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Cylindrical Bending of A Hardcover Book with Internal Friction

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To have a better protection, strong toughness and good flexibility, all lives and plants must have skins, similarly, all books should have covers. In this paper, we follow in the footsteps of Poincloux et al. [Phy.Rev.Lett. 126, 218004 (2021)] and extend their centerline-based theory from books without covers to hardcover books with internal friction. Our investigations show that the hardcover are more essential than the core layers in terms of bending response as well as energy absorption. The central goal of studying the covered book is not only to predict the bending deformation of the books, but also as a representative case to help finding some clue on the universal behaviours of multilayered architectures with internal friction.

INTRODUCTION

To have a better protection, toughness, strength, shock absorption and at same time still to maintain a good flexibility, layered or laminated architectures with internal sliding features are essential mechanism in natural and man-made structural system [1, 2]. For example, scaled skins are a very common structure in both the animal kingdom and engineering applications, such as lizards, fish, leaf springs, scaled armour, pangolin and books [1–13] as shown in Fig.1.

FIG. 1: Layered architectures of fish scales, leaf springs, scaled armour, pangolin and books

In the deformation of the layered structures/system, the interactions between layers play a central role in controlling the overall mechanical performance of the system, in particular the interlayered friction is crucial to the response of elastic system. As pointed by Poincloux et al. [1], it is a great challenging to predict how the microscopic architecture and interlayer interactions of a layered mechanical system give rise to a specific macroscopic constitutive response, especially for large deformations.

For a multilayers plate with n layers having equal thickness h, the total bending stiffness is nB for zero interfacial friction, and n^3B for infinite interfacial friction [14]. What is the bending stiffness of a multilayers with internal friction ?

$$\text{Bending stiffness} = \begin{cases} nB & \text{with zero friction} \\ What\,? & \text{with finite friction} \\ n^3B & \text{with infinite friction} \end{cases}$$

Although we can't give a definitive answer to this question yet, we can still estimate that the stiffness of the multilayers must be bounded by scale of nB and n^3B, where the bending stiffness of a single layer, $B = \frac{Eh^3W}{12(1-\nu^2)}$.

The problem is how to quantitatively determine the effective bending stiffness of the multilayers with internal friction. Generally speaking, there are two approaches to deal with this kind of problem, one is "top-down" and another is "bottom-up" [10, 12, 13]. The "top-down" is treating the multilayers as a single system and divide it into several layers, formulating each layer and solving differential equation systems under both boundary conditions and interlayers conditions. Obviously, the interlayers conditions are difficulty to be defined due to the sliding and discontinuation nature between layers. The "bottom-up" can be viewed as the inversion of the "top-down" and starting from a single layer and building up layer by layer, clearly the the "bottom-up" approach still faces the same challenge as the "top-down", i.e., the problem of how to define the interlayer conditions. In addition to the difficulty of the interlayers conditions, both approaches face a same problem, that is, there are too many unknowns in the governing equations, which rapidly escalate with the increase of numbers of layers.

Based on the above analysis, it seems that neither approach is suitable. In order to resolve the issue, we must find alternative way, and where is the way out?

Before Poincloux et al. [1], although there were some studies in other problem involving friction [15–17], howev-

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er, there is no general solution method to tackle friction, especially when it couples with other ingredients, such as elasticity, nonlinear geometry, and multiplicity of contacts. Poincloux et al. [1] have made a breakthrough and studied the multilayers plate with n layers having same thickness h by performing precision nonlinear bending tests of a multilayered stack of elastic plates interacting solely through friction. They developed a centerline-based theory by using dimension reduction procedure to describe the stack as a nonlinear planar rod with internal shear. They considered the coupling between the nonlinear geometry and the elasticity of the stacked plates, treating the interlayer friction perturbatively. This model yields predictions for the stack's mechanical response in three-point bending that are in excellent agreement with their experiments. Remarkably, they found that the energy dissipated during deformation can be rationalized over 3 orders of magnitude, including the regimes of a thick stack with large deflection [1].

The success story of Poincloux et al. [1] suggests that they may have found a new way out. So, what exactly is their method? Although they have not provided clear statement on their method, here we attempt to summarize their method based on our own understanding. Poincloux's method can be interpreted as a modified "bottom-up". This method consists of two steps, the first step does not consider internal friction, but calculate the deformation of the laminated structure and other physical quantities; the second step introduces friction. The key to the success of this method is the first step, the basic idea of which can be understood in this way, that is, firstly select a thickness middle line as the backbone line, then start with a single layer analysis, and then carry out continuation extension in the thickness direction; when encountering a distance away from the centerline (backbone) in the process of continuous extension y, averaging within each layer by $\bar{\Phi}(s) \approx \frac{1}{b-a}\int_a^b \Phi(s,\eta)d\eta$, $\eta \in [a,b]$, and then adding up the averaging results. It is worth to mention that the thickness parameter y is not a coordinate and has nothing to do with the coordinates x_1, x_2 as shown in Fig.2.

The beauty of Poincloux's modelling is converting the governing ordinary differential equations into a single ordinary differential equation, which reduce the difficulties of the problem dramatically.

As we known, in nature, all lives and plants must have skins, similarly, all books should have covers. Although Poincloux et al. [1] have not studied the book with covers, their method is quite general and able to be used to treat other similar problems. To fill up the gap, in this paper, we follow in the footsteps of Poincloux et al. [1] and extend their centerline-based theory from books without covers to the hardcover book with internal friction. Our later investigations show that the hardcover are more essential than the core layers in terms of both bending response and energy absorption, which clearly indicates the necessariness of studying the hardcover book. The central goal of studying the covered book is not only to predict the bending deformation of the book, but also as a representative case to help finding some clue on the universal behaviours of multilayered architectures. Therefore, it is worth to study the book mechanics problem even more further.

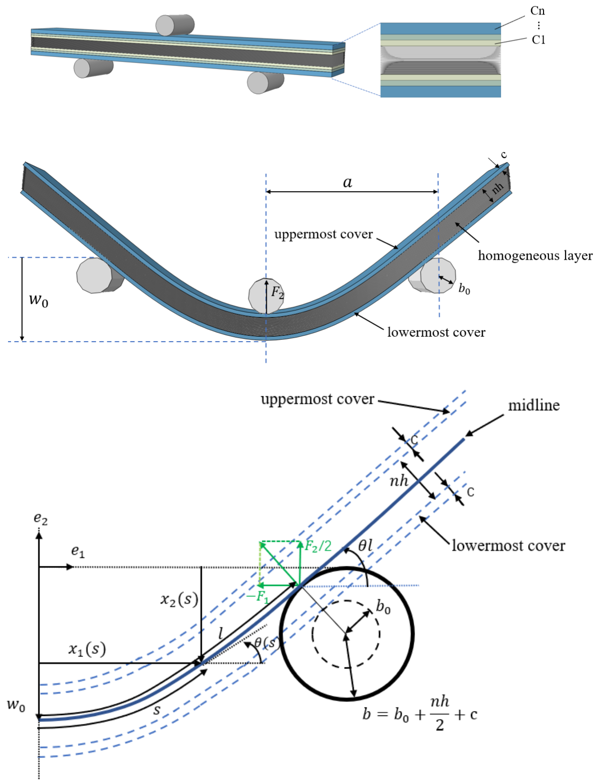

FIG. 2: Modelling of a book with hardcover and coordinates.

In this paper, Section 1 we first highlight the layered architectures and its challenges. Section 2 we introduce assumptions of theory and derive all formulations. In Section 3, we discuss the power of bending and energy dissipation. Section 4 we reduce the general formulations to small deflection. Section 5 we present discussions of some numerical studies. Finally, conclusions are drawn.

FORMULATIONS OF A BOOK WITH INTERLAYER FRICTION

Problem and assumptions: Consider a hardcover book in Fig.2 whose layout is denoted $[c|\underbrace{h...h}_{n}|c]$. The book length is L and width is W, top/bottom hardcover thickness are c. The plate has n core layers and each layer thickness is h, hence book stack dimensions is $L \times W \times (nh + 2c)$, where $nh + 2c$ is total plate thick-

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ness (height). The Young's modular and Poisson ratio of both the core sheet and hardcover are E_c, E and ν_c, ν, respectively.

For investigation of the stack with interlayer friction, following assumptions will be adopted: 1. The Kirchhoff hypothesis is applied; 2. Each layer is inextensible; 3. No delamination; 4. Interlayers can slide with friction.

Interlayers can slide with friction consists of the following contents: (1) Displacements along arc direction on the interface are discontinue (1) Displacement along arc direction of each layer are different; (3) Both shear strain and shear stress on the interface are also discontinue.

The typical multilayer is illustrated in Fig.3.

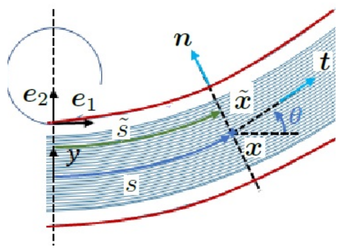

FIG. 3: Tangent angle θ versus arc length $\tilde{s}$ along the upper- and lowermost

As we stated in the introduction, Poincloux's method is starting from the single layer and then representing all offset layer's quantities in terms of centerline's ones. These can be done is because of applying the assumptions of the theory, we can find the multilayers deform collectively with same rotation $\theta(s)$ and transverse displacement $w(s)$. Then by summing all averaged quantities, we can get the resultant quantities of the book, such as moment and bending energy etc. Since the book is subjected to a transverse load that is uniform at any section parallel to the s-axis. In such a case, the deflection w and other quantities of the book are functions of only s. Therefore, all derivatives with respect to the width are zero. In such cases, the deflected surface of the book is cylindrical, and it is referred to as the cylindrical bending, which can be formulated by theory of rod or beam in plane stress along the direction of width.

Single layer formulations: The centerline of each layer is represented by an inextensible curve $\boldsymbol{x}(s)$ with arch length s and curvature $\kappa(s)$, where s is reserved for arch lengths measured along the layer's centerline, whereas $\tilde{s}$ pertains to the arch length along a off-centerline. The transverse coordinate y varies from $-h/2$ at the lowermost plate to $h/2$ at the uppermost one.

Assume $\boldsymbol{x}(s)$ as a reference (middle) centerline and $\boldsymbol{n}(s)$ as the unit normal vector to the centerline of inextensible planar curve. The unit tangent of the centerline is given by

$$\boldsymbol{t} = \frac{d\boldsymbol{x}}{ds}, \quad |\boldsymbol{t}| = 1, \tag{1}$$

which is orthogonal to the normal, ie., $\boldsymbol{t} \cdot \boldsymbol{n} = 0$. The curvature of the reference (middle) inextensible curve is

$$\kappa(s) = \frac{d\theta}{ds} = |\frac{d^2\boldsymbol{x}}{ds^2}| = |\frac{d\boldsymbol{t}}{ds}|, \tag{2}$$

where θ is denoted as the angle between $\boldsymbol{t}$ and horizontal axis x_1. From the planar Frenet's frame formula, we have $\frac{d\boldsymbol{t}}{ds} = -\kappa\boldsymbol{n}$ and $\frac{d\boldsymbol{n}}{ds} = \kappa\boldsymbol{t}$.

The displacement of points with the off the centerline y is $\boldsymbol{u}^y(s) = \boldsymbol{u}(s) + (\boldsymbol{n} - \boldsymbol{N})y$, where $\boldsymbol{u}(s)$ is displacement of the points on the centerline. With the Kirchhoff hypothesis, $\boldsymbol{n} - \boldsymbol{N} = \theta\boldsymbol{e}_s$, where $\theta(s)$ is rotation of the normal of the centerline and equals to the tangent of the line, i.e., $\theta = |\frac{d\boldsymbol{t}}{ds}|$, therefore we have displacement components in both arch direction $u^y(s) = u(s) + \theta y$ and in y direction $w^y(s) = w(s)$.

Owing to the assumption of inextensible planar curve, $\frac{du}{ds} = 0$, hence the strain is

$$\varepsilon = y\frac{d\theta}{ds} = y(\kappa - \kappa_0,) \tag{3}$$

where the initial curvature κ_0.

Plane stress applies to a sheet of material in which the stress in the thickness direction is much lower than the stresses within the plane [14]. The stress in the thickness direction is taken as zero. The offset centerline stress is expressed as

$$\sigma = \frac{E}{1-\nu^2}\varepsilon = \frac{E}{1-\nu^2}y(\kappa - \kappa_0). \tag{4}$$

The bending moment of a typical layer with thickness h is defined by $M_\ell = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} y(\sigma W dy)$, where $y \in [-\frac{h}{2}, \frac{h}{2}]$ and lower index L denotes as "layer", hence

$$M_L = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} y(\frac{E}{1-\nu^2}y\kappa W dy) = B(\kappa - \kappa_0), \tag{5}$$

where the bending stiffness of a single layer is $B = \frac{Eh^3W}{12(1-\nu^2)}$. The relation reveals that the bending moment is linear proportional to the curvature.

The bending energy of the layer with thickness h is defined as $\Pi_\ell = \frac{1}{2}\int M\kappa ds = \frac{B}{2}\int(\kappa-\kappa_0)^2 ds$, hence

$$\Pi_L = \frac{B}{2}\int(\kappa - \kappa_0)^2 ds. \tag{6}$$

This relation indicates the bending energy has linkage with the square of the curvature. In this study, the initial curvature is assumed as zero, namely $\kappa_0 = 0$.

Multilayers formulations: Once we have the single layer formulations, we can formulate the multilayers structure. If we extend the thickness coordinate y to multilayers thickness, the final position of a point belonging to

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the layer offset by y from the stack's centerline is written as

$$\tilde{\boldsymbol{x}}(s,y)=\boldsymbol{x}(s)+\boldsymbol{n}(s)y, \tag{7}$$

where $\boldsymbol{x}(s)=x_1(s)\boldsymbol{e}_1+x_2(s)\boldsymbol{e}_2$ is a reference inextensible planar curve of centerline, $\boldsymbol{n}(s)$ is the unit normal vector to the centerline(Notes: To prevent too many notations, we use the same notion as the single layer, but with different meaning, heren $\boldsymbol{x}(s)$ is the centerline (backbone) curves of multilayers, which was denoted as $\boldsymbol{x}_{bb}$ in Poincloux et al. [1]).

The centerline of the stack is represented an inextensible curve $\boldsymbol{x}(s)$ with arch length and curvature $\kappa(s)$, where s is reserved for arch lengths measured along the stack's centerline, whereas $\tilde{s}$ pertains to the arch length along a specific layer. The transverse coordinate y varies from $-(nh+c)/2$ at the lowermost plate to $(nh+c)/2$ at the uppermost one.

The arc length $\tilde{s}$ on the offset curve is given by $d\tilde{s}=\sqrt{d\tilde{s}^2}=\sqrt{d\tilde{\boldsymbol{x}}(s,y)\cdot d\tilde{\boldsymbol{x}}(s,y)}=ds\sqrt{(\boldsymbol{t}+y\frac{d\boldsymbol{n}}{ds})\cdot(\boldsymbol{t}+y\frac{d\boldsymbol{n}}{ds})}$. Applying the Frenet's formula, we have

$$d\tilde{s}=ds\sqrt{1+2y\kappa+y^2\kappa^2}=(1+y\kappa)ds, \tag{8}$$

Interlayer shear is measured by $d\tilde{s}-ds=1+y\kappa$ due to the combined effects of curvature and sheet inextensibility.

The tangent of offset curves is defined as $\tilde{\boldsymbol{t}}=\frac{d\tilde{\boldsymbol{x}}}{d\tilde{s}}$, which leads to $\tilde{\boldsymbol{t}}=(1+y\kappa)^{-1}(\boldsymbol{t}+y\frac{d\boldsymbol{n}}{ds})=(1+yk)^{-1}(1+y\kappa)\boldsymbol{t}=\boldsymbol{t}$. This indicate that the offset curves remains parallel to the centerline $\boldsymbol{x}$. The curvature of offset curves is given by

$$\tilde{\kappa}=\frac{d\theta}{d\tilde{s}}=\frac{d\theta}{(1+yk)ds}=\frac{\kappa}{1+y\kappa}. \tag{9}$$

The bending moment of the layer offset y are

$$M_L=B\tilde{\kappa}=B\frac{\kappa}{1+y\kappa} \tag{10}$$

The bending moment: The bending moment of the stack is the summation of hardcover and core layer's bending moment, i.e., $M=M_{\mathrm{Bottom}}+\sum M_\ell+M_{Top}$, where $M_{\mathrm{Bottom}}=\frac{E_cc^3W}{12(1-\nu_c^2)}\tilde{\kappa}_{\mathrm{Bottom}}=\frac{E_cc^3W}{12(1-\nu_c^2)}(\frac{\kappa}{1+y\kappa})_{\mathrm{Bottom}}$, $M_{\mathrm{Core}}=\frac{Eh^3W}{12(1-\nu^2)}\tilde{\kappa}_{\mathrm{Core}}=\frac{Eh^3W}{12(1-\nu^2)}(\frac{\kappa}{1+y\kappa})_{\mathrm{Core}}$, and $M_{\mathrm{Top}}=\frac{E_cc^3W}{12(1-\nu_c^2)}\tilde{\kappa}_{\mathrm{Top}}=\frac{E_cc^3W}{12(1-\nu_c^2)}(\frac{\kappa}{1+y\kappa})_{\mathrm{Top}}$.

The problem we are facing is that the bending moments are the function of y within each layer, according to Poincloux's method, we can get rid of it by averaging them in the domain of y as follows:

$$\bar{\Phi}(s)\approx\frac{1}{y_2-y_1}\int_{y_1}^{y_2}\Phi(s,y)dy,\quad y\in[y_1,y_2]. \tag{11}$$

Therefore, we have

$$\begin{aligned}M_{\mathrm{Bottom}}&=\frac{E_cc^3W}{12(1-\nu_c^2)}(\frac{\kappa}{1+y\kappa})_{\mathrm{Bottom}}\\&\approx\frac{E_cc^3W}{12(1-\nu_c^2)}\left(\frac{1}{c}\int_{-\frac{nh}{2}-c}^{-\frac{nh}{2}}\frac{\kappa}{1+y\kappa}dy\right)\\&=\frac{B_c}{c}\ln\frac{1-\frac{nh}{2}\kappa}{1-(\frac{nh}{2}+c)\kappa},\end{aligned} \tag{12}$$

where hardcover's bending stiffness is $B_c=\frac{E_cc^3W}{12(1-\nu_c^2)}$. In the same way, we have

$$\begin{aligned}M_{\mathrm{Top}}&=\frac{E_cc^3W}{12(1-\nu_c^2)}(\frac{\kappa}{1+y\kappa})_{\mathrm{Top}}\\&\approx\frac{E_cc^3W}{12(1-\nu_c^2)}\left(\frac{1}{c}\int_{\frac{nh}{2}}^{\frac{nh}{2}+c}\frac{\kappa}{1+y\kappa}dy\right)\\&=\frac{B_c}{c}\ln\frac{1+(\frac{nh}{2}+c)\kappa}{1+\frac{nh}{2}\kappa},\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\sum M_L=\sum B(\frac{\kappa}{1+y\kappa})_\ell\\&\approx B[\frac{1}{h}\int_{-\frac{nh}{2}}^{-\frac{nh}{2}+h}\frac{\kappa}{1+y\kappa}dy+\frac{1}{h}\int_{-\frac{nh}{2}+h}^{-\frac{nh}{2}+2h}\frac{\kappa}{1+y\kappa}dy\\&+...+\frac{1}{h}\int_{\frac{nh}{2}-2h}^{\frac{nh}{2}-h}\frac{\kappa}{1+y\kappa}dy+\frac{1}{h}\int_{\frac{nh}{2}-h}^{\frac{nh}{2}}\frac{\kappa}{1+y\kappa}dy]\\&=\frac{B}{h}\int_{-\frac{nh}{2}}^{\frac{nh}{2}}\frac{\kappa}{1+y\kappa}dy=\frac{B}{h}\ln\frac{1+\frac{nh}{2}\kappa}{1-\frac{nh}{2}\kappa},\end{aligned} \tag{14}$$

Hence, we have the total resultant bending moment as follows

$$\begin{aligned}M=&\frac{B}{h}\ln\frac{1+\frac{nh}{2}\kappa}{1-\frac{nh}{2}\kappa}-\frac{B_c}{c}\ln\frac{1+\frac{nh}{2}\kappa}{1-\frac{nh}{2}\kappa}\\&+\frac{B_c}{c}\ln\frac{1+(\frac{nh}{2}+c)\kappa}{1-(\frac{nh}{2}+c)\kappa}.\end{aligned} \tag{15}$$

If $c=h$, for a book-like-plates with $n+2$ layers, the bending moment is $M=\frac{B}{h}\ln\frac{1+\frac{(n+2)h}{2}\kappa}{1-\frac{(n+2)h}{2}\kappa}$, and for the plate with n layers, $M=\frac{B}{h}\ln\frac{1+\frac{nh}{2}\kappa}{1-\frac{nh}{2}\kappa}$.

The relation between M and κ is nonlinear, hence the incremental stiffness is defined as $K_{\mathrm{Nlin}}=\frac{dM}{d\kappa}$, hence

$$\begin{aligned}K_{\mathrm{Nlin}}=&\frac{B}{h}\frac{nh}{1-(\frac{nh}{2}\kappa)^2}-\frac{B_c}{c}\frac{nh}{1-(\frac{nh}{2}\kappa)^2}\\&+\frac{B_c}{c}\frac{nh+2c}{1-[(\frac{nh}{2}+c)\kappa]^2}.\end{aligned} \tag{16}$$

In the linear case of small deflection with small curvature $1-(\frac{nh}{2}\kappa)^2\approx1$, the above stiffness can be approximated as $K_{\mathrm{Lin}}\approx nB+2B_c$, which will be used in the last section.

The bending energy: The bending moment of the book is the summation of hardcover and core layer's bending energy, i.e., $\Pi_B = \Pi_{\text{Bottom}} + \sum \Pi_\ell + \Pi_{Top}$, where $\Pi_{\text{Bottom}} = \frac{1}{2}(\int \tilde{M}\tilde{\kappa})_{\text{Bottom}} d\tilde{s} = \frac{1}{2}(\int B_c\tilde{\kappa}^2)_{\text{Bottom}} d\tilde{s} = \frac{1}{2}[B_c \int (\frac{\kappa}{1+y\kappa})^2(1+y\kappa)ds]_{\text{Bottom}} = \frac{1}{2}[B_c \int \frac{\kappa^2}{1+y\kappa}ds]_{\text{Bottom}}$, $\Pi_{\text{Core}} = \frac{1}{2}[B_c \int \frac{\kappa^2}{1+y\kappa}ds]_{\text{Core}}$, and $\Pi_{\text{Top}} = \frac{1}{2}[B_c \int \frac{\kappa^2}{1+y\kappa}ds]_{\text{Top}}$.

The problem we are facing is that the bending energy are also the function of y within each layer, in the same way, we can also get rid of it by averaging method as stated before. Therefore, we have

$$\begin{aligned}\Pi_{\text{Bottom}} &= \frac{1}{2}[B_c \int \frac{\kappa^2}{1+y\kappa}ds]_{\text{Bottom}} \\ &\approx \frac{1}{2}B_c \int \left[\left(\frac{1}{c}\int_{-\frac{nh}{2}-c}^{-\frac{nh}{2}} \frac{\kappa^2}{1+y\kappa}dy\right)\right] ds \\ &= \frac{B_c}{c}\int \kappa \ln \frac{1-\frac{nh}{2}\kappa}{1-(\frac{nh}{2}+c)\kappa}ds, \end{aligned} \tag{17}$$

Similarly, we have

$$\begin{aligned}\Pi_{\text{Top}} &= \frac{1}{2}[B_c \int \frac{\kappa^2}{1+y\kappa}ds]_{\text{Top}} \\ &\approx \frac{1}{2}B_c \int \left[\left(\frac{1}{c}\int_{\frac{nh}{2}}^{\frac{nh}{2}+c} \frac{\kappa^2}{1+y\kappa}dy\right)\right] ds \\ &= \frac{B_c}{c}\int \kappa \ln \frac{1+(\frac{nh}{2}+c)\kappa}{1+\frac{nh}{2}\kappa}ds, \end{aligned} \tag{18}$$

and

$$\begin{aligned}&\sum \Pi_L = \sum \frac{1}{2}[B \int \frac{\kappa^2}{1+y\kappa}ds]_\ell \\ &\approx \frac{1}{2}B\int [\frac{1}{h}\int_{-\frac{nh}{2}}^{-\frac{nh}{2}+h} \frac{\kappa^2}{1+y\kappa}dy + \frac{1}{h}\int_{-\frac{nh}{2}+h}^{-\frac{nh}{2}+2h} \frac{\kappa^2}{1+y\kappa}dy \\ &+ ... + \frac{1}{h}\int_{\frac{nh}{2}-2h}^{\frac{nh}{2}-h} \frac{\kappa^2}{1+y\kappa}dy + \frac{1}{h}\int_{\frac{nh}{2}-h}^{\frac{nh}{2}} \frac{\kappa^2}{1+y\kappa}dy]ds \\ &= \frac{B}{h}\int \left(\int_{-\frac{nh}{2}}^{\frac{nh}{2}} \frac{\kappa}{1+y\kappa}dy\right) ds \\ &= \frac{1}{2}\frac{B}{h}\int \kappa \ln \frac{1+\frac{nh}{2}\kappa}{1-\frac{nh}{2}\kappa}ds, \end{aligned} \tag{19}$$

Hence, we have the total bending energy

$$\begin{aligned}\Pi_B = \frac{1}{2}\int_{-\ell}^{\ell} \kappa &\left[\frac{B}{h}\ln\frac{1+\frac{nh}{2}\kappa}{1-\frac{nh}{2}\kappa} - \frac{B_c}{c}\ln\frac{1+\frac{nh}{2}\kappa}{1-\frac{nh}{2}\kappa}\right. \\ &\left.+\frac{B_c}{c}\ln\frac{1+(\frac{nh}{2}+c)\kappa}{1-(\frac{nh}{2}+c)\kappa}\right] ds, \end{aligned} \tag{20}$$

Notice the bending moment in Eq.15, we have a simpler form of bending energy as follows

$$\Pi_B = \frac{1}{2}\int_{-\ell}^{\ell} M\kappa ds, \tag{21}$$

If $c = h$, for a book-like-plates with n layers, the bending energy is $\Pi_B = \frac{B}{2h}\int \kappa \ln \frac{1+\frac{nh}{2}\kappa}{1-\frac{nh}{2}\kappa}ds$, which was obtained by [1].

Equilibrium equation: To derive equilibrium equation of the stack, we can write down its total energy $\Pi = \Pi_B - \Pi_P$, where the work done by force F_1 and F_2 is

$$\Pi_P = 2\int_0^\ell \left(-F_1\cos\theta + \frac{F_2}{2}\sin\theta\right) ds. \tag{22}$$

From variational principle, $\delta\Pi = \delta\Pi_B - \delta\Pi_P = 0$, and take into account of the symmetry of the deformation, we can get

$$\delta\frac{1}{2}\int_0^\ell M\kappa ds - \int_0^\ell [F_1\sin\theta + \frac{F_2}{2}\cos\theta]\delta\theta ds = 0. \tag{23}$$

Since $\kappa = \frac{d\theta}{ds}$, then $\delta k = \delta\frac{d\theta}{ds} = \frac{d}{ds}(\delta\theta)$. Executing the above variational and integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\left[(M+\kappa\frac{dM}{d\kappa})\delta\theta\right]_0^\ell - \frac{1}{2}\int_0^\ell [\frac{dM}{ds} + \frac{d}{ds}(\kappa\frac{dM}{d\kappa})]\delta\theta ds \\ &-\int_0^\ell [F_1\sin\theta(s) + \frac{F_2}{2}\cos\theta(s)]\delta\theta ds = 0 \end{aligned} \tag{24}$$

Owing to the arbitrary nature of the variational $\delta\theta$, the above variational will give us equilibrium equation

$$\delta\theta: \frac{dM}{ds} + \frac{d}{ds}(\kappa\frac{dM}{d\kappa}) + 2F_1\sin\theta(s) + F_2\cos\theta(s) = 0, \tag{25}$$

and boundary condition at $s = 0$ and $s = L$

$$\left[(M+\kappa\frac{dM}{d\kappa})\delta\theta\right]_0^\ell = 0. \tag{26}$$

Since $\frac{dM}{d\kappa} = \frac{B}{h}\frac{nh}{1-(\frac{nh}{2}\kappa)^2} - \frac{B_c}{c}\frac{nh}{1-(\frac{nh}{2}\kappa)^2} + \frac{B_c}{c}\frac{nh+2c}{1-[(\frac{nh}{2}+c)\kappa]^2}$, we have $\frac{dM}{ds} + \frac{d}{ds}(\kappa\frac{dM}{d\kappa}) = 2\left\{\frac{2B}{h}\frac{\frac{nh}{2}\theta''}{[1-(\frac{nh}{2}\theta')^2]^2} - \frac{2B_c}{c}\frac{\frac{nh}{2}\theta''}{[1-(\frac{nh}{2}\theta')^2]^2} - \frac{2B_c}{c}\frac{(\frac{nh}{2}+c)\theta''}{[1-[(\frac{nh}{2}+c)\theta']^2]^2}\right\}$,, where $\theta' = \frac{d\theta(s)}{ds}$ and $\theta'' = \frac{d^2\theta(s)}{ds^2}$.

The equilibrium equation in Eq.25 can be expressed as follows

$$\begin{aligned}&\underbrace{\frac{h}{c}\frac{(n+2c/h)B_c\theta''}{[1-[(\frac{nh}{2}+c)\theta']^2]^2}}_{\text{Top}} + \underbrace{\frac{nB\theta''}{[1-(\frac{nh}{2}\theta')^2]^2}}_{\text{Core}} \\ &\underbrace{-\frac{h}{c}\frac{nB_c\theta''}{[1-(\frac{nh}{2}\theta')^2]^2}}_{\text{Bottom}} + F_1\sin\theta + \frac{F_2}{2}\cos\theta = 0, \end{aligned} \tag{27}$$

where the underbraces indicate the contribution from top cover, core and bottom cover, respectively. This differential equation is a strong nonlinear and has to be solved

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numerically. We have written a Matlab code to find its numerical solutions.

Owing to the symmetric deformation, the rotation $\theta = 0$ at $s = 0$, however at $s = L$ the rotation is unknown, therefore the boundary conditions in Eq.26 will be simplified to $\theta(0) = 0$, $\left(M + \kappa \frac{dM}{d\kappa}\right)_{s=\ell} = 0$. Since at $s = \ell$, the end is fee and we can enforce the boundary condition $\theta'(\ell) =$, i.e., $\kappa(\ell) = 0$, which leads to $M(\ell) = 0$, therefore we have boundary conditions

$$\theta(0) = 0,\ \theta'(\ell) = 0. \tag{28}$$

Eq.27 is the equilibrium equation of the books with hardcover. For the case of $c = h$ and $n \to n-2$, the above equilibrium equation will be reduced to the equilibrium equation derived by Poincloux et al. [1].

To find the centerline curves, we need to reconstruct it from the rotation by $\frac{dx_1(s)}{ds} = \cos\theta(s)$, $\frac{dx_2(s)}{ds} = \sin\theta(s)$. The boundary conditions are $(\theta, x_1, x_2)|_{s=0} = (0, 0, -w_0)$ and $(\theta', x_1, x_2)|_{s=\ell} = (0, a - b\sin\theta(\ell), b(\cos\theta(\ell) - 1))$, with $b = b_0 + nh/2$ as the effective radius of the support. At $s = \ell$, the balance relation is satisfied: $-F_1\cos\theta(\ell) + F_2/2\sin\theta(\ell) = 0$.

Interfacial stress in terms of backbone solution: Now we need to analysis the interfacial interactions so that we introduce the fiction. It is clear that each layer must be in a balance state if the stack in equilibrium balance. Poincloux et al. [1] derived all formulation in this subsection based on $d\tilde{s} = (1 - y\kappa)ds$).. For self-contained purpose, we reformulate the relevant quantities based on $d\tilde{s} = (1 + y\kappa)ds$).

Equilibrium equations of a layer

$$\frac{\partial \tilde{M}(\tilde{s}, y)}{\partial \tilde{s}} - \tilde{Q}(\tilde{s}, y) = 0 \tag{29}$$

$$\frac{\partial \tilde{N}(\tilde{s}, y)}{\partial \tilde{s}} + \tilde{\kappa}(\tilde{s}, y)\tilde{Q}(\tilde{s}, y) = 0 \tag{30}$$

$$\frac{\partial \tilde{Q}(\tilde{s}, y)}{\partial \tilde{s}} - \tilde{\kappa}(\tilde{s}, y)\tilde{N}(\tilde{s}, y) + \tilde{p_n}(\tilde{s}, y) = 0. \tag{31}$$

Shear force in a layer

$$\begin{aligned}\tilde{Q}(\tilde{s}, y) &= \frac{\partial \tilde{M}(\tilde{s}, y)}{\partial \tilde{s}} = \frac{1}{1 + y\kappa}\frac{\partial \tilde{M}(\tilde{s}, y)}{\partial s} \\ &= \frac{1}{1 + y\kappa}\frac{\partial B\tilde{\kappa}(\tilde{s}, y)}{\partial s} = \frac{B}{1 + y\kappa}\frac{\partial}{\partial s}\left(\frac{\kappa}{1 + y\kappa}\right) \\ &= \frac{B}{1 + y\kappa}\left(\frac{1}{1 + y\kappa}\frac{\partial \kappa}{\partial s} + \frac{y\kappa}{(1 + y\kappa)^2}\frac{\partial \kappa}{\partial s}\right) \\ &= \frac{B}{(1 + y\kappa)^3}\frac{d\kappa}{ds} = \frac{B}{(1 + y\kappa)^3}\theta''.\end{aligned} \tag{32}$$

Membrance force in a layer

$$\begin{aligned}\frac{\partial \tilde{N}(\tilde{s}, y)}{\partial \tilde{s}} &= -\tilde{\kappa}(\tilde{s}, y)\tilde{Q}(\tilde{s}, y) = -\frac{\kappa}{1 + y\kappa}\tilde{Q}(\tilde{s}, y) \\ &= -\frac{\kappa}{1 + y\kappa}\frac{B}{(1 + y\kappa)^3}\frac{d\kappa}{ds}\end{aligned} \tag{33}$$

leads to

$$\begin{aligned}\frac{\partial \tilde{N}(s, y)}{\partial s} &= -\frac{B\kappa}{(1 + y\kappa)^3}\frac{d\kappa}{ds} = -\frac{B}{y^2}\frac{y\kappa}{(1 + y\kappa)^3}\frac{d(y\kappa)}{ds} \\ &= -\frac{B}{y^2}\frac{d}{ds}\left[\frac{1}{2}\left(\frac{y\kappa}{1 + y\kappa}\right)^2\right] = -\frac{B}{2}\frac{d}{ds}\left(\frac{\kappa}{1 + y\kappa}\right)^2.\end{aligned} \tag{34}$$

Integration respect to s, and taking into account of jump condition $\|\tilde{N}\|_\ell = 0$, we have

$$\tilde{N}(s, y) = -\frac{B}{2}\left(\frac{\kappa}{1 + y\kappa}\right)^2 + \underbrace{C(y)}_{=0} \tag{35}$$

Layer normal stress: Let us evaluate the normal stress applied by the neighbors to a given layer $\tilde{p_n}d\tilde{s}$ for $s < \ell$, we have

$$\begin{aligned}\tilde{p_n}d\tilde{s} &= \tilde{p_n}(1 + y\kappa)ds \\ &= -\left[\frac{\partial \tilde{Q}(\tilde{s}, y)}{\partial \tilde{s}} - \tilde{\kappa}(\tilde{s}, y)\tilde{N}(\tilde{s}, y)\right](1 + y\kappa)ds \\ &= -\left[\frac{\partial \tilde{Q}}{\partial s} - \kappa\tilde{N}\right]ds \\ &= -B\left[\frac{\partial}{\partial s}\left(\frac{1}{(1 + y\kappa)^3}\frac{d\kappa}{ds}\right) + \frac{\kappa}{2}\left(\frac{\kappa}{1 + y\kappa}\right)^2\right]ds,\end{aligned} \tag{36}$$

hence

$$\begin{aligned}\tilde{p_n}(1 + y\kappa) &= -B\left[\frac{\partial}{\partial s}\left(\frac{1}{(1 + y\kappa)^3}\frac{d\kappa}{ds}\right) + \frac{\kappa}{2}\left(\frac{\kappa}{1 + y\kappa}\right)^2\right] \\ &= -B\left[\frac{1}{(1 + y\kappa)^3}\frac{d^2\kappa}{ds^2} - \frac{3y}{(1 + y\kappa)^4}\left(\frac{d\kappa}{ds}\right)^2 + \frac{\kappa}{2}\left(\frac{\kappa}{1 + y\kappa}\right)^2\right],\end{aligned} \tag{37}$$

If denote $\Sigma(s, y)$ as the normal stress at the plate-plate interfaces. The normal force applied by the plate above the plate having coordinate y, over an interface element with length $d\tilde{s}$, is there $-\Sigma(s, y + h/2)$, the net force experienced by the plate from the adjacent plates is therefore $\Sigma(s, y + h/2) - \Sigma(s, y - h/2) = h\frac{\partial(1+y\kappa)\Sigma}{\partial y} = \tilde{p_n}d\tilde{s}$, which can be rewritten as

$$h\frac{\partial[(1 + y\kappa)\Sigma]}{\partial y} = \tilde{p_n}(1 + y\kappa) \tag{38}$$

This equation can be integrated with respect to y, using the free boundary conditions at top and bottom of the stack $\Sigma(s, \pm(nh/2 + c)) = 0$, if we start integration from the bottom cover at $y = -(\frac{nh}{2} + c)$, this yields the normal stress in the elastic backbone solution as

$$\begin{aligned}\Sigma(s, y) &= \frac{1}{h(1 + y\kappa)}\int_{-nh/2-c}^{y}(\tilde{p_n}(s, \xi)(1 + \xi\kappa))_{\text{Bottom}}\,d\xi, \\ &\quad y \in [-nh - c, -nh/2],\end{aligned} \tag{39}$$

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$$\Sigma(s,y) = \frac{1}{h(1+y\kappa)}\int_{-nh/2-c}^{-nh/2}(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Bottom}}\,d\xi + \frac{1}{h(1+y\kappa)}\int_{-nh/2}^{y}(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Core}}\,d\xi,\quad y\in[-nh/2-c, nh/2], \tag{40}$$

$$\Sigma(s,y) = \frac{1}{h(1+y\kappa)}\int_{-nh/2-c}^{-nh/2}(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Bottom}}\,d\xi + \frac{1}{h(1+y\kappa)}\int_{-nh/2}^{nh/2}(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Core}}\,d\xi + \frac{1}{h(1+y\kappa)}\int_{nh/2}^{y}(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Top}}\,d\xi,\quad y\in[nh/2-c, nh/2+c], \tag{41}$$

where $(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Bottom}} = -B_c[\frac{1}{(1+\xi\kappa)^3}\frac{d^2\kappa}{ds^2} - \frac{3\xi}{(1+\xi\kappa)^4}(\frac{d\kappa}{ds})^2 + \frac{\kappa}{2}(\frac{\kappa}{1+\xi\kappa})^2]$, $(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Core}} = -B[\frac{1}{(1+\xi\kappa)^3}\frac{d^2\kappa}{ds^2} - \frac{3\xi}{(1+\xi\kappa)^4}(\frac{d\kappa}{ds})^2 + \frac{\kappa}{2}(\frac{\kappa}{1+\xi\kappa})^2]$, and $(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Top}} = -B_c[\frac{1}{(1+\xi\kappa)^3}\frac{d^2\kappa}{ds^2} - \frac{3\xi}{(1+\xi\kappa)^4}(\frac{d\kappa}{ds})^2 + \frac{\kappa}{2}(\frac{\kappa}{1+\xi\kappa})^2]$.

Normal stress-singular contribution from two rollers: The expression in Eq.41 for the normal stress is valid away from the points $s=-\ell,0,\ell$, where point-like forces applied. The singular normal stress at the point of indentation $s=0$ is not needed because the sliding velocity of the plates is zero there by symmetry, implying that there is no frictional dissipation.

Due to the symmetry of deformation, we only need to derive the singular force at the roller $s=\ell$, where the loint-like net normal force $\tilde{p}_n^D$ is applied to each plate, and leads to the following balance of force and moments,

$$\|\tilde{Q}\|_\ell + \tilde{p}_n^D = 0,\ \|\tilde{N}\|_\ell = 0,\ \|\tilde{M}\|_\ell = 0, \tag{42}$$

where $\|f\|_\ell = f(\ell^+,y) - f(\ell^-,y)$ denotes the discontinuity of a function f across the point $s=\ell$, namely a jump at $s=\ell$. $\|\tilde{N}\|_\ell = 0$ has been used to determine $\tilde{N}$.

Since $M = B\tilde{\kappa}$, thus $\|M\|_\ell = M(\ell^+) - M(\ell^-) = 0$ implies $\kappa((\ell^+)) - \kappa(\ell^-) = 0$. Since beyond the point at $s=\ell$, the plates have no deformation but rigid rotation, it means that $\kappa(\ell^+) = 0$, therefore $\kappa(\ell^-) = 0$ and $\frac{d}{ds}\kappa(\ell^+) = 0$.

Substituting $\tilde{Q} = \frac{B}{(1+y\kappa)^3}\frac{d\kappa}{ds}$ into the Eq.42, we have $\|\tilde{Q}\|_\ell = \|\frac{B}{(1+y\kappa)^3}\frac{d\kappa}{ds}\|_\ell = \left(\frac{B}{(1+y\kappa)^3}\frac{d\kappa}{ds}\right)_{\ell+} - \left(\frac{B}{(1+y\kappa)^3}\frac{d\kappa}{ds}\right)_{\ell^-} = B\left(\frac{d\kappa}{ds}\right)_{\ell^+} - B\left(\frac{d\kappa}{ds}\right)_{\ell^-} = -B\left(\frac{d\kappa}{ds}\right)_{\ell^-}$, we have

$$-B\left(\frac{d\kappa}{ds}\right)_{\ell^-} + \tilde{p}_n^D = 0 \tag{43}$$

In the same way, we have the normal stress at ℓ, namely $\left(h\frac{\partial\Sigma(1+y\kappa)}{\partial y}\right)_\ell = (\tilde{p}_n(1+y\kappa))_\ell$, we have

$$\begin{aligned} -\frac{nh}{2} - c < y < -\frac{nh}{2} &: c\frac{\partial\Sigma}{\partial y} = \tilde{p}_n,\\ -\frac{nh}{2} < y < \frac{nh}{2} &: h\frac{\partial\Sigma}{\partial y} = \tilde{p}_n,\\ \frac{nh}{2} < y < \frac{nh}{2} + c &: c\frac{\partial\Sigma}{\partial y} = \tilde{p}_n, \end{aligned} \tag{44}$$

and surface boundary conditions $\Sigma^D(-nh/2-c) = |\boldsymbol{F}|$ and $\Sigma^D(nh/2+c) = 0$.

Similar to the treatment by Poincloux et al.,[1], for the book with two hardcover, the singular contribution to the transverse stress at $s=\ell$ can be obtained by

$$\Sigma^D(y) = |\boldsymbol{F}|(\frac{1}{2} - \frac{y}{nh+2c}), \tag{45}$$

which satisfies the surface conditions at both bottom and top covers, namely $\Sigma^D(\pm(nh/2+c)) = 0$, where $\boldsymbol{F} = -F_1\boldsymbol{e}_1 + \frac{F_2}{2}\boldsymbol{e}_2$ is the point-like force applied by the rollers below y and $|\boldsymbol{F}| = \sqrt{F_1^2 + \frac{F_2^2}{4}}$.

Sliding velocity: Similar to the treatment by Poincloux et al.,[1], we have the sliding displacement $u(s,y) = \tilde{s} - s$, then $\frac{\partial u}{\partial s} = \frac{\partial\tilde{s}}{\partial s} - 1 = (1+y\kappa) - 1 = y\kappa = y\frac{d\theta}{ds}$, hence

$$u(s,y) = y\theta(s). \tag{46}$$

Applying this to a single layer with thickness h, we have relative displacement at the interface

$$\delta(s,y) = \tilde{s}(s, y+\frac{h}{2}) - s(s, y-\frac{h}{2}) = h\theta. \tag{47}$$

The time derivative of this relation gives the relative sliding velocity

$$\dot{\delta} = (s,y) = h\dot{\theta}(s). \tag{48}$$

POWER DISSIPATION BY FRICTION FORCES

In order to take into account the contribution of the interlayers friction, we need to investigate the energy dissipation caused by the friction. Without loss generality, we assume the dynamical process is quasi-static.

The interlayer energy dissipation between the two rollers is

$$\mathcal{P}_1 = \int_{-L}^{L}\mu(\Sigma)|\dot{\delta}|d\tilde{s} = \mu\int_{-L}^{L}|\dot{\delta}|(\Sigma)(1+y\kappa)ds \tag{49}$$

From Eq.38, $h\frac{\partial[(1+y\kappa)\Sigma]}{\partial y} = \tilde{p_n}(1+y\kappa)$, we have $h(1+y\kappa)\Sigma = \int\tilde{p_n}(1+y\kappa)dy$ for a plate with thickness h. Av-

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eraging along the y, we have

$$\begin{aligned}\mathcal{P}_1 &= \int_{-L}^{L} \mu(\Sigma)|\dot{\delta}|d\tilde{s} = \mu\int_{-L}^{L}|\dot{\delta}|\frac{1}{h}\left(\int \tilde{p_n}(1+y\kappa)dy\right)ds\\ &= \mu\int_{-L}^{L} c|\dot{\theta}|\frac{1}{c}\left(\int_{-nh/2-c}^{-nh/2}\tilde{p_n}(1+y\kappa)dy\right)ds\\ &+ \mu\int_{-L}^{L} h|\dot{\theta}|\frac{1}{h}\left(\int_{-nh/2}^{nh/2}\tilde{p_n}(1+y\kappa)dy\right)ds\\ &+ \mu\int_{-L}^{L} c|\dot{\theta}|\frac{1}{c}\left(\int_{nh/2-c}^{nh/2}\tilde{p_n}(1+y\kappa)dy\right)ds\\ &= \mu\int_{-L}^{L}|\dot{\theta}|(J_{\text{Bottom}}+J_{\text{Core}}+J_{\text{Top}})ds\end{aligned} \tag{50}$$

Taking into account symmetry and $\theta = 0$ if $s \in [\ell, L]$, we have

$$\mathcal{P}_1 = 2\int_0^{\ell}\mu|\dot{\theta}(s)|R(s)ds \tag{51}$$

where $R(s) = J_{\text{Bottom}} + J_{\text{Core}} + J_{\text{Top}}$ and

$$\begin{aligned}J_{\text{Bottom}} &= \int_{-nh/2-c}^{-nh/2}(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Bottom}}\,d\xi,\\ J_{\text{Core}} &= \int_{-nh/2}^{nh/2}(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Core}}\,d\xi,\\ J_{\text{Top}} &= \int_{nh/2}^{nh/2+c}(\tilde{p_n}(s,\xi)(1+\xi\kappa))_{\text{Top}}\,d\xi,\end{aligned} \tag{52}$$

We can complete the above integrations and give us

$$\begin{aligned}J_{\text{Bottom}} &= -B_c(c_{11}\kappa'' - c_{12}\kappa'^2 + c_{13}\kappa^3),\\ J_{\text{Core}} &= -B(c_{21}\kappa'' - c_{22}\kappa'^2 + c_{23}\kappa^3),\\ J_{\text{Top}} &= -B_c(c_{31}\kappa'' - c_{32}\kappa'^2 + c_{33}\kappa^3),\end{aligned} \tag{53}$$

where the coefficients $c_{11} = -\frac{1}{2\kappa}(-\frac{h\kappa n}{2}+1)^{-2} + \frac{1}{2\kappa}[(-\frac{nh}{2}-c)\kappa+1]^{-2}$, $c_{12} = \frac{1}{2\kappa^2}(\frac{3h\kappa n}{2}-1)(-\frac{h\kappa n}{2}+1)^{-3} + \frac{1}{2\kappa^2}[3(\frac{nh}{2}+c)\kappa+1][(-\frac{nh}{2}-c)\kappa+1]^{-3}$, $c_{13} = -\frac{1}{\kappa}(-\frac{h\kappa n}{2}+1)^{-1} + \frac{1}{\kappa}((-\frac{nh}{2}-c)\kappa+1)^{-1}$, $c_{21} = -\frac{1}{2\kappa}(\frac{h\kappa n}{2}+1)^{-2} + \frac{1}{2\kappa}(-\frac{h\kappa n}{2}+1)^{-2}$, $c_{22} = -\frac{1}{2\kappa^2}(\frac{3h\kappa n}{2}+1)(\frac{h\kappa n}{2}+1)^{-3} - \frac{1}{2\kappa^2}(\frac{3h\kappa n}{2}-1)(-\frac{h\kappa n}{2}+1)^{-3}$, $c_{23} = -\frac{1}{2\kappa^2}(\frac{3h\kappa n}{2}+1)(\frac{h\kappa n}{2}+1)^{-3} - \frac{1}{2\kappa^2}(\frac{3h\kappa n}{2}-1)(-\frac{h\kappa n}{2}+1)^{-3}$, $c_{31} = -\frac{1}{2\kappa}((\frac{nh}{2}+c)\kappa+1)^{-2} + \frac{1}{2\kappa}(-\frac{h\kappa n}{2}+1)^{-2}$, $c_{32} = -\frac{1}{2\kappa^2}(3(\frac{nh}{2}+c)\kappa+1)((\frac{nh}{2}+c)\kappa+1)^{-3} - \frac{1}{2\kappa^2}(\frac{3h\kappa n}{2}-1)(-\frac{h\kappa n}{2}+1)^{-3}$, $c_{33} = -\frac{1}{\kappa}((\frac{nh}{2}+c)\kappa+1)^{-1} + \frac{1}{\kappa}(-\frac{h\kappa n}{2}+1)^{-1}$.

Collecting the above results, we can rewrite $R(s)$ as

$$\begin{aligned}R(s) &= [(c_{11}+B_{31})B_c + c_{21}B]\theta'''\\ &- [(c_{12}+B_{32})B_c + c_{22}B]\theta''\\ &+ [(c_{13}+B_3)B_c + c_{23}B]\theta'\end{aligned} \tag{54}$$

The power $\mathcal{P}_2$ dissipated by friction caused by the Dirac-like contribution at $s = \ell$ gives

$$\begin{aligned}\mathcal{P}_2 &= \mu\Sigma^D|\dot{\delta}| = \mu|\boldsymbol{F}||\dot{\delta}|(\frac{1}{2}-\frac{y}{nh+2c})\\ &\approx \mu|\boldsymbol{F}|[\frac{1}{c}\int_{-nh/2-c}^{-nh/2}c|\dot{\theta}|(\frac{1}{2}-\frac{y}{nh+2c})dy\\ &+ \frac{1}{h}\int_{-nh/2}^{nh/2}h|\dot{\theta}|(\frac{1}{2}-\frac{y}{nh+2c})dy\\ &+ \frac{1}{c}\int_{nh/2}^{nh/2+c}c|\dot{\theta}|(\frac{1}{2}-\frac{y}{nh+2c})dy]\\ &= \mu|\boldsymbol{F}||\dot{\theta}|\int_{-nh/2-c}^{nh/2+c}(\frac{1}{2}-\frac{y}{nh+2c})dy\\ &= \mu(\frac{nh}{2}+c)|\boldsymbol{F}||\dot{\theta}\end{aligned} \tag{55}$$

Finally, the power dissipated by friction in the entire s-tack is

$$\begin{aligned}\mathcal{P}_\mu &= \mathcal{P}_1 + 2\mathcal{P}_2\\ &= \mu(nh+2c)|\boldsymbol{F}|\dot{\theta} + 2\int_0^{\ell}\mu|\dot{\theta}(s)|R(s)ds,\end{aligned} \tag{56}$$

where the factor 2 is because there are two rollers.

The first term in Eq.56 represents the dissipation in the stack by the pointlike contact force at the supports, while the second term is the dissipation everywhere elase in the stack. By symmetry, there is no sliding hence no dissipation at the poking point at $s = 0$. The poking force is then derived by a globe balance of power as

$$F_2\dot{w}_0 = -\dot{\mathcal{E}} + P_\mu, \tag{57}$$

hence $F_2 = (-\dot{\mathcal{E}} + P_\mu)/\dot{w}_0$. Whereas $\dot{w}_0$ and $\dot{\mathcal{E}}$ change sign between loading and unloading, P_μ does not change, implying that F_2 is different during the loading and unloading phases.

THE SMALL DEFLECTION AND ITS GENERALIZATION

For small deflection $w_0/a \ll 1$ and slender stack $nh/a \ll 1$. In this case, we have approximations $\cos\theta \approx 1$, $\sin\theta \approx \theta$ and $\ell = a$. The linearized equilibrium equation and boundary conditions can be obtained as follows

$$(nB + 2B_c)\theta'' + \frac{F_2}{2} = 0, \quad \frac{dx_2}{ds} = \theta, \tag{58}$$

$$x_2(0) = -w_0,\ x_2(a) = 0,\ \theta(0) = 0,\ \frac{d\theta}{ds}(a) = 0. \tag{59}$$

The solutions are

$$\begin{aligned}\theta(s) &= \frac{F_2a^2}{2}\frac{1}{nB+2B_c}\left[\frac{s}{a}-\frac{1}{2}(\frac{s}{a})^2\right],\\ x_2(s) &= \frac{F_2a^3}{4}\frac{1}{nB+2B_c}\left[-[1-(\frac{s}{a})^2]+\frac{1}{3}[1-(\frac{s}{a})^3]\right].\end{aligned} \tag{60}$$

Notice $x_2(0) = -w_0$, we get a linear indentation relation $F_2 = \frac{6}{a^3}(nb + 2B_c)w_0$. This linear law gives the elastic energy of the stack, namely $\mathcal{E} = \frac{1}{2}\int_0^{w_0} F_2 dw_0 = \frac{3}{a^3}(nB + 2B_c)w_0^2$, and its power $\dot{\mathcal{E}} = \frac{6}{a^3}(nB + 2B_c)w_0\dot{w}_0$.

Similar to the point of view Poincloux [1], for linear case, one finds $\tilde{p}_n = 0$ and each layer is in equilibrium and gives normal stress $\Sigma(s, y) = 0$. Therefore, the power dissipation of interlayeres is zero, namely $\mathcal{P}_1 = 0$. The only power dissipation is from rollers, reads $\mathcal{P}_\mu = \mu(nB + 2B_c)\frac{F_2}{2}\theta(a)$, notice $\theta(a) = \frac{F_2 a^2}{4(nb+2B_c)} = \frac{3}{2a}w_0$, hence the total power dissipation is

$$\mathcal{P}_\mu = 2\mathcal{P}_2 = \frac{9}{2}\mu\frac{nh + 2c}{a^4}(nB + 2B_c)w_0|\dot{w}_0| \quad (61)$$

From power energy balance, $F_2\dot{w}_0 = -\dot{\mathcal{E}} + \mathcal{P}_\mu$, we have the indentation force at rollers in loading (+) and unloading (-) as

$$F_2 = \frac{6}{a^2}K_{\mathrm{Lin}}\frac{w_0}{a}. \quad (62)$$

where the incremental stiffness with internal friction is

$$K_{\mathrm{Lin}} = (nB + 2B_c)\left(1 \pm \frac{3}{4}\mu\frac{nh + 2c}{a}\right). \quad (63)$$

This indicates the stiffness caused by the internal friction is stiffening upon loading and softening upon unloading.

If we denote the bending stiffness of the backbone without friction as

$$K_{m,bb} = nB + 2B_c, \quad (64)$$

then we have the bending stiffness ratio influenced by the internal friction as follows

$$\frac{K_{\mathrm{Lin}}}{K_{m,bb}} - 1 = \pm\frac{3}{4}\mu\frac{nh + 2c}{a}. \quad (65)$$

The energy dissipation during one cycle of loading and unloading is given by

$$\begin{aligned} D_{\mathrm{Lin}} &= \int_0^{w_0^{\max}} F_2^+ dw_0 - \int_0^{w_0^{\max}} F_2^- dw_0 \\ &= \frac{9}{2}\mu\left(\frac{w_0^{\max}}{a}\right)^2\left(\frac{nh + 2c}{a}\right)(nB + 2B_c). \end{aligned} \quad (66)$$

If the book covers are layout in symmetric to the centerline of the stack, i.e., $[c_1|c_2...|c_k|\underbrace{h...h}_{n}|c_k|...|c_2|c_1]$, the above results in small deflection can be approximately generalized, because the balance equation in this case is

$$(nB + 2\sum_{i=1}^{k} c_i)\theta'' + \frac{F_2}{2} = 0, \quad \frac{dx_2}{ds} = \theta, \quad (67)$$

$$x_2(0) = -w_0,\ x_2(a) = 0,\ \theta(0) = 0,\ \frac{d\theta}{ds}(a) = 0. \quad (68)$$

hence

$$F_2 = \frac{6}{a^2}K_{\mathrm{Lin}}\frac{w_0}{a}. \quad (69)$$

where the incremental stiffness with internal friction is

$$K_{\mathrm{Lin}} = \left(nB + 2\sum_{i=1}^{k} B_i\right)\left(1 \pm \frac{3}{4}\mu\frac{nh + 2\sum_{i=1}^{k} c_i}{a}\right). \quad (70)$$

This indicates the stiffness caused by the internal friction is stiffening upon loading and softening upon unloading.

The energy dissipation during one cycle of loading and unloading is given by

$$\begin{aligned} D_{\mathrm{Lin}} &= \int_0^{w_0^{\max}} F_2^+ dw_0 - \int_0^{w_0^{\max}} F_2^- dw_0 \\ &= \frac{9}{2}\mu\left(\frac{w_0^{\max}}{a}\right)^2\left(\frac{nh + 2\sum_{i=1}^{k} c_i}{a}\right)(nB + 2\sum_{i=1}^{k} B_i). \end{aligned} \quad (71)$$

where $B_i = \frac{Ec_i^3 W}{12(1-\nu_i^2)}$.

NUMERICAL SIMULATIONS

For numerical validation, we take the same data from Poincloux et al. [1] and carry out some comparisons studies for different combination of parameters. Data for all numerical calculations are: length $L = 110$ [mm], width $W = 30$ [mm], thickness $h = 0.286$ [mm], the Young modulus $E = 2.4$ GPa, the Passion ratio $\nu = 0.44$ and friction coefficient $\mu = 0.52$.

Our formulations are numerically calculated by our own Matlab code, which uses the function *ode*45 to compute the solution of Eq.27. The FEM results are simulated by ABAQUS.

Case 1 as shown in Fig.4: When assuming that the total thickness of the structure of the book is unchanged, as shown in Fig.4, in the case of $c/h = 10$ and $n = 15$, our results are compared with Poincloux et al. [1] who set $n = 35$, and it is found that the angle of θ of two studies are almost overlap on each other. This means that the relationship between its angles of θ will be consistent with the arc length $\tilde{s}$, when the total thickness of the book structure is equal. The geometric reason behind this consistent is coming from the Kirchhoff assumption on the normal vector.

Case study 2 as shown in Fig.5: Finite element modeling of a book with $c/h = 15$ and $n = 10$ is performed and comparison with the results calculated by formulation in this paper. The FEM results and our results were obtained by ABAQUS and our own Matlab code, respectively. The figure shows that the results are in excellent agreement with each other.

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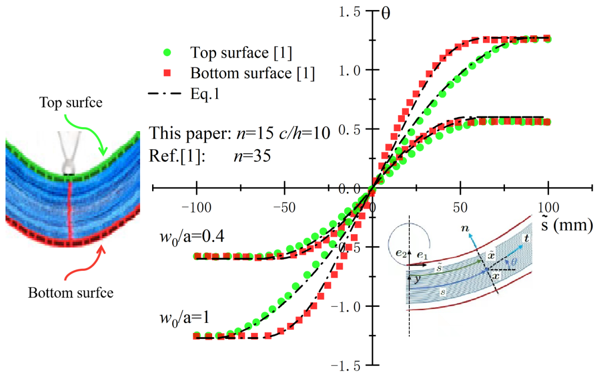

FIG. 4: Schematic diagram of the geometric quantities used in Eq. 7. Tangent angle $\theta(s$ versus arc length $\tilde{s}$ along the upper- and lowermost plates (green and red symbols, respectively). The predictions (dashed lines) were obtained by integrating Eq.27.

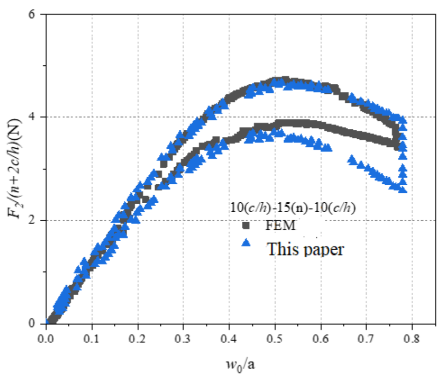

FIG. 5: Validation by FEM

Case study 3 as shown in Fig.6: Assume total thickness of the book is unchange as $n + 2c/h = 35$, from the Fig. 6, we see that, as the thickness of the cover increases, although the total thickness has not changed and the number of layers decreases, its energy consumption capacity still increases, that is, the area of the parcel of its loading-unloading curve increases with the increase of the thickness of the cover.

CONCLUSIONS

We clearly summarized the method proposed by Poincloux et al.[1] and clarified the process of deriving governing equations, in particular, we revealed that the key points to the success of their method were the averaging calculations. In the light of the breakthrough work by Poincloux et al.[1], we successfully generalized the formulations from books without covers to the books with hardcover and obtained the exact solution of smal-

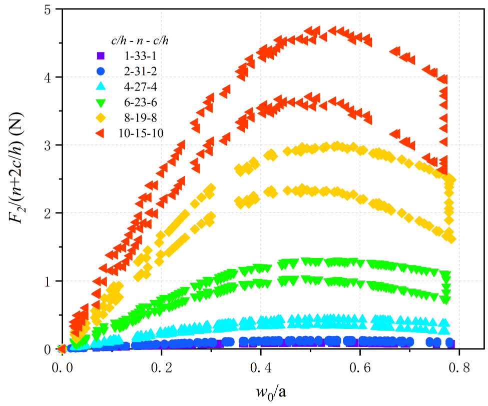

FIG. 6: Fixed total book thickness, the thickness of both covers and core vary.

l deflection of the hardcover book. Numerical analysis found that the hardcover of the books had a great impact on their bending ability as well as energy dissipation. Our investigations shown that the hardcover are more essential than the core layers in terms of bending response as well as energy absorption. When considering internal friction, the energy dissipation within per loading-unloading cycle can be made to vary by a considerable amount. The study here helps to understand the mechanical interactions behavior in between of geometry, friction and elasticity. Our research shows that Poincloux's method is a fair general and worth applying to other similar studies where the friction must be considered [18, 19].

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