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Solving Yang-Baxter Matrix Equation via Extremal Ranks of Partial Banded Block Matrix

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Abstract: The equation $ABA = BAB$, where B is unknown matrix, is the original Yang-Baxter matrix equation for an arbitrary square matrix A. In this work, we establish the formulas of the extremal ranks of a 3×3 partial banded block matrix

$$\begin{bmatrix} M_{11} & M_{12} & X \\ M_{12}^* & M_{22} & M_{23} \\ X & M_{23}^* & M_{33} \end{bmatrix}$$

where X is a variant complex matrix subject to the linear matrix equation $AXA = T$, where A and T are Hermitian matrices. To clarify the practical aspects of the reached results, we provide a necessary and sufficient condition for the solvability of the Yang Baxter matrix equation $AXA = XAX = T$ over the complex field $\mathbb{C}$.

Keywords: Yang-Baxter matrix equation; Moore-Penrose inverse; extremal ranks; Hermitian; Inertia

MSC: 15A24; 15A09; 15A03; 15B57.

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1. Introduction

Over this work, all $t \times s$ complex matrices and all $s \times s$ Hermitian complex matrices are denoted, respectively, by $\mathbb{C}^{t\times s}$ and $\mathbb{C}_h^{s\times s}$, the symbols $r(N)$, N^* and I_s stand for the rank of N, conjugate transpose of N and the identity matrix of order s, respectively. We characterize the Moore Penrose inverse of a matrix, Penrose [18] defined the Moore-Penrose inverse of a matrix A through the following four conditions:

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

where X represents the Moore-Penrose inverse of A, denoted by A^+. A matrix X that satisfies all four conditions is unique. For more studies on the Moore-Penrose generalized inverse see ([2,3]). We denote $E_N = I_t - NN^+$, $F_N = I_s - N^+N$ the two orthogonal projectors induced by $N \in \mathbb{C}^{t\times s}$. The inertia of $N \in \mathbb{C}_h^{s\times s}$ is defined to be the triplet $In(N) = \{i_+(N), i_-(N), i_0(N)\}$. Where $i_+(N)$, $i_-(N)$ and $i_0(N)$ are the number of positive, negative and zero eigenvalues of N counted with their multiplicities.

This study aims to establish the well-known quadratic matrix problem

$$ABA = BAB \tag{1}$$

where A is a given square complex matrix, and B is unknown. This equation is frequently referred to as the Yang-Baxter-like matrix equation, given that it is related to the classical Yang-Baxter equation, initiated separately by Yang [25] and Baxter [1] in statistical physics. Regarding current developments to solve the original Yang-Baxter problem or the contemporary quantum Yang–Baxter equation, we refer to ([12–14]).

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The solvability of matrix equations is a significant topic in matrix theory. The Yang-Baxter matrix equation presents a considerable challenge owing to its nonlinear characteristics and the complexity involved in determining its solutions. Because it is equal to solving a general polynomial system of nonlinear equations since it applies to any matrix A, see ([4–11,16]). For instance, in [7] Ding and Rhee established a set of solutions to (1) via spectral projectors without any assumptions on the given matrix A, then in [8] Ding and Zhang continue to investigate spectral solutions of the equation (1). Utilizing projections of the generalized eigenspace onto its subspaces. Dinčić and Djordjević [5] illustrated that the two trivial solutions of (1) can be path-connected within the solution set if A is singular. Moreover, they showed that every nontrivial non-commuting solution is always contained in some path-connected subset of the solution set, then they presented a new approach for obtaining infinitely many new nontrivial non-commuting solutions.

Regarding the optimization theory, the equation (1) Has practical relevance in optimization problems involving matrix ranks and inertias. Recently, for an arbitrary square matrix A, Lifang et al.[15] solved the rank optimization problem $\min_{ABA=L} r(L - BAB)$, where the two matrices A and L are connected via the relation $L = ABA = BAB$, they derived rank formulas that could assist in identifying all the solutions of (1).

In matrix theory, the minimal and maximal rank of matrix pencils constitute a significant problem. Recently, Tian examined the extremal ranks of certain matrix pencils over the field of complex numbers and produced numerous rank formulas with practical applications, see ([20–23]). For instance, Tian [22] investigated the minimal rank of a 3×3 partial banded block matrix

$$\begin{bmatrix} D & D & * \\ D & * & D \\ * & D & D \end{bmatrix}$$

with three independent variant matrices. In the present paper, we calculate the extremal ranks of a kind of 3×3 Hermitian partial banded block matrix

$$M(X) = \begin{bmatrix} M_{11} & M_{12} & X \\ M_{12}^* & M_{22} & M_{23} \\ X & M_{23}^* & M_{33} \end{bmatrix} \tag{2}$$

where $M_{ii} \in \mathbb{C}_h^{n\times n}$ for $i = \overline{1,3}$, $M_{12}, M_{23} \in \mathbb{C}^{n\times n}$ are known, $X \in \mathbb{C}_h^{n\times n}$ is a variant matrix.

Motivated by the works above, In this work, we introduce the Hermitian matrix $T \in \mathbb{C}_h^{n\times n}$ via the relation

$$T = AXA = XAX \tag{3}$$

to establish new solvability conditions for the Yang Baxter matrix equation (3), where $A \in \mathbb{C}_h^{n\times n}$ is a given Hermitian matrix, and $X \in \mathbb{C}_h^{n\times n}$ is unknown. The main contributions of this paper are reflected in the following problems:

Problem 1. Let $M(X)$ be as given in (2), and assume that $AXA = T$ has a solution X. Provide specific algebraic formulas for finding the following extremal ranks

$$\max_{AXA=T} r(M(X)) \text{ and } \min_{AXA=T} r(M(X)) \tag{4}$$

Problem 2. Let A and T be as given in (3). Based on the obtained algebraic formulas in problem 1,
i) Provide sufficient and neceassary conditions for the existence of X such that the Yang-Baxter matrix equation (3) holds.
ii) Provide sufficient and necessary conditions of (3) to hold for all solutions X of $AXA = T$.

The structure of this paper is as follows. Section 2 presents fundamental formulas regarding the rank and inertia of matrices and matrix expressions. Section 3 examines the optimization problems in (4). Section 4, outlines the solvability conditions of (3). Section 5 summarizes the previous sections.

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2. Preliminaries

Regarding the importance of inertia and rank formulas in developing the content of the current paper. This section presents some essential results about the inertias and rank formulas of matrix expressions and block matrices.

Lemma 1. *[17] Let $H \in \mathbb{C}^{s\times r}$, $L \in \mathbb{C}^{s\times k}$, $C \in \mathbb{C}^{l\times r}$, $D \in \mathbb{C}^{l\times k}$. Then,*

$$r(H) = r\begin{bmatrix} H & L \end{bmatrix} - r(E_H L),\, r(L) = r\begin{bmatrix} H & L \end{bmatrix} - r(E_L H), \tag{5}$$

$$r(CF_H) = r\begin{bmatrix} H \\ C \end{bmatrix} - r(H),\, r(HF_C) = r\begin{bmatrix} H \\ C \end{bmatrix} - r(C) + r(HF_C), \tag{6}$$

$$r(C) + r(E_L H F_C) = r\begin{bmatrix} H & L \\ C & 0 \end{bmatrix} - r(L), \tag{7}$$

$$r\begin{bmatrix} H & L & 0 \\ C & 0 & R \\ 0 & N & 0 \end{bmatrix} = r\begin{bmatrix} H & LF_N \\ E_R C & 0 \end{bmatrix} + r(N) + r(R). \tag{8}$$

Lemma 2. *[19] Let $C \in \mathbb{C}_h^{t\times t}$, $M \in \mathbb{C}_h^{s\times s}$, $D \in \mathbb{C}^{t\times s}$. Then,*

$$i_\pm\begin{bmatrix} C & 0 \\ 0 & M \end{bmatrix} = i_\pm(C) + i_\pm(M),\ i_\pm\begin{bmatrix} 0 & D \\ D^* & 0 \end{bmatrix} = r(D).$$

Lemma 3. *[19] Let $A \in \mathbb{C}_h^{t\times t}$, $B \in \mathbb{C}^{t\times s}$. Then,*

$$i_\pm\begin{bmatrix} A & BF_P \\ F_P B^* & 0 \end{bmatrix} = i_\pm\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - r(P). \tag{9}$$

Lemma 4. *[18] Let $N \in \mathbb{C}^{l\times s}$, $L \in \mathbb{C}^{t\times q}$, and $D \in \mathbb{C}^{l\times q}$ are given, and $X \in \mathbb{C}^{s\times t}$ is an unknown matrix, the matrix equation $NXL = D$ is consistent if and only if*

$$NN^+D = D \text{ and } DL^+L = D$$

In that case the general solution is given by

$$X = N^+DL^+ + F_N V + UE_L. \tag{10}$$

where V, U are arbitrary.

Lemma 5. *[24] Let*

$$\Psi(X_1, X_2) = A - H_1X_1C_1 - H_2X_2C_2 - (H_1X_1C_1)^* - (H_2X_2C_2)^*$$

where $A \in \mathbb{C}_h^t$, $H_i \in \mathbb{C}^{t\times s_i}$, $C_i \in \mathbb{C}^{l_i\times t}$ *are given, and* $X_i \in \mathbb{C}^{s_i\times l_i}$, $i = 1,2$, *are variant matrices, and let* $R(H_2) \subseteq R(H_1)$, $R(C_1^*) \subseteq R(H_1)$, $R(C_2^*) \subseteq R(H_1)$. *Denote*

$$N = \begin{bmatrix} A & H_2 & C_1^* & C_2^* \\ C_1 & 0 & 0 & 0 \end{bmatrix}, N_1 = \begin{bmatrix} A & H_2 & C_1^* & C_2^* \\ H_2^* & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 \end{bmatrix}, N_2 = \begin{bmatrix} A & H_2 & C_1^* & C_2^* \\ C_1 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 \end{bmatrix}$$
$$\Gamma = \begin{bmatrix} A & H_1 \\ C_1 & 0 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} A & H_2 & C_1^* \\ H_2^* & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix}, \Gamma_2 = \begin{bmatrix} A & C_1^* & C_2^* \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix}$$

Then, the optimal ranks of $\Psi(X_1, X_2)$ *are given by*
(i)

$$\max_{X_i} r[\Psi(X_1, X_2)] = \min\left\{ r\begin{bmatrix} A & H_1 \end{bmatrix}, r(N), r(\Gamma_1), r(\Gamma_2) \right\} \tag{11}$$
$$\min_{X_i} r[\Psi(X_1, X_2)] = 2r\begin{bmatrix} A & H_1 \end{bmatrix} - 2r(\Gamma) + 2r(N) + \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \tag{12}$$

where

$$\varepsilon_1 = r(\Gamma_1) - 2r(N_1), \varepsilon_2 = r(\Gamma_2) - 2r(N_2),$$
$$\varepsilon_3 = i_+(\Gamma_1) + i_-(\Gamma_2) - r(N_1) - r(N_2),$$
$$\varepsilon_4 = i_-(\Gamma_1) + i_+(\Gamma_2) - r(N_1) - r(N_2).$$

We have the following result regarding the definitions of the rank of a matrix.

Lemma 6. *[19] Let* Υ *be a set consisting of matrices over* $\mathbb{C}^{k\times s}$*. Then,*
a) $0 \in \Upsilon$ *iff* $\min_{X\in\Upsilon} r(X) = 0$,
b) $\Upsilon = \{0\}$ *iff* $\max_{X\in\Upsilon} r(X) = 0$.

3. Extremal Ranks of Matrix Pencil (2) Subject to $AXA = T$

In this section, we calculate the maximal and minimal ranks of (2) subject to the consistent equation $AXA = T$. The following Theorem provides the main result of this paper.

Theorem 1. *The extremal ranks of the matrix pencil (2) subject to the consistent matrix equation* $AXA = T$ *are given by*
i)

$$\max_{AXA=T} r[M(X)] = \min\left\{ \begin{array}{c} n + r\begin{bmatrix} M_{12}^* & M_{22} & M_{23} \\ A^+TA^+ & M_{23}^* & M_{33} \end{bmatrix}, 3n + r\begin{bmatrix} M_{12}^* & M_{22} & M_{23}A \\ AM_{11} & AM_{12} & T \end{bmatrix} - 2r(A), \\ 2n + r\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}, 4n + r\begin{bmatrix} M_{22} & M_{23}A & M_{12}^*A \\ AM_{23}^* & AM_{33}A & T \\ AM_{12} & T & AM_{11}A \end{bmatrix} - 4r(A) \end{array} \right\} \tag{13}$$

ii)

$$\min_{AXA=T} r[M(X)] = 4n + 2r\begin{bmatrix} M_{12}^* & M_{22} & M_{23} \\ A^+TA^+ & M_{23}^* & M_{33} \end{bmatrix} - 2r\begin{bmatrix} M_{12}^* & M_{22} & M_{23}A \\ A^+TA^+ & M_{23}^* & M_{33}A \end{bmatrix} + 2r\begin{bmatrix} M_{12}^* & M_{22} & M_{23}A \\ AM_{11} & AM_{12} & T \end{bmatrix} - 2r(A) + \max\{\tau_1, \tau_2, \tau_3, \tau_4\} \tag{14}$$

such that

$$\tau_1 = r\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} - 2r\begin{bmatrix} M_{12}^* & M_{22} \\ AM_{11} & AM_{12} \end{bmatrix} + 2r(A) - 4n,$$

$$\tau_2 = r\begin{bmatrix} M_{22} & M_{23}A & M_{12}^*A \\ AM_{23}^* & AM_{33}A & T \\ AM_{12} & T & AM_{11}A \end{bmatrix} - 2r\begin{bmatrix} M_{22} & M_{23}A & M_{12}^*A \\ AM_{12} & T & AM_{11}A \end{bmatrix} + 2r(A) - 4n,$$

$$\tau_3 = i_+\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} + i_-\begin{bmatrix} M_{22} & M_{23}A & M_{12}^*A \\ AM_{23}^* & AM_{33}A & T \\ AM_{12} & T & 0 \end{bmatrix} - r\begin{bmatrix} M_{12}^* & M_{22} \\ AM_{11} & AM_{12} \end{bmatrix} - r\begin{bmatrix} M_{22} & M_{23}A & M_{12}^*A \\ AM_{12} & T & AM_{11}A \end{bmatrix} + 2r(A) - 4n,$$

$$\tau_4 = i_-\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} + i_+\begin{bmatrix} M_{22} & M_{23}A & M_{12}^*A \\ AM_{23}^* & AM_{33}A & T \\ AM_{12} & T & 0 \end{bmatrix} - r\begin{bmatrix} M_{12}^* & M_{22} \\ AM_{11} & AM_{12} \end{bmatrix} - r\begin{bmatrix} M_{22} & M_{23}A & M_{12}^*A \\ AM_{12} & T & AM_{11}A \end{bmatrix} + 2r(A) - 4n.$$

Proof. It follows from Lemma 4 that, Equation $AXA = T$ is consistent iff $AA^+T = T$ and $TA^+A = T$, in this case, its general Hermitian solution is given by

$$X = A^+TA^+ + F_AU + U^*E_A \tag{15}$$

where $U \in \mathbb{C}^{n\times n}$ is arbitrary.
Setting (15) into (2) yields

$$\begin{aligned} M(X) &= \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ + F_AU + U^*E_A \\ M_{12}^* & M_{22} & M_{23} \\ A^+TA^+ + F_AU + U^*E_A & M_{23}^* & M_{33} \end{bmatrix} \\ &= \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ \\ M_{12}^* & M_{22} & M_{23} \\ A^+TA^+ & M_{23}^* & M_{33} \end{bmatrix} + \left(\begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} U^* \begin{bmatrix} 0 & 0 & E_A \end{bmatrix}\right) + \left(\begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} U^* \begin{bmatrix} 0 & 0 & E_A \end{bmatrix}\right)^* \\ &+ \begin{bmatrix} 0 \\ 0 \\ I_n \end{bmatrix} U^* \begin{bmatrix} E_A & 0 & 0 \end{bmatrix} + \left(\begin{bmatrix} 0 \\ 0 \\ I_n \end{bmatrix} U^* \begin{bmatrix} E_A & 0 & 0 \end{bmatrix}\right)^* \\ &= G + J_1U^*\Gamma_1 + J_2U^*\Gamma_2 + (J_1U^*\Gamma_1)^* + (J_2U^*\Gamma_2)^*, \end{aligned} \tag{16}$$

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where

$$G = \begin{bmatrix} M_{11} & M_{12} & A^{+}TA^{+} \\ M_{12}^{*} & M_{22} & M_{23} \\ A^{+}TA^{+} & M_{23}^{*} & M_{33} \end{bmatrix},\ J_1 = \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix}$$
$$\Gamma_1 = \begin{bmatrix} 0 & 0 & E_A \end{bmatrix}, J_2 = \begin{bmatrix} 0 \\ 0 \\ I_n \end{bmatrix}, \Gamma_2 = \begin{bmatrix} E_A & 0 & 0 \end{bmatrix}. \tag{17}$$

It is clear that

$$R(J_2) \subseteq R(J_1), R(\Gamma_1^{*}) \subseteq R(J_1), R(\Gamma_2^{*}) \subseteq R(J_1)$$

Then, applying Lemma 5 to (16) yields

$$\max_{X_i} r[M(X)] = \min\left\{ r\begin{bmatrix} G & J_1 \end{bmatrix}, r\begin{bmatrix} G & J_2 & \Gamma_1^{*} & \Gamma_2^{*} \\ \Gamma_1 & 0 & 0 & 0 \end{bmatrix}, r\begin{bmatrix} G & J_2 & \Gamma_1^{*} \\ J_2^{*} & 0 & 0 \\ \Gamma_1 & 0 & 0 \end{bmatrix}, r\begin{bmatrix} G & \Gamma_1^{*} & \Gamma_2^{*} \\ \Gamma_1 & 0 & 0 \\ \Gamma_2 & 0 & 0 \end{bmatrix} \right\}$$
$$\min_{X_i} r[M(X)] = 2r\begin{bmatrix} G & J_1 \end{bmatrix} - 2r\begin{bmatrix} G & J_1 \\ \Gamma_1 & 0 \end{bmatrix} + 2r\begin{bmatrix} G & J_2 & \Gamma_1^{*} & \Gamma_2^{*} \\ \Gamma_1 & 0 & 0 & 0 \end{bmatrix} + \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$

Where

$$\alpha_1 = r\begin{bmatrix} G & J_2 & \Gamma_1^{*} \\ J_2^{*} & 0 & 0 \\ \Gamma_1 & 0 & 0 \end{bmatrix} - 2r\begin{bmatrix} G & J_2 & \Gamma_1^{*} & \Gamma_2^{*} \\ J_2^{*} & 0 & 0 & 0 \\ \Gamma_1 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_2 = r\begin{bmatrix} G & \Gamma_1^{*} & \Gamma_2^{*} \\ \Gamma_1 & 0 & 0 \\ \Gamma_2 & 0 & 0 \end{bmatrix} - 2r\begin{bmatrix} G & J_2 & \Gamma_1^{*} & \Gamma_2^{*} \\ \Gamma_1 & 0 & 0 & 0 \\ \Gamma_2 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_3 = i_{+}\begin{bmatrix} G & J_2 & \Gamma_1^{*} \\ J_2^{*} & 0 & 0 \\ \Gamma_1 & 0 & 0 \end{bmatrix} + i_{-}\begin{bmatrix} G & \Gamma_1^{*} & \Gamma_2^{*} \\ \Gamma_1 & 0 & 0 \\ \Gamma_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} G & J_2 & \Gamma_1^{*} & \Gamma_2^{*} \\ J_2^{*} & 0 & 0 & 0 \\ \Gamma_1 & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} G & J_2 & \Gamma_1^{*} & \Gamma_2^{*} \\ \Gamma_1 & 0 & 0 & 0 \\ \Gamma_2 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_4 = i_{-}\begin{bmatrix} G & J_2 & \Gamma_1^{*} \\ J_2^{*} & 0 & 0 \\ \Gamma_1 & 0 & 0 \end{bmatrix} + i_{+}\begin{bmatrix} G & \Gamma_1^{*} & \Gamma_2^{*} \\ \Gamma_1 & 0 & 0 \\ \Gamma_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} G & J_2 & \Gamma_1^{*} & \Gamma_2^{*} \\ J_2^{*} & 0 & 0 & 0 \\ \Gamma_1 & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} G & J_2 & \Gamma_1^{*} & \Gamma_2^{*} \\ \Gamma_1 & 0 & 0 & 0 \\ \Gamma_2 & 0 & 0 & 0 \end{bmatrix}.$$

From (17), we get

$$\max_X r[M(X)]$$
$$= \min\left\{ \begin{array}{l} r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & I_n \\ M_{12}^* & M_{22} & M_{23} & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & 0 \end{bmatrix}, r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 & 0 \end{bmatrix}, \\ r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & E_A & 0 & 0 \end{bmatrix}, r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 \\ E_A & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \right\} \tag{18}$$

$$\min_X r[M(X)] = 2r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & I_n \\ M_{12}^* & M_{22} & M_{23} & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & 0 \end{bmatrix} - 2r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & I_n \\ M_{12}^* & M_{22} & M_{23} & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & 0 \\ 0 & 0 & E_A & 0 \end{bmatrix} \tag{19}$$
$$+ 2r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 & 0 \end{bmatrix} + \max\{\beta_1, \beta_2, \beta_3, \beta_4\}$$

where

$$\beta_1 = r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & E_A & 0 & 0 \end{bmatrix} - 2r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & E_A & 0 & 0 & 0 \end{bmatrix}$$

$$\beta_2 = r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 \\ E_A & 0 & 0 & 0 & 0 \end{bmatrix} - 2r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 & 0 \\ E_A & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\beta_3 = i_+ \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & E_A & 0 & 0 \end{bmatrix} + i_- \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 \\ E_A & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$- r \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & E_A & 0 & 0 & 0 \end{bmatrix} - r \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 & 0 \\ E_A & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\beta_4 = i_- \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & E_A & 0 & 0 \end{bmatrix} + i_+ \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 \\ E_A & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$- r \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & E_A & 0 & 0 & 0 \end{bmatrix} - r \begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 & 0 \\ E_A & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying Lemmas 1, 2 and 3, and matrix rank transformations, then simplifying by $TA^+A = T$ and $AA^+T = T$, we get

$$\begin{aligned} r\begin{bmatrix} G & J_2 & L_1^* & L_2^* \\ L_1 & 0 & 0 & 0 \end{bmatrix} &= r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & 0 & 0 & F_A \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 & 0 \\ A^+TA^+ & M_{23}^* & M_{33} & I_n & F_A & 0 \\ 0 & 0 & E_A & 0 & 0 & 0 \end{bmatrix} \\ &= n + r\begin{bmatrix} M_{11} & M_{12} & A^+TA^+ & I & 0 \\ M_{12}^* & M_{22} & M_{23} & 0 & 0 \\ 0 & 0 & I & 0 & A \\ 0 & 0 & 0 & A & 0 \end{bmatrix} - 2r(A) \\ &= 3n + r\begin{bmatrix} M_{12}^* & M_{22} & M_{23}A \\ AM_{11} & AM_{12} & T \end{bmatrix} - 2r(A) \end{aligned}$$

Similarly, we can get the other rank equalities and substitution them into (18) and (19) we obtain (13) and (14). □

4. Solvability Conditions for the System (3)

In this part, we apply Theorem 1 to provide a practical solvability condition to the Yang-Baxter matrix equation (3). Moreover, we get the following result.

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Theorem 2. *Let A and T be as given in (3). Then the extremal ranks of $T - XAX$ subject to the consistent matrix equation $AXA = T$ is given by*

a)

$$\max_{AXA=T} r(T - XAX) = \min\left\{\begin{array}{c} n, n + r\begin{bmatrix} AT & T \end{bmatrix} - r(A), \\ r(T) + r(A), 2n + r\begin{bmatrix} A & T \\ T & ATA \end{bmatrix} - 3r(A) \end{array}\right\}. \tag{20}$$

b)

$$\min_{AXA=T} r(T - XAX) = \max\left\{\begin{array}{c} 2r\begin{bmatrix} AT & T \end{bmatrix} + r(T) - r(A) - 2r(AT), \\ 2r\begin{bmatrix} AT & T \end{bmatrix} + r\begin{bmatrix} A & T \\ T & ATA \end{bmatrix} - 2r\begin{bmatrix} T & ATA \end{bmatrix} - r(A), \\ 2r\begin{bmatrix} AT & T \end{bmatrix} - r\begin{bmatrix} T & ATA \end{bmatrix} + i_+(T) - i_+(A) - r(AT), \\ 2r\begin{bmatrix} AT & T \end{bmatrix} - r\begin{bmatrix} T & ATA \end{bmatrix} + i_-(T) - i_-(A) - r(AT) \end{array}\right\}. \tag{21}$$

Proof. a) Notice that

$$r(T - XAX) = r\begin{bmatrix} T & 0 & X \\ 0 & -A & I_n \\ X & I_n & 0 \end{bmatrix} - 2n \tag{22}$$

Hence,

$$\max_{AXA=T} r(T - XAX) = \max_{X} r\begin{bmatrix} T & 0 & X \\ 0 & -A & I_n \\ X & I_n & 0 \end{bmatrix} - 2n \tag{23}$$

It follows from Theorem 1, that

$$\max_{X} r\begin{bmatrix} T & 0 & X \\ 0 & -A & I_n \\ X & I_n & 0 \end{bmatrix} = \min\left\{\begin{array}{c} n + r\begin{bmatrix} 0 & -A & I_n \\ A^{+}TA^{+} & I & 0 \end{bmatrix}, 3n + r\begin{bmatrix} 0 & -A & A \\ AT & 0 & T \end{bmatrix} - 2r(A), \\ 2n + r\begin{bmatrix} T & 0 \\ 0 & -A \end{bmatrix}, 4n + r\begin{bmatrix} -A & A & 0 \\ A & 0 & T \\ 0 & T & ATA \end{bmatrix} - 4r(A) \end{array}\right\}$$

Using elementary block matrix operation, we can get

$$\max_{X} r\begin{bmatrix} T & 0 & X \\ 0 & -A & I_n \\ X & I_n & 0 \end{bmatrix} = \min\left\{\begin{array}{c} 3n, 3n + r\begin{bmatrix} AT & T \end{bmatrix} - r(A), \\ 2n + r(T) + r(A), 4n + r\begin{bmatrix} A & T \\ T & ATA \end{bmatrix} - 3r(A) \end{array}\right\} \tag{24}$$

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Then, combining (24) with (23) yields (20).
b) From (22) we have,

$$\min_{AXA=T} r(T-XAX) = \min_{X} r\begin{bmatrix} T & 0 & X \\ 0 & -A & I_n \\ X & I_n & 0 \end{bmatrix} - 2n \tag{25}$$

Further, it follows from Theorem 1 that

$$\begin{aligned}\min_{X} r\begin{bmatrix} T & 0 & X \\ 0 & -A & I_n \\ X & I_n & 0 \end{bmatrix} &= 4n + 2r\begin{bmatrix} 0 & -A & I_n \\ A^{+}TA^{+} & I & 0 \end{bmatrix} - 2r\begin{bmatrix} 0 & -A & A \\ A^{+}TA^{+} & I & 0 \end{bmatrix} \\ &+ 2r\begin{bmatrix} 0 & -A & A \\ AT & 0 & T \end{bmatrix} - 2r(A) + \max\{\delta_1, \delta_2, \delta_3, \delta_4\}\end{aligned}$$

where

$$\delta_1 = r\begin{bmatrix} T & 0 \\ 0 & -A \end{bmatrix} - 2r\begin{bmatrix} 0 & -A \\ AT & 0 \end{bmatrix} + 2r(A) - 4n$$

$$\delta_2 = r\begin{bmatrix} -A & A & 0 \\ A & 0 & T \\ 0 & T & ATA \end{bmatrix} - 2r\begin{bmatrix} -A & A & 0 \\ 0 & T & ATA \end{bmatrix} + 2r(A) - 4n$$

$$\delta_3 = i_{+}\begin{bmatrix} T & 0 \\ 0 & -A \end{bmatrix} + i_{-}\begin{bmatrix} -A & A & 0 \\ A & 0 & T \\ 0 & T & 0 \end{bmatrix} - r\begin{bmatrix} 0 & -A \\ AT & 0 \end{bmatrix} - r\begin{bmatrix} -A & A & 0 \\ 0 & T & ATA \end{bmatrix} + 2r(A) - 4n$$

$$\delta_4 = i_{-}\begin{bmatrix} T & 0 \\ 0 & -A \end{bmatrix} + i_{+}\begin{bmatrix} -A & A & 0 \\ A & 0 & T \\ 0 & T & 0 \end{bmatrix} - r\begin{bmatrix} 0 & -A \\ AT & 0 \end{bmatrix} - r\begin{bmatrix} -A & A & 0 \\ 0 & T & ATA \end{bmatrix} + 2r(A) - 4n$$

Using elamantary block matrix operations, we can get

$$\min_{X} r\begin{bmatrix} T & 0 & X \\ 0 & -A & I_n \\ X & I_n & 0 \end{bmatrix} = 6n - 2r(A) + 2r\begin{bmatrix} AT & T \end{bmatrix} + \max\{\delta_1, \delta_2, \delta_3, \delta_4\} \tag{26}$$

with

$$\begin{aligned}\delta_1 &= r[T] + r(A) - 2r[AT] - 4n, \\ \delta_2 &= r\begin{bmatrix} A & T \\ T & ATA \end{bmatrix} - 2r\begin{bmatrix} T & ATA \end{bmatrix} + r(A) - 4n, \\ \delta_3 &= i_{+}[T] + i_{-}(A) - r[AT] - r\begin{bmatrix} T & ATA \end{bmatrix} + r(A) - 4n, \\ \delta_4 &= i_{-}[T] + i_{+}(A) - r[AT] - r\begin{bmatrix} T & ATA \end{bmatrix} + r(A) - 4n.\end{aligned}$$

Further, combining (26) with (25), leads to (21). □

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Corollary 1. *Let A and T be given matrices, such that $AXA = T$ is consistent, Then*
a) The Yang-Baxter matrix equation $AXA = XAX = T$ is consistent if and only if the following equalities are all satisfied:

$$\begin{gathered} r\begin{bmatrix} AT & T \end{bmatrix} = r(AT), \quad r(T) = r(A), \\ r\begin{bmatrix} ATA & T \end{bmatrix} = r\begin{bmatrix} AT & T \end{bmatrix}, \\ r\begin{bmatrix} A & T \\ T & ATA \end{bmatrix} = r(A), \\ i_+(T) = i_+(A), \;\; i_-(T) = i_-(A). \end{gathered} \tag{27}$$

b) All the solutions X of the matrix equation $AXA = T$ are solutions of the equation $XAX = T$ if and only if

$$r(A) = n,$$

$$\text{or } A = 0 \text{ and } T = 0,$$

$$\text{or } r\begin{bmatrix} A & T \\ T & ATA \end{bmatrix} = 3r(A) - 2n.$$

Proof. It is obvious that Equation (3) is consistent if and only if $AXA = T$ is consistent and

$$\min_{AXA=T} r(T - XAX) = 0$$

According to (21), Equation (3) is consistent if and only if

$$2r\begin{bmatrix} AT & T \end{bmatrix} + r(T) - r(A) - 2r(AT) = 0$$

and

$$2r\begin{bmatrix} AT & T \end{bmatrix} + r\begin{bmatrix} A & T \\ T & ATA \end{bmatrix} - 2r\begin{bmatrix} T & ATA \end{bmatrix} - r(A) = 0,$$

and

$$2r\begin{bmatrix} AT & T \end{bmatrix} - r\begin{bmatrix} T & ATA \end{bmatrix} + i_+(T) - i_+(A) - r(AT) = 0,$$

and

$$2r\begin{bmatrix} AT & T \end{bmatrix} - r\begin{bmatrix} T & ATA \end{bmatrix} + i_-(T) - i_-(A) - r(AT) = 0.$$

Further, we can write these equalities respectively as

$$\left(2r\begin{bmatrix} AT & T \end{bmatrix} - 2r(AT)\right) + (r(T) - r(A)) = 0,$$

$$\left(2r\begin{bmatrix} AT & T \end{bmatrix} - 2r\begin{bmatrix} T & ATA \end{bmatrix}\right) + \left(r\begin{bmatrix} A & T \\ T & ATA \end{bmatrix} - r(A)\right) = 0,$$

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$$\left(2r\begin{bmatrix} AT & T \end{bmatrix} - r\begin{bmatrix} T & ATA \end{bmatrix} - r(AT)\right) + (i_+(T) - i_+(A)) = 0,$$

$$\left(2r\begin{bmatrix} AT & T \end{bmatrix} - r\begin{bmatrix} T & ATA \end{bmatrix} - r(AT)\right) + (i_-(T) - i_-(A)) = 0.$$

Since the rank of a matrix is a nonnegative integer, it implies that conditions in (27) all hold.
b) It is the immediate result of (20).
□

Example 1. *Let* $A = T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$*, with* $r(A) = r(T) = 1$

Further, we have

$$r = \begin{bmatrix} AT & T \end{bmatrix} = r\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = r(AT)$$

$$r = \begin{bmatrix} ATA & T \end{bmatrix} = r\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} AT & T \end{bmatrix}$$

$$r\begin{bmatrix} A & T \\ T & ATA \end{bmatrix} = r\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = r(A)$$

$i_+(T) = i_+(A) = 1,$ *and* $i_-(T) = i_-(A) = 0$

Then, according to conditions (27) in Corollary 1, the Yang Baxter matrix equation $AXA = XAX = T$ *is consistent and the following matrix*

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \tag{28}$$

is one of the solutions set.

5. Conclusion

In this paper, we derived the extremal ranks of a kind of 3×3 Hermitian partial banded block variant matrix (2) subject to the consistent matrix equation $AXA = T$. Moreover, To show the practical importance of the obtained results, Theorem 2 is utilized to establish the necessary and sufficient conditions for the solvability of the Yang-Baxter matrix equation. $AXA = XAX = T$.

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