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Generalizing the Mean

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1 Preliminary Definitions

Suppose (X, d) is a metric space and $E \subseteq X$. Let $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an **(exact) dimension function** (or **gauge function**) which is monotonically increasing, strictly positive, and right continuous [12]. If

$$\mu_\delta^h(E) = \inf\left\{\sum_{i=1}^{\infty} h(\text{diam}(C_i)) : \text{diam}(C_i) \leq \delta, E \subseteq \bigcup_{i=1}^{\infty} C_i\right\} \tag{1}$$

where *diam* is the diameter of a set and:

$$\mu^h(E) = \sup_{\delta>0} \mu_\delta^h(E) \tag{2}$$

is the *Hausdorff Outer Measure*, we define h so $\mu^h(E)$ is strictly positive and finite for a majority (but not all) "nice" sets (that's sets that are measurable in the sense of Caratheodory [8]). (For easy reading, I will sometimes say such sets are just "measurable").

Now when $f : A \to \mathbb{R}$, and A is a bounded subset of $\mathbb{R}^d$, the average with respect to the Hausdorff Measure is:

$$m_f(A) := \frac{1}{\mu^h(A)} \int_A f(x)\, d\mu^h \tag{3}$$

And when A is unbounded and $t \in \mathbb{R}^+$, $m_f(A)$ can be adjusted as:

$$m'_f(A) := \lim_{t\to\infty} \frac{1}{\mu^h(A \cap [-t,t])} \int_{A\cap[-t,t]} f(x)\, d\mu^h \tag{4}$$

where we add $[-t, t]$ so when $A = \mathbb{R}$, the density of positive real numbers is:

$$\frac{\mu^h(\mathbb{R}^+ \cap [-t,t])}{\mu^h(\mathbb{R} \cap [-t,t])} = \frac{\mu^h((0,t])}{\mu^h([-t,t])} = 1/2$$

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and the density of negative real numbers is

$$\frac{\mu^h\left(\mathbb{R}^- \cap [-t,t]\right)}{\mu^h(\mathbb{R}\cap[-t,t])} = \frac{\mu^h([-t,0))}{\mu^h([-t,t])} = 1/2$$

which is intuitive since $[-t,t]$ has a mid-point of zero that's neither positive nor negative.

2 Motivation for Extending the Mean From the Hausdorff Measure and Fractal Setting to the Non-Fractal Setting

The function $m'_f(A)$ gives a satisfying average that is unique for a majority measurable A in the sense of Caratheodory. Despite this, there's measurable A without meaningful gauge functions since they're either σ-finite with respect to the counting-measure (e.g. Countably-Infinite sets) or their gauge function doesn't exist (e.g. the Liouville Numbers [7]). In these cases, $m'_f(A)$ can't exist as $\mu^h(A)$ is neither positive nor finite.

While there are methods to extending $m'_f(A)$, I haven't found a constructive extension which gives a unique, satisfying average for all functions defined on measurable sets in the sense of Caratheodory but with no meaningful gauge function.

One extension uses non-standard measure theory [11] but isn't unique as it requires ultra-filters, Zorn's Lemma and equivalent principles.

Other methods extend $m'_f(A)$ to A in the fractal setting ([5],[6]) but does not work for non-fractal, measurable A.

Additional options can be found in the work of Attila Losonczi (e.g. [1]) where he provides all averages and their properties but *I'm unsure if the averages he mentions are unique and satisfying for nowhere-continuous f which has a domain dense in an interval but with no meaningful gauge function.*

For example, consider $f : \mathbb{Q}\cap[0,1] \to \mathbb{R}$ and

$$f(x) = \begin{cases} 2 & x \in \left\{a^2 : a \in \mathbb{Q}\right\} \cap [0,1] \\ 1 & x \in \left(\mathbb{Q} \setminus \left\{a^2 : a \in \mathbb{Q}\right\}\right) \cap [0,1] \end{cases} \tag{5}$$

In this case, is the average 1, 2 or a value in between?

Note we must choose a unique, satisfying average for the cases that aren't covered; since, for the cases already covered, mathematicians choose $m'_f(A)$, or

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the averages in [5] and [6] than other averages.

3 Question 1

How do we find a constructive extension of $m'_f(A)$, [5] and [6] (with as many properties as the average can have from [2], [3] and [4]) which gives a unique, satisfying average for nowhere continuous functions defined on non-fractal, measurable sets with no meaningful gauge function?

3.1 Possible Answer

I believe [9] constructs a Generalized Hausdorff Measure that gives a unique, satisfying average for functions defined on *uncountable*, non-fractal measurable sets but I'm not sure if this measure could do the same for functions defined on measurable, countably infinite sets.

Are my assumptions correct?

4 Attempt To Answer The Other Half Of Question 1

Since I don't fully understand uncountable, measurable sets with no gauge function I will define a unique, satisfying average for f defined on countably infinite subsets of the real numbers (e.g. equation [5]). (I hope this is compatible with $m'_f(A)$, [11], [5], [6], and [9] and have as many properties as Losonczi listed in [2], [3] and [4]).

Note there are already methods to averaging over a countably infinite set; however, I would like to generalize them to give more satisfying averages to choose from.

4.1 Purpose of Changing the Current Definition of Average on Countably Infinite Sets

Suppose $f : A \to \mathbb{R}$ and A is a countably infinite, bounded subset of $\mathbb{R}$.

If $t \in \mathbb{N}$ and $\{a_n\}_{n=1}^{\infty}$ is an enumeration of A, the average of f is:

$$\lim_{t\to\infty} \frac{f(a_1) + f(a_2) + \cdots + f(a_t)}{t} \tag{6}$$

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where different enumerations of a function's domain could possibly give different averages: for instance nowhere continuous functions defined on countable sets dense in an interval)

A structure, however, (see Section 4.2) is a generalization of an enumeration that allows more satisfying averages to choose from.

Since different structures of the function's domain give different averages, I want to create a choice function that picks a unique class of equivalent structures (see section 4.3) such that it gives a satisfying average similar to the Hausdorff Measure for fractals.

For specific examples of A (see section 4.4), I would like to find the most natural or satsifying choice function which chooses the structures I believe would give the most satisfying average (*if it exists*). (If it does not exist, then I'd like to:

1. choose an alternate structure where the average does exist or
2. is undefined if no structure gives a defined average.

4.2 Defining Structures

Suppose $F_1, F_2, ...$ are a sequence of finite subsets of A where

1. $F_1 \subset F_2 \subset ...$
2. $\bigcup_{n=1}^{\infty} F_n = A$.

We denote the sequence of subsets as a **structure** of A which has the form $\{F_n\}$.

An example of a structure, such as when $A = \left\{\frac{1}{m} : m \in \mathbb{N}\right\}$,
is $\{F_n\}_{n\in\mathbb{N}} = \left\{\left\{\frac{1}{m} : m \in \mathbb{N}, m \le n\right\}\right\}_{n\in\mathbb{N}}$.

As mentioned earlier, the structure F_n generalizes the enumeration since as n increases by one, if $|F_n|$ increases by one, then $\{F_n\}$ behaves as an enumeration.

Further, there may be multiple structures of A e.g. for $A = \left\{\frac{1}{m} : m \in \mathbb{N}\right\}$, a second structure of the set is
$\{F_n\}_{n\in\mathbb{N}} = \left\{\left\{\frac{1}{2m} : m \in \mathbb{N}, m \le n\right\} \cup \left\{\frac{1}{2m+1} : m \in \mathbb{N}, m \le 2n\right\}\right\}_{n\in\mathbb{N}}$.

4.3 Defining Equivalent and Non-Equivalent Structures

Suppose we have two structures of A, $\{F_n\}$ and $\{F'_j\}$

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Structures are non-equivalent if there exists a function $f : A \to \mathbb{R}$ where, using the monotonic convergence theorem (if f is bounded) and the rigorous definition of limits of sequences (if unbounded):

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{x\in F_n} f(x) \neq \lim_{j\to\infty} \frac{1}{|F_j'|} \sum_{x\in F_j'} f(x) \tag{7}$$

Otherwise if for all functions $f : A \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{x\in F_n} f(x) = \lim_{j\to\infty} \frac{1}{|F_j'|} \sum_{x\in F_j'} f(x) \tag{8}$$

Then the structures $\{F_n\}$ and $\{F_j'\}$ are equivelant.

4.4 Specific Structures of Specific Countably Infinite A That My Choice Function Should Choose

Suppose the average of $f : A \to \mathbb{R}$ for countably infinite A, from structure $\{F_n\}$ of A, (using the equations in [7] and [8]) is:

$$\hat{m}_f\left(\{F_n\}, A\right) = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{x\in F_n} f(x) \tag{9}$$

Then, for specific A, if $\{F_n''\}$ is the set of equivalent structures I want the choice function to choose, then:

1. When $A = \mathbb{Z}$, $\{F_n''\}$ should equal $\{m \in \mathbb{Z} : -n \le m \le n\}$

2. When $p \in 2\mathbb{N}+1$, $A = \{\sqrt[p]{r} : r \in \mathbb{Q}\}$ $\{F_n''\}$ should equal:

$$\left\{\sqrt[p]{m/n!} : m \in \mathbb{N}, \lceil -n \cdot n! \rceil \le m \le \lfloor n \cdot n! \rfloor\right\}$$

   if $\hat{m}_f\left(\{F_n''\}, A\right)$ is defined and finite. This would give a satisfying average. (I don't know the structure the choice function should choose if $\hat{m}_f\left(\{F_n''\}, A\right)$ is not defined and finite. I will attempt to answer this in the following sections.)

3. When $A = \{1/m : m \in \mathbb{N}\}$ and $[\times]$ is the nearest integer function, $\{F_n''\}$ should be $\{1/\left[2^n/m\right] : m \in \mathbb{N}, 1 \le m \le 2^n\}$ if $\hat{m}_f\left(\{F_n''\}, A\right)$ is defined and finite.

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4. When A is almost nowhere dense (e.g. $\left\{\frac{1}{m} : m \in \mathbb{N}\right\}$), $\{F_n''\}$ should be points with the smallest 1-d Euclidean Distance from each point in $C_n = \{m/2^n : -n \cdot 2^n \leq m \leq n \cdot 2^n\}$ (unless the point in C_n is a limit point of A where minimum distance won't exist) such that $\hat{m}_f(\{F_n''\}, A)$ is defined and finite.

(For other countably infinite A, I am unsure what the choice function should choose. I wish for a unique set of equivalent structures.)

4.4.1 Reasons For The Choices in 4.4

For cases with a known and desired set of equivalent structures, the reason for choosing them is that they give an intuitive $\hat{m}_f\left(\{F_n''\}, A\right)$ when f is nowhere continuous.

For example, for equation [5], with a domain of rationals between 0 and 1:

$$f(x) = \begin{cases} 2 & x \in \left\{a^2 : a \in \mathbb{Q}\right\} \cap [0,1] \\ 1 & x \in \left(\mathbb{Q} \setminus \left\{a^2 : a \in \mathbb{Q}\right\}\right) \cap [0,1] \end{cases}$$

an intuitive average takes the arithmetic mean of the function on a finite, *evenly distributed*, *high density* sample of the domain under certain rules. The sample of the domain should contain a finite number of unique, non-simplified fractions that have the same positive-integer denominator (i.e. a common multiple of the first n positive integers, for example $n!$) and all positive-integer numerators between 0 and $n!$. As the denominator becomes larger the fractions began "covering" the entire domain of equation [5]. The result is the following:

$$\frac{1}{n!} \sum_{m=1}^{n!} f\left(\frac{m}{n!}\right) \tag{10}$$

As $n \to \infty$, the result is 1.

Note with equation [5]'s domain (which is $\mathbb{Q} \cap [0,1]$); because the $\{F_n''\}$ I want is $\{m/n! : m \in \mathbb{N}, 1 \leq m \leq n!\}$, using equation [9], we get that $\hat{m}_f\left(\{F_n''\}, A\right)$ should equal 1. This is the same as the intuitive average in the previous paragraph.

If the domain of f, when $p \in 2\mathbb{N}+1$, is $\{\sqrt[p]{a} : a \in \mathbb{Q} \cap [0,1]\}$ we apply a similar method but this time the numerator and common denominator are p-roots of the integer. Note $\hat{m}_f\left(\{F_n''\}, A\right)$ should give the same result as this method.

Also suppose $f : \left\{\frac{1}{m} : m \in \mathbb{N}\right\} \to \mathbb{R}$ and $A = \left\{\frac{1}{m} : m \in \mathbb{N}\right\}$, where $\{F_n''\} = \{1/[2^n/m] : m \in \mathbb{N}, 1 \leq m \leq 2^n\}$ and

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$$f(x) = \begin{cases} 1/\sqrt{x} & x \in \left\{1/(2^j) : j \in \mathbb{N}\right\} \\ 1 & \text{otherwise} \end{cases} \tag{11}$$

If we use the most natural structure of A (i.e. $\{F_n\} = \left\{\frac{1}{m} : m \in \mathbb{N}, m \leq n\right\}$), $\hat{m}_f(\{F_n\}, A) = 1$ but the values of $1/\sqrt{x}$, for $x \in \left\{1/2^j : j \in \mathbb{N}\right\}$, are *significantly* larger than 1. Therefore, it could be reasonable that $1/\sqrt{x}$ should have more weight on the average.

Using a calculator, I found $\hat{m}_f(\{F''_n\}, A)$ is approximately 2.707107; however, note for $f : \left\{\frac{1}{m} : m \in \mathbb{N}\right\} \to \mathbb{R}$, if we replace $1/\sqrt{x}$ with $1/x$:

$$f(x) = \begin{cases} 1/x & x \in \left\{1/(2^j) : j \in \mathbb{N}\right\} \\ 1 & \text{otherwise} \end{cases} \tag{12}$$

then $\hat{m}_f(\{F''_n\}, A) = \infty$.

Using the choice function in the section 4.6, it may be possible to get a unique, finite $\hat{m}_f(\{F''_n\}, A)$ as long as there exists an $\{F_n\}$ where $\hat{m}_f(\{F_n\}, A)$ exists.

4.5 Using Discrepancy to Define A Choice Function

4.5.1 Defining Equidistribution For Structures

Older definitions of discrepancy and equidistribution (on enumerations) are shown in articles [13] and [10]

As with structures $\{F_n\}$, we say it's **equidistributed** or **uniformly distributed** on $A_t = [\inf(A \cap [-t,t]), \sup(A \cap [-t,t])]$, if for any sub-interval $[c,d]$ of A_t we have:

$$\lim_{t\to\infty}\lim_{n\to\infty} \frac{|F_n \cap [c,d]|}{|F_n|} = \frac{d-c}{\ell(A_t)} \tag{13}$$

where $\ell(A_t)$ is the length of the interval A_t

We add $[-t,t]$ so when A has no infima or suprema, the limit on the left side of equation [13] exists.

Note current measures of **discrepancy** measure the maximum point of density deviation from a uniform or equidistributed sample

$$\sup_{\inf(A\cap[-t,t])\leq c\leq d\leq \sup(A\cap[-t,t])} \left| \frac{|F_n \cap [c,d]|}{|F_n|} - \frac{d-c}{\ell(A_t)} \right| \tag{14}$$

with more rigorous definitions deriving from articles [13] and [10] (we replace $\{a_1, ..., a_N\}$ with F_n and N with $|F_n|$). Unfortunately the discrepancy of most structures converges to zero as $n \to \infty$ making it impossible to find a structure with a lower discrepancy compared to the rest.

One solution is finding a $\{F_n\}$ where the lower bound of its' discrepancy converges to zero the fastest. Unfortunately, I'm unconfident with current measures as most *calculate the maximum point of density deviation rather than the overall deviation from an equidistributed structure*).

4.5.2 Defining A Precise Form Of Discrepancy

Below are steps to measuring the *overall deviation* of a structure from an equidistributed structure).

1. Arrange the values in F_n from least to greatest and take the absolute difference between consecutive elements. Call this ΔF_n. (Note ΔF_n is **not a set** since if absolute differences repeat, *we don't delete the repeating differences.*)

   1.1 **Example:** If $A = \left\{\frac{1}{m} : m \in \mathbb{N}\right\}$ and $\{F_n\}_{n\in\mathbb{N}} = \left\{\left\{\frac{1}{m} : m \in \mathbb{N}, m \le n\right\}\right\}_{n\in\mathbb{N}}$ then
   $F_4 = \{1, 1/2, 1/3, 1/4\}$

   Arranging F_4 from least to greatest gives us $\{1/4, 1/3, 1/2, 1\}$

   Therefore, $\Delta F_4 = \{|1/4 - 1/3|, |1/2 - 1/3|, |1/2 - 1|\} = \{1/12, 1/6, 1/2\}$. (None of the differences here are the same, but there are examples, such as the one below, where at least two of the differences are equivalent.)

   1.2 **Example:** If $A = \mathbb{Q}\cap[0,1]$ and $\{F_n\}_{n\in\mathbb{N}} = \left\{\left\{\frac{j}{k} : j, k \in \mathbb{N}, k \le n, 0 \le j \le k\right\}\right\}_{n\in\mathbb{N}}$ then the elements of F_4, arranged from least to greatest is,
   $F_4 = \{0, 1/4, 1/3, 1/2, 2/3, 3/4, 1\}$ and

   $\Delta F_4 = \{|0 - 1/4|, |1/4 - 1/3|, |1/2 - 1/3|, |2/3 - 1/2|, |3/4 - 2/3|, |1 - 3/4|\} =$

   $\{1/4, 1/12, 1/6, 1/6, 1/12, 1/4\}$. (Here the difference $1/4$ repeats two times but we do not delete the second $1/4$)

2. Divide ΔF_t by the sum of all its elements so we get a distribution where all the elements sum to 1. We shall call this $\Delta F_n / \sum_{x\in\Delta F_n} x$ or *the information probability of the structure*

   2.1 From example 1.1 note $\sum_{x\in\Delta F_3} x = 1/2 + 1/6 + 1/12 = 3/4$ and $\Delta F_3 / \sum_{x\in\Delta F_3} x = 4/3 \cdot \{1/2, 1/6, 1/12\} = \{2/3, 2/9, 1/9\}$.

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Note the elements in this set sum to 1 and act as a probability distribution (despite not being actual probabilities)

3. Since the elements of information probability always sum to 1, we can calculate its deviance from a discrete uniform distribution using Entropy which is written as

$$E(F_n) = -\sum_{j \in \Delta F_n / \sum\limits_{x \in \Delta F_n} x} j \log j \tag{15}$$

(Note the smaller the *deviation from a disrete uniform distribution*, the greater the entropy of the information probability and the lower the structure's *discrepancy*. Moreover, if $E(F_n) \to \infty$ as $n \to \infty$, we say $\{F_n\}$ is *equidistributed*).

3.1 From $\Delta F_3 / \sum\limits_{x \in \Delta F_3} x$, in example 2.1, $E(F_3)$ is the same as

$$\begin{aligned} -\sum_{j \in \{2/3, 2/9, 1/9\}} j \log j &= -\left(2/3 \log(2/3) + 2/9 \log(2/9) + 1/9 \log(1/9)\right) \\ &\approx .369 \end{aligned}$$

4.6 Defining The Choice Function

Inorder to get my results from Section 4.4, if $g : A \to \mathbb{R}$ is the identity function, we should adjust:

$$T(F_n) = 2^{\hat{m}_g(\{F_n\}, A)} \left(2^{E(F_n)} + |F_n|\right) \tag{16}$$

and also adjust the equations below (where $\mathbb{S}'(A)$ is the set of structures of A; where, if $\{F_j\} \in S'(A)$ then $\hat{m}_f(\{F_j\}, A)$ is finite and defined and finite)

$$\left|\overline{F''_n}\right| = \inf\left\{|F_j| : j \in \mathbb{N}, \{F_j\} \in \mathbb{S}'(A), T(F_j) \geq T(F''_n)\right\} \tag{17}$$

$$\left|\underline{F''_n}\right| = \sup\left\{|F_j| : j \in \mathbb{N}, \{F_j\} \in \mathbb{S}'(A), T(F_j) \leq T(F''_n)\right\} \tag{18}$$

to choose $C_1 : \mathbb{R}^3 \to \mathbb{R}$ and $C_2 : \mathbb{R}^3 \to \mathbb{R}$ such that:

$$C_1\left(|F''_n|, \left|\overline{F''_n}\right|, \left|\underline{F''_n}\right|\right) \leq |F''_n| \leq C_2\left(|F''_n|, \left|\overline{F''_n}\right|, \left|\underline{F''_n}\right|\right) \tag{19}$$

or otherwise

$$\sum_{n=1}^{z} C_1\left(|F''_n|, \left|\overline{F''_n}\right|, \left|\underline{F''_n}\right|\right) \leq \sum_{n=1}^{z} |F''_n| \leq \sum_{n=1}^{z} C_2\left(|F''_n|, \left|\overline{F''_n}\right|, \left|\underline{F''_n}\right|\right) \tag{20}$$

5 Question 2

What are the most elegant choices for C_1 and C_2 (which for each of the A listed in Section 4.4) give the $\{F''_n\}$ required?

6 Generalized Mean

If $f : A \to \mathbb{R}$, A is a subset of $\mathbb{R}$, and $\text{avg}_f(A)$ is a unique, satisfying average of f defined on sets measurable in the sense of Caratheodory, then $\text{avg}_f(A)$ should be defined as:

$$\text{avg}_f(A) := \begin{cases} m'_f(A) \ \text{(See eq: [4])} & A \text{ has a gauge function} \\ \text{Averages in [5], [6]} & A \text{ is fractal but has no gauge function} \\ \hat{m}_f\left(\{F''_n\}, A\right) & A \text{ is countably infinite, non fractal-like and for} \\ & \text{at least one structure, } \hat{m}_f(\{F_n\}, A) \text{ is defined} \\ \text{Average from [9]} & A \text{ is uncountable and non-fractal with} \\ \ \text{or Unknown} & \text{no gauge function} \\ \text{Undefined} & \text{Satisfying average cannot exist e.g. there is} \\ & \text{no } \{F_n\} \text{ where } \hat{m}_f(\{F_n\}, A) \text{ exists} \end{cases} \tag{21}$$

And an example where the average is unknown is for nowhere continuous f defined on Liouville Numbers [12].

7 Question 3

Does [9] give a satisfying average when A is uncountable and non-fractal with no gauge function; or, is $avg_f(A)$ is unknown for this case?

8 Question 4:

Can we unite the peice-wise average in Section 6 into a elegant, *non-peicewise* mean?

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