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Article

On the Fractional Derivative Duality in Some Transforms

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Abstract: The duality is one of the most interesting properties of the Laplace and Fourier transforms associated to the integer order derivative. Here, we will generalize it for fractional derivatives and extend the results to the Mellin, Z and discrete-time Fourier transforms. The scale and nabla derivatives are used.

Keywords: Liouville derivative; scale derivative; Hadamard derivative; Laplace transform; Mellin transform; Z transform; Fourier transform

MSC: 44A05; 26A33

1. Introduction

One of the characteristic properties of the classic Fourier transform (FT) is the duality between direct and inverse, which allows the derivative of the transform to be expressed in such a way that the transform of the derivative takes on the same type of expression. To understand this idea, consider the Fourier transform of an absolutely integrable function, $f(t)$

$$F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t$$

and inverse

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}\mathrm{d}\omega$$

Let D represent the usual derivative and $n \in \mathbb{N}$. As $D_\omega^n e^{-i\omega t} = (-it)^n e^{-i\omega t}$, we have

$$F^{(n)}(i\omega) = \int_{-\infty}^{\infty} (-it)^n f(t)e^{-i\omega t}\mathrm{d}t \tag{1}$$

and, similarly, $D_t^n e^{i\omega t} = (-i\omega)^n e^{i\omega t}$,

$$f^{(n)}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^n F(i\omega)e^{i\omega t}\mathrm{d}\omega \tag{2}$$

These relations clearly express the duality property of FT and the associated usual derivative: deriving t (ω) in one domain corresponds to multiplying ω (t) in the other domain. This is the property that we want to generalize to fractional orders and to other transforms. We will consider: Laplace (LT), Mellin, Z, and discrete Fourier transforms. To do this, we introduce the necessary fractional derivatives: Liouville, Hadamard, and discrete, nabla and bilinear. In the Section 2 we introduce the necessary properties and the fractional derivatives to use. The integer order case of duality is dealt with in Section 3 and the fractional cases are studied in Section 4.

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2. Suitable derivatives

2.1. Derivative requirements

The most important transforms have exponentials or powers as kernels. Therefore, if we want to compute derivatives of the corresponding transforms and continue having a transform of the same type, we must have derivatives verifying (Liouville requirement)

$$D_z^\alpha e^{az} = a^\alpha e^{az} \tag{3}$$

or (Hadamard requirement)

$$\mathfrak{D}_z^\beta z^b = b^\beta z^b \tag{4}$$

for suitable orders $\alpha, \beta \in \mathbb{R}$ and complex variable $z \in \mathbb{C}$. For simplicity matter and later utility, we will assume that $a, b \in \mathbb{R}$ also.

2.2. Liouville type derivatives

The Liouville type derivatives can be expressed in a general unified way [1,2] assuming both summation and integral formulations.

Definition 1. *We will consider the Grünwald-Letnikov (GL) derivatives that we define by:*

$$D_\pm^\alpha f(z) := \lim_{h\to 0^+} (\pm h)^{-\alpha} \sum_{n=0}^{+\infty} \frac{(-\alpha)_n}{n!} f(z \mp nh). \tag{5}$$

where we denoted by $(a)_n$, $n = 1, 2, \cdots$ the Pochhammer symbol for the rising factorial

$$(a)_0 = 1, \ (a)_n = \prod_{k=0}^{n-1} (a+k).$$

To avoid confusion of symbols for different derivatives, we will write frequently $D_\pm^\alpha f(t) = f_\pm^{(\alpha)}(t)$. If necessary we will put the independent variable in the subscript.

Theorem 1. *Let $f(z) = e^{az}$. Then, [2,3]*

$$D_\pm^\alpha e^{az} = a^\alpha e^{az}, \tag{6}$$

if $\pm Re(a) > 0$, while both diverge with $\pm Re(a) < 0$, unless $\alpha \in \mathbb{Z}$.

For applications to the Laplace transform, we need to compute derivatives of e^{st}, $t \in \mathbb{R}, s \in \mathbb{C}$. We obtain

- derivative in t

$$D_{t\pm}^\alpha e^{st} = s^\alpha e^{st}, \quad \pm Re(s) > 0; \tag{7}$$

  It is useful for the derivative computation of the inverse LT.
- derivative in s

$$D_{s\pm}^\alpha e^{-st} = (-t)^\alpha e^{-st}, \quad \mp t > 0; \tag{8}$$

  This derivative allows us to express the derivative of the direct LT.

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2.3. Hadamard type derivatives

Definition 2. *The Hadamard type derivatives are scale-invariant and verify the above requirements too. Similarly to the Liouville's type, we define the stretching (+) and shrinking (-) GL-type derivatives by [4]*

$$\mathfrak{D}^{\alpha}_{v\pm}x(v) = \lim_{q\to 1^+} \ln^{-\alpha}(q^{\pm 1}) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} x(vq^{\mp n}), \tag{9}$$

where $q > 1$.

Theorem 2.

$$\mathfrak{D}^{\alpha}_{v\pm}v^b = b^{\alpha}v^b \tag{10}$$

provided that $\pm Re(v) > 0$. *If* $\pm Re(v) < 0$, *they diverge.*

Similarly to the LT case, we need to compute derivatives of $\tau^v, \tau \in \mathbb{R}^+, v \in \mathbb{C}$. We obtain

- derivative in τ
$$\mathfrak{D}^{\alpha}_{\tau\pm}\tau^v = v^{\alpha}\tau^v, \quad \pm Re(v) > 0; \tag{11}$$
It is useful for the derivative computation of the inverse MT.
- derivative in v
$$D^{\alpha}_{v\pm}\tau^{-v} = (-\ln\tau)^{\alpha}\tau^{-v}, \quad \tau^{\mp 1} > 1; \tag{12}$$
This derivative allows us to express the derivative of the direct MT.

2.4. Discrete-time derivatives

2.4.1. Fractional nabla and delta derivatives

In the following we consider that our domain is the *time scale* or *time sequence*

$$\mathbb{T}_h = (h\mathbb{Z}) = \{\ldots, -nh\ldots, -2h, -h, 0, h, 2h, \ldots, nh, \ldots\}$$

with $h \in \mathbb{R}^+$, that is called graininess or sampling interval [5,6].

Definition 3. *Let* $f(t)$ *be a function defined on* $\mathbb{T}$. *Set* $t = nh$. *We define the nabla derivative by:*

$$\nabla f(t) = \frac{f(t) - f(t-h)}{h} \tag{13}$$

and the delta derivative by

$$\Delta f(t) = \frac{f(t+h) - f(t)}{h}. \tag{14}$$

The corresponding fractional derivatives read [6]:
nabla

$$\nabla^{\alpha} f(t) = \frac{\sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(t-nh)}{h^{\alpha}} \tag{15}$$

and delta

$$\Delta^{\alpha} f(t) = \frac{\sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(t+nh)}{h^{\alpha}}, \tag{16}$$

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Theorem 3. *The eigenvalue of these derivatives is s^α and the corresponding eigenfunctions are the nabla and delta exponentials given by [6]*

$$e_\nabla(kh,s) = \frac{1}{(1-sh)^k} \tag{17}$$

and

$$e_\Delta(kh,s) = (1+sh)^k. \tag{18}$$

These results were generalized for irregular time sequences in [6,7]. In the following, we will continue with the nabla derivative.

Definition 4. *With the nabla exponential, we can define the nabla Laplace transform [6] through*

$$F_\nabla(s) = h \sum_{k=-\infty}^{+\infty} f(kh) e_\nabla(-kh,s). \tag{19}$$

with its inverse transform being given by

$$f(kh) = -\frac{1}{2\pi j}\oint_\gamma F_\nabla(s) e_\nabla((k+1)h,s)\, ds, \tag{20}$$

where the integration path, γ, is any simple closed contour in a region of analyticity of the integrand that includes the point $s = \frac{1}{h}$. The simplest path is a circle with centre at $s = \frac{1}{h}$.

Corollary 1. *Let $z^{-1} = 1 - sh$. Then*

$$\nabla^\alpha z^n = \left[\frac{1-z^{-1}}{h}\right]^\alpha z^n, \tag{21}$$

The proof is immediate.
With this change we entered in the framework of the Z transform.

2.4.2. Forward and backward derivatives based on the bilinear transformation

The Tustin transformation is usually expressed by [8,9]

$$s = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}, \tag{22}$$

where s is the derivative operator associated with the (continuous-time) LT and z^{-1} the delay operator tied with the Z transform.

Definition 5. *Let $x(nh)$ be a discrete-time function, we define the order 1 forward or nabla bilinear derivative $\nabla_b x(nh)$ of $x(nh)$ as the solution of the difference equation*

$$\nabla_b x(nh) + \nabla_b x(nh-h) = \frac{2}{h}\left[x(nh) - x(nh-h)\right]. \tag{23}$$

Similarly, we define the order 1 backward or delta bilinear derivative $\Delta_b x(nh)$ of $x(nh)$ as the solution of

$$\Delta_b x(nh+h) + \Delta_b x(nh) = \frac{2}{h}\left[x(nh+h) - x(nh)\right]. \tag{24}$$

Definition 6. *We define the nabla bilinear derivative (∇_b) as an elemental DT system such that*

$$\nabla_b z^n = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} z^n. \tag{25}$$

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The transfer function of such derivative, $H_b(z)$, is defined by

$$H_\nabla(z) = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}, \qquad |z| > 1. \tag{26}$$

For the backward bilinear derivative a transfer function is defined similarly.

Let $\alpha \in \mathbb{R}$. The $\alpha-$order nabla bilinear fractional derivative is a discrete-time linear system with transfer function

$$H_\nabla(z) = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha, \qquad |z| > 1, \tag{27}$$

such that

$$\nabla_b^\alpha z^n = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha z^n, \qquad |z| > 1. \tag{28}$$

With this formulation we entered again in the context of the Z transform.

3. Main transforms and integer order derivatives

3.1. Continuous-Time Laplace and Fourier Transforms

Definition 7. *The direct LT is given by [10]*

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,\mathrm{d}t,\ s \in \mathbb{C} \cap \mathcal{R}_c. \tag{29}$$

while the inverse LT (synthesis equation) reads

$$f(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,\mathrm{d}s,\ t \in \mathbb{R}, \tag{30}$$

where $a \in \mathbb{R}$ must be located inside the region of convergence, $\mathcal{R}_c$, (ROC) of $F(s)$. The right hand side represents the Bromwich integral. In the following we will denote by γ the integration path.

We can obtain existence conditions for the BLT, from those of the FT [11–13]. Let $f(t)$ be a function

- piecewise continuous,
- with bounded variation,
- locally integrable (in the sense that the function is absolutely integrable in any real interval $[a, b]$, so that $\int_a^b |f(t)|\,\mathrm{d}t < \infty$),
- of exponential order,

then there exists the BLT of $f(t)$.

Remark 1. *Loosely speaking, a function of exponential order is the one that does not "grow faster" than given exponentials, as $t \to \pm\infty$. This means two things. First, that there are real constants A, $a > 0$ such that $|f(t)| < A \cdot e^{at}$, when t is large and negative (say, for $t < t_1 \in \mathbb{R}$). Second, that there are real constants B, $b > 0$ such that $|f(t)| < B \cdot e^{bt}$, when t is large (say, for $t > t_2 \in \mathbb{R}$). It also has to be true that $b < a$. We are interested in dealing with functions for which $b < 0$ and $a > 0$ so that the function has Fourier transform.*

Under the stated conditions, the integral in (29) converges absolutely and uniformly in a vertical strip in the complex plane defined by $b < Re(s) < a$, where $F(s)$ is analytic. This strip is called *region of convergence* (ROC), and the values of the constants a and b are the abscissas of convergence. It can be shown that:

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1. If $f(t)$ is absolutely integrable and of finite duration, then the ROC is the entire s-plane, since the Laplace transform is finite and $F(s)$ exists for any s.
2. If $f(t)$ is right-sided (i.e., exists with $t \geq t_0 \in \mathbb{R}$) and $Re(s) = a \in \mathcal{R}_c$, then any s to the right of a is also in $\mathcal{R}_c$.
3. If $f(t)$ is left-sided (i.e., exists with $t \leq t_0 \in \mathbb{R}$) and $Re(s) = a \in \mathcal{R}_c$, then any s to the left of a is also in $\mathcal{R}_c$.
4. A function $f(t)$ is absolutely integrable (satisfying the Dirichlet conditions and having the Fourier transform) if and only if the ROC of the corresponding Laplace transform $F(s)$ includes the imaginary axis, since $Re(s) = 0$ and $s = i\omega$.
5. A given complex variable function only can define univocaly a LT if it has attached a suitable ROC.
6. If $F(s) = \mathcal{L}[f(t)]$, then $\mathcal{L}[f(-t)] = F(-s)$.
7. If the region of convergence of $F(s)$ includes the frontiers $b \leq Re(s) \leq a$, then $F(s)$ is completely defined in that region by the values at that lines, $F(a + i\tau)$ and $F(b + i\tau)$, $\tau \in \mathbb{R}$.
8. $F(s)$ is bounded in the strip $a + \epsilon \leq Re(s) \leq b - \epsilon$, with $\epsilon > 0$.

As the integer order GL derivative of an exponential exists for any value, provided that $s \neq 0$, we obtain easily

Theorem 4.

$$F^{(n)}(s) = \int_{-\infty}^{\infty} (-t)^n f(t) e^{-st} \, \mathrm{d}t, \; s \in \mathbb{C}. \tag{31}$$

and

$$f^{(n)}(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s) s^n e^{st} \, \mathrm{d}s, \; t \in \mathbb{R}, \tag{32}$$

From these relations we obtain the corresponding properties that we introduced above for the FT.

3.2. The Mellin transform

Definition 8. *Let us define the Mellin transform by*

$$G(v) = \int_{0}^{\infty} g(u) u^{-v-1} du, \tag{33}$$

which is modifed version of the usual Mellin transform (MT). This has a parameter sign change $-v \rightarrow v$ relatively to the current [14]. The idea is to keep the parallelism with the LT so that the properties related to the transform domain are essentially the same. In fact, it results from the change $e^t \rightarrow u$. The inverse Mellin transform related to (10) is

$$g(\tau) = \mathcal{M}^{-1}[X(v)] = \frac{1}{2\pi i} \int_{\gamma} G(v) \tau^v dv, \quad \tau \in \mathbb{R}^+,$$

where γ is vertical straight line in the ROC of the transform.

As the MT results from the LT through an exponential variable change, the convergence properties are easily deduced. In particular, the integral in (33) converges absolutely and uniformly in a vertical strip in the complex plane defined by $b < Re(v) < a$, where $G(v)$ is analytic.

Theorem 5. *The duality in the MT is expressed by*

$$\mathfrak{D}^n_{\pm} g(\tau) = \frac{1}{2\pi i} \int_{\gamma} (\pm v)^n G(v) \tau^v dv, \quad \tau \in \mathbb{R}^+, \tag{34}$$

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and

$$G^{(n)}(v) = \int_0^\infty (-\ln(u))^n g(u) u^{-v-1}\, du, \; s \in \mathbb{C}. \tag{35}$$

The proof is immediate using of the scale-derivative in the first and the classic derivative in the inverse and direct MT.

3.3. On the Z and Discrete-Time Fourier Transforms

Definition 9. *Let $x(n)$ denote any function defined on $\mathbb{T}$, leaving implicit the graininess, unless it is convenient to display it. The Z transform (ZT) is defined by*

$$X(z) = \mathcal{Z}\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(nh)\, z^{-n}, \quad z \in \mathbb{C}. \tag{36}$$

The inverse ZT can be obtained by the integral defined by

$$x(nh) = \frac{1}{2\pi i} \oint_\gamma X(z) z^{n-1} \mathrm{d}z, \tag{37}$$

where γ is a circle centred at the origin, located in the ROC of the transform, and taken in a counterclock wise direction.

In some scientific domains, as Geophysics, z instead of z^{-1} is used. In some areas, the ZT is often called "generating function" or "characteristic function". The existence conditions of the ZT are similar to those of the bilateral LT [8,9,15]. They can be stated as follows.

Definition 10. *A discrete-time signal $x(n)$ is called an* exponential order signal *if there exist integers n_1 and n_2, and positive real numbers a, b, A, and B, such that $A\, a^{n_1} < |x(n)| < B\, b^{n_2}$ for $n_1 < n < n_2$.*

For these signals the ZT exists and the ROC is an annulus centred at the origin, generally delimited by two circles of radius r_- and r_+, such that $r_- < |z| < r_+$. However, there are some cases where the annulus can become infinite:

- If the signal is right (i.e., $x(n) = 0,\ n < n_0 \in \mathbb{Z}$), then the ROC is the exterior of a circle centered at the origin ($r_+ = \infty$): $|z| > r_-$.
- If the signal is left (i.e., $x(n) = 0,\ n > n_0 \in \mathbb{Z}$), then the ROC is the interior of a circle centered at the origin ($r_- = 0$): $|z| < r_+$.
- If the signal is a pulse (i.e., non null only on a finite set), then the ROC is the whole complex plane, possibly with the exception of the origin. In the ROC, the ZT defines an analytical function.

If the ROC contains the unit circle, then by making $z = e^{i\omega},\ |\omega| < \pi,\ i = \sqrt{-1}$, we obtain the discrete-time Fourier transform, which we will shortly call Fourier transform (FT). This means that not all signals with ZT have FT. The signals with ZT and FT are those for which the ROC is non-degenerate and contains the unit circle ($r_- < 1,\ r_+ > 1$). For some signals, such as sinusoids, the ROC degenerates in the unit circumference ($r_- = r_+ = 1$), and there is no ZT.

In such situation the integral in (37) converges uniformly. The calculation uses the Cauchy's theorem of complex variable functions [15].

To treat the duality, we must note the importance of the unit circle that suggests the use of the scale-derivative $\mathfrak{D}_\pm$ according to the ROC: (+) for $|z| > 1$ and (-) for $|z| < 1$.

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Theorem 6. *Using the stretching and shrinking derivatives we obtain*

$$\mathfrak{D}^n_{z\pm}X(z) = \sum_{k=-\infty}^{\infty}(-k)^n x(kh)\, z^{-k}, \tag{38}$$

for $|z|^{\pm 1} > 1$. *Concerning the inverse ZT, we use the nabla derivative to get*

$$\nabla^n x(kh) = -\frac{1}{2\pi i}\oint_\gamma X(z)\left[\frac{1-z^{-1}}{h}\right]^n z^{k-1}\mathrm{d}z \tag{39}$$

For both, the proofs are immediate.

Definition 11. *For functions that have a ROC including the unit circle or for functions having a degenerate ROC, as it is the case of the periodic signals, it is preferable to work with the discrete-time Fourier transform that can be obtained from the Z transform through the transformation* $z = e^{i\omega h}$, $|\omega h| < \pi$

$$X(e^{i\omega}) = \sum_{k=-\infty}^{\infty} x(kh)e^{-i\omega hk} \tag{40}$$

with the inversion integral

$$x(k) = \frac{1}{2\pi h}\int_{-\pi/h}^{\pi/h} X(e^{i\omega})e^{i\omega hk}\mathrm{d}\omega \tag{41}$$

To obtain the duality, we must note that we made an exponential transformation to pass from (36) to (40). Therefore, the Liouville derivative must be used. We can state:

Theorem 7.

$$X^{(n)}(e^{i\omega}) = \sum_{k=-\infty}^{\infty}(-ikh)^n x(kh)e^{-i\omega hk} \tag{42}$$

and

$$\nabla^n x(kh) = \frac{1}{2\pi h}\int_{-\pi/h}^{\pi/h}\left[\frac{1-e^{-i\omega h}}{h}\right]^n X(e^{i\omega})e^{i\omega hk}\mathrm{d}\omega \tag{43}$$

Remark 2. *The change from the nabla derivative stated in (13) to the corresponding bilinear implies to change of the factor* $\left[\frac{1-z^{-1}}{h}\right]$ *by* $\left[\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right]$, *so that we obatain alternative derivative properties in (39) and (43).*

4. Main transforms and non integer order derivatives

4.1. Laplace transform

We reproduce here the results stated in theorem 4

$$F^{(n)}(s) = \int_{-\infty}^{\infty}(-t)^n f(t)e^{-st}\,\mathrm{d}t,\ s\in\mathbb{C}.$$

and

$$h^{(n)}(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}F(s)s^n e^{st}\,\mathrm{d}s,\ t\in\mathbb{R},$$

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As it is clear, the substitution $n \to \alpha$ creates problems, since the complex variable expression s^α is no longer a function, it involves setting a branchcut line. The simplest way is to choose the negative or positive real semi-axis. The results in section 2 allow us to write:

1. Right function case ($Re(s) > 0$)

$$F_-^{(\alpha)}(s) = \int_0^\infty (-t)^\alpha f(t) e^{-st}\, \mathrm{d}t, \tag{44}$$

and

$$f_+^{(\alpha)}(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s) s^\alpha e^{st}\, \mathrm{d}s,\ a > 0. \tag{45}$$

2. Left function case ($Re(s) < 0$)

$$F_+^{(\alpha)}(s) = \int_{-\infty}^0 (-t)^\alpha f(t) e^{-st}\, \mathrm{d}t, \tag{46}$$

and

$$f_-^{(\alpha)}(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s) s^\alpha e^{st}\, \mathrm{d}s,\ a < 0. \tag{47}$$

3. Two-sided function case ($|Re(s)| < b$)
The above relations suggest we introduce the two-sided fractional derivative [16]:

$$D_\theta^\alpha f(t) := \lim_{h \to 0^+} h^{-\alpha} \sum_{n=-\infty}^{+\infty} (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha+\theta}{2} - n + 1\right) \Gamma\left(\frac{\alpha-\theta}{2} + n + 1\right)} f(t - nh), \tag{48}$$

for which

$$D_\theta^\alpha e^{\pm i\omega t} = |\omega|^\alpha e^{\mp i\theta \frac{\pi}{2} \operatorname{sgn}(\omega)} e^{\pm i\omega t}.$$

Assuming that $|\theta| \neq |\alpha|$, then

$$D_\theta^\alpha F(i\omega) = \int_{-\infty}^\infty |t|^\alpha e^{-i\theta \frac{\pi}{2} \operatorname{sgn}(t)} f(t) e^{-i\omega t}\, \mathrm{d}t, \tag{49}$$

and

$$D_\theta^\alpha f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty |\omega|^\alpha e^{i\theta \frac{\pi}{2} \operatorname{sgn}(\omega)} F(i\omega) e^{i\omega t}\, \mathrm{d}\omega. \tag{50}$$

In particular, we obtain

$$D_0^\alpha F(i\omega) = \int_{-\infty}^\infty |t|^\alpha f(t) e^{-i\omega t}\, \mathrm{d}t, \tag{51}$$

and

$$D_0^\alpha f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty |\omega|^\alpha F(i\omega) e^{i\omega t}\, \mathrm{d}\omega. \tag{52}$$

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4.2. Mellin transform

The MT case is very similar to the LT case. The relations are easily obtained from those in the previous sub-section, having in mind the situations where the Liouville derivative is substituted by Hadamard's. The results in section 2 allow us to write:

1. Stretching case ($Re(v) > 0$)

$$G_{-}^{(\alpha)}(v) = \int_{1}^{\infty} (-\ln\tau)^{\alpha} g(\tau)\tau^{-v-1}\,\mathrm{d}\tau, \tag{53}$$

and

$$\mathfrak{D}_{\tau+}^{\alpha} g(\tau) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} v^{\alpha} G(v)\tau^{v}\,\mathrm{d}s,\ a > 0. \tag{54}$$

2. Schrinking case ($Re(v) < 0$)

$$G_{+}^{(\alpha)}(v) = \int_{0}^{1} (\ln\tau)^{\alpha} g(\tau)\tau^{-v-1}\,\mathrm{d}\tau, \tag{55}$$

and

$$\mathfrak{D}_{\tau-}^{\alpha} g(\tau) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} v^{\alpha} G(v)\tau^{v}\,\mathrm{d}v,\ a < 0. \tag{56}$$

3. Bilateral scale case ($|Re(v)| < b$)
The above relations suggest we introduce the two-sided scale derivative by:

$$\mathfrak{D}_{\tau,\theta}^{\alpha} g(\tau) = \lim_{q\to 1^{+}} \ln(q)^{-\alpha} \sum_{n=-\infty}^{+\infty} (-1)^{n} \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha+\theta}{2} - n + 1\right)\Gamma\left(\frac{\alpha-\theta}{2} + n + 1\right)} g(\tau q^{-n}), \tag{57}$$

for which

$$\mathfrak{D}_{\tau,\theta}^{\alpha} \tau^{i\omega} = |\omega|^{\alpha} e^{-i\theta\frac{\pi}{2}\operatorname{sgn}(\omega)} \tau^{i\omega}.$$

Assuming that $|\theta| \neq |\alpha|$, then

$$D_{\theta}^{\alpha} G(i\omega) = \int_{0}^{\infty} |\ln\tau|^{\alpha} e^{i\theta\frac{\pi}{2}\operatorname{sgn}(\ln\tau)} g(\tau)\tau^{-i\omega-1}\,\mathrm{d}\tau, \tag{58}$$

and

$$\mathfrak{D}_{\tau,\theta}^{\alpha} g(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^{\alpha} e^{i\theta\frac{\pi}{2}\operatorname{sgn}(\omega)} G(i\omega)\tau^{i\omega t}\,\mathrm{d}\omega. \tag{59}$$

In particular, we obtain

$$D_{0}^{\alpha} G(i\omega) = \int_{0}^{\infty} |\ln\tau|^{\alpha} g(\tau)\tau^{-i\omega-1}\,\mathrm{d}\tau, \tag{60}$$

and

$$\mathfrak{D}_{0}^{\alpha} g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^{\alpha} G(i\omega) e^{i\omega t}\,\mathrm{d}\omega. \tag{61}$$

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4.3. Z and Discrete-Time Fourier Transforms

Theorem 38 expresses the duality of the ZT for the integer order case. The situation here is similar to the one we found in the LT and MT cases, having to consider separately the three cases, corresponding to the exterior of the unit circle,$|z| > 1$, the unity disk, $|z| < 1$, and the annulus containing the unit circle.

1. Right sequence case ($|z| > 1$)

$$\mathfrak{D}^{\alpha}_{z+}X(z) = \sum_{k=0}^{\infty}(-k)^{\alpha}x(kh)\,z^{-k}, \tag{62}$$

and

$$\nabla^{\alpha}x(kh) = -\frac{1}{2\pi i}\oint_{\gamma} X(z)\left[\frac{1-z^{-1}}{h}\right]^{\alpha} z^{k-1}\mathrm{d}z \tag{63}$$

where γ is a closed path lying outside the unity circle.

2. Left sequence case ($|z| < 1$)

$$\mathfrak{D}^{\alpha}_{z-}X(z) = \sum_{k=-\infty}^{-1}(-k)^{\alpha}x(kh)\,z^{-k}, \tag{64}$$

and

$$\nabla^{\alpha}x(kh) = -\frac{1}{2\pi i}\oint_{\gamma} X(z)\left[\frac{1-z^{-1}}{h}\right]^{\alpha} z^{k-1}\mathrm{d}z \tag{65}$$

where γ is now inside the unity circle.

3. Two-sided function case ($r_- < |z| < r_+$)

Definition 12. *As above, attending to the relation [17]:*

$$(1-z)^{a}(1-z^{-1})^{b} = \sum_{n=-\infty}^{+\infty}(-1)^{n}\frac{\Gamma\left(a+b+1\right)}{\Gamma\left(a-n+1\right)\Gamma\left(b+n+1\right)}z^{-n}, \tag{66}$$

and considering again two real parameters α, the derivative order, and theta the asymmetry parameter such that $\alpha > -1$ if $\theta \neq \pm\alpha$, or $\alpha \in \mathbb{R}$ if $\theta = \pm\alpha$, we define a discrete-time two-sided derivative by

$$\Theta^{\alpha}_{\theta}x(kh) := h^{-\alpha}\sum_{n=-\infty}^{+\infty}(-1)^{n}\frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\frac{\alpha+\theta}{2}-n+1\right)\Gamma\left(\frac{\alpha-\theta}{2}+n+1\right)}x(kh-nh). \tag{67}$$

Theorem 8. *Let $x(kh) = e^{i\omega hk}$. Then, [18]*

$$\Theta^{\alpha}_{\theta}e^{i\omega hk} = \left|2\sin(\omega h/2)\right|^{\alpha} e^{i\theta\frac{\pi}{2}\,\mathrm{sgn}(\omega)}e^{i\omega hk} \tag{68}$$

The proof comes from left side in (66) by letting $z = e^{i\omega hk}$.
Assuming that $|\theta| \neq |\alpha|$, then

Theorem 9.

$$D^{\alpha}_{\theta}X(e^{i\omega}) = \sum_{k=-\infty}^{\infty}|kh|^{\alpha}e^{i\theta\frac{\pi}{2}\,\mathrm{sgn}(k)}x(kh)e^{-i\omega hk} \tag{69}$$

and

$$\Theta^{\alpha}_{\theta}x(kh) = \frac{1}{2\pi h}\int_{-\pi/h}^{\pi/h}\left|2\sin(\omega h/2)\right|^{\alpha} e^{i\theta\frac{\pi}{2}\,\mathrm{sgn}(\omega)}X(e^{i\omega})e^{i\omega hk}\mathrm{d}\omega \tag{70}$$

If h is very small, $2\sin(\omega h/2) \approx \omega h$ making (70) more similar to (69).

In particular, we obtain

$$D_0^\alpha X(e^{i\omega}) = \sum_{k=-\infty}^{\infty} |kh|^\alpha x(kh) e^{-i\omega hk} \tag{71}$$

and

$$\Theta_0^\alpha x(kh) = \frac{1}{2\pi h} \int_{-\pi/h}^{\pi/h} |2\sin(\omega h/2)|^\alpha \, e^{i\omega hk} \mathrm{d}\omega \tag{72}$$

Remark 3. *As stated in the previous section, the change from the nabla derivative stated in (13) to the corresponding bilinear implies to substitute the factor* $\left[\frac{1-z^{-1}}{h}\right]$ *by* $\left[\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right]$, *so that we obtain alternative derivative properties in (63) and (70). Relatively to (70) the substitution consists in* $2\sin(\omega h/2) \to \tan(\omega h/2)$.

5. Conclusions

The duality property of the Laplace and Fourier transforms associated to the integer order derivative was reviewed. We generalized it for fractional derivatives and extended the results to the Mellin, Z and discrete-time Fourier trasforms. To do it, we used the scale and discrete derivatives.

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