they are related to the components of the field strength tensor as

$$(\{\hat{D}_0, \hat{D}_i\}\phi)^a = -ef^{abc}\hat{F}^b_{0i}\phi^c; \quad (\{\hat{D}_i, \hat{D}_j\}\phi)^a = -ef^{abc}\hat{F}^b_{ij}\phi^c. \tag{196}$$

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Here F^a_{0i} and F^a_{ij} are exactly same as the ones we construct using the Feynman's approach in eq.(83) and eq.(87).

We have derived the Yang-Mills equations for three choices of rainbow functions valid up to first order in the rainbow parameters, given by

$$\epsilon^{\alpha\mu\nu\rho}\Big(1-\frac{\lambda E_o}{E_P}x_\sigma\partial^\sigma\Big)(D_\mu F_{\nu\rho})^a=0;\ \ \Big(1-\frac{\lambda E_o}{E_P}x_\nu\partial^\nu\Big)(D_\mu F^{\alpha\mu})^a=0, \tag{197}$$

$$\epsilon^{\alpha\mu\nu\rho}\Big(1-\frac{\beta E_o}{2E_P}x_0\partial^0\Big)(D_\mu F_{\nu\rho})^a=0;\ \ \Big(1-\frac{\beta E_o}{2E_P}x_0\partial^0\Big)(D_\mu F^{\alpha\mu})^a=0, \tag{198}$$

$$\epsilon^{\alpha\mu\nu\rho}\Big(1+\frac{\eta E_o}{2E_P}x_m\partial^m\Big)(D_\mu F_{\nu\rho})^a=0;\ \ \Big(1+\frac{\eta E_o}{2E_P}x_m\partial^m\Big)(D_\mu F^{\alpha\mu})^a=0, \tag{199}$$

respectively. As in Maxwell's theory, here also the solutions to Yang-Mills equations in the Minkowski space-time will be solutions in RST as well, as the above equations are written in terms of Yang-Mills equations in Minkowski space-time.

We then derived the components of the Lorentz force equation for an isospin-carrying particle moving in the presence of a Yang-Mills field in RST. Here we have generalised Wong's equation, the equation describing the time evaluation of $su(N)$ generators, to RST and obtained the Poisson bracket between velocities and Lie algebra valued functions in RST. This is used for defining all the quantities appearing in the Lorentz force equation in RST. We derived the non-Abelian field strength tensor, Yang-Mills equations, Lagrangian, and the components of Lorentz force in RST for different choices of rainbow functions. In the limit $f, g \to 1$, all these expressions, as expected, go to the well-known Minkowski space-time form.

Next, we have investigated the time-dependent A-B effect in RST for both Abelian and non-Abelian gauge fields. For this, we have used the explicit form of the Abelian/non-Abelian field strength tensors we have constructed in section 5. We generalise the expression for the A-B phase factor in Minkowski space-time and obtain the corresponding RST expression. Then we use the field strength tensor for each choice of rainbow functions, which we derived earlier, and obtain the A-B phase. We have explicitly calculated the A-B phase for different time-dependent gauge field configurations. For both Abelian and non-Abelian cases, we showed that the time-dependent A-B phase is, in general, non-zero. This non-vanishing A-B phase is a drastic deviation from the Minkowski space-time, where the A-B phase is identically zero. Thus, the time-dependent A-B phase provides a clear, experimentally measurable signal to validate the rainbow gravity proposal.

In the Minkowski space-time, for a charged particle interacting with an Abelian field, when $\vec{E}=-\partial_t\vec{A}$ and $\vec{B}=\vec{\nabla}\times\vec{A}$, where we have assumed $A_0=0$, it was shown that the time-dependent A-B phase due to electric and magnetic field cancels each other [29]. We have derived the A-B phase in the energy-dependent space-time and shown that the time-dependent A-B phase is non-zero. For the Abelian case, we have considered different choices of time-dependent potentials given in eq.(152), eq.(158), and eq.(160). In the case of the time-dependent potential given in eq.(152), the electric field ($\vec{E}=-\partial_t\vec{A}$) and magnetic field ($\vec{B}=\vec{\nabla}\times\vec{A}$) are non-zero, and we showed that for two choices of rainbow functions, we get a non-zero value for the A-B phase, which depends on the rainbow parameters. Note that here, taking the Minkowski limit, i.e., when the rainbow parameters (β and η) go to zero, we get the well-known Minkowski space-time result, where the A-B phase is zero. For the potential given in eq.(158), the electric field and magnetic field are non-zero, but constants. Here, we have shown that for two choices of rainbow functions, the A-B phase is non-zero and is modified due to the RST. For the third choice of Abelian potential (see eq.(160)), for which $\vec{E}=-\partial_t\vec{A}\neq 0$, but $\vec{B}=0$, we get a non-zero A-B phase for all three choices of rainbow functions. We have also shown that for this choice of Abelian potential, the A-B phase is getting modified for two choices of rainbow functions (see eq.(161) and eq.(162)), whereas it is non-zero but not modified [29] for the third choice of rainbow function (see eq.(163)), as the contribution to the A-B phase due to RST coming from first and third terms of eq.(150) cancels each other.

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We find the A-B phase for a time-independent Abelian potential of the form,

$$A_0 = A_3 = 0,\ \ A_1 = -\frac{By}{2},\ \ A_2 = \frac{Bx}{2}, \tag{200}$$

for which electric field is $\vec{E} = -\partial_t \vec{A} = 0$ and magnetic field is $\vec{B} = B\hat{k}$, to be

$$\hat{\alpha}_{AB} = \frac{e}{\hbar}\int \frac{\vec{B}\cdot d\vec{S}}{g^2} = \frac{e}{\hbar}\frac{\Phi}{g^2}. \tag{201}$$

This takes the value $\frac{e}{\hbar}(1-\frac{2\lambda E_o}{E_P})\Phi$, $\frac{e}{\hbar}\Phi$ and $\frac{e}{\hbar}(1+\frac{\eta E_o}{E_P})\Phi$, respectively for the three choices of rainbow functions. Note that, here for the second choice of rainbow function, the A-B phase is non-zero but not modified, whereas for the other two choices, the A-B phase has the well-known Minkowski part and a rainbow-dependent part.

We have also analysed the A-B phase for non-Abelian potentials and obtained the A-B phase for three different choices of rainbow functions. We choose $A_0^a = 0$, which gives $\vec{E}^a = -\partial_t \vec{A}^a$ and $\vec{B}^a = (\vec{\nabla}\times\vec{A})^a - \frac{e}{2}f^{abc}A^b \times A^c$, and showed that the A-B phase for a non-Abelian time-dependent potential is non-zero in the RST (see eq.(178)). For illustration, we considered the time-dependent potential given in eq.(179), where the electric field and magnetic field are non-zero. We have shown that for this potential, for two choices of rainbow functions, the A-B phase is non-zero. Note that here in the limit when rainbow parameters (β, η) go to zero, we get the Minkowski result, i.e., zero A-B phase. We have also studied the potential in eq.(185), where both the electric field and magnetic field are non-zero, and we get a non-zero value for the A-B phase for two choices of rainbow functions. For the third choice of potential, eq.(186), which produces zero magnetic field and $\vec{E}^a = -\partial_t \vec{A}^a \neq 0$, the A-B phase is non-zero for all three choices of rainbow functions, but for the third choice, the modification due to rainbow parameters is zero. Note that in all these cases, the Minkowski limit (where the rainbow parameters go to zero, equivalently $f, g \to 1$), we showed that the A-B phase matches with the corresponding results in Minkowski space-time.

The non-Abelian A-B effect in RST can also be studied for a time-independent potential of the form [37],

$$\begin{aligned}(A_0)^a &= (A_r)^a = (A_\theta)^a = 0, (A_\phi)^a = \mathcal{K}^a\frac{e(1-cos\theta)}{rsin\theta}; 0<r, 0\le\theta<\frac{\pi}{2}+\rho, 0\le\phi<2\pi \\ (A_0)^a &= (A_r)^a = (A_\theta)^a = 0, (A_\phi)^a = -\mathcal{K}^a\frac{e(1+cos\theta)}{rsin\theta}; 0<r, \frac{\pi}{2}-\rho<\theta\le\pi, 0\le\phi<2\pi\end{aligned} \tag{202}$$

Note that here for both the regions, $0\le\theta<\frac{\pi}{2}+\rho$ and $\frac{\pi}{2}-\rho<\theta\le\pi$, the electric field is zero and the magnetic field is given as, $\vec{B} = \frac{e}{r^2}\hat{r}$. The corresponding A-B phase is,

$$\begin{aligned}\oint \hat{A}_\mu d\hat{x}^\mu &= \int \frac{1}{2}\hat{F}_{\mu\nu}d\hat{x}^\mu \wedge d\hat{x}^\nu + ef^{abc}\int \hat{A}^b_\mu \hat{A}^c_\nu I^a d\hat{x}^\mu \wedge d\hat{x}^\nu \\ &= \int \frac{\vec{B}\cdot d\vec{S}}{g^2} + k\int x_\alpha \partial^\alpha \vec{B}\cdot\frac{d\vec{S}}{g^2}.\end{aligned} \tag{203}$$

Here we have used the fact that the potential has only one non-zero component. For the choice of rainbow function (eq.(1)) we get from the above equation

$$\oint \hat{A}_\mu d\hat{x}^\mu = (1-\frac{2\lambda E_o}{E_P})\int \frac{e}{r^2}\hat{r}\cdot d\vec{S} + \frac{\lambda E_o}{E_P}\int \frac{2e}{r^2}\hat{r}\cdot d\vec{S} = \Phi, \tag{204}$$

where we have used $\vec{B} = \frac{e}{r^2}\hat{r}$. For the other two choices of rainbow functions also, the time-independent A-B phase is obtained as $\oint \hat{A}_\mu d\hat{x}^\mu = \Phi$, where $\Phi = \int \vec{B}.d\vec{S} = \frac{4\pi e R^2}{r^2}$ is the magnetic flux. Note that, for all three choices of rainbow functions, we obtain a non-zero A-B phase; however, this A-B phase is not modified by RST.

Let us consider another time-independent potential.

$$A_0^a = A_3^a = 0, \;\; A_1^a = -\frac{K^a y}{2}, \;\; A_2^a = \frac{K^a x}{2} \tag{205}$$

Here the electric field is $\vec{E}^a = -\partial_t \vec{A}^a = 0$ and magnetic field is $\vec{B}^a = K^a \hat{k}$, which makes the general expression for A-B phase to be $\oint \hat{A}_\mu d\hat{x}^\mu = \int \frac{\vec{B}\cdot\vec{dS}}{g^2}$, where other terms are zero as we have used the facts that the magnetic field is constant and f^{abc} is anti-symmetric. Here the A-B phase is obtained as $(1 - \frac{2\lambda E_o}{E_P})\Phi$, Φ and $(1 + \frac{\eta E_o}{E_P})\Phi$ respectively, for the three choices of rainbow functions.

In this paper, we have derived Maxwell's equations and Yang-Mills field equations in RST. The expression for the field strength tensor in RST shows that the "electric" and "magnetic" fields get modified in RST and were written in terms of the corresponding Minkowski ones and the rainbow parameters. Using this, we have studied the A-B effect and showed that compared to Minkowski space-time, where the time-dependent A-B phase is zero, in the energy-dependent RST, the time-dependent A-B phase is non-zero. This is interesting as it paves the way for experimental verification of the RST proposal.

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