
On the Cross-Scale Prospects of the Logarithmically Corrected Gravitational Potential: From Black Hole Singularities to Galactic Rotation

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Posted Date: 14 April 2026

doi: 10.20944/preprints202512.1663.v14

Keywords: black hole; singularity; general relativity; gravitational potential; quantum gravity; event horizon telescope (EHT); Metric; Geodesic; AdS/CFT correspondence; quantum vortex; Logarithmic Correction; Einstein Field Equations; Black Hole Shadow; Sgr A*; M87*; NGC315; high-velocity stars; S62; S4714; S4711; S4716; dark matter; galaxy rotation curves; milky way galaxy; Andromeda Galaxy; NGC2974 galaxy; gravitational lensing; bullet cluster; equivalence principle; MOND; modified gravity



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Article

On the Cross-Scale Prospects of the Logarithmically Corrected Gravitational Potential: From Black Hole Singularities to Galactic Rotation

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Abstract

We derive an effective gravitational potential $\Phi_{halo}(r) \sim -[\ln(r/r_*) + 1]/r$ from the asymptotic behavior of dark matter halo models. At microscopic scales, the logarithmic term changes sign, producing repulsion that prevents matter from collapsing into a singularity. The corresponding logarithmically corrected Schwarzschild metric yields parameter-free, a priori predictions for the shadows of Sgr A* and M87* that agree with Event Horizon Telescope observations. Six falsifiable predictions for unobserved black holes, particularly NGC315, can discriminate this metric from the Kerr solution; we also make falsifiable predictions for the periastron precession of stars S4711 and S4716 based on the same framework (Appendix D). On galactic scales, the same logarithmic term fits rotation curves of the Milky Way, Andromeda, and NGC2974 using only ordinary matter, and passes the Bullet Cluster lensing test. Tidal effects in the Solar System are far below current experimental limits, ensuring consistency with the equivalence principle and parameterized post-Newtonian tests. We further derive the modified field equations via coarse-grained variation (Appendix B) from the effective action of a quantum vortex background, thus providing a more complete theoretical bridge to the modified Poisson equation and metric used in the main text. This effective theoretical framework indicates that various gravitational phenomena from black holes to galaxies may share a common quantum topological origin. It provides a unified, testable alternative to the dark matter problem, and also points out a potential path for the observable detection of quantum gravity effects.

Keywords: black hole; singularity; general relativity; gravitational potential; quantum gravity; Event Horizon Telescope (EHT); metric; geodesic; AdS/CFT correspondence; quantum vortex; logarithmic correction; einstein field equations; black hole shadow; Sgr A*; M87*; NGC315; high-velocity stars; S62; S4714; S4711; S4716; dark matter; galaxy rotation curves; milky way galaxy; andromeda galaxy; NGC2974 galaxy; gravitational lensing; bullet cluster; equivalence principle; MOND; modified gravity

1. Introduction

Modern astrophysics and gravitational theory have long been faced with two major cross-scale challenges. At the microscopic scale of black holes, the singularity predicted by classical General Relativity (GR) features infinite curvature, which violates the requirement of finiteness for physical quantities in quantum mechanics, and the "information paradox" caused by Hawking radiation remains unsolved. At the macroscopic scale of galaxies, the observed rotation speeds of peripheral stars and gas are much higher than the limit that can be maintained by the gravity of visible matter. The mainstream Λ CDM model has to rely on the unproven hypothesis of dark matter halos, and there is tension between its small-scale predictions and observations.

Traditional theories provide fragmented explanations for these two challenges: black hole physics relies on the Kerr metric (which requires posterior fitting of spin and inclination), while galactic dynamics depends on the dark matter hypothesis, both lacking a unified physical core. More critically, these theories either suffer from internal incompleteness (e.g., the singularity) or lack a

direct physical carrier (e.g., dark matter particles). In this work, we adopt a bottom-up effective field theory approach. Without presupposing the specific form of the microscopic mechanism, we derive a logarithmically corrected gravitational potential from the universal asymptotic behavior of dark matter halo models, and on this basis, deduce cross-scale observable effects. The microscopic picture presented in the Appendix is for heuristic discussion only and does not serve as a theoretical premise.

We identify a universal logarithmic asymptotic behavior of the extra gravitational potential from dark matter halo models:

$$\Phi_{halo}(r) \sim -\frac{\ln(r/r_*) + 1}{r}$$

Through the analysis of this potential, we further discover a physical picture: at distances approaching the black hole singularity ($r < r_* \approx 8.792 \times 10^{-11} \text{m}$), the negative contribution of $\ln(r/r_*)$ renders the extra gravitational potential repulsive, preventing the gravitational collapse of matter into a singularity. In the strong gravitational field near black holes and the weak gravitational field of galaxies, the positive contribution of $\ln(r/r_*)$ provides an additional attractive gravitational force (calculable solely from the mass of ordinary matter), thereby maintaining the high orbital speed of stars and the flattening of rotation curves.

This mechanism requires no artificial mathematical renormalization, nor does it introduce new unobservable entities (such as higher-dimensional strings or dark matter particles). It achieves two key goals solely through the dynamics of ordinary matter: (1) singularity resolution; (2) flattening of galactic rotation curves. To explain this mechanism, we also provide a speculative (for discussion) microscopic physical picture in the Appendix: relying on quantum vortices (a physical carrier with solid experimental support) and the AdS/CFT correspondence (a mathematical structure originated from string theory to describe non-local entanglement), we unify black hole physics and galactic dynamics under a single mechanism. This provides a potential solution to the cross-scale gravitational challenges based solely on general relativity and quantum mechanics in four-dimensional spacetime, without relying on conventional pre-assumed physical models (such as the hypothesis of higher-dimensional strings or dark matter particles that have eluded detection for decades).

2. Universal Logarithmic Asymptotics of Dark-Matter Halo Models

A wide variety of dark-matter halo profiles have been proposed to explain the flat rotation curves of galaxies, including cuspy profiles derived from N-body simulations and phenomenological cored profiles motivated by observations. Despite their apparent diversity, we show that all commonly used halo models converge asymptotically to the same effective gravitational behavior, characterized by a logarithmic potential. This universality strongly suggests that the logarithmic term represents the true physical content of halo modeling, while the detailed density profiles merely encode different regularizations of the same asymptotic structure.

2.1. General Condition for Flat Rotation Curves

For a test particle on a circular orbit, the centripetal acceleration satisfies $\frac{v^2(r)}{r} = g(r) = \frac{GM(r)}{r^2}$, where $M(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$. A flat rotation curve: $v(r) \rightarrow v_0 = \text{const}$, implies $g(r) \sim \frac{v_0^2}{r}$,

$M(r) \sim \frac{v_0^2}{G} r$. Differentiating, $\rho(r) = \frac{1}{4\pi r^2} \frac{dM}{dr} \sim \frac{1}{r^2}$. However, no realistic halo model maintains $\rho \sim r^{-2}$ at arbitrarily large radii, as this would lead to divergent total mass. Consequently, all viable models steepen to $\rho(r) \sim r^{-3}$ ($r \gg r_s$), which leads to

$$M(r) \sim \ln r, \quad g(r) \sim \frac{\ln r}{r^2} \quad (1)$$

(Note: In this and all subsequent sections, the expression $\ln r$ is understood as $\ln(r/r_*)$ where r_* is a universal reference scale. As shown in Section 3.2, the choice of r_* does not affect any physical observable because it can be absorbed into the renormalization of the Newtonian mass parameter.)

This logarithmic behavior is therefore not model-dependent but a mathematical consequence of mass convergence combined with extended flat rotation curves. In addition, we note that the gravitational acceleration with logarithmic behavior ($g(r) \sim \frac{\ln r}{r^2}$) will have a sign reversal of the logarithmic term " $\ln r$ " at extremely short (microscopic) distances, which is likely to form a dynamic mechanism that repels classical gravity ($g(r) \sim \frac{1}{r^2}$) to naturally "desingularize".

2.2. Cuspy Halo Models: NFW and Einasto [1,2,3]

2.2.1. NFW Profile

The NFW profile: $\rho(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2}$, satisfies $\rho(r) \sim r^{-3}$ ($r \gg r_s$).

Integrating:

$$M(r) \propto \ln\left(1 + \frac{r}{r_s}\right), \text{ and thus } g(r) = \frac{GM(r)}{r^2} \sim \frac{\ln r}{r^2}.$$

The logarithmic term therefore arises inevitably from the outer density tail, not from any detailed inner structure.

2.2.2. Einasto Profile

The Einasto profile: $\ln \rho(r) = \ln \rho_0 - \left(\frac{r}{r_0}\right)^\alpha$, $\alpha \ll 1$,

admits the expansion:

$$\left(\frac{r}{r_0}\right)^\alpha \simeq 1 + \alpha \ln \frac{r}{r_0} + \mathcal{O}(\alpha^2).$$

Hence: $\rho(r) \approx \rho_0 r^{-\alpha}$, which again steepens toward an effective r^{-3} behavior at large radii, yielding $M(r) \sim \ln r$.

The shape parameter α merely controls how rapidly the logarithmic regime is approached.

2.3. Cored Halo Models: Burkert and Pseudo-Isothermal

Cored profiles replace the inner cusp with a constant-density core but retain the same outer asymptotics.

For example, the Burkert profile: $\rho(r) = \frac{\rho_0}{(1+r/r_0)(1+(r/r_0)^2)}$, satisfies $\rho(r) \sim r^{-3}$ ($r \gg r_0$),

leading again to

$$M(r) \sim \ln r, \quad g(r) \sim \frac{\ln r}{r^2}.$$

Thus, core formation modifies only the inner boundary conditions, leaving the outer logarithmic behavior intact.

2.4. Self-Interacting and Wave Dark Matter [4,5]

Self-interacting dark matter (SIDM) and fuzzy/wave dark matter (FDM) models generate cores through microphysical mechanisms (collisions or quantum pressure). Nevertheless, in all cases the outer halo relaxes to an NFW-like tail, $\rho(r) \rightarrow r^{-3}$, ensuring $M(r) \sim \ln r$ and $g(r) \sim \frac{\ln r}{r^2}$.

Hence, these models do not introduce new large-scale gravitational behavior, but merely regulate the inner halo.

2.5. Universality of the Logarithmic Potential

Since $g(r) = \frac{d\Phi}{dr}$, the asymptotic form $g(r) \sim \frac{\ln r}{r^2}$ corresponds to an effective potential:

$$\Phi_{\text{halo}}(r) \sim -\frac{\ln r + 1}{r} \quad (2)$$

We emphasize that this logarithmic potential is not a peculiarity of any specific halo model, but a universal asymptotic structure shared by all viable dark-matter halo parametrizations.

3. Effective Theory: Logarithmically Corrected Gravitational Potential

3.1. Modified Poisson Equation

Based on the mathematical asymptotic behavior of dark matter halo density ($\rho(r) \sim r^{-3}$) summarized in Section 2, inspired by quantum tornado experiments (where vortices emerge in environments mimicking the vicinity of black holes) [6], and combined with the divergent behavior of GR itself near the singularity ($R_{trt}^r \propto r^{-3}$), we introduce a modified Poisson equation. Given the effective theory positioning of this work, the speculative microscopic derivation is detailed in Appendix A and B; this effective framework does not depend on the microscopic derivation, but is validated by consistency with observational data:

$$\nabla^2 \Phi = 4\pi G \left(M\delta^3(r) + \frac{kG_h M^2}{4\pi G r^3} \right) \quad (3)$$

where $M\delta^3(r)$ is the source term of the classical gravitational point mass ($\delta^3(r)$ is the three-dimensional Dirac delta function), and $\frac{kG_h M^2}{4\pi G r^3}$ is the additional gravitational source term (derived from the dark matter halo density $\rho(r) \sim r^{-3}$), which we tentatively describe as the "quantum gravity" (i.e., logarithmically corrected gravity) source term in this paper.

k is the relative intensity factor of non-local entanglement ($k = \frac{M_{BH,ref}}{M_{BH,topo}}$, $M_{BH,ref}$: reference black hole mass, $M_{BH,topo}$: topological black hole mass (equal to the target black hole mass in the strong-field regime: $M_{BH,topo} = M$). Usually, we take the supermassive black hole Sgr A* at the center of the Milky Way as the reference: $k = \frac{M_{SgrA}}{M_{BH,topo}}$. If we take the central black hole of other galaxies as the reference, we need to transform the reference G_h accordingly. For example, taking M87* as the reference: $G_{h,M87*} = \frac{M_{SgrA}}{M_{M87*}} G_h$, then $k_{M87*} = \frac{M_{M87*}}{M_{BH,topo}}$, so $k_{M87*} G_{h,M87*} = \frac{M_{M87*}}{M_{BH,topo}} \cdot \frac{M_{SgrA}}{M_{M87*}} G_h = k G_h$, which shows that the value of kG_h is independent of the selected reference black hole $M_{BH,ref}$.

$$G_h = \frac{\hbar c^2 G^3}{8} \quad (4)$$

This expression is derived from the theoretical framework based on quantum vortices and nested AdS/CFT correspondence [7] in Appendix B, rather than a posteriori fitting (especially the subsequent multi-scale observational tests can hardly be completed by fitting this fixed value). Although the derivation in the Appendix is not yet rigorous, it provides a first-principles motivation for the value of G_h .

G_h is the effective coupling constant describing the strength of the logarithmic correction. Within the effective theory framework, its dimensionality is determined by the dimensional matching between the correction term $\frac{kG_h M^2}{4\pi G r^3}$ and the Newtonian term $M\delta^3(r)$ in the Poisson equation. From the standard Poisson equation $\nabla^2 \Phi = 4\pi G \rho$, both the correction term and the Newtonian term represent mass density, with the dimensionality of $kg \cdot m^{-3}$, thus $[G_h] = kg^{-2}m^3s^{-2}$. For consistency with subsequent observational calculations, we adopt the numerical form of G_h as $G_h = \hbar c^2 G^3 / 8 \approx 3.5224 \times 10^{-49} kg^{-2}m^3s^{-2}$, where $\hbar$, c , and G are the reduced Planck constant, the speed of light in vacuum, and Newton's gravitational constant, respectively. This expression links the correction term to fundamental physical constants, while its dimensionality is determined by the self-consistency of the effective theory and does not rely on the "dimensional

compactification" hypothesis (speculative discussion in Appendix B). At the effective theory level, G_h should be regarded as a phenomenological parameter describing the strength of the logarithmic correction, whose value can be cross-validated by subsequent multi-scale observations (e.g., black hole shadows, velocities of high-speed stars, galactic rotation curves).

3.2. Corrected Gravitational Potential with Logarithmic Term

Solving the above modified Poisson equation under static spherical symmetry, we obtain the general solution:

$$\Phi(r) = -\frac{GM}{r} - \frac{kG_h M^2 (\ln r + 1)}{r} - \frac{C_1}{r} + C_2$$

With the boundary condition that $\Phi(r) \rightarrow 0$ as $r \rightarrow \infty$, we have $C_2 = 0$, and the term $-C_1/r$ can be absorbed into the classical Newtonian term $-GM/r$. The expression is thus simplified to

$$\begin{aligned} \Phi(r) &= -\frac{GM}{r} - \frac{kG_h M^2 (\ln r + 1)}{r} \\ \Leftrightarrow \Phi(r) &= -\frac{GM}{r} - \frac{kG_h M^2 [\ln(r/r_*) + 1]}{r} \end{aligned} \quad (5)$$

This equation consists of two terms:

- **Classical gravitational term** $-\frac{GM}{r}$: Dominates conventional gravitational effects, consistent with Newtonian gravity and the weak-field approximation of general relativity.
- **Quantum gravitational logarithmic term** $-\frac{kG_h M^2 (\ln r + 1)}{r}$ (consistent with Equation (2)): Serves as the core cross-scale correction term. Its effect depends on the magnitude of the distance r —exhibiting repulsive behavior at short distances (black hole "singularity" scale) and gravitational enhancement at long distances (galaxy scale). Essentially, it is likely a macroscopic manifestation of nonlocal entanglement of quantum vortices under the hierarchical nested structure ($AdS_2/CFT_1 \subseteq AdS_3/CFT_2 \subseteq AdS_4/CFT_3$).

• **Note on scale normalization**: The logarithmic term is written with a reference scale r_* to ensure a dimensionless argument. This scale is defined by the condition $\Phi(r_*) = 0$, which gives $\ln(r_*/r_*) = 0$ and therefore r_* cancels out in the definition. Expanding the term:

$$-\frac{kG_h M^2 [\ln(r/r_*) + 1]}{r} = -\frac{kG_h M^2 (\ln r + 1)}{r} + \frac{kG_h M^2 \ln r_*}{r}$$

The last term $\propto 1/r$ is indistinguishable from a renormalization of the Newtonian mass: $GM_{eff} = GM - kG_h M^2 \ln r_*$. In any astronomical observation where GM is calibrated dynamically (e.g., via orbital motions or lensing), the observable quantity is GM_{eff} , not the bare GM . Consequently, the value of r_* does not affect any physical prediction. The calculation results obtained by normalizing r_* to 1m and using $\ln(r/1)$ are completely equivalent to those obtained by strictly using $\ln(r/r_*)$. In all practical calculations (black hole shadow, high-velocity stars, galaxy rotation curves, gravitational lensing, etc.), directly substituting the numerical value of r in meters and taking its natural logarithm $\ln r$ is equivalent to $\ln(r/1)$. This convention applies throughout the paper and will not be reiterated in subsequent sections.

• If the quantum gravitational effect under non-local entanglement is not considered ($k = 0$, i.e., ignoring the black hole: $M_{BH,ref} = 0$), the gravitational potential automatically degenerates into the classical gravitational potential: $\Phi(r) = -\frac{GM}{r}$, and the framework also naturally degrades to the classical gravitational framework.

3.3. Cross-Scale Physical Nature of the Logarithmic Term

The unique properties of the logarithmic term $\ln r$ are the key to realizing "short-range repulsion and long-range attraction":

- When $r \rightarrow 0$ (black hole core region): $\ln r$ tends to negative infinity, and the quantum gravitational term transforms into a strong repulsive potential. When $r < r_* = e^{-1 - \frac{G}{kG_h M}} = e^{-1 - \frac{G}{G_h M_{SgrA*}}} \approx 8.792 \times 10^{-11} \text{ m}$ (in the strong field regime, $k = M_{BH,ref}/M_{BH,topo} = M_{SgrA*}/M$), $\Phi(r) > 0$ in the total potential, dynamically preventing matter from collapsing into a singularity.

- When r is sufficiently large (galactic peripheral region): $\ln r$ is a positive finite value, and the quantum gravitational term provides additional gravity logarithmically dependent on distance, compensating for the insufficient gravity of visible matter and maintaining the stable rotational velocity of stars.

This characteristic stems from the monotonicity and boundary behavior of the logarithmic function. No additional adjustment of physical mechanisms is required; a single mathematical form can adapt to the scale transition from the microscopic to the macroscopic, reflecting the simplicity and self-consistency of the theory.

4. Black Hole Scale Application: Physical Avoidance of Singularity, Shadow Prediction and High-Speed Stars

4.1. Physical Avoidance of Singularity

In the core region of a black hole, the quantum repulsive potential dominated by the logarithmic term plays a central role:

- Suppression of curvature divergence:** The repulsive potential prevents matter from reaching $r = 0$ and avoids the divergence of spacetime curvature, thus realizing the dynamical avoidance of singularities without the need for renormalization.
- A potential mechanism for resolving the information paradox:** The repulsive potential excites virtual particles from vacuum fluctuations into real particles, which tunnel out of the black hole horizon through the nested AdS/CFT correspondence ($AdS_2/CFT_1 \subseteq AdS_3/CFT_2 \subseteq AdS_4/CFT_3$). These real particles carry information away from the black hole, and the black hole loses mass synchronously. This mechanism is conducive to making black hole physics satisfy the unitarity of quantum mechanics, namely the principle of information conservation.

4.2. Logarithmically Corrected Schwarzschild Metric and A Priori Prediction of Black Hole Shadows

Based on the corrected gravitational potential, the quantum-corrected Schwarzschild metric is derived (substituting $\Phi(r)$ into the relationship between the metric and gravitational potential under the weak-field approximation of general relativity):

$$ds^2 = -A(r)c^2 dt^2 + B(r)dr^2 + r^2 d\Omega^2$$

$$A(r) \approx 1 + \frac{2\Phi(r)}{c^2} = 1 - \frac{2GM}{c^2 r} - \frac{2kG_h M^2 (\ln r + 1)}{c^2 r} \quad (6)$$

$$B(r) \approx 1 - \frac{2\Phi(r)}{c^2} = 1 + \frac{2GM}{c^2 r} + \frac{2kG_h M^2 (\ln r + 1)}{c^2 r} \quad (7)$$

Analysis of Metric Singularity Resolution

For motion in the equatorial plane with $\theta = \pi/2$ and $\dot{\theta} = 0$, the Lagrangian is given by: $L = -Ac^2\dot{t}^2 + B\dot{r}^2 + r^2\dot{\phi}^2$. From Killing symmetry, the conserved energy and angular momentum are obtained as: $E = A(r)c^2\dot{t}$ and $L = r^2\dot{\phi}$ (with respect to the affine parameter λ). The general

"first integral" radial equation is derived (where $\sigma = 1$ for timelike geodesics (proper time τ); $\sigma = 0$ for null geodesics (affine parameter λ)):

$$\dot{r}^2 = \frac{E^2 - A(r) \left(\frac{c^2 L^2}{r^2} + \sigma c^4 \right)}{ABc^2}$$

along with: $\dot{t} = \frac{E}{Ac^2}$ and $\dot{\phi} = \frac{L}{r^2}$. We define the effective potential as: $V_\sigma(r) = A(r) \left(\sigma c^4 + \frac{c^2 L^2}{r^2} \right)$.

Analysis of timelike geodesics ($\sigma = 1$): the effective potential $V_{timelike}(r) = A(r) \left(c^4 + \frac{c^2 L^2}{r^2} \right)$ tends to $+\infty$ as $r \rightarrow 0$ (since $A(r) \approx 1 - \frac{2GM}{c^2 r} - \frac{2kG_h M^2 (\ln r + 1)}{c^2 r} \rightarrow +\infty$). For any finite energy E and angular momentum L , the radial equation

$$\dot{r}^2 = \left(\frac{dr}{d\tau} \right)^2 = \frac{E^2 - A(r) \left(\frac{c^2 L^2}{r^2} + c^4 \right)}{ABc^2} = \frac{E^2 - V_{timelike}(r)}{ABc^2}$$

must satisfy $E^2 = V_{timelike}(r_{min})$ at some $r_{min} > 0$ —a bounce occurs, meaning the particle cannot physically reach $r = 0$. The analysis for null geodesics ($\sigma = 0$) is essentially the same (the effective potential $V_{null}(r) = A(r) \left(\frac{c^2 L^2}{r^2} \right) \rightarrow +\infty$ as $r \rightarrow 0$), so light cannot physically reach $r = 0$ either.

Note: Although this corrected metric is derived from the weak-field approximation of GR, it should not be understood merely as an extrapolation of a weak-field potential construction. Instead, it should be interpreted as the strong-field extended solution of the corrected field equations corresponding to the effective action variation in Appendix B, under the structure of holographic screen renormalization group flow (Appendix C). Moreover, its dynamical singularity avoidance mechanism is self-consistent with the analysis of the corrected gravitational potential.

This metric does not require fitting of black hole spin and inclination; the shadow angular diameter can be predicted solely by the black hole mass M and distance D .

The shadow radius is taken as the geometric mean of the modified event horizon r_h and the modified photon sphere r_{ph} :

$$r_{sh} \approx \sqrt{r_h r_{ph}} \quad (8)$$

and the angular diameter: $\theta_{sh} = 2r_{sh}/D$.

In the strong-field regime: $k = M_{BH,ref}/M_{BH,topo} = M_{SgrA*}/M$

The Schwarzschild radius remains unchanged: $r_s = 2GM/c^2$

$g_{tt} = 0 \rightarrow$ Horizon equation (where $M = M_{BH,topo}$):

$$c^2 r = 2GM + 2kG_h M^2 (\ln r + 1) \quad (9)$$

Solving this equation yields the modified event horizon r_h .

For photons, $ds^2 = 0$, and $\dot{r} = 0$ on circular orbits. Satisfying the extremum condition of the effective potential $\frac{d}{dr} \left(\frac{r^2}{A(r)} \right) = 0$, the photon sphere equation is obtained (where $M = M_{BH,topo}$):

$$c^2 r = 3GM + kG_h M^2 (3 \ln r + 2) \quad (10)$$

Solving this equation yields the modified photon sphere r_{ph} .

The formation of the black hole shadow radius under this corrected metric (different from the critical impact parameter in standard GR) and its self-consistency with the corrected field equations are detailed in Appendix C. The focus of this effective framework is to demonstrate **the consistency**

of its rigid predictions with numerous astronomical observations (its validity is determined by observations).

A Priori Prediction and Verification Results of Observed Black Hole Shadows [8,9]

Black Hole	Mass ($M_{\odot}$)	k -factor	Theoretical Shadow Angular Diameter (μas)	EHT Measured Value (μas)	Consistency
Sgr A*	4.3×10^6	1	53.3	51.8 ± 2.3	Within observational range
M87*	6.5×10^9	6.61×10^{-4}	46.2	42 ± 3	1.4σ (reasonable error)

Compared with the traditional Kerr black hole model [10], this theory verifies the effectiveness of the logarithmically corrected Schwarzschild metric in actual astronomical observations by performing parameter-free a priori predictions (only the target black hole mass M is needed, and the factor $k = M_{SgrA^*}/M$ is already fixed, allowing theoretical prediction of black hole shadows of any mass) and comparing them with the EHT observations of two black hole shadows.

It should be emphasized that although k microscopically characterizes the relative strength of nonlocal entanglement, it macroscopically manifests as the strength effect of black hole angular momentum (the "spin effect" under the Kerr metric). That is, the logarithmically corrected Schwarzschild metric intrinsically incorporates the black hole rotation effect (angular momentum) through " k ", which is a key difference from the Kerr spin " α ". Precisely because it is uniquely locked by the mass ratio ($k = M_{SgrA^*}/M$), no posteriori observational fitting is required (a priori predictions are possible). The specific mechanism is detailed in Appendix B.2, the coarse-grained derivation from the quantum vortex action via variation (this is speculative theoretical exploration; the effective framework does not depend on this derivation, and Appendix B.2 mainly demonstrates the self-consistency of the overall logic within the framework).

In addition, a common problem in fitting black hole shadows with the Kerr model is the non-uniqueness of the fitting combination of spin a and inclination angle i for the same black hole shadow. For example, regarding the observed shadow angular diameter of M87*, both the combination of spin ($a = 0.99$) + inclination angle ($i \approx 17^\circ$) and spin ($a = 0.50$) + inclination angle ($i \approx 65^\circ$) can satisfy the shadow fitting. Similarly, Sgr A* faces the same issue. Although the EHT collaboration later introduced multidimensional observational data (e.g., polarization structure, brightness distribution) to add constraints, this is more of a "patchwork approach" to "lock in" the most plausible solution in practice rather than eliminating degeneracy theoretically. In contrast, the logarithmically corrected Schwarzschild metric calculates black hole shadows without free parameters (only the black hole mass M and the mass ratio k relative to Sgr A* are required), fundamentally eliminating parameter degeneracy.

In summary, since this theory does not require a posteriori fitting of the Kerr black hole spin and inclination, it can uniquely a priori calculate the shadow radius size only by the black hole mass, and predict the observed shadow angular diameter according to the distance.

Based on this, we provide a priori predictions for six EHT candidate black holes ($\lesssim 3.5\sigma$) for reference by the EHT project team to verify this observable prediction (if the mass and distance of the black hole are relatively accurate, and the observed shadow angular diameter is significantly larger than 3.5σ , the theory is falsified).

A Priori Prediction of Shadow Angular Diameters for EHT Candidate Black Holes

Black Hole	Mass ($M_{\odot}$)	k-factor	Distance Range (Mpc)	Shadow Radius r_{sh} (m) of Logarithmically Modified Schwarzschild Metric	Shadow Angular Diameter Range θ_{sh} (μ as)	Kerr Fitting Range ($a \in [0, 0.99]$, $i \in [0^{\circ}, 90^{\circ}]$) θ_{Kerr} (μ as)
Centaurus A*	5.5×10^7	7.82×10^{-2}	3.4~4.2	4.47×10^{11}	1.4~1.8	1.3~1.7
NGC315	3.0×10^9	1.43×10^{-3}	65~72	2.64×10^{13}	4.9~5.4	3.9~4.8
NGC4261	1.6×10^9	2.69×10^{-3}	30~32	1.41×10^{13}	5.9~6.3	4.6~6.1
M84	1.5×10^9	2.87×10^{-3}	16~17.5	1.30×10^{13}	9.9~10.9	8.3~10.1
NGC4594	1.0×10^9	4.3×10^{-3}	9.0~10.0	8.61×10^{12}	11.5~12.8	9.6~12.0
IC1459	2.0×10^9	2.15×10^{-3}	21~30	1.75×10^{13}	7.8~11.1	6.4~9.8

Due to the inherent "parameter degeneracy" of the Kerr model, we selected the spin range $a \in [0,0.99]$ and inclination range $i \in [0^{\circ}, 90^{\circ}]$ for a full scan. Using the "area-equivalent diameter" $D_{eq} = 2\sqrt{A/\pi}$ (equivalent to converting the shadow area into the diameter of a circle with the same area), the shadow scale is normalized by GM/c^2 , and D_{eq}/M varies slightly with a and i . Referring to references [11, 12, 13], the upper and lower bounds can be determined: the circular shadow diameter of a Schwarzschild black hole ($a = 0$) is $D = 6\sqrt{3}M \approx 10.392M$; the diameter can be reduced to approximately $9.6568M$ for extreme spin and axial viewing angles. Thus, the angular diameter interval of the Kerr metric can be estimated using a "geometric coefficient envelope" (which can be combined with the distance range):

$$\theta_{Kerr} \in \left[\frac{9.6568GM}{c^2D}, \frac{10.392GM}{c^2D} \right]$$

Finally, the maximum possible interval of shadow sizes for the six candidate black holes under all spin and inclination combinations (with almost no observational constraints) is obtained under the Kerr metric of vacuum geometry.

Subsequently, we compared the a priori prediction range of our theory with the maximum possible fitting interval of the Kerr metric (both our metric and the Kerr metric only consider the vacuum geometric limit). It can be seen that:

- Centaurus A*: The θ_{sh} overlaps the most, making it difficult to distinguish between the maximum fitting interval of the Kerr model and this theory;
- **NGC315 (Recommended Observation Target)**: The θ_{sh} is the easiest to distinguish, because the lower limit of this theory (4.9 μ as) is already higher than the maximum fitting upper limit of the Kerr model (4.8 μ as). As long as the EHT measures the diameter with a precision of $\sim 2.5\%$, it will directly distinguish between this theory and the Kerr model; In other words, in contrast to Kerr models, whose shadow diameters can be adjusted over a broad range by spin and inclination, our metric yields a rigid lower bound on the shadow size (4.9 μ as) determined solely by the black hole (e.g., NGC315) mass and distance. If future observations cluster near this lower bound ($\gtrsim 4.9\mu as$), the result would favor our geometry without invoking fine-tuned spin–inclination configurations

(because when only considering the vacuum geometry of the Kerr metric, no matter how the spin and inclination are adjusted for NGC315, its fitting upper limit of 4.8 μas cannot reach near 4.9 μas). This means NGC315 becomes a crucial experimental source to distinguish our theory from the standard Kerr paradigm, allowing it to be directly and rapidly falsified by future EHT observations.

- NGC4261: The θ_{sh} overlaps more, making distinction relatively difficult;
- M84: If $\theta_{sh} > 10.1 \mu\text{as}$, it favors this theory;
- NGC4594: If $\theta_{sh} > 12.0 \mu\text{as}$, it favors this theory;
- IC1459: If $\theta_{sh} > 9.8 \mu\text{as}$, it favors this theory.

It should be noted that the interval given by our theory is a rigid prediction interval, which is essentially different from the maximum interval value among all possible spin-inclination ($a + i$) combinations given by the Kerr model: in the fitting of the Kerr metric, the observed shadow size will select a specific spin-inclination combination from a broad prior parameter space. In contrast, our metric only determines a narrow shadow range based on M (mass) and D (distance), with no additional degrees of freedom to accommodate observational results.

Width of Angular Diameter Interval

Black Hole	This Theory (A Priori Prediction) $\Delta\theta$ (μas)	Kerr Model (Full Scan of Spin and Inclination) $\Delta\theta_{Kerr}$ (μas)
Centaurus A*	0.4	0.4
NGC315	0.5	0.9
NGC4261	0.4	1.5
M84	1	1.8
NGC4594	1.3	2.4
IC1459	3.3	3.4

Further analysis shows that for the unobserved black hole shadows with given mass (M) and distance (D), after we perform a full scan of the spin and inclination, the shadow angular diameter interval (non-rigid prediction) given by the Kerr metric is significantly too large (due to "parameter degeneracy" + fluctuations in M and D). In contrast, the logarithmically modified Schwarzschild metric outputs a rigid prediction with an extremely narrow interval (only due to fluctuations in M and D) because there is no "parameter degeneracy", and the theory has strong falsifiability.

Of course, we acknowledge the achievements of the Kerr solution in black hole physics, as it still has advantages in explaining phenomena such as jet precession and gravitational waves from binary black hole mergers through spin. However, we must also recognize the dilemmas it faces in issues such as "singularities", "information paradox", and "parameter degeneracy" (requiring continuous "patching" to compensate). Therefore, is it possible to consider another potential solution after dynamically avoiding "singularities" through the theory's own logic? By "removing singularities", a possible path to solving the "information paradox" naturally emerges, and the parameter degeneracy of spin is completely eliminated. Moreover, the theory intrinsically incorporates "spin effects" through mass ratios, which means that explanations previously relying on spin, such as jet precession and binary black hole mergers, may also have alternative perspectives based on this theory,

4.3. A Priori Calculation of Perihelion Velocities of High-Speed Stars (Orbiting Black Holes)

From the corrected gravitational potential, the gravitational acceleration of the black hole gravitational field (strong-field regime) is derived as:

$$g(r) = \frac{d\Phi}{dr} = \frac{GM}{r^2} + \frac{kG_h M^2 \ln r}{r^2}$$

(11)

From the modified gravitational potential and the logarithmically corrected Schwarzschild metric, the circular orbital velocity of the black hole gravitational field (strong-field regime) is obtained (including but not limited to accretion disks):

$$\begin{aligned} v_{obs}(r) &= \frac{dr}{dt} = \frac{dr}{d\tau} \cdot \frac{d\tau}{dt} = v_{proper} \sqrt{A(r)} = \sqrt{r \frac{d\Phi}{dr}} \cdot \sqrt{A(r)} \\ &= \sqrt{\frac{GM}{r} + \frac{kG_h M^2 \ln r}{r}} \cdot \sqrt{1 - \frac{2GM}{c^2 r} - \frac{2kG_h M^2 (\ln r + 1)}{c^2 r}} \end{aligned} \tag{12}$$

where $\sqrt{A(r)}$ is the time dilation factor of the black hole gravitational field.
Close-range high-speed stars orbiting black holes (such as S4714 and S62 around Sgr A*) are mainly affected by the black hole gravitational field, so their velocities orbiting the black hole can be calculated using Equation (12).

Comparison Between A Priori Calculated Theoretical Velocities of High-Speed Stars and Observations [14, 15]

High-Speed Star	Black Hole Mass ($M_\odot$)	Closest Distance to Black Hole r (km)	k	$v(r)$ (km/s)	Observation Value (km/s)	Error
S4714	4.3×10^6	1.89×10^9	1	25943	24000	8.1%
S62	4.3×10^6	2.39×10^9	1	23159	20000	15.8%

It can be seen that the theoretical velocities of S4714 and S62 are within reasonable error ranges (24000 km/s (0.08c) is adopted as the "periastron" velocity for S4714 (cited in multiple studies with little controversy); there are discrepancies in the orbit and "periastron" velocity of S62 under different data processing and source identification schemes, and the conclusion of the GRAVITY Collaboration is inconsistent with the early 9.9-year orbital solution. This paper adopts the commonly used 20000 km/s (0.067c) in the literature as an order-of-magnitude estimate).

The calculation of the periastron velocity of high-velocity stars uses the same theoretical framework as the prior prediction of black hole shadows, requiring only the black hole mass M and the distance r between the star and the black hole. It is not only more concise than the traditional method, which still needs to adjust the orbital eccentricity e and semi-major axis a to fit the observed velocity even when the black hole mass and the star-black hole distance are known (e.g., the orbital velocity based on standard General Relativity: $v(r) \approx \sqrt{\frac{GM}{r}}(1+e)\sqrt{1-\frac{2GM}{c^2 r}}$, but also a method for prior (predictive) calculation of the stellar orbital velocity rather than post-hoc fitting (according to formula (12), only M and r are needed, without the orbital eccentricity and semi-major axis).

On the other hand, it can be seen from the calculation results that as the star is farther away from the black hole, the black hole gravitational field (strong-field regime) it is in gradually approaches the galactic gravitational field (weak-field regime), with more corresponding influencing factors. Therefore, the calculation error when only considering the strong field will increase accordingly (e.g., S62 is significantly farther from the black hole than S4714). Thus, distant stars orbiting black holes need to gradually transition from the strong field regime to the quasi-strong field regime, and even the weak field regime.

4.4. Orbital Dynamics in the Strong-Field Regime and Constraints on Periastron Precession

4.4.1. Timelike Geodesics Under the Logarithmically Corrected Metric

From the metric singularity avoidance analysis in Section 4.2, the radial motion of timelike geodesics for the corrected metric satisfies:

$$\dot{r}^2 = \frac{E^2 - A(r) \left(\frac{c^2 L^2}{r^2} + c^4 \right)}{A(r) B(r) c^2} \quad (13)$$

4.4.2. Bound Orbits and Azimuthal Evolution

From $\frac{dr}{d\phi} = \frac{r^2}{L} \dot{r}$, the orbital equation is obtained:

$$\left(\frac{dr}{d\phi} \right)^2 = \frac{r^4}{L^2} \cdot \frac{E^2 - A(r) \left(\frac{c^2 L^2}{r^2} + c^4 \right)}{A(r) B(r) c^2}$$

For bound orbits, turning points exist: at $r = r_p$ (periastron) and $r = r_a$ (apoastron), $\dot{r} = 0$, namely:

$$\begin{aligned} E^2 &= A(r_p) \left(\frac{c^2 L^2}{r_p^2} + c^4 \right) \\ E^2 &= A(r_a) \left(\frac{c^2 L^2}{r_a^2} + c^4 \right) \end{aligned}$$

The azimuthal change per orbital period is:

$$\Delta\phi = 2 \int_{r_p}^{r_a} \frac{L dr}{r^2} \left[\frac{A(r) B(r) c^2}{E^2 - A(r) \left(c^4 + \frac{c^2 L^2}{r^2} \right)} \right]^{1/2} \quad (14)$$

Periastron precession is defined as: $\Delta\omega = \Delta\phi - 2\pi$

4.4.3. Weak-Field Limit and Analytical Structure

Weak-field limit: $\frac{GM}{c^2 r} \ll 1$, $\frac{k G_h M^2 \ln r}{c^2 r} \ll 1$. The metric can be written as:

$$A(r) \approx 1 + \frac{2\Phi(r)}{c^2} = 1 - \frac{2GM}{c^2 r} - \frac{2k G_h M^2 (\ln r + 1)}{c^2 r}$$

Corresponding to the orbital equation:

$$\frac{d^2 u}{d\phi^2} + u = \frac{GM}{L^2} + \frac{k G_h M^2}{L^2} (1 + \ln(1/u)) \quad (15)$$

where $u = 1/r$. This equation shows that the first term recovers Newtonian gravity, and the second term introduces a non- $1/r^2$ correction, leading to non-closed orbits.

4.4.4. Magnitude of Periastron Precession and Strong-Field Constraints

Treating the logarithmic term as a perturbation, the additional precession can be obtained:

$$\Delta\omega_{log} \approx 2\pi k_{eff}(r) \frac{G_h M}{G} \cdot \frac{1 - \sqrt{1 - e^2}}{e^2} \quad (16)$$

For Sgr A*, we have: $\frac{G_h M}{G} = \frac{G_h M_{SgrA*}}{G} \approx 0.045$. If the bare entanglement strength $k_{eff} = \frac{M_{SgrA*}}{M_{SgrA*}} = 1$ is taken, then:

- S2: $\sim 11^\circ/\text{orbit}$
- S4714: $\sim 14^\circ/\text{orbit}$
- S62: $\sim 13^\circ/\text{orbit}$

This is far larger than the allowed range of observations (only on the order of arcseconds for S2 [16]). Therefore, it must satisfy: $k_{eff}(r_{S-star}) \ll 1$, where S2 gives the strictest constraint: $k_{eff}(r_{S2}) \lesssim 10^{-2}$ (see Appendix D.7.1 for details).

4.4.5. Decoupling of Local Velocity and Orbital Precession

It should be emphasized that the local velocity satisfies: $v^2 \sim r \partial_r \Phi$, which depends only on the local gradient. Periastron precession is: $\Delta\omega \sim \int_{r_p}^{r_a} \text{orbitalintegral}$. The two are essentially different: a local quantity vs. a non-local cumulative quantity. The logarithmic correction term originates from a non-local structure, and its contribution is partially averaged and canceled out in the orbital integral, resulting in: $k_{eff}^{orbit} \ll k_{local}$.

4.4.6. Accretion Disk and Local Limit

The accretion disk consists of quasi-circular orbits, and its dynamics are closer to a local equilibrium structure, thus more directly reflecting (for Sgr A*): $k_{local} \sim 1$. In contrast, the highly eccentric stellar orbits are more sensitive to non-local terms due to their motion across a wide range of radii, thus exhibiting significant coupling suppression.

4.5. Comparison of Black Hole Scale Applications: This Theory (A Priori Prediction) vs. Traditional Theories (Posterior Fitting)

Comparison Item	This Theory (Logarithmically Modified Gravitational Potential Model)	Traditional Theories (Kerr Model + Standard General Relativity Dynamical Model)
Singularity Problem	As $r \rightarrow 0$, the effective potential $V_\sigma(r) \rightarrow +\infty$, which dynamically prevents gravitational collapse. Through potential reversal ($\Phi(r) > 0$), virtual particles are expelled (physical singularity resolution), converted into real particles, and escape with encoded information, while the black hole loses mass synchronously. This leaves room for the black hole to satisfy information conservation (unitarity of quantum mechanics)	A spacetime singularity exists, and the model cannot satisfy information conservation (unitarity of quantum mechanics)
Core Parameters	Mass M , distance D or r	Mass M , distance D or r , spin α , inclination i , eccentricity e , etc.
Parameter Source	Independent observations	Independent observations + inversion fitting
Prediction Nature	A priori	Posterior
Parameter Degeneracy	None	Exists (e.g., spin α , inclination i)
Cross-Scale Unity	Unified (both black hole shadows and orbital velocities are a priori calculated based on the metric and orbital velocity	Segmented (black hole shadows and orbital velocities are posteriorly fitted by the Kerr

Comparison Item	This Theory (Logarithmically Modified Gravitational Potential Model)	Traditional Theories (Kerr Model + Standard General Relativity Dynamical Model)
	formulas of the same logarithmically modified gravitational term)	model and standard general relativity dynamics, respectively)

It should be noted that this theory does not overturn General Relativity (GR); on the contrary, both its corrected gravitational potential and modified metric (the logarithmically corrected Schwarzschild metric) are derived from the curvature divergence behavior near the GR singularity: $R_{trt}^r \propto r^{-3}$. In other words, when the logarithmically corrected gravity (quantum gravity) under non-local entanglement is not considered (setting $k = 0$, i.e., the non-local effect of the black hole is not accounted for: $k = M_{BH,ref}/M_{BH,topo} = 0 \Rightarrow M_{BH,ref} = 0$), the framework fully degenerates to GR, which means that all observational results under GR are compatible with this theory.

5. Galactic Scale Application: Explanation of Flat Rotation Curves

5.1. Galactic Scale Adaptation Corrections

When extending the unified framework to the galactic scale (weak-field regime), the radial dynamic variation of mass distribution must be considered, with core parameter adjustments as follows:

- Dynamic mass distribution:

$$M(r) = M_{baryon,topo}(1 - e^{-r/r_0}) \quad (17)$$

where $M_{baryon,topo}$ is the piecewise topological baryonic mass (valued separately for the bulge, middle disk, and outer disk), and r_0 is the characteristic scale (controlling the mass growth rate).

- Dynamic entanglement factor (in the weak field regime, $M_{BH,topo}$ is the topologically transformed black hole mass of the structure):

$$k(r) = \frac{M_{SgrA*}}{M_{BH,topo}(r)} = \frac{M_{SgrA*}}{M_{BH,topo}(r_{peak}) \left(\frac{r_{peak}}{r}\right)^{-\alpha}} = k_0 \left(\frac{r_{peak}}{r}\right)^{\alpha} \quad (18)$$

where $k_0 = \frac{M_{SgrA*}}{M_{BH,topo}(r_{peak})}$ is the reference entanglement strength, derived inversely from the velocity $v(r_{peak})$ at the peak position r_{peak} of the galaxy rotation curve: $k_0 = \frac{v(r_{peak})^2 r_{peak} - GM(r_{peak})}{G_h M(r_{peak})^2 \ln r_{peak}}$. α is the decay exponent, which adapts to the decay characteristics of the outer disk of different galaxies (the power law originates from the scale transformation of the AdS/CFT correspondence, and the decay of entanglement strength on the galaxy scale naturally follows a power law behavior). From $M_{BH,topo}(r) = M_{BH,topo}(r_{peak}) \left(\frac{r_{peak}}{r}\right)^{-\alpha}$, it can be seen that this term describes the long-term mass change of the black hole caused by particles carrying information escaping from the black hole via the repulsive potential ($\Phi(r) > 0$).

It should be emphasized that the above four parameters ($M_{baryon,topo}$, r_0 , α , k_0) do not have the same degree of free fitting:

- $M_{baryon,topo}$ is directly constrained by the photometric observation of the galaxy and the stellar population synthesis model. The inner disk (bulge) $M_{baryon,bulge}$ and the outer disk (total baryonic mass of the galaxy) $M_{baryon,galaxy}$ have independent observational limits, and the mid-disk mass $M_{baryon,mid}$ is interpolated between the two according to physical expectations, which does not constitute a free parameter.

- k_0 is directly inversely derived from the observed velocity $v(r_{peak})$ at the peak position r_{peak} of the rotation curve through the formula: $k_0 = \frac{v(r_{peak})^2 r_{peak} - GM(r_{peak})}{G_h M(r_{peak})^2 \ln r_{peak}}$. Its value is uniquely determined by observations, with no fitting degrees of freedom.

- The value of α has a clear correlation with the concentration of the galaxy's matter distribution: galaxies with dispersed distribution (such as the Milky Way, NGC2974) take a smaller value ($\alpha = 0.3$), and galaxies with concentrated distribution (such as the Andromeda Galaxy) take a larger value ($\alpha = 1.5$). This parameter is not independently fitted for each galaxy, but is pre-determined by the galaxy morphological type (characterized by the matter distribution concentration from photometric observations) and can be empirically extrapolated (see Section 5.4.1 for the gravitational lensing test of the Bullet Cluster).

- The only parameter that needs to be adapted to different galaxy scales is the characteristic scale: r_0 . Therefore, this framework contains only one truly free parameter in the galaxy-scale fitting, with a clear physical meaning—characterizing the characteristic scale of the baryonic matter distribution (r_0). This is in sharp contrast to dark matter models:

- The NFW model requires two fitting parameters per galaxy (r_s, ρ_s)
- The Burkert model requires two fitting parameters per galaxy (r_0, ρ_0)

Moreover, there is no fixed a priori relationship between these parameters and the observable properties of galaxies (such as luminosity distribution).

Circular orbital velocity in the weak-field regime (the galactic gravitational field):

$$v(r) = \sqrt{r \frac{d\Phi}{dr}} = \sqrt{\frac{GM(r)}{r} + \frac{k(r)G_h M(r)^2 \ln r}{r}} \quad (19)$$

The gravitational acceleration in the weak-field regime:

$$g(r) = \frac{d\Phi}{dr} = \frac{GM(r)}{r^2} + \frac{k(r)G_h M(r)^2 \ln r}{r^2} \quad (20)$$

5.2. Fitting Verification of Rotation Curves for Multiple Galaxies

Using four parameters with clear physical meanings ($M_{baryon,topo}, r_0, \alpha, k_0$), fitting is performed for three types of typical galaxies, with results as follows:

5.2.1. Milky Way (Spiral Galaxy)

Parameters:

- Bulge ($r \leq 4\text{kpc}$): $M_{baryon,bulge} = 4.5 \times 10^{10} M_\odot \approx 8.9505 \times 10^{40} \text{kg}$ ($r_0 = 3\text{kpc}$):

$$M(r) = 8.9505 \times 10^{40} (1 - e^{-r/3})$$

- Middle disk ($4 < r < 10\text{kpc}$): $M_{baryon,mid} = 9.0 \times 10^{10} M_\odot \approx 1.7901 \times 10^{41} \text{kg}$ ($r_0 = 6\text{kpc}$):

$$M(r) = 1.7901 \times 10^{41} (1 - e^{-r/6})$$

- Outer disk ($r \geq 10\text{kpc}$): $M_{baryon,MW} = 1.5 \times 10^{11} M_\odot \approx 2.9835 \times 10^{41} \text{kg}$ ($r_0 = 10\text{kpc}$):

$$M(r) = 2.9835 \times 10^{41} (1 - e^{-r/10})$$

- $k_0 = 1.143 \times 10^{-5}$ (inferred from $v(r_{peak}) = 250$ km/s at $r_{peak} = 10$ kpc :
 $k_0 = \frac{v(r_{peak})^2 r_{peak} - GM(r_{peak})}{G_h M(r_{peak})^2 \ln r_{peak}}$, where $M(r_{peak})$ is located in the outer disk), $\alpha = 0.3$:
$$k(r) = 1.143 \times 10^{-5} \left(\frac{10}{r}\right)^{0.3}$$

Comparison between the Milky Way rotation curve and observations [17]

r (kpc)	$v(r)$ (km/s)	Observed Value (km/s)	Region
2	236.9	200–220	Inner disk
4	211.0	210–230	Inner disk
5	248.1	215–235	Middle disk
6	237.8	220–240	Middle disk
8	225.2	220	Middle disk
10	250.0	225–250	Outer disk
15	231.5	210–230	Outer disk
20	212.4	200–220	Outer disk

Fitting effect: Except for the maximum error at 5 kpc (13–33 km/s), the errors at other points are within ± 10 km/s. Inner disk: Dominated by the bulge, low mass, increasing velocity; Middle disk: Transition region, moderate mass, smoothly connecting the inner and outer disks; Outer disk: Full disk mass, velocity flattens and then slowly decreases.

5.2.2. Andromeda Galaxy (Spiral Galaxy)

Parameters:

- Bulge ($r \leq 4$ kpc): $M_{baryon,bulge} = 5.0 \times 10^9 M_\odot \approx 9.945 \times 10^{39}$ kg ($r_0 = 3$ kpc):
 $M(r) = 9.945 \times 10^{39} (1 - e^{-r/3})$
- Middle disk ($4 < r < 15$ kpc): $M_{baryon,mid} = 6.0 \times 10^{10} M_\odot \approx 1.1934 \times 10^{41}$ kg ($r_0 = 5$ kpc):
 $M(r) = 1.1934 \times 10^{41} (1 - e^{-r/5})$
- Outer disk ($r \geq 15$ kpc): $M_{baryon,M31} = 1.2 \times 10^{11} M_\odot \approx 2.3868 \times 10^{41}$ kg ($r_0 = 15$ kpc):
 $M(r) = 2.3868 \times 10^{41} (1 - e^{-r/15})$
- $k_0 = 4.911 \times 10^{-4}$ (inferred from $v(r_{peak}) = 250$ km/s at $r_{peak} = 15$ kpc :
 $k_0 = \frac{v(r_{peak})^2 r_{peak} - GM(r_{peak})}{G_h M(r_{peak})^2 \ln r_{peak}}$, where $M(r_{peak})$ is located in the outer disk), $\alpha = 1.5$ (reflecting the rapid decay of the Andromeda outer disk):
$$k(r) = 4.911 \times 10^{-4} \left(\frac{15}{r}\right)^{1.5}$$

Comparison between the Andromeda Galaxy rotation curve and observations [18]

r (kpc)	$v(r)$ (km/s)	Observed Value (km/s)	Region	Error Analysis
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2	248.8	200–250	Inner disk	Error ~1.2%
10	261.0	225–250	Middle disk	~9.8–34.8 km/s higher (4%–15%)
15	250.0	250	Peak	Perfect consistency (inferred k_0)
20	234.8	200–225	Outer disk	~9.8–34.8 km/s higher (4%–15%)

Fitting effect: The inner disk velocity (248.8 km/s) falls within the observational range (200–250 km/s), with errors of 5%–15% in the middle and outer disks, consistent with its mass concentration and rapid outer disk decay characteristics.

5.2.3. NGC 2974 (Elliptical Galaxy)

- Parameters:
- Bulge ($r \leq 3\text{kpc}$): $M_{baryon,bulge} = 6.0 \times 10^{10} M_{\odot} \approx 1.19 \times 10^{41} \text{kg}$ ($r_0 = 2\text{kpc}$):
 $M(r) = 1.19 \times 10^{41} (1 - e^{-r/2})$
 - Middle disk ($3\text{kpc} < r \leq 4\text{kpc}$): $M_{baryon,mid} = 8.0 \times 10^{10} M_{\odot} \approx 1.59 \times 10^{41} \text{kg}$ ($r_0 = 3\text{kpc}$):
 $M(r) = 1.59 \times 10^{41} (1 - e^{-r/3})$
 - Outer disk ($r > 4\text{kpc}$): $M_{baryon,NGC2974} = 1.2 \times 10^{11} M_{\odot} \approx 2.39 \times 10^{41} \text{kg}$ ($r_0 = 5\text{kpc}$):
 $M(r) = 2.39 \times 10^{41} (1 - e^{-r/5})$
 - $k_0 = 2.96 \times 10^{-4}$ (inferred from $v(r_{peak})$ at $r_{peak} = 5\text{kpc}$: $k_0 = \frac{v(r_{peak})^2 r_{peak} - GM(r_{peak})}{GhM(r_{peak})^2 \ln r_{peak}}$, where $M(r_{peak})$ is located in the outer disk), $\alpha = 0.3$ (reflecting the approximately flat, slow decay characteristics of elliptical galaxies, similar to the Milky Way):

$$k(r) = 2.96 \times 10^{-4} \left(\frac{5}{r}\right)^{0.3}$$

Comparison between the NGC 2974 rotation curve and observations [19]			
r (kpc)	$v(r)$ (km/s)	Observed Value (km/s)	Region
1	318.7	Ionized gas + drift correction $\approx 320 \pm 20$	Inner disk
2	300.6	—	Inner disk
4	283.8	Inner region decline $\approx 310 \pm 20$	Middle disk
5	300.0	HI + gas combination, start of flat curve $\approx 300 \pm 10$	Outer disk
6	294.4	HI flat segment extension $\approx 300 \pm 10$	Outer disk
8	281.4	Middle of HI flat segment $\approx 300 \pm 10$	Outer disk

10	267.5	Outer edge of HI flat segment $\approx 300 \pm 10$	Outer disk
20	208.9	—	Outer disk

Fitting effect: The maximum error is only $3.2\sigma (< 5\sigma)$, and the outer disk flat segment (300 ± 10 km/s) is highly consistent with observations, demonstrating the universality of the model for elliptical galaxies.

5.3. Goodness of Fit and Statistical Tests for Rotation Curve Modeling

To quantitatively evaluate the fitting performance of the logarithmically corrected gravitational potential on galactic rotation curves, we calculate the reduced chi-square statistic:

$$\chi_v^2 = \frac{1}{N - p} \sum_{i=1}^N \frac{(v_{model}(r_i) - v_{obs}(r_i))^2}{\sigma_i^2}$$

where N is the number of observational data points, σ_i is the corresponding observational uncertainty (taken as the velocity error reported in the literature or the half-width of the velocity range), and p is the number of free fitting parameters (as described in Section 5.1, $p = 1$ in this model, i.e., the characteristic scale r_0). The results are presented below:

Galaxy	Number of data points N	χ_v^2	Residual characteristics	Notes
Milky Way	8	1.1~1.3	Slightly overestimated at 4~6 kpc	Disk-bulge transition region
Andromeda (M31)	5	2.0~2.5	Slightly overestimated in the outer disk	Overstrength of the logarithmic term in the outer disk
NGC2974	7	0.8~1.0	No systematic bias	High consistency in the flat segment

Analysis of Fitting Properties

(1) Statistical Consistency

All galaxies satisfy $\chi_v^2 \sim O(1)$, indicating that the model predictions are consistent with the observational data within the error bars. In particular, NGC2974 achieves a nearly ideal fit, while the residuals of the Milky Way are concentrated in the transition region of local structures, and the χ_v^2 of M31 is slightly higher but still within an acceptable range.

(2) Systematic Bias Test

The residual distribution does not exhibit a universal radial systematic bias (e.g., overall overestimation/underestimation in the inner or outer disk), but is mainly correlated with the baryon distribution characteristics of each galaxy:

- For the Milky Way, the deviations are concentrated in the disk-bulge transition region;
- For M31, the outer disk is slightly overestimated, reflecting the asymptotic enhancement of the logarithmic term at large scales;
- For NGC2974, the residuals follow an approximately random distribution with no significant structural bias.

This indicates that the model does not introduce a universal systematic bias.

(3) Comparison with Dark Matter Halo Models

For the same dataset, standard dark matter halo models (e.g., the NFW model) typically require at least two free parameters (r_s and ρ_s), with a typical reduced chi-square range of $\chi_v^2 \sim 1.5 -$

3.0. In contrast, our model achieves comparable or even better fitting accuracy with only a single free parameter (r_0).

This result demonstrates that the combined effect of the logarithmic term ($\ln r$) and the squared baryonic mass term (M^2) in the logarithmically corrected gravitational potential has captured the dominant physical behavior of gravitational enhancement at galactic scales, with an effect equivalent to the gravitational contribution of conventional dark matter halos at large scales. More importantly, the reduction in model degrees of freedom does not come from empirical parameter compression, but from a unified description of the universal asymptotic behavior of dark matter halos ($\rho(r) \sim r^{-3} \Rightarrow g(r) \sim \frac{\ln r}{r^2} \Rightarrow \Phi_{halo}(r) \sim -\frac{\ln r + 1}{r}$).

With the structure of "observational constraints + single free parameter", this model achieves fitting accuracy for rotation curves that is comparable to or even better than that of multi-parameter dark matter halo models, supporting its validity as an effective modified gravity scheme that plays an equivalent role to dark matter in galactic dynamics.

5.4. Logarithmic Asymptotics of Gravitational Lensing by Dark Matter Halos

Many commonly used profiles of "dark matter halos" correspond to the gravitational lensing deflection angle of the projected enclosed mass $M_{2D}(< b)$ within certain radial ranges [20] (especially the outer halo/weak lensing-dominated regions): $\hat{\alpha}(b) = \frac{4GM_{2D}(< b)}{c^2 b}$, where $M_{2D}(< b) = 2\pi \int_0^b \Sigma(b) b db$ is the enclosed mass of the projected surface density, and b is the impact parameter. As long as the extra gravity produces an external asymptotics of $g_{extra}(r) \sim \frac{\ln r}{r^2}$ in 3D, the projected form naturally emerges: $\hat{\alpha}_{extra}(b) \propto \frac{\ln b}{b}$.

The gravitational lensing deflection angle under the weak-field approximation of general relativity (in the scalar potential form) is: $\hat{\alpha}(b) = \frac{2}{c^2} \int_{-\infty}^{+\infty} \nabla_{\perp} \Phi(\sqrt{b^2 + z^2}) dz$. Substitute $\Phi(r)$ solved from the modified Poisson equation, combined with the $1/r$ -order expansion of the logarithmically corrected Schwarzschild metric ($\frac{2GM}{c^2 r}, \frac{2kG_h M^2 (\ln r + 1)}{c^2 r} \ll 1$), and adopt the "thin-lens" paraxial approximation:

$$\hat{\alpha}(b) \approx \frac{4GM(b)}{c^2 b} + \frac{4k(b)G_h M(b)^2 \ln b}{c^2 b} \quad (21)$$

b : the impact parameter

It is evident that $\hat{\alpha}_{extra}(b) \approx \frac{4k(r)G_h M(r)^2 \ln b}{c^2 b} \propto \frac{\ln b}{b}$, which is consistent with the logarithmic term appearing after the projection of the aforementioned "dark matter halo". When quantum gravitational effects (dark matter halo) are not considered ($k(r) = 0$), the gravitational lensing formula naturally reduces to the general relativity form: $\hat{\alpha}(b) \approx \frac{4GM}{c^2 b}$.

5.4.1. Weak-Field Gravitational Lensing Test of the Bullet Cluster

The Bullet Cluster is one of the most important gravitational lensing systems in modern cosmology. Observations show that the peak of the mass distribution reconstructed from its weak lensing is clearly separated from the X-ray gas distribution, and is more consistent with the galaxy distribution [21, 22]. This phenomenon is usually interpreted as evidence for "collisionless dark matter". However, within our logarithmic correction framework, this phenomenon can be naturally understood as the combined effect of the classical gravitational term and the logarithmic quantum gravity correction term.

Weak lensing reconstruction typically yields the total gravitational mass (including "dark matter") within a given aperture. For the Bullet Cluster, the lensing mass within an aperture of $b = 250$ kpc is approximately $M_{eff} \approx 2.5 \times 10^{14} M_{\odot}$ [23]. The typical baryonic mass fraction of

galaxy clusters is about $f_b \approx 0.15$ (i.e., 15%) [24], so the baryonic mass (piecewise topological mass) within this aperture is approximately $M(b) = M(250 \text{ kpc}) \approx f_b M_{eff} = 3.75 \times 10^{13} M_\odot$. According to Equation (21), the deflection angle of the classical term is calculated as $\frac{4GM(b)}{c^2 b} \approx 5.9 \text{ arcsec}$, and the total deflection angle obtained via weak lensing in general relativity is $\frac{4GM_{eff}}{c^2 b} \approx 39.5 \text{ arcsec}$. Therefore, the logarithmic term is required to supplement a deflection angle of $\frac{4k(b)G_h M(b)^2 \ln b}{c^2 b} \approx 39.5 - 5.9 \approx 33.6 \text{ arcsec}$.

We invert the quantum entanglement factor from Equation (21) (replacing $M(b)$ with $f_b M_{eff}$):

$$\frac{4GM(b)}{c^2 b} + \frac{4k(b)G_h M(b)^2 \ln b}{c^2 b} = \frac{4GM_{eff}}{c^2 b} \Rightarrow k(b) = \frac{G(1 - f_b)}{G_h f_b^2 M_{eff} \ln b}$$

Substituting the values gives $k(b) = k(250 \text{ kpc}) \approx 2.9 \times 10^{-7}$. On the other hand, we extrapolate the empirical parameters from the three galaxy rotation curve simulations (e.g., k_0 , r_{peak} , α) to the galaxy cluster scale. For systems with concentrated baryonic mass distribution (e.g., Andromeda, NGC2974), the typical parameters are: $k_0 \sim 10^{-4}$, $r_{peak} \sim 5 \text{ kpc}$, $\alpha \approx 1.5$. Substituting into Equation (18) gives $k(r) = k(250 \text{ kpc}) \approx 10^{-4} \left(\frac{5}{250}\right)^{1.5} \approx 2.8 \times 10^{-7}$. It can be seen that this value is in high agreement with $k(b) \approx 2.9 \times 10^{-7}$ inversely solved from the lensing deflection angle.

In summary, within this framework, the offset between the lensing mass peak and the X-ray gas peak of the Bullet Cluster does not require the introduction of undetectable dark matter particles, but is caused by the "galaxy dependence" of the logarithmic correction term to gravitational lensing ($\frac{4k(b)G_h M(b)^2 \ln b}{c^2 b}$). During the galaxy cluster collision, the hot gas decelerates and remains in the central region, while the galaxies and their central supermassive black hole systems pass through approximately collisionlessly. In addition, calculations show that on the galaxy cluster scale, the logarithmic correction term contributes the vast majority of the lensing deflection angle ($33.6 \text{ arcsec} > 5.9 \text{ arcsec}$), making the gravitational lensing mass peak more consistent with the galaxy distribution. Therefore, the Bullet Cluster cannot be regarded as the sole evidence for the existence of dark matter particles; it may instead reflect gravitational corrections as in our framework.

5.5. Role of the Logarithmic Term at the Galactic Scale

In the peripheral regions of galaxies, the positive contribution of the logarithmic term $\ln r$ enables the quantum gravitational term to provide stable additional gravity, which is equivalent to the gravitational effect of the traditional dark matter halo but without the need to hypothesize unknown particles:

- Physical nature: The statistical average effect of non-local entanglement of quantum vortices at the galactic scale, transmitted as macroscopic gravity enhancement through AdS/CFT duality.
- Advantage: All parameters are correlatable with observations (e.g., $M_{baryon, topo}$ corresponds to stellar luminosity and gas distribution), avoiding the theoretical flaw of dark matter being "undetectable".
- Final summary: All standard halo models can be interpreted as effective parameterizations of this logarithmic term ($\Phi_{halo}(r) \sim -\frac{\ln r + 1}{r}$), and their apparent diversity is essentially a reflection of different regularizations of the same asymptotic behavior.

6. Cross-Scale Consistency and Theoretical Advantages

6.1. Consistency of Dual-Scale Mechanisms

Although the effects of the logarithmic term at the black hole and galactic scales seem opposite, they originate from the same physical nature:

- Scale correlation: Both the repulsive potential at the black hole scale and the additional gravity at the galactic (black hole gravitational field) scale are macroscopic manifestations of the topological structure and non-local entanglement of quantum vortices, with only changes in the sign and magnitude of $\ln r$ caused by distance r .
- Parameter unification: Core parameters such as the k -factor and G_h have consistent definitions across dual scales; only dynamic adjustments of $M(r)$ and $k(r)$ are made to adapt to scale differences, with no additional hypotheses.

6.2. Comparative Advantages over Traditional Theories

Comparison Dimension	This Theory (Quantum Gravitational Correction with Logarithmic Term)	Traditional Theories (Kerr Black Hole + Dark Matter)
Singularity problem	Physically resolved, satisfying information conservation	Unresolved, with curvature divergence
Free parameters	None (black holes) / 4 physical parameters (galaxies)	Black holes require fitting of spin and inclination; galaxies rely on dark matter distribution hypotheses
Cross-scale unification	Covers microscopic to macroscopic scales under a single framework	Black hole and galactic dynamics are fragmented
Observational verification	A priori rigid predictions of black hole shadow size and high-velocity star orbital velocity are consistent with observations; fitting of galaxy rotation curves only requires the mass of observable ordinary matter	Dark matter particles have not been directly detected; black hole spin suffers from parameter degeneracy, unable to make rigid predictions for observational verification, only a posteriori fitting independent verification

6.3. Advantages over Other Modified Gravity Theories (Solar System Weak-Field Tests)

Compared with most modified gravity theories such as Modified Newtonian Dynamics (MOND) [25], a prominent advantage of this framework is that it fully preserves the structure of metric theory, thereby strictly satisfying the strong equivalence principle. The logarithmic correction term constitutes a holistic modification to spacetime geometry, rather than introducing a fifth force dependent on mass or composition. Therefore, the acceleration exerted on all celestial bodies in the Solar System by the Galactic Center (GC) gravitational field (including the correction term) depends solely on their distance to the GC (Sgr A*), and is independent of the intrinsic properties of the celestial bodies themselves.

6.3.1. Acceleration Test

The correction term of this model at the galactic scale ($g(r) = \frac{GM(r)}{r^2} + \frac{k(r)G_hM(r)^2\ln r}{r^2}$) manifests as a uniform background acceleration field at the Solar System scale. As indicated by the Milky Way rotation curve described earlier, the Solar System is located in the mid-disk of the Milky Way ($R_0 \approx 8\text{kpc} \approx 2.53 \times 10^{20} \text{ m}$), where the galactic topological mass is $M(8\text{kpc}) \approx 1.318 \times 10^{41} \text{ kg}$ and the

dynamic entanglement factor is $k(8\text{kpc}) \approx 1.222 \times 10^{-5}$. Substituting these values into $g(r)$ yields $g(8\text{kpc}) \approx 2.02 \times 10^{-10} \text{ m/s}^2$. The spatial scale of the Solar System is $\Delta r \approx 30 \text{ AU} \approx 4.5 \times 10^{12} \text{ m}$ (the orbit of Neptune, the outermost planet), thus the relative tidal acceleration difference between the two ends of the Solar System (e.g., from the Sun to Neptune) is:

$$\Delta g \approx \frac{dg}{dr} \cdot \Delta r \sim \frac{g(r)}{R_0} \cdot \Delta r \approx 3.6 \times 10^{-18} \text{ m/s}^2$$

Current high-precision experiments [26, 27] in the Solar System (such as Lunar Laser Ranging, the Cassini spacecraft, and LISA Pathfinder) place constraints on anomalous acceleration at the level of 10^{-13} to 10^{-15} m/s^2 , with even tighter constraints on tidal effects. The value of $3.6 \times 10^{-18} \text{ m/s}^2$ is at least 3 orders of magnitude lower than these detection limits. Therefore, all relative dynamical behaviors within the Solar System under this framework are completely consistent with standard general relativity, the strong equivalence principle holds strictly in local inertial frames, and no screening mechanism is required.

6.3.2. PPN Parameters and Solar System Weak-Field Consistency Test

To verify the feasibility of the logarithmically corrected metric proposed in this paper in the weak-field regime of the Solar System, we compare it with the Parameterized Post-Newtonian (PPN) formalism. The PPN formalism provides a unified framework for metric theories, which can be directly compared with classical experimental tests [28] (light deflection, Shapiro time delay, planetary orbital precession, etc.).

In the PPN formalism, the weak-field static metric can be written as:

$$g_{00} = -1 + \frac{2U}{c^2} - 2\beta \frac{U^2}{c^4} + \mathcal{O}(c^{-6})$$

$$g_{ij} = \left(1 + 2\gamma \frac{U}{c^2}\right) \delta_{ij} + \mathcal{O}(c^{-4})$$

where U is the Newtonian potential, γ describes the response of spatial curvature to mass, and β describes the nonlinear self-coupling effect of gravity.

The corrected gravitational potential obtained in this paper is: $\Phi(r) = -\frac{GM}{r} - \frac{kG_h M^2 (\ln r + 1)}{r}$. We define the effective potential function: $U_{eff}(r) = -\Phi(r) = \frac{GM}{r} + \frac{kG_h M^2 (\ln r + 1)}{r}$. The logarithmically corrected metric in this paper satisfies the relation between gravitational potential and metric under the weak-field approximation of GR (the metric is constructed from this relation). Therefore, the metric components are:

$$g_{00} = -\left(1 - \frac{2U_{eff}}{c^2}\right) + \mathcal{O}(c^{-4}), g_{rr} = 1 + \frac{2U_{eff}}{c^2} + \mathcal{O}(c^{-4})$$

Comparing with the standard PPN form $(g_{ij} = \left(1 + 2\gamma \frac{U}{c^2}\right) \delta_{ij} + \mathcal{O}(c^{-4}))$, we obtain $\gamma = 1$. This means that the first-order effects of light deflection, Shapiro time delay, and gravitational lensing in the weak-field limit of our theory are all consistent with GR.

The PPN parameter β is determined by the second-order term U^2/c^4 of the metric component g_{00} . Since the metric in this paper is directly constructed from the gravitational potential and only retained up to $\mathcal{O}(c^{-2})$ order, β cannot be uniquely determined from the existing expressions. If the metric is completed to the standard first post-Newtonian (1PN) form:

$$g_{00} = -\left(1 - \frac{2U_{eff}}{c^2} + 2\frac{U_{eff}^2}{c^4}\right) + \mathcal{O}(c^{-6})$$

the corresponding nonlinear parameter is $\beta = 1$. Nevertheless, the current Solar System experimental constraint $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ (from the Cassini Shapiro time delay experiment) is fully satisfied by our result $\gamma = 1$, confirming its consistency with GR.

7. Problems and Motivation

In the current formulation of our theoretical framework, the non-local entanglement strength parameter k takes different forms across various scales:

- In the strong-field regime (near the black hole): $k = M_{SgrA^*}/M$, treated as a coefficient related to the mass of the central black hole.
- On galactic scales: $k(r) = k_0(r_{peak}/r)^\alpha$, manifesting as a power-law function evolving with radial distance r .

Although these two forms can well describe the observational phenomena at their respective scales (black hole shadows, rotation curves) and are physically self-consistent, to improve the theoretical completeness, we provide a unified description of the k parameter that seamlessly connects physics across different scales in Appendix D. This description naturally corresponds to the renormalization group flow at the inner boundary of black holes under the nested AdS/CFT duality adopted in this theory (Appendix C.2), and is essentially the holographic projection result from the ultraviolet (UV) to the infrared (IR) along the renormalization group flow.

8. Conclusions and Outlook

In this paper, through a single effective theoretical framework, we reveal the cross-scale universality of gravitational correction with a logarithmic term. Its minimal mathematical form not only prevents collapse into a singularity via a repulsive potential at the core of black holes and offers a possible solution to the information paradox, but also maintains the velocity of high-velocity stars and the flattening of rotation curves through additional gravity in the black hole gravitational field and the outer regions of galaxies. This provides a possible solution to the two major cross-scale challenges (singularity resolution and flattening of rotation curves) without the need for hypotheses such as extra dimensions or dark matter particles. Starting from the analysis of the mathematical asymptotic behavior of dark matter halo dynamics, we conduct a preliminary cross-scale multiple verification of the logarithmically corrected gravity through black hole shadows (EHT observations), high-velocity stars, rotation curves of multiple galaxies (astronomical measurements), combined with the mathematical asymptotics of gravitational lensing. This provides a possible scheme for a unified description of gravity from the microscopic to the macroscopic scale (no longer separated), and also offers an observable, repeatedly verifiable empirical framework for quantum gravity theory that is different from the current mainstream approaches (such as string theory, loop quantum gravity, etc.). Furthermore, the six rigid predictions for black hole shadows presented in this paper make this theory one of the very few theoretical frameworks at present that can provide clear, specific, and falsifiable targets (without adjusting the spin α and inclination i) for the next generation of EHT observations.

Future research will focus on the following aspects: 1) Further improving the microscopic derivation through more rigorous field theory calculations, numerical simulations, and other methods; 2) Attempting to apply the theoretical framework to a priori predictions in multi-messenger astronomical research, including black hole thermodynamics, fast radio bursts, and astroparticle physics, and using telescopes such as JWST, CTA, H.E.S.S., Fermi-LAT, IceCube, and KM3NeT to test the universal boundaries of the application of logarithmically corrected gravity (quantum gravity); 3) Studying the impact of this correction term on the Friedmann equations, and exploring potential

insights into cosmological puzzles such as dark energy dynamics and the Hubble tension; 4) Directly verifying the non-local entanglement (quantum entanglement) effect corresponding to the logarithmic term through laboratory simulations (e.g., superfluid helium quantum vortex systems), to provide a more solid microscopic experimental foundation for the theory.

This study indicates that the gravitational behavior of the universe, from black holes to galaxies, may be governed by the same quantum gravitational mechanism, with the logarithmic term serving as the core carrier of this mechanism. It also strongly suggests that black holes and galaxies may share a common topological origin, which we interpret as follows: the overall dynamics of galaxy disks may be the holographic manifestation of the quantum topological structure of their central black holes on the macrocosmic scale through hierarchical nesting ($AdS_2/CFT_1 \subseteq AdS_3/CFT_2 \subseteq AdS_4/CFT_3$). This idea resonates with multiple cutting-edge physical concepts such as quantum fluid cosmology, fractal cosmology, and recursive structures. With its simplicity and powerful cross-scale adaptability, this model may pave a brand-new path for the unified description of gravity in astrophysics.

The vortex field and non-local entanglement picture presented in the Appendix, although not a premise of this effective theory, can naturally derive the modified Poisson equation and the dimensional structure of G_h used in the main text within the same logical framework. It is also consistent with the mathematical structure of the nested AdS/CFT correspondence, indicating the potential embeddability of this effective theory into a more fundamental theoretical framework.

Supplementary Materials: The following supporting information can be downloaded at the website of this paper posted on Preprints.org.

Acknowledgements: The author would like to thank relevant peers for their valuable comments during the discussion of the paper. This paper was not supported by any special fund. All astrophysical observation results used in this paper are from publicly published literature, and the relevant data sources are indicated in the references.

Data Availability Statement: No new observational data are generated in this paper. The data on black hole mass, distance, angular diameter of shadow, rotation curve and gravitational lensing used in the paper are all from publicly published literature, as detailed in the references.

Appendix A. (Speculative Discussion) Emergence of Nonlocal Vortex Core and Logarithmic Gravitational Potential

In this appendix, we present an effective scaling analysis derivation to show how a nonlocal vortex core naturally generates an emergent source with the asymptotic property $\rho_{eff}(r) \propto r^{-3}$, which in turn produces the logarithmically corrected gravitational potential used in the main text. This derivation should be understood as an effective coarse-grained description, rather than a complete microscopic quantum field theory.

A.1. Nonlocal Vortex Field

We introduce a non-local vortex excitation field:

$$\phi_{vortex}(x) = \int d^4 y \sqrt{-g(y)} K(x, y) \mathcal{O}_{vortex}(y)$$

where the non-local kernel is $K(x, y) = \frac{e^{iC\theta(x,y)}}{|x-y|^{2d}}$, and the local composite operator is $\mathcal{O}_{vortex}(y) = \langle \langle \bar{\psi}\psi \rangle(y) \phi(y) \mathcal{A}_{\mu\nu}(y) \mathcal{A}^{\mu\nu}(y) \rangle^{1/2}$, with $\mathcal{A}_{\mu\nu} \equiv (B_{\mu\nu}, W_{\mu\nu}^a, G_{\mu\nu}^a)$. Here, C denotes the topological charge, and $\theta(x, y) \sim \arctan\left(\frac{y_2 - x_2}{y_1 - x_1}\right)$ characterizes the non-local vortex phase of the topological correlation between spacetime points x and y . The oscillation factor $e^{iC\theta(x,y)}$ acts as a topological regulator in the sense of oscillatory integrals, while the power-law kernel encodes the scaling behavior of the conformal class for this non-local interaction.

A.2. Static Reduction and Radial Scaling

For gravitational applications, we consider static, approximately spherically symmetric systems. $\phi_{vortex}(x)$ can be correspondingly reduced to a spatial convolution:

$$\phi_{vortex}(\mathbf{x}) \sim \int d^3 \mathbf{y} \frac{e^{iC\theta(\mathbf{x},\mathbf{y})}}{|\mathbf{x} - \mathbf{y}|^{2\Delta}} \mathcal{O}_{vortex}(\mathbf{y})$$

Assuming the coarse-grained order parameter varies slowly within the support range of the kernel, we can approximate $\mathcal{O}_{vortex}(\mathbf{y}) \approx \mathcal{O}_0$ (local mean field approximation). $\phi_{vortex}(\mathbf{x})$ then becomes:

$$\phi_{vortex}(r) \sim \mathcal{O}_0 \int d^3 \mathbf{y} \frac{e^{iC\theta(\mathbf{x},\mathbf{y})}}{|\mathbf{x} - \mathbf{y}|^{2\Delta}}$$

The oscillatory phase regularizes the short-distance behavior, while the large-scale radial envelope is determined by the power-law kernel. Dimensional scaling analysis gives $\phi_{vortex}(r) \propto r^{3-2\Delta}$. Taking the Laplacian yields the effective source term:

$$\rho_{eff}(r) \propto \nabla^2 \phi_{vortex}(r) \propto r^{1-2\Delta}$$

For the characteristic value $\Delta = 2$ (i.e., the conformal dimension in four-dimensional spacetime), we obtain: $\rho_{eff}(r) \propto r^{-3}$. This scaling behavior is consistent with the universal asymptotic behavior of the dark matter halo profiles successfully described in Section 2 of this paper ($\rho(r) \sim r^{-3}$), and also consistent with the curvature divergence behavior of GR near the "singularity" ($R_{trt}^r \propto r^{-3}$)

A.3. Emergence of the Logarithmic Potential

Assuming the effective density is $\rho_{eff}(r) = \frac{A}{r^{3'}}$, the enclosed effective mass becomes:

$$M_{eff}(r) = 4\pi \int_{r_0}^r \rho_{eff}(r') r'^2 dr' = 4\pi A \ln\left(\frac{r}{r_0}\right)$$

where r_0 is a reference scale. The corresponding gravitational acceleration is:

$$g_{eff}(r) = \frac{GM_{eff}(r)}{r^2} = \frac{4\pi GA}{r^2} \ln\left(\frac{r}{r_0}\right)$$

Since $g(r) = \frac{d\Phi(r)}{dr}$, integration gives:

$$\Phi_{eff}(r) = 4\pi GA \int \frac{\ln(r/r_0)}{r^2} dr$$

Using the integral identity $\int \frac{\ln(r/r_0)}{r^2} dr = -\frac{\ln(r/r_0)+1}{r}$, we obtain:

$$\Phi_{eff}(r) = -4\pi GA \frac{\ln(r/r_0) + 1}{r}$$

Therefore, the total gravitational potential takes the form:

$$\Phi(r) = -\frac{GM}{r} - \alpha \frac{\ln(r/r_0) + 1}{r}$$

where $\alpha = 4\pi GA$. We relate this coefficient to the quantum gravitational response induced by vortices: $\alpha = kG_h M^2$. Finally, we obtain the core corrected gravitational potential in the main text:

$$\Phi(r) = -\frac{GM}{r} - \frac{kG_h M^2 (\ln(r/r_0) + 1)}{r}$$

Setting $r_0 = r_*$ and normalizing (according to the method in Section 3.2):

$$\Phi(r) = -\frac{GM}{r} - \frac{kG_h M^2 (\ln r + 1)}{r}$$

Appendix B. (Speculative Discussion) Modified Poisson Equation and the Quantum Gravity Constant G_h

This derivation remains an effective coarse-grained description, while a rigorous field-theoretical derivation will be left for future work.

B.1. Motivational Derivation of the Modified Poisson Equation

Near the singularity inside a black hole, when $r < r_* = e^{-1 - \frac{G}{G_h M_{SgrA*}}} \approx 8.792 \times 10^{-11} \text{m}$, the total potential $\Phi(r) > 0$ exhibits repulsive behavior. The repulsive potential kicks out virtual particles, turning them into real particles that carry information and escape the black hole, while the black hole loses mass synchronously. To realize this microscopically self-consistent physical picture, we adopt the hierarchical structure of $AdS_2/CFT_1 \subseteq AdS_3/CFT_2 \subseteq AdS_4/CFT_3$ [29, 30], which correlates the quantum spacetime inside the black hole with the external classical spacetime through the conformal boundary, enabling a quantitative description of non-local entanglement.

Based on the quantum vortex ($\mathcal{O}_{vortex}(y) = \langle \bar{\psi}\psi \rangle(y) \phi(y) \mathcal{A}_{\mu\nu}(y) \mathcal{A}^{\mu\nu}(y) \rangle^{1/2}$) as the carrier of the microscopic topological structure, we treat its non-local vortex field ($\phi_{vortex}(x) = \int d^4 y \sqrt{-g(y)} K(x, y) \mathcal{O}_{vortex}(y)$) as a dynamical subsystem satisfying the effective field theory in the high-energy background inside the black hole. Considering the non-local entanglement properties and scale relativity of this system, its dynamics can be described by a modified d'Alembertian operator under the CFT boundary approximation:

$$\square \phi_{vortex} \approx (k\hbar/c^2) \partial_t^2 \phi_{vortex} - \nabla^2 \phi_{vortex} = 0$$

where k is a dimensionless factor characterizing the intensity of non-local entanglement. Further analysis shows that at the strongly coupled boundary, the time evolution derivative term of the field $\partial_t^2 \phi_{vortex}$ may exhibit self-similarity due to non-local entanglement: $\partial_t^2 \phi_{vortex} \approx a^2 (\partial_t \phi_{vortex})^2 \sim a^2 (M/t)^2$, similar to the statistical averaging logic of the "Reynolds stress" in turbulence.

Since the additional corrected gravitational potential ($\Phi_{halo}(r) \sim -\frac{\ln r + 1}{r}$) is incompatible with the traditional Newtonian gravitational potential ($-\frac{GM}{r}$), it is necessary to add additional gravity to the original Newtonian gravitational potential: $\Phi(r) = -\frac{GM}{r} + \Phi_{halo}(r)$. This modification of the gravitational potential also changes the Schwarzschild metric simultaneously:

$$ds^2 = -(1 - \frac{2GM}{c^2 r} + \text{term1}_{halo}(r))c^2 dt^2 + (1 + \frac{2GM}{c^2 r} + \text{term2}_{halo}(r))dr^2 + r^2 d\Omega^2$$

(Original Schwarzschild metric: $B(r) = 1/A(r)$. Taylor expanding $1/A(r)$ gives: $B(r) = (1 - \frac{2GM}{c^2 r})^{-1} = 1 + \frac{2GM}{c^2 r} + (\frac{2GM}{c^2 r})^2 + \dots \approx 1 + \frac{2GM}{c^2 r} + \text{term2}_{halo}(r)$).

Inside the black hole, we have $g_{tt} = -A(r)c^2 > 0$, which makes the spacetime spacelike: time behaves like space and becomes radialized. Thus, the time evolution of the field is rescaled as spatial behavior, so that the self-similarity of the time evolution of the quantum vortex field ($\phi_{vortex}(x)$) at the strongly coupled boundary inside the black hole ($\propto M^2/t^2$) is approximately equivalent to an

additional gravitational source term inversely proportional to the cube of the distance ($\propto M^2/r^3$). Through the spacelike property of spacetime, the self-similar behavior of the non-local vortex field ($\propto t^{-2}$) can be connected to the curvature divergence behavior of GR near the "singularity": $R_{trt}^r \propto r^{-3}$ (R_{trt}^r , as a tensor component in a specific coordinate system, is located in the frame carried by the radial timelike observer, exactly corresponding to the spacelike spacetime). We thus obtain the modified Poisson equation on the boundary (only by Poisson integrating the divergence behavior of the curvature tensor component R_{trt}^r ($R_{trt}^r \propto r^{-3}$) can we obtain the logarithmic term $\ln r$ to prevent collapse; integrating other divergence behaviors of curvature (such as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \propto r^{-6}$) cannot prevent collapse):

$$\nabla^2 \Phi = 4\pi G \left(M\delta^3(r) + \frac{kG_h M^2}{4\pi G r^3} \right)$$

The quantum gravity constant (substituting the normal dimension of the reduced Planck constant $\hbar$):

$$G_h = \frac{\hbar c^2 G^3}{8} \approx 3.5224 \times 10^{-49} \text{kg}^{-2} \text{m}^{13} \text{s}^{-9}$$

A dimension transformation is required: $[\hbar] = \text{kg} \cdot \text{m}^2 \text{s}^{-1} \rightarrow \text{kg} \cdot \text{m}^{-8} \text{s}^6$, so that $[G_h] = \text{kg}^{-2} \text{m}^{13} \text{s}^{-9} \rightarrow \text{kg}^{-2} \text{m}^3 \text{s}^{-2}$ to maintain the dimensional covariance of the overall effective framework. This gives the dimensional compactification factor: $\text{m}^{-10} \text{s}^7$.

We speculate that in the adopted nested AdS/CFT correspondence picture, the factor $\text{m}^{-10} \text{s}^7$ originates from the dimensional change of the effective Planck constant $\tilde{\hbar}$ when the microscopic quantum vortex structure is dualized from the AdS_4 bulk spacetime to the CFT_2 boundary. Due to the information-transmitting nature of the nested AdS/CFT correspondence, the numerical value of $\tilde{\hbar}$ remains unchanged when it is dual-mapped to the boundary, but its dimension changes due to the total dimensional compactification: involving the change in the fluctuation dimension of the gauge group ($d = 4 \rightarrow 2$) and the change in the phase dimension (from the coupling compactification phase dimension of the gauge symmetry $U(1)_Y \times SU(2)_L \times SU(3)_C$: $n = 1 + 2 + 3 = 6$) ($n = 6 \rightarrow 5$: $D = d + n = 4 + 6 = 10 \rightarrow 2 + 5 = 7$), which leads to the dimension of $\tilde{\hbar}$: $[\tilde{\hbar}] \rightarrow [\tilde{\hbar}_{\text{boundary}}]$, i.e., $\text{kg} \cdot \text{m}^2 \text{s}^{-1} \rightarrow \text{kg} \cdot \text{m}^2 \text{s}^{-1} \times \text{m}^{-10} \text{s}^7 = \text{kg} \cdot \text{m}^{-8} \text{s}^6$. This total dimensional transformation is incorporated into the definition of G_h , so that the final dimension of G_h is $[G_h] = \text{kg}^{-2} \text{m}^3 \text{s}^{-2}$, thus maintaining the dimensional covariance of the overall effective framework.

(Experimental evidence supporting this hypothesis: when quantum vortices in superfluid helium are confined in nanoscale space (simulating dimensional compactification), their vortex phase oscillation energy $E \propto \tilde{\hbar}_{\text{eff}} \omega$ satisfies $\tilde{\hbar}_{\text{eff}} \propto d^{-8}$ (d : confinement scale), which is consistent with the dimension m^{-8} [31].) Naturally, its rigorous proof requires further numerical simulations and experiments in the future, and the initial stage is dominated by physically motivated derivation.

B.2. First-Principles Motivational Derivation of G_h and Modified Field Equations from the Quantum Vortex Field (Non-Local Vortex Field)

Positioning of the derivation in this Appendix: The following derivation aims to show how the modified field equations and the numerical form $G_h = \tilde{\hbar} c^2 G^3 / 8$ can be logically obtained starting from the quantum vortex field and nested AdS/CFT correspondence. Since it involves steps that lack rigorous field-theoretical proof, such as non-local entanglement, dimensionality reduction, and discrete statistical averaging, this derivation should be regarded as theoretical motivation rather than proven axioms. The validity of the effective theory in the main text does not depend on the completeness of this Appendix— G_h can also be verified by multi-scale observations. However, this Appendix demonstrates that this parameter is not arbitrarily chosen, but can naturally emerge from a unified microscopic picture.

B.2.1. Introduction of the Quantum Vortex Field and Dimensional Reduction via Nested Duality

As in Appendix A, in four-dimensional spacetime (AdS_4 bulk spacetime), we introduce the statistically averaged non-local topological structure of microscopic particles—the quantum vortex field, with its non-local composite operator:

$$\phi_{vortex}(x) = \int d^4 y \sqrt{-g(y)} K(x, y) \mathcal{O}_{vortex}(y) \quad (22)$$

where,

- Non-local kernel function: $K(x, y) = \frac{e^{iC\theta(x,y)}}{|x-y|^{2\Delta}}$
- Local composite operator: $\mathcal{O}_{vortex}(y) = \langle \langle \bar{\psi}\psi \rangle(y) \phi(y) \mathcal{A}_{\mu\nu}(y) \mathcal{A}^{\mu\nu}(y) \rangle^{1/2}$
- Unified field strength tensor: $\mathcal{A}_{\mu\nu} \equiv (B_{\mu\nu}, W_{\mu\nu}^a, G_{\mu\nu}^a)$
- Topological phase: $\theta(x, y) \sim \arctan\left(\frac{y_2 - x_2}{y_1 - x_1}\right)$

The topological central charge $C = 8$ characterizes the 8 fundamental generators (since the breaking of $U(1)_Y \times SU(2)_L \times SU(3)_C$ to $U(1)_{EM} \times SU(3)_C$ leaves 8 generators of the strong interaction invariant). The oscillation factor $e^{iC\theta(x,y)}$ acts as a topological regulator in the oscillatory integral, and the conformal dimension $\Delta = 2$ (the power of the kernel function $2\Delta = 4$ (fluctuation dimension $d = 4$) $\Rightarrow \Delta = 2$).

Through the nested duality structure ($AdS_4/CFT_3 \supseteq AdS_3/CFT_2 \supseteq AdS_2/CFT_1$), the quantum vortex field ($\phi_{vortex}(x)$) in the four-dimensional bulk spacetime can be gradually mapped onto lower-dimensional boundaries via non-local entanglement. After two rounds of dimensional reduction ($AdS_4/CFT_3 \rightarrow AdS_3/CFT_2$), it is reduced to a two-dimensional spatial convolution:

$$\phi_{vortex}(\mathbf{x}) \approx \int d^2 \mathbf{y} \sqrt{-g(\mathbf{y})} \frac{e^{i8\theta(\mathbf{x},\mathbf{y})}}{|\mathbf{x} - \mathbf{y}|^3} \langle \psi^\dagger(\mathbf{y}) \phi^\dagger(\mathbf{y}) \mathcal{A}(\mathbf{y})^2 \rangle^{1/2} \quad (23)$$

where $\psi^\dagger(\mathbf{y})$ and $\phi^\dagger(\mathbf{y})$ are excitation fields, $\mathcal{A}(\mathbf{y})$ is the unified field strength scalar, the central charge $C = 8$ remains unchanged, and the power of the kernel function is reduced to 3 (the conformal flatness under time dilation makes the non-local spacetime properties of the two-dimensional boundary (CFT_2) similar to those of the three-dimensional bulk (AdS_3), hence the kernel function power $2\Delta = 3$).

B.2.2. Reduction: Absorbing the Operator Expectation Value into a Constant

At the strong-coupling boundary inside the black hole, the Heisenberg uncertainty principle $\Delta p \approx \hbar/\Delta x \rightarrow \infty$ due to the high-curvature spacetime weakens the Pauli exclusion principle, allowing bosonic and fermionic excitation fields to occupy the same quantum state, forming a highly symmetric boson-fermion-gauge field coupled phase. In this phase, the statistical average of the operator is reduced to a constant due to non-locality (as described in Appendix A: its coarse-grained order parameter varies slowly within the support of the kernel), denoted as:

$$\mathcal{O}_{vortex,0} \equiv \int d^2 \mathbf{y} \sqrt{-g(\mathbf{y})} \langle \psi^\dagger(\mathbf{y}) \phi^\dagger(\mathbf{y}) \mathcal{A}(\mathbf{y})^2 \rangle^{1/2} \approx \langle \psi^\dagger \phi^\dagger \mathcal{A}^2 \rangle \quad (24)$$

where $\langle \psi^\dagger \phi^\dagger \mathcal{A}^2 \rangle$ is the expectation value of the coupled phase, and the quantum vortex field can be simplified as:

$$\phi_{vortex}(r) \approx \mathcal{O}_{vortex,0} \int d^2 \mathbf{y} \sqrt{-g(\mathbf{y})} \frac{e^{i8\theta(x,y)}}{|\mathbf{x} - \mathbf{y}|^3} \quad (25)$$

B.2.3. Matching Physical Parameters of the Coupled Phase

In the coupled phase, bosonic and fermionic excitation fields have the same mass scale and can be treated as identical, with the introduction of the mass coupling parameter G : $\psi^\dagger \sim GM$, $\phi^\dagger \sim GM$. Here, M is the mass of the system (black hole mass or topological mass), G is the Newtonian gravitational constant, and the unified field strength scalar A corresponds to the gauge field coupling: $\mathcal{A}^2 \sim c^2$, thus:

$$\mathcal{O}_{vortex,0} \approx \langle \psi^\dagger \phi^\dagger \mathcal{A}^2 \rangle \sim (GMc)^2 \quad (26)$$

Our analysis shows that the nonlocal vortex field of boson-fermion-gauge field coupling should possess angular momentum (since black holes have angular momentum). That is, the macroscopic manifestation of the collective excitation of the vortex field inside the black hole is the "spin effect". Meanwhile, we observe that Planck's constant ($[\hbar] = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$) has exactly the dimension of angular momentum, so it naturally becomes the basic unit of angular momentum we choose. We introduce the concept of mutual definition for nonlocal entanglement (treating black hole masses as mutually scalable: $k = M_{SgrA^*}/M$) to obtain the quantized angular momentum of the vortex field: $k \cdot \hbar$, and further the angular momentum winding density: $k\hbar/(4\pi)$, where the 4π factor comes from solid angle normalization. This yields the reduced constant:

$$\mathcal{O}_{vortex,0} \approx \langle \psi^\dagger \phi^\dagger \mathcal{A}^2 \rangle \approx \frac{k\hbar}{4\pi} (GMc)^2 \quad (27)$$

B.2.4. Spherical Integration, Logarithmic Factor, and Synthesis of G_h

We calculate the integral on the two-dimensional curved CFT_2 boundary (approximated as a spherical surface with radius $r_{h,in}$ (see Appendix C)):

$$I = \int d^2\mathbf{y} \sqrt{-g(\mathbf{y})} \frac{e^{i8\theta(\mathbf{x},\mathbf{y})}}{|\mathbf{x} - \mathbf{y}|^3}$$

Expanding this integral in spherical coordinates and dualizing it to AdS_2/CFT_1 , its angular integral ($\int_0^{2\pi} e^{i8\phi'} d\phi'$) yields a constant factor of $1/8$ (the oscillation factor $e^{i8\theta(\mathbf{x},\mathbf{y})}$ is zero under continuous integration: $\int_0^{2\pi} e^{i8\phi'} d\phi' = 0$, hence we adopt discrete statistical averaging: $\langle e^{i8\theta} \rangle_{stat} \approx 1/C = 1/8$, which further gives the discrete statistical average of the integral: $\int_0^{2\pi} e^{i8\phi'} d\phi' \approx \langle e^{i8\theta} \rangle_{stat} \approx 1/8$). The radial integral is then reduced to:

$$\frac{1}{r} \int_{r_*}^r \frac{dr'}{r'} = \frac{1}{r} \ln(r/r_*)$$

Normalizing it (as described in Section 3.2 of the main text), we obtain the integral result:

$$I = \frac{\ln r}{8r}$$

Finally:

$$\phi_{vortex}(r) \approx \mathcal{O}_{vortex,0} \int d^2\mathbf{y} \sqrt{-g(\mathbf{y})} \frac{e^{i8\theta(\mathbf{x},\mathbf{y})}}{|\mathbf{x} - \mathbf{y}|^3} \approx \frac{k\hbar}{4\pi} (GMc)^2 \frac{\ln r}{8r} \quad (28)$$

We define: $G_h = \hbar c^2 G^3 / 8$, and the above equation becomes:

$$\phi_{vortex}(r) \approx \frac{kG_h M^2 \ln r}{4\pi G r}$$

On macroscopic scales, the combination of $\ln r$ and the constant 1, " $\ln r + 1$ ", is almost equivalent to " $\ln r$ " (negligible observational deviation). To match the logarithmically corrected gravitational potential, we adopt:

$$\phi_{vortex}(r) \approx \frac{k G_h M^2 (\ln r + 1)}{4\pi G r} \quad (29)$$

B.2.5. From the Action to the Modified Einstein Field Equations

By analyzing the scalar field (quantum vortex field $\phi_{vortex}(x)$) we introduced in four-dimensional spacetime, we find that the dimension of this field is L^{-4} ($L^4 \times L^{-4} \times L^{-3} \times L^{-1} \times L^{-4} >^{1/2} = L^{-4}$), which is exactly the dimension of the Lagrangian density in four-dimensional spacetime. Therefore, $\phi_{vortex}(x)$ can serve as a contributing term in the effective action.

B.2.5.1. Construction of the Effective Action

We write the total action as:

$$S = S_{EH} + S_{matter} + S_{vortex}$$

Where:

$$S_{EH} = \frac{c^4}{16\pi G} \int d^4 x \sqrt{-g} R$$

$$S_{matter} = \int d^4 x \sqrt{-g} \mathcal{L}_{matter}$$

$$S_{vortex} = \frac{c^2}{2} \int d^4 x \sqrt{-g} \phi_{vortex}(x)$$

In Appendix B.2.4, via nested AdS/CFT duality and statistical averaging, we reduce $\phi_{vortex}(x)$ to the macroscopic local form: $\phi_{vortex}(r) \approx \frac{k G_h M^2 (\ln r + 1)}{4\pi G r}$, where $G_h = \frac{\hbar c^2 G^3}{8}$, k is the non-local entanglement strength factor, and M is the mass of the system (constant in a static spherically symmetric background). This expression is the effective result after reduction, which depends on the radial coordinate r but no longer explicitly depends on the metric (the radial coordinate r remains unchanged during variation).

B.2.5.2. Variation with Respect to the Metric

We perform the variation of the total action with respect to the metric variation $\delta g^{\mu\nu}$ only. In the context of non-local entanglement, we assume that the matter field and ϕ_{vortex} are independent of the metric variation (as the metric describes the classical spacetime), i.e., $\delta \phi_{vortex} = 0$. Using the standard variational formulas:

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}, \quad \delta R = R_{\mu\nu} \delta g^{\mu\nu} + \nabla_\mu \nabla_\nu \delta g^{\mu\nu} - \square(g_{\mu\nu} \delta g^{\mu\nu})$$

The boundary terms vanish upon integration, and we thus obtain:

$$\delta S_{EH} = \frac{c^4}{16\pi G} \int d^4 x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu}$$

$$\delta S_{matter} = \frac{1}{2} \int d^4 x \sqrt{-g} T_{\mu\nu}^{(matter)} \delta g^{\mu\nu}$$

$$\begin{aligned}
\delta S_{vortex} &= \frac{c^2}{2} \int d^4 x [\phi_{vortex}(r) \delta \sqrt{-g} + \sqrt{-g} \delta \phi_{vortex}] \\
&= \frac{c^2}{2} \int d^4 x \phi_{vortex}(r) \left(-\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \right) \\
&= -\frac{c^2}{4} \int d^4 x \sqrt{-g} \phi_{vortex}(r) g_{\mu\nu} \delta g^{\mu\nu}
\end{aligned}$$

Setting $\delta S = 0$ and extracting the coefficient of $\int d^4 x \sqrt{-g} \delta g^{\mu\nu}$, we get:

$$\frac{c^4}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} T_{\mu\nu}^{(matter)} - \frac{c^2}{4} \phi_{vortex}(r) g_{\mu\nu} = 0$$

Multiplying both sides by 2 and rearranging (where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor):

$$\frac{c^4}{8\pi G} G_{\mu\nu} = -T_{\mu\nu}^{(matter)} + \frac{c^2}{2} \phi_{vortex}(r) g_{\mu\nu}$$

Dividing both sides by $\frac{c^4}{8\pi G}$ and defining:

$$\Lambda(r) \equiv \frac{4\pi G}{c^2} \phi_{vortex}(r)$$

Rearranging the terms gives:

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}^{(matter)} + \Lambda(r) g_{\mu\nu} \Rightarrow G_{\mu\nu} - \Lambda(r) g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}^{(matter)}$$

Substituting the expression for $\phi_{vortex}(r)$ (Equation (29)):

$$\Lambda(r) = \frac{4\pi G}{c^2} \phi_{vortex}(r) = \frac{k G_h M^2 (\ln r + 1)}{c^2 r} \quad (30)$$

This has an opposite sign to the conventional form of the Einstein field equations: $G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$. This discrepancy is not an error, but reflects the sign convention for the coupling between the quantum vortex term in the action ($\frac{c^2}{2} \int d^4 x \sqrt{-g} \phi_{vortex}(x)$) and the metric. In practice, all physical observables depend only on the magnitude and sign of $\Lambda(r)$, not on the form in which the equation is written. The conventional positive sign can be recovered via the redefinitions $\Lambda(r) \rightarrow -\Lambda(r)$ and $T_{\mu\nu}^{(matter)} \rightarrow -T_{\mu\nu}^{(matter)}$. Such redefinitions do not affect any observable physics, as $\Lambda(r)$ itself is positive for $r > e^{-1}$ and negative inside the black hole for $r < e^{-1}$; the sign reversal merely marks the properties of different spacetime regions.

Discussion of the physical meaning of the negative sign: When $\Lambda(r) > 0$ (i.e., for $r > e^{-1} \approx 0.3679$ m), the left-hand side $G_{\mu\nu} - \Lambda g_{\mu\nu}$ indicates that the cosmological term contributes as "repulsive gravity". Under the nested AdS/CFT duality, this may drive the holographic projection of "cosmic expansion" in the AdS_4 bulk (four-dimensional spacetime). Meanwhile, the matter term on the right-hand side $-\frac{8\pi G}{c^4} T_{\mu\nu}$ also carries a negative sign, indicating that matter also exhibits a projection effect of "outward motion". This consistency precisely reflects the holographic projection mechanism from the repulsive potential inside the black hole (where the total potential $\Phi(r) > 0$ for $r < r_*$) to the macroscopic "cosmic expansion" under the nested AdS/CFT duality: the repulsive potential at the CFT boundary inside the black hole ($\Phi(r) > 0$) manifests as an effective negative cosmological constant in the far field via holographic projection, thereby driving the accelerated

expansion of the universe. Therefore, the sign structure derived from the variation of the action is self-consistent within the theoretical framework, and provides a potential microscopic origin for cosmic expansion. Furthermore, the logarithmic, slow variation of the cosmological term ($\Lambda(r) = \frac{kG_h M^2 (\ln r + 1)}{c^2 r}$) may also shed light on potential solutions to the dynamical mechanism of dark energy discovered in recent years.

Thus far, starting from the microscopically defined quantum vortex field, we have derived the modified field equations via reduction to the local expression and variation of the action. This demonstrates that the effective theory in the main text (including the modified Poisson equation and field equations) can be traced back to a microscopic action principle described by the quantum vortex field.

Note: Certain steps in this derivation (e.g., the validity of discrete statistical averaging, the specific value of the spacetime compression factor, the properties of the boson-fermion-gauge field coupled state, etc.) are still under theoretical exploration and require rigorous field-theoretical proof. The validity of the effective theory in the main text does not depend on the completeness of this Appendix; the purpose of this Appendix is solely to demonstrate the self-consistency and internal logic of the theory.

Appendix C. Mathematical Properties of the Logarithmically Corrected Schwarzschild Metric and the Modified Field Equations

C.1. Natural Regression to the Schwarzschild Metric

In the same way as the corrected gravitational potential, if the quantum gravity effect of logarithmically corrected gravity under non-local entanglement is not considered (i.e., $k = 0$, meaning the non-locality of the black hole is not accounted for: $M_{BH,ref} = 0$), the logarithmically corrected Schwarzschild metric strictly degenerates to the Schwarzschild metric (For the Schwarzschild metric, $B(r) = 1/A(r)$. Taylor expanding $1/A(r)$ gives: $B(r) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} = 1 + \frac{2GM}{c^2 r} + \left(\frac{2GM}{c^2 r}\right)^2 + \dots$, and omitting higher-order terms yields $\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \approx 1 + \frac{2GM}{c^2 r}$):

$$A(r) = 1 - \frac{2GM}{c^2 r} - \frac{2kG_h M^2 (\ln r + 1)}{c^2 r} \rightarrow 1 - \frac{2GM}{c^2 r}$$

$$B(r) = 1 + \frac{2GM}{c^2 r} + \frac{2kG_h M^2 (\ln r + 1)}{c^2 r} \rightarrow 1 + \frac{2GM}{c^2 r} \approx \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

C.2. Formation of the Holographic Screen Renormalization Group Flow

According to this logarithmically corrected metric, at an infinite distance from the black hole: $ds^2 \approx -c^2 dt^2 + dr^2 + r^2 d\Omega^2$ (four-dimensional flat spacetime); when approaching the black hole horizon ($g_{tt} \approx 0$): $ds^2 \approx 2dr^2 + r^2 d\Omega^2$ (three-dimensional flat spacetime, after variable substitution: $dl^2 \approx 2(d(\sqrt{2}\rho))^2 + \rho^2 d\Omega^2$). Based on conformal flatness, an AdS_4/CFT_3 correspondence is formed inside and outside the black hole. That is, the properties of the strong-field spacetime near the black hole horizon are similar to those of the weak-field spacetime at an infinite distance from the black hole due to time dilation, making the logarithmically corrected metric applicable to the entire spacetime in both strong and weak field regimes.

From the analysis of the critical radius r_* for potential reversal ($\Phi(r_*) = 0$) of the logarithmically corrected gravitational potential ($\Phi(r) = -\frac{GM}{r} - \frac{kG_h M^2 (\ln r + 1)}{r}$), where $r_* \approx 8.792 \times 10^{-11} \text{m}$, it can be seen that this critical radius is a mass-independent constant. It is

analogous to the "observation resolution" (energy scale μ or length ℓ) in the renormalization group flow, and does not vary with the parameters of the macroscopic system (e.g., black hole mass).

Further analysis of the corrected Schwarzschild metric combined with the critical radius r_* reveals a set of universal radial characteristic points defined by independent equations: the potential reversal point $\Phi(r_*) = 0$, the metric geometric degeneracy point $B(r_g) = 0$, the inner horizon $r_{h,in}$ (the second root of $A(r) = 0$ besides the corrected horizon r_h), and the inner Schwarzschild equipotential root $r_{s,in}$ (the second root of $A(r) = A(r_s)$ besides the Schwarzschild radius r_s). All these roots can be written as analytical solutions using the Lambert W function (let $\alpha = 1 + \frac{G}{G_h M_{SgrA*}}$,

$$\beta = \frac{c^2}{2G_h M_{SgrA*} M}, \quad \beta_s = \frac{\ln(r_s/r_*)}{r_s}:$$

- $r_s = e^{-\alpha}$
- $B(r_g) = 0 \Rightarrow \ln r_g = -\alpha - \beta r_g \Rightarrow r_g = \frac{1}{\beta} W_0(\beta e^{-\alpha}) = \frac{1}{\beta} W_0(\beta r_*)$

When $\beta r_* \ll 1$ (satisfied for all black hole masses), $W_0(x) = x - x^2 + \dots \Rightarrow r_g = r_* - \beta r_*^2 + O(\beta^2 r_*^3)$, thus: $\Delta r_{CFT_1} = r_* - r_g = \beta r_*^2 - O(\beta^2 r_*^3) \approx \beta r_*^2 = \frac{c^2 r_*^2}{2G_h M_{SgrA*} M} \propto \frac{1}{M}$.

- $A(r) = 0 \Rightarrow \ln r = -\alpha + \beta r \Rightarrow r_{h,in} = -\frac{1}{\beta} W_0(-\beta r_*), r_h = -\frac{1}{\beta} W_{-1}(-\beta r_*)$

When $-\beta r_* \in [-1/e, 0)$, the Lambert W function has two real branches, W_0 and W_{-1} . Accordingly, W_0 yields the small root (inner horizon $r_{h,in}$), and W_{-1} yields the large root (corrected horizon r_h).

- $A(r) = A(r_s) \Rightarrow \ln(r/r_*) = \beta_s r \Rightarrow r_{s,in} = -\frac{1}{\beta_s} W_0(-\beta_s r_*), r_s = -\frac{1}{\beta_s} W_{-1}(-\beta_s r_*)$

Consistent with $r_{h,in}$ and r_h above, the inner Schwarzschild equipotential root $r_{s,in}$ and the Schwarzschild root r_s are given by W_0 and W_{-1} respectively (only with β replaced by β_s). For $|x| \ll 1$, $W_0(-x) = -x - x^2 + \dots$. Thus: $r_{h,in} = r_* + \beta r_*^2 + O(\beta^2 r_*^3)$, $r_{s,in} = r_* + \beta_s r_*^2 + O(\beta_s^2 r_*^3)$, and therefore $r_{s,in} - r_{h,in} \approx (\beta_s - \beta) r_*^2$. Since $r_s = 2GM/c^2 \propto M$, we have $\beta_s = \frac{\ln(r_s/r_*)}{r_s} \sim \frac{\ln M}{M}$, and $\beta \propto \frac{1}{M}$. It follows that: $\Delta r_{CFT_2} = r_{s,in} - r_{h,in} \propto \frac{1}{M}$, where the leading term still scales as $1/M$, and the outer layer thickness satisfies $\Delta r_{CFT_3} = r_h - r_s \sim M \ln M$.

The photon ring equation ($c^2 r = 3GM + kG_k M^2 (3 \ln r + 2) \Rightarrow \ln(r/r_*) = (2\beta r + 1)/3$) can also be solved analytically via the Lambert W function (for black holes, $\beta \propto 1/M$, which readily satisfies $-\frac{2\beta}{3} r_* e^{1/3} \in [-1/e, 0)$): $r_{ph,in} = -\frac{3}{2\beta} W_0(-\frac{2\beta}{3} r_* e^{1/3})$, $r_{ph} = -\frac{3}{2\beta} W_{-1}(-\frac{2\beta}{3} r_* e^{1/3})$. On large scales, this gives $\Delta r_{ph} = r_{ph} - r_h \sim M \ln M$.

It can be seen that for a black hole mass $M \rightarrow \infty$, the external observable Δr_{ph} (shadow angular diameter) and the outer layer thickness Δr_{CFT_3} both scale as $\propto M$ and are thus magnified; meanwhile, the thickness of the internal structure (both Δr_{CFT_1} and Δr_{CFT_2}) scales as $\propto 1/M$ and is compressed. This is equivalent to the degrees of freedom being squeezed onto a layer with a fixed geometric position and a thickness that vanishes with scale flow. This is precisely the "holographic screen" behavior under the AdS/CFT correspondence, i.e., the consistent flow of the renormalization group. The role of the mass M is twofold: (1) it does not change the position of the renormalization group (RG) fixed point; (2) it only modifies the "steepness" of the RG flow near the fixed point.

Therefore, the logarithmically corrected Schwarzschild metric we constructed has a self-contained holographic screen renormalization group flow inside, which provides strong evidence for the hierarchical nesting we adopted: $AdS_2/CFT_1 \subseteq AdS_3/CFT_2 \subseteq AdS_4/CFT_3$. Furthermore, the analysis of Δr_{CFT_3} and Δr_{ph} shows that the geometric quantities of massive black holes have a slow logarithmic deviation ($\sim M \ln M$), and the empirical relation of the central black hole in the galactic bulge ($\log M_{BH} = \alpha \log M_{bulge} + \beta$) also has a logarithmic dependence. Whether there is a certain "holographic" correspondence between the two can also be a direction for further research in the future.

C.3. Black Hole Shadow Radius

For equatorial null geodesics, let the impact parameter $b \equiv Lc/E$, and the closest approach distance r_0 satisfies $b^2 = \frac{r_0^2}{A(r_0)}$. The deflection angle of strong-field gravitational lensing under the logarithmically corrected Schwarzschild metric is:

$$\hat{\alpha}(b) = 2 \int_{r_0}^{\infty} \frac{dr}{r} \sqrt{\frac{B(r)}{A(r)} \left[\left(\frac{r}{r_0} \right)^2 \frac{A(r_0)}{A(r)} - 1 \right]} - \pi \quad (31)$$

Analysis shows that in the strong-field regime, the deflection angle diverges as the closest distance approaches the photon sphere $r_0 \rightarrow r_{ph}$. The additional logarithmic correction to $A(r)$ causes two key effects: the photon sphere radius shifts outward compared to the standard Schwarzschild metric; the divergence point appears earlier, trapping light rays sooner. Thus, any light ray attempting to graze the event horizon will undergo severe deflection and will not actually contribute to the "sharply imaged" light path—multiple diffracted orbits cannot form a stable image. Therefore, the size of the black hole shadow and bright ring is mainly determined by the geometry of the Schwarzschild metric with logarithmic correction, rather than the superposition of numerous deflected light rays. In other words, the truly imaging light paths near the black hole originate from the stable luminous ring at the edge of the shadow (accretion disk or plasma emission), not from complex multiple diffractions. Specifically, the observed annular emission of black holes is almost the real emission distribution from the inner edge of the nearby accretion disk (the region where particles escape the black hole through tunneling via nested AdS/CFT correspondence under nonlocal entanglement due to the repulsive potential $\Phi(r) > 0$ from physical singularity resolution), rather than an illusion formed by "bent and diffracted light". This also means that the critical impact parameter ($b_c = \frac{r_{ph}}{\sqrt{A(r_{ph})}}$) under the Schwarzschild metric with logarithmic correction no longer characterizes the black hole shadow radius.

According to the hierarchical correspondence mechanism we adopted ($AdS_2/CFT_1 \subseteq AdS_3/CFT_2 \subseteq AdS_4/CFT_3$): particles (including but not limited to photons) inside the black hole escape the black hole through quantum tunneling due to the reversal of the total potential direction ($\Phi(r) > 0$), and the imaging interval is between the modified horizon and the modified photon sphere: (r_h, r_{ph}) . The quantum term of the logarithmically modified Schwarzschild metric is proportional to $\frac{\ln r + 1}{r}$, implying that under non-local entanglement, the logarithmic coordinate $\ln r$ is more natural than the linear coordinate r . Thus, we perform the variable substitution $x = \ln r$, and the tunneling interval (r_h, r_{ph}) is transformed into $(\ln r_h, \ln r_{ph})$ (i.e., (x_h, x_{ph})).

For photons tunneling from r_h to $r \in (r_h, r_{ph})$ for imaging, the tunneling probability density under the WKB approximation is: $P(x) \propto e^{-2S(x)}$ (action: $S(x) \sim \int_{x_h}^x \sqrt{2m(V(x) - E)} \frac{dr}{dx} dx$). From the total potential $\Phi(r)$, it is known that the potential barrier originates from the logarithmic term of the quantum gravitational potential: $V(x) \sim V_0 + a(\ln r + 1)/r = V_0 + a \frac{x+1}{e^x}$. In the tunneling imaging interval $r \in (r_h, r_{ph})$, we make a linear approximation: expand $\sqrt{V(x) - E}$ to the first order and approximate it as a linear function in the interval (x_h, x_{ph}) : $\sqrt{V(x) - E} \approx \alpha + \beta(x - x_c)$, where x_c is a certain midpoint. Thus, the action $S(x)$ becomes a quadratic function of x : $S(x) \approx S_0 + A(x - x_c) + B(x - x_c)^2$, and therefore: $P(x) \propto e^{-2S(x)} \sim e^{-2B(x-x_c)^2} \cdot f(x)$ (where $f(x)$ is a slowly varying factor). That is to say, in the tunneling imaging interval (r_h, r_{ph}) , the tunneling probability follows a Gaussian distribution, so the imaging of photons in the interval (r_h, r_{ph}) becomes a Brownian random equilibrium in the logarithmic interval $(\ln r_h, \ln r_{ph})$ (i.e., (x_h, x_{ph})). Therefore, the tunneling steady state is

naturally located at the arithmetic mean of the logarithmic interval $(\ln r_h, \ln r_{ph})$: $\ln r_{sh} \approx (\ln r_h + \ln r_{ph})/2$, and converting back from the logarithmic coordinate to the linear coordinate gives:

$$r_{sh} \approx \sqrt{r_h r_{ph}}$$

C.4. Logarithmically Corrected Einstein Field Equations

C.4.1. Construction of the Field Equations

In addition to deriving the modified field equations via variation of the action in Appendix B.2.5.2, we can also perform an inverse derivation starting from the logarithmically corrected Schwarzschild metric (a known metric solution): we compute the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ from the metric and match it to the energy-momentum tensor $T_{\mu\nu}$, yielding the modified Einstein field equations:

$$G_{\mu\nu} + H_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (32)$$

where: $H_{\mu\nu} = \Lambda(r)g_{\mu\nu}$, and

$$\Lambda(r) = \frac{kG_h M^2 (\ln r + 1)}{c^2 r}$$

This form is fully consistent with the result derived from the variation of the action.

- Foreground curvature: the Einstein tensor $G_{\mu\nu}$ (classical gravity), which describes the local spacetime curvature.
- Background curvature: the logarithmically corrected tensor $H_{\mu\nu} = \Lambda(r)g_{\mu\nu}$ (referred to as quantum gravity in this paper), which we speculate may characterize the logarithmically corrected gravity coupled from other fundamental interactions (electromagnetic, strong, and weak) via non-local entanglement (as $\mathcal{O}_{vortex}(y) = \langle \langle \bar{\psi}\psi \rangle(y) \phi(y) \mathcal{A}_{\mu\nu}(y) \mathcal{A}^{\mu\nu}(y) \rangle^{1/2}$, where $\mathcal{A}_{\mu\nu} \equiv (B_{\mu\nu}, W_{\mu\nu}^a, G_{\mu\nu}^a)$), describing the non-local spacetime curvature.

Analysis of the corrected field equations shows that when $r \in (0, 1/e)$, $\Lambda(r) < 0$, making the central region of the black hole near the critical radius r_* an anti-de Sitter spacetime. This provides a necessary condition for the nested AdS/CFT correspondence [32] inside and outside the black hole (evidence from the field equations).

C.4.2. Tensor Self-Consistency, Stress Decomposition, Global Asymptotic Structure and Linear Perturbation

C.4.2.1. Tensor Self-Consistency: Definition and Conservation

For the metric:

$$ds^2 = -A(r)c^2 dt^2 + B(r)dr^2 + r^2 d\Omega^2$$

where $A(r) = 1 - u(r)$, $B(r) = 1 + u(r)$, and $u(r) = \frac{2GM}{c^2 r} + \frac{2kG_h M^2 (\ln r + 1)}{c^2 r}$, the Einstein tensor $G^\mu{}_\nu$ is a closed-form function $G^\mu{}_\nu[u, u', u'']$. By the Bianchi identity $\nabla_\mu G^\mu{}_\nu = 0$, we define $T^{eff\ \mu}{}_\nu = \frac{c^4}{8\pi G} G^\mu{}_\nu$, which satisfies $\nabla_\mu T^{eff\ \mu}{}_\nu = 0$. Therefore, this model is a strictly self-consistent theory with conserved quantities at the tensor level, requiring no additional constraints.

C.4.2.2. Pure Trace-Trace-Free Decomposition

We define $\Lambda_{eff}(r) = \frac{1}{4}G^\mu{}_\nu$, such that the Einstein tensor can be decomposed as:

$$G^\mu{}_\nu = \Lambda_{eff}(r)\delta^\mu{}_\nu + \widetilde{G}^\mu{}_\nu$$

where $\widetilde{G}^\mu{}_\nu = 0$. This corresponds to the decomposition of the energy-momentum tensor:

$$T^{eff} = T^{(\Lambda)} + T^{(aniso)}$$

with $T^{(\Lambda)\mu}{}_\nu = \frac{c^4}{8\pi G} \Lambda_{eff}(r) \delta^\mu_\nu$ and $T^{(aniso)\mu}{}_\nu = \frac{c^4}{8\pi G} \widetilde{G}^\mu{}_\nu$. This decomposition shows that the pure trace term corresponds to the "geometric vacuum self-response", while the trace-free term corresponds to the "anisotropic shear stress", avoiding the interpretation of all exotic components as matter fields.

C.4.2.3. Three-Segment Asymptotic Structure

- **Critical radius of potential reversal ($r = r_*$):** From $\Phi(r_*) = 0 \Rightarrow u(r_*) = 0$, we obtain:

$$\Lambda_{eff}(r_*) = \frac{1}{4r_*^3} \left(\frac{2kG_h M^2}{c^2} \right),$$

$$\widetilde{G}^t_t = \frac{3}{4r_*^3} \left(\frac{2kG_h M^2}{c^2} \right), \widetilde{G}^r_r = -\frac{5}{4r_*^3} \left(\frac{2kG_h M^2}{c^2} \right), \widetilde{G}^\theta_\theta = \frac{1}{4r_*^3} \left(\frac{2kG_h M^2}{c^2} \right).$$

All tensor components are finite at this radius, as is the curvature (the Kretschmann scalar $K(r_*) = \frac{13}{r_*^6} \left(\frac{2kG_h M^2}{c^2} \right)^2$). The Null Energy Condition (NEC) is violated in this region, with the violation originating entirely from the trace-free shear component, consistent with the general property of all singularity-resolution boundary models.

- **Far field ($r \rightarrow \infty$):** $\Lambda_{eff}(r) \sim \frac{1}{4r^3} \left(\frac{2kG_h M^2}{c^2} \right)$, and $\widetilde{G}^\mu{}_\nu \sim \mathcal{O}(r^{-3})$. The correction term therefore produces no long-range r^0 or r^{-1} contamination, and its impact on the Solar System is suppressed to below the precision limit of current precision weak-field experiments (see Section 6.3 for details).

- **Near the event horizon ($r \rightarrow r_h$):** $A(r_h) = 0 \rightarrow u(r_h) = 1$, and expansion gives $\Lambda_{eff}(r) \sim \frac{1}{16(r-r_h)^2}$. This enhancement is not a "divergence of bulk density", but a screen/thin-shell enhancement corresponding to the CFT_3 boundary.

C.4.2.4. Localization of Energy Conditions

- At $r = r_*$: The NEC and Weak Energy Condition (WEC) are violated, with the violation sourced from the trace-free shear stress.
- In the far field: $\rho_{eff} \rightarrow 0^+$, and the violation of the NEC/WEC is asymptotically restored.
- Near the event horizon: Holographic screen enhancement occurs, which is a boundary effect rather than a property of ordinary matter.

In summary, the effective energy-momentum tensor of this model can be rigorously decomposed into a pure trace geometric effect and a trace-free anisotropic shear stress. At the critical radius of potential reversal ($r = r_*$), all curvature scalars and stress components are finite, and the NEC/WEC violation is entirely borne by the trace-free shear stress. In the far field, all corrections decay as r^{-3} , introducing no long-range contamination. Near the event horizon ($r = r_h$), the pure trace term is enhanced in a $1/(r - r_h)^2$ form, manifesting the CFT_3 holographic screen structure. Therefore, the logarithmically corrected metric and field equations are self-consistent and conserved at the tensor level, with a clear physical partition of spacetime regions.

C.4.2.5. Linear Perturbation

On macroscopic scales, the propagation properties of gravitational waves can be obtained through linear perturbation analysis of the background metric. We write the logarithmically

corrected Schwarzschild metric as: $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, where $g_{\mu\nu}^{(0)}$ is the background static spherically symmetric metric, and $h_{\mu\nu}$ denotes the perturbation of the propagating gravitational waves. In the weak-field regime (with $u(r) = \frac{2GM}{c^2 r} + \frac{2kG_h M^2 (\ln r + 1)}{c^2 r}$ as defined in Section C.4.2.1), the deviation of the background metric from flat spacetime is of order $u(r) \sim O(r^{-1}) \ll 1$, given that $r \sim 10^{11} - 10^{20} \text{ m}$ for macroscopic astronomical scales. Further calculation of the effective curvature source term corresponding to this metric gives $\Lambda_{eff}(r) \sim O(r^{-3})$, meaning the additional curvature source decays rapidly as r^{-3} with distance.

Under the Lorenz gauge: $\partial^\mu \bar{h}_{\mu\nu} = 0$, the linearized field equation reads: $\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}^{eff}$. Since $T_{\mu\nu}^{eff} \sim r^{-3}$ (negligibly small on macroscopic scales), far from the strong-field regime we have $\square \bar{h}_{\mu\nu} \ll 10^{-25} \text{ m}^{-2} \approx 0$, which is almost completely consistent with GR. This implies that the gravitational wave signals observed by existing experiments such as LIGO/Virgo/KAGRA will not produce observable deviations from our theory, including propagation speed, polarization degrees of freedom, transverse traceless property, and other characteristics.

Appendix D. Unified Running of Non-Local Entanglement Strength: $k_{eff}(r, \mathcal{O})$

We propose that k should not be regarded as a fixed coefficient or a simple radial function, but rather as an "effective coupling constant" analogous to the renormalization group running in quantum field theory. Its effective value depends simultaneously on the spatial scale where the observation occurs and the type of the observation itself. We therefore upgrade its definition to:

$$k_{eff} = k_{eff}(r, \mathcal{O})$$

where r represents the spatial scale (from the ultraviolet (UV) to the infrared (IR)), and $\mathcal{O}$ represents the type of observation (e.g., local measurements and orbit-averaged measurements). This upgrade transforms the parameter k from an input quantity to an output quantity determined by the theoretical structure.

D.1. Unified Running Equation

Based on the principle of minimal phenomenology, we propose the following unified running equation for k_{eff} :

$$k_{eff}(r, \mathcal{O}) = k_{bare}(r) [1 + \lambda_{nl} \mathcal{F}(\mathcal{O}, r)]^{-1} \quad (33)$$

This equation consists of two key components:

1. **Spatial running part** $k_{bare}(r)$ (bare coupling running with space), which inherits the form $k(r) = k_0 \left(\frac{r_{peak}}{r} \right)^\alpha$ from the galactic scale. This form originates from the mass evolution of topological black holes and has successfully fitted galactic rotation curves. Combined with the strong-field region, we need to consider that as the distance to the black hole decreases, $k(r) \rightarrow k = M_{SgrA^*}/M$. Therefore, we define.

$$k_{bare}(r) = \frac{(M_{SgrA^*}/M)}{1 + (r/r_{sf})^\beta} = \frac{M_{SgrA^*}}{M[1 + (r/r_{sf})^\beta]} \quad (34)$$

This form satisfies the following asymptotic behaviors:

- $r < r_{sf}$ (strong field/UV): $k_{bare}(r) \rightarrow M_{SgrA^*}/M$. For the center of the Milky Way ($M = M_{SgrA^*}$), this limit recovers to 1.

- $r_{sf} < r < r_{qsf}$ (quasi-strong field/VIS $\rightarrow$ weak field/IR): $k_{bare}(r) \sim k \left(\frac{r_{sf}}{r} \right)^\beta \rightarrow k_{bare}(r) \sim k \left(\frac{r_{qsf}}{r} \right)^\alpha$.

- When $r \gg r_{qsf} > r_{sf}$ (weak field/IR $\rightarrow$ rotation curve):

$$k_{bare}(r) \sim k\left(\frac{r_{qsf}}{r}\right)^\alpha \rightarrow k(r) = k_0 \left(\frac{r_{peak}}{r}\right)^\alpha$$

Thus, both r_{sf} and r_{qsf} characterize spatial scale transitions ($UV \rightarrow VIS \rightarrow IR$).

2. **Observation type correction term $\mathcal{F}(\mathcal{O}, r)$:** This is a new core element introduced to characterize the differences in nonlocal entanglement effects due to different observation methods:

$$\mathcal{F}(\mathcal{O}, r) = \begin{cases} 0, & \text{for local observations (e.g., periastron velocity)} \\ \left(\frac{\Delta r_{orbit}}{r}\right)^n, & \text{for orbital observations (e.g., orbital precession)} \end{cases} \quad (35)$$

where Δr_{orbit} (characteristic orbital scale) for an elliptical orbit is $\Delta r_{orbit} = r_a - r_p = 2ae$, with r_p (periastron) $= a(1 - e)$ and r_a (apastron) $= a(1 + e)$. λ_{nl} and n are undetermined nonlocal correction parameters.

Physical meaning: The introduction of this term is based on the following physical picture: Local instantaneous measurements (such as velocities) are almost unaffected by nonlocal averaging effects; while observables such as orbital precession involve averaging over the entire orbit, their effective coupling strength is significantly weakened, and the degree of weakening is related to the ratio of the orbital scale to the observation point ($\Delta r_{orbit}/r$). This essentially corresponds to the difference between local operator one-point functions and nonlocal orbital-averaged correlation functions in holographic duality — nonlocal measurements are diluted by the averaging effect of holographic entanglement entropy.

D.2. Strong-Field Limit: Recovery of Established Empirical Laws

In the strong-field limit where stellar motion is extremely close to the black hole ($r \rightarrow r_p, \Delta r_{orbit} \ll r$), we have $\mathcal{F} \rightarrow 0$. In this case, Equation (33) reduces to:

$$k_{loc}(r) \approx k_{bare} \approx \frac{M_{SgrA^*}}{M}$$

This is exactly consistent with the law found in Section 4.3 when analyzing the motion of high-speed stars near Sgr A* — namely, that local velocity measurements require $k \approx 1$. This empirical law becomes a natural theoretical inference within this unified framework.

D.3. Prediction and Interpretation for Orbital Precession Observations

For orbital precession observations, since $\mathcal{F} > 0$, Equation (33) gives:

$$k_{eff}^{orbit} = \frac{k_{bare}(r)}{1 + \lambda_{nl}(\Delta r_{orbit}/r)^n} < k_{bare}(r)$$

This directly predicts and explains why the effective k values required to fit the orbital precession of stars such as S2, S62, and S4714 are less than 1 and differ from each other. The physical origin lies in the varying strength of the non-local averaging effect for different orbits.

D.4. Unified Description of the Stellar Orbital System at the Galactic Center

Based on equation (33), we can give a unified and self-consistent description of the observations of several typical stellar orbits at the center of the Milky Way:

- S4714 ($r_p \approx 1.9 \times 10^{12}\text{m}, r_a \approx 2.5 \times 10^{14}\text{m}$): its $\frac{\Delta r_{orbit}}{r_p} = \frac{2e}{1-e} \approx 131.33$
- S62 ($r_p \approx 2.4 \times 10^{12}\text{m}, r_a \approx 2.18 \times 10^{14}\text{m}$): its $\frac{\Delta r_{orbit}}{r_p} = \frac{2e}{1-e} \approx 81.33$

- S2 ($r_p \approx 1.8 \times 10^{13} \text{m}$, $r_a \approx 2.74 \times 10^{14} \text{m}$): its $\frac{\Delta r_{orbit}}{r_p} = \frac{2e}{1-e} \approx 15.24$

For the two high-velocity stars S4714 and S62, since r_p is extremely close to the black hole, their bare coupling can be approximated as the maximum saturated entanglement: $k_{bare}(r_p) = \frac{1}{1+(r_p/r_{sf})^\beta} \Rightarrow$ because $r_p \ll r_{sf}$, so $k_{bare}(r_p) \approx k = \frac{M_{SgrA*}}{M_{SgrA*}} = 1$. Therefore: $k_{eff}^{S62} > k_{eff}^{S4714}$. For S2, since $r_p \gg r_{sf}$ leads to $k_{bare}(r_p) \ll 1$, although $\left(\frac{\Delta r_{orbit}}{r_p}\right)_{S2} < \left(\frac{\Delta r_{orbit}}{r_p}\right)_{S62}$ or $\left(\frac{\Delta r_{orbit}}{r_p}\right)_{S4714}$, the increase in the orbital average term $\left(\frac{1}{1+\lambda_{nl}(\Delta r_{orbit}/r_p)^n}\right)$ is not enough to offset the decrease in the bare coupling of S2, leading to a clear theoretical prediction hierarchy:

$$k_{eff}^{S62} > k_{eff}^{S4714} > k_{eff}^{S2}$$

Then, according to the orbital precession angle formula (the eccentricities of the three stars are not much different):

$$\Delta\omega_{log} \approx 2\pi k_{eff} \frac{G_h M}{G} \cdot \frac{1 - \sqrt{1 - e^2}}{e^2} \Rightarrow \Delta\omega_{log} \propto k_{eff}$$

we can naturally derive the prediction hierarchy of the periastron precession angles of the three stars:

$$\Delta\omega_{log}^{S62} > \Delta\omega_{log}^{S4714} > \Delta\omega_{log}^{S2}$$

This relationship can be directly tested by high-precision observations (actual observational results: S62 precession angle 9.9° , S4714 precession angle 1.83° , S2 precession angle $12'$).

Conclusion: Based on the eccentricities of stars orbiting the central black hole at high speeds, the hierarchical relationship of periastron orbital precession among the high-velocity star population can be predicted:

- When the eccentricities are very close (such as S4714 and S62), the orbital average term $\left(\frac{1}{1+\lambda_{nl}(\Delta r_{orbit}/r_p)^n}\right)$ plays a dominant role, and this value $\left(\frac{\Delta r_{orbit}}{r_p} = \frac{2e}{1-e}\right)$ can be directly compared, with orbital precession being inversely proportional to it.
- When the eccentricities are not very close (such as S2 versus S4714 and S62), the bare coupling $\left(k \left(\frac{r_{sf}}{r_p}\right)^\beta\right)$ plays a dominant role, k_{eff} decays rapidly as a power law, and $\Delta\omega_{log}$ follows this rapid decay (e.g., S2 decays to the arcminute level).

D.5. Natural Connection to Galactic Scale Physics

A key advantage of this unified framework is that it automatically transitions to the galactic scale. When $r \gg r_{BH}$ (the gravitational influence scale of the black hole) and involves longer-period "orbital" averages reflected in galactic rotation curves, the effect of the nonlocal correction term $\mathcal{F}$ tends to saturate or merge with the spatial running part, such that $k_{eff}(r, \mathcal{O}) \rightarrow k(r) = k_0 \left(\frac{r_{peak}}{r}\right)^\alpha$. This means that the form of $k(r)$ required to describe galactic rotation curve flattening is no longer an independent assumption, but a natural manifestation of this unified running equation in the large-scale, strong nonlocal average limit. The theory thus achieves a seamless connection from the strong field to the galactic scale without any formula switching.

D.6. Conclusion and Summary of the Physical Picture

We propose a unified running equation for the effective parameter $k_{eff}(r, \mathcal{O})$ of nonlocal entanglement strength. This framework reconstructs k as an effective coupling constant that depends simultaneously on scale and observation type, whose behavior is analogous to renormalization group flow:

- In the ultraviolet (UV) local limit (e.g., near the black hole event horizon), the effective coupling is strong (e.g., $k_{eff} \sim 1$ near Sgr A*), dominating the high-speed motion at periastron.
- In the intermediate region (VIS), the effective coupling runs down rapidly due to the power-law decay of local coupling, leading to significant differences in orbital precession data for different stars.
- As it transitions to the infrared (IR) nonlocal scale, the effective coupling runs slower due to the "observation averaging" effect, and finally reproduces the form of $k(r)$ required to describe galactic rotation curve flattening on large scales.

This upgrade evolves the theory from a modified gravity model for specific phenomena into a cross-scale framework with an inherent running structure that can unify the description of gravitational phenomena from the vicinity of black holes to galactic scales.

D.7. Observational Constraints and Parameter Determination

Based on the established unified description ($k_{eff}(r, \mathcal{O})$), we explicitly decompose the effective coupling (entanglement) into two parts: the "bare coupling" that depends only on scale, and the "response function" that depends on the type of observation:

$$k_{eff}(r, \mathcal{O}) = k_{bare}(r) \mathcal{R}_{\mathcal{O}}(r) \quad (36)$$

where $k_{bare}(r)$ describes the pure scale running from the ultraviolet (UV) to the infrared (IR), and $\mathcal{R}_{\mathcal{O}}(r)$ characterizes the modulation of the effective coupling by different observation types.

We define the bare coupling as follows:

$$k_{bare}(r) = \begin{cases} \frac{k}{1 + (r/r_{sf})^\beta} & (\text{strongfield} \rightarrow \text{quasi} - \text{strongfield}) \\ \frac{k}{1 + (r/r_{qsf})^\alpha} & (\text{quasi} - \text{strongfield} \rightarrow \text{weakfield}) \end{cases} \quad (37)$$

where $k = M_{SgrA^*}/M$, r_{sf} is the transition scale from strong field to quasi-strong field, r_{qsf} is the transition scale from quasi-strong field to weak field, M is the central black hole mass of the studied system (galaxy), and M_{SgrA^*} is the central black hole mass of the Milky Way. This form satisfies the asymptotic behaviors described earlier:

Different types of observations have different sensitivities to non-local effects, which is described by the response function $\mathcal{R}_{\mathcal{O}}(r)$.

- **Local observations:** Such as periastron velocity, photon ring radius, etc. These observables are directly determined by the local gradient of the potential field and are almost unaffected by orbital averaging effects. Hence, their response function is:

$$\mathcal{R}_{local}(r) = 1, \Rightarrow k_{eff}^{local}(r) = k_{bare}(r)$$

- **Orbital precession observations:** The orbital precession angle is a non-local quantity integrated along the entire orbit, and its effective coupling is suppressed by the orbital radial span. The response function is:

$$\mathcal{R}_{orbit}(r) = [1 + \lambda_{nl} \mathcal{F}(\mathcal{O}, r)]^{-1} = \frac{1}{1 + \lambda_{nl} (\Delta r_{orbit}/r)^n} \quad (38)$$

For periastron orbital precession response:

$$\mathcal{R}_{orbit}(r_p) = \frac{1}{1 + \lambda_{nl}(\Delta r_{orbit}/r_p)^n}$$

where $r_p = a(1 - e)$ is the periastron distance, $\Delta r_{orb} = r_a - r_p = 2ae$ is the orbital radial span. Defining the dimensionless orbital shape factor $X \equiv \Delta r_{orb}/r_p = 2e/(1 - e)$, the effective coupling for orbital precession is:

$$k_{eff}^{orbit}(r_p) = \frac{k_{bare}(r_p)}{1 + \lambda_{nl}X^n} \quad (39)$$

D.7.1. Constraints on S-Stars at the Galactic Center

For the central black hole of the Milky Way ($M = M_{SgrA*}$), at the periastron of the orbits of S4714 and S62 ($r_p \ll r_{sf}$), we have $k_{bare}(r_p) \approx 1$. Equation (39) simplifies to:

$$k_{eff}^{orbit} \simeq \frac{1}{1 + \lambda_{nl}X^n}$$

Substituting the observed precessions of S4714, S62, and S2 from Section 4.4 (S4714: $e \approx 0.985$, upper limit of $\Delta\omega_{log}$ is 1.83° ; S62: $e \approx 0.976$, upper limit of $\Delta\omega_{log}$ is 9.9° ; S2: $e \approx 0.88$, upper limit of $\Delta\omega_{log}$ is $12'$) into the additional precession formula (16):

$$\Delta\omega_{log} \approx 2\pi k_{eff} \frac{G_h M}{G} \cdot \frac{1 - \sqrt{1 - e^2}}{e^2}$$

we invert to obtain $k_{eff}^{obs}(S4714) \lesssim 0.13$, $k_{eff}^{obs}(S62) \lesssim 0.74$, and $k_{eff}^{obs}(S2) \lesssim 0.018$. Using their orbital eccentricities (from $X = 2e/(1 - e)$): $X_{S4714} \approx 131.33$, $X_{S62} \approx 81.33$, $X_{S2} \approx 15.24$), we solve the simultaneous equations from S4714 and S62 data to get:

$$\lambda_{nl} \approx 6.28 \times 10^{-13}, \quad n \approx 6$$

Note that since the literature gives the total precession ($\Delta\omega$) rather than the direct logarithmic correction value ($\Delta\omega_{log}$), the substituted k_{eff} values (e.g., $k_{eff}(S4714) \sim 0.13$) are only effective scalar orders of magnitude used to constrain the running model. Substituting the solved nonlocal parameters λ_{nl} and n into formula (39) and inverting using $k_{eff}(S2) \sim 0.018$, we obtain:

$$k_{bare,S2}(r_p) \approx 0.018$$

It can be seen that for the high-velocity stars closest to Sgr A*, the bare coupling at periastron is close to saturation, and the differences in orbital precession mainly come from the nonlocal average term (i.e., the orbital average $\mathcal{R}_{orbit}(r)$); while for more distant stars, the bare coupling has decayed significantly ($k_{bare}(r) = \frac{k}{1+(r/r_{sf})^\beta}$ decays as a power law), leading to k_{eff}^{orbit} and $k_{bare}(r_p)$ approximately coinciding, and the orbital average correction becomes secondary.

D.7.2. The Weak-Field Running Relation Cannot Uniquely Determine r_{qsf}

At the weak-field galactic scale ($r \gg r_{qsf}$), the bare coupling reduces to $k_{bare}(r) \approx (M_{SgrA*}/M)(r_{qsf}/r)^\alpha$. If we match this form with the empirical formula $k(r) = k_0(r_{peak}/r)^\alpha$ describing galactic rotation curves, we obtain:

$$k_0 \left(\frac{r_{peak}}{r} \right)^\alpha = \frac{M_{SgrA*}}{M} \left(\frac{r_{qsf}}{r} \right)^\alpha \Rightarrow r_{qsf} = r_{peak} \left(\frac{M}{M_{SgrA*}} k_0 \right)^{1/\alpha}$$

However, if we interpret the mass M as the value of the topological black hole mass at r_{qsf} :
 $M_{BH,topo}(r_{qsf}) = M_{BH,topo}(r_{peak}) \left(\frac{r_{peak}}{r_{qsf}} \right)^{-\alpha}$, substituting gives: $r_{qsf} = r_{peak} \left(\left(\frac{r_{peak}}{r_c} \right)^{-\alpha} \right)^{1/\alpha} = r_{qsf}$. It can be seen that this relation reduces to an identity, indicating that the transition scale r_c cannot be uniquely determined solely by the weak-field power-law relation and topological mass running.

On the other hand, if we take M directly as the target black hole mass according to the strong-field saturation value (taking M_{SgrA*} for the Milky Way as an example), then: $r_{qsf} = r_{peak} k_0^{1/\alpha}$. Substituting the Milky Way parameters from Section 5.2.1 ($k_0 \sim 10^{-5}$, $\alpha \sim 0.3$), we get: $r_{qsf} \sim 10^{-16} \text{ kpc} \sim 10^4 \text{ m} \ll r_h$ (horizon radius) $\sim 10^{10} \text{ m}$, which is clearly unreasonable.

In summary, the transition scale r_{qsf} must be independently determined by additional physical matching conditions that connect the strong and weak fields, such as the precession observations of S-stars at the Galactic center and other observational phenomena that are simultaneously sensitive to both the strong-field saturation limit and the weak-field power-law transition. This analysis also clarifies the independent parameter status of r_{qsf} in the theory, which is a derived physical quantity related to the specific structure of the galaxy, not a fundamental universal constant.

D.7.3. Constraints on Transition Scales r_{qsf} and r_{sf}

Physical logic: For local periastron velocities, the closer to the black hole, the closer to the local limit ($k \approx 1$). Therefore, S4714 is closer to this local limit (where nonlocal averaging has not fully developed) than S62 and S2.

Introduce the local quasi-strong field saturation condition (saturation degree: $\eta \in (0, \frac{M_{SgrA*}}{M})$), from the quasi-strong field $\rightarrow$ weak field part of formula (37), we have:

$$k_{bare}(r_p) = \frac{k}{1 + (r_p/r_{qsf})^\alpha} \geq \eta \Rightarrow r_{qsf} \geq r_p \left(\frac{\eta}{\frac{M_{SgrA*}}{M} - \eta} \right)^{1/\alpha} \quad (40)$$

We need to select the high-velocity star "closest to local saturation" (S4714 for the Milky Way ($M = M_{SgrA*}$, $\alpha \sim 0.3$, $r_{p,S4714} \approx 1.89 \times 10^{12} \text{ m}$)) to constrain the η range:

From the observed periastron velocity of S4714 (24000 km/s), we invert using the velocity formula (equation (12): $v_{obs}(r) = \sqrt{\frac{GM}{r} + \frac{kG_h M^2 \ln r}{r}} \sqrt{1 - \frac{2GM}{c^2 r} - \frac{2kG_h M^2 (\ln r + 1)}{c^2 r}}$) to get: $k = \eta_{best} \approx 0.73$. Considering observational errors and environmental perturbations (orbit reconstruction, projection effects, gas distribution, etc.), we obtain a reasonable interval: $0.75 \lesssim \eta_{S4714} \lesssim 0.85$. Therefore:

- $\eta \approx 0.75 \Rightarrow r_{qsf} \cong 7.4 \times 10^{13} \text{ m}$
- $\eta \approx 0.80 \Rightarrow r_{qsf} \cong 1.9 \times 10^{14} \text{ m}$
- $\eta \approx 0.85 \Rightarrow r_{qsf} \cong 1.1 \times 10^{15} \text{ m}$

Conclusion: The transition scale between quasi-strong field and weak field for Sgr A* (constrained by S4714) is: $10^{14} \text{ m} \lesssim r_{qsf} \lesssim 10^{15} \text{ m}$. Taking the intermediate value:

$$r_{qsf} \cong 4.5 \times 10^{14} \text{ m}$$

In addition, similarly, from the strong field $\rightarrow$ quasi-strong field part of formula (37) ($k_{bare}(r) = \frac{k}{1 + (r/r_{sf})^\beta}$), by substituting the local saturation degree η of S4714 at periastron and

$k_{bare,S2}(r_p)$ solved in Section D.7.1, we can solve for r_{sf} and β simultaneously (for Sgr A*, $k = 1$):

$$r_{sf} \sim 3.1 \times 10^{12} \text{m} - 3.7 \times 10^{12} \text{m}, \quad \beta \sim 2.3 - 2.5$$

Taking the intermediate value:

$$r_{sf} \cong 3.4 \times 10^{12} \text{m}, \quad \beta \cong 2.4$$

By determining the two transition scales r_{sf} and r_{qsf} , we can see that:

- $r \ll r_{sf}$: Local dominance (strong field)—similar to GR strong field superimposed with a "dark matter" dense region, which can be tested by observing the density of "dark matter" in this region.
- $r_{sf} < r < r_{qsf}$: Transition region (quasi-strong field)—nonlocal effects begin to appear, and the "dark matter" density decreases somewhat.
- $r \gg r_{qsf}$: Nonlocal dominance (weak field)—the "dark matter" density is basically stable, similar to a gentle effect region.

The three-region division basically effectively corresponds to the nested duality ($AdS_2/CFT_1 \subseteq AdS_3/CFT_2 \subseteq AdS_4/CFT_3$) that runs through the whole text, as well as its renormalization group flow behavior.

In addition, according to the definition of bare entanglement strength in this paper: $k = k_{bare}^{max} = \frac{M_{SgrA*}}{M}$, we have: $r_{qsf} \geq r_p \left(\frac{\eta}{k_{bare}^{max} - \eta} \right)^{1/\alpha}$, $\eta \in (0, k_{bare}^{max})$. Therefore, a more general conclusion is obtained: the maximum value of local coupling is determined by the central black hole mass:

$$M = M_{SgrA*} \Rightarrow k_{bare}^{max} = 1$$

$$M > M_{SgrA*} \Rightarrow k_{bare}^{max} < 1 \quad (\text{strong - field suppression})$$

$$M < M_{SgrA*} \Rightarrow k_{bare}^{max} > 1 \quad (\text{strong - field enhancement})$$

Therefore, r_{qsf} not only characterizes the spatial scale transition, but also encodes the "strong-field saturation capability" of black holes of different masses. Its essence is—how far from the black hole can a coupling strength close to the bare entanglement (k_{bare}^{max}) be maintained.

D.7.4. Application of the Unified Running Model

D.7.4.1. Orbital Precession of High-Velocity Stars near Supermassive Black Holes

Based on the previously calculated bare coupling attenuation parameters extending from the strong field to the quasi-strong field near Sgr A* ($r_{sf} \cong 3.4 \times 10^{12} \text{m}$, $\beta \cong 2.4$) combined with the formula $k_{eff}(r, \mathcal{O}) = k_{bare}(r) \mathcal{R}_0(r)$, we can make falsifiable order-of-magnitude a priori predictions (not exact values) for the periastron orbital precession of any S star in the fast-moving S-star population whose periastron lies within the transition scale between the quasi-strong field and the weak field ($r_{qsf} \cong 4.5 \times 10^{14} \text{m}$).

- Taking S4711 as an example (periastron $r_{p,S4711} \cong 144 \text{AU} \approx 2.15 \times 10^{13} \text{m}$), which is located in the quasi-strong field near Sgr A* ($r_{sf} < r_{p,S4711} < r_{qsf}$). Substituting into the "strong field $\rightarrow$ quasi-strong field" part of formula (37) ($k_{bare}(r) = \frac{k}{1+(r/r_{sf})^\beta}$, and for Sgr A* $k = M_{SgrA*}/M = 1$), we obtain the bare coupling at periastron:

$$k_{bare,S4711}(r_p) \cong 0.012$$

Based on the verified empirical relationship for S2, which is also located in the quasi-strong field ($k_{bare} \approx k_{eff}$), we get $k_{bare,S4711}(r_p) \approx k_{eff,S4711}(r_p) \cong 0.012$. The observational paper on

S4711 [33] gives a period of 7.6 years and a periastron of 144 AU. From Kepler's relations, its semi-major axis is estimated to be about 629 AU, so the eccentricity is $e_{S4711} \approx 1 - 144/629 \approx 0.771$. Substituting into the additional precession formula (for Sgr A*, $\frac{G_h M}{G} = \frac{G_h M_{SgrA*}}{G} \approx 0.045$):

$$\Delta\omega_{log}(S4711) \approx 2.2 \times 10^{-3} \text{rad/orbit} \approx 7.4' \text{/orbit}$$

Conclusion: The orbital precession of S4711 at periastron is expected to be on the order of 7'–11' per orbit. No mature observational results are available for comparison at present, so this order of magnitude constitutes a falsifiable prediction of orbital precession for this model.

- Taking S4716 as another example, using its published orbital parameters [34] ($a \approx 1.93 \times 10^{-3} \text{pc}$, $e \approx 0.756$), which correspond to a periastron of $r_{p,S4716} \approx 97 - 100 \text{AU}$, we substitute to calculate $k_{bare,S4716}(r_p)$ (still using $r_{sf} \cong 3.4 \times 10^{12} \text{m}$ and $\beta \cong 2.4$) and obtain $k_{eff,S4716}(r_p) \cong 0.029$. Using the extra precession formula, we further calculate the orbital precession of S4716 at periastron: $\Delta\omega_{log}(S4716) \approx 5.0 \times 10^{-3} \text{rad/orbit} \approx 17' \text{/orbit}$. Thus, the expected order of magnitude is 17'–25' per orbit. No mature observational results are available for comparison at present, so this value constitutes the second falsifiable prediction of orbital precession for this model.

D.7.4.2. The Pericenter Velocity of High-Velocity Stars near Supermassive Black Holes

As the radial distance from the black hole increases, the mass M in the classical term and the squared mass M^2 in the logarithmic correction term transition from the bare black hole mass (e.g., M_{SgrA*}) to the topological structure (baryonic matter) mass $M(r)$. To reflect the physical picture where $M(r)$ increases with distance, inspired by Section 5 of this paper, we define (where r denotes the radial distance from the black hole):

$$M(r) = M_{SgrA*} \cdot e^{r/r_{ss}} \quad (41)$$

This formula yields the limiting behavior: $r \rightarrow 0 \Rightarrow M(r) \rightarrow M_{SgrA*}$

Here, r_{ss} is the characteristic scale of the gravitational field—projected from the steady-state tunneling scale at the AdS/CFT boundary (e.g., the steady-state tunneling in AdS_4/CFT_3 : $r_{sh} \approx \sqrt{r_h r_{ph}}$ (see Appendix C.3)) to the bulk (corresponding to a steady-state characteristic scale in the range $r_{sf} < r < r_{qsf}$): $r_{ss} \approx \sqrt{r_{sf} r_{qsf}} \approx 3.912 \times 10^{13} \text{m}$ (previously calculated: $r_{sf} \cong 3.4 \times 10^{12} \text{m}$, $r_{qsf} \cong 4.5 \times 10^{14} \text{m}$).

In this way, we can perform an a priori calculation (requiring only the distance r) of the periastron velocity $v_{obs}(r_p)$ of high-velocity stars in the quasi-strong-field regime of Sgr A*, using the strong-field velocity formula for black holes with logarithmic correction:

$$v_{obs}(r) = \sqrt{\frac{GM(r)}{r} + \frac{k_{bare}(r)G_h M(r)^2 \ln r}{r}} \cdot \sqrt{1 - \frac{2GM(r)}{c^2 r} - \frac{2k_{bare}(r)G_h M(r)^2 (\ln r + 1)}{c^2 r}}$$

Comparison of Theoretically Calculated and Observed Velocities of S2, S4711 and S4716

Star	Periastron r_p	Topological mass $M(r)$	Bare coupling $k_{bare}(r)$	Theoretical velocity $v_{obs}(r)$	Observed velocity v_{true}	Error $\frac{ v_{true}-v_{obs} }{v_{true}}$
S2	$1.8 \times 10^{13} \text{m}$	$1.355 \times 10^{37} \text{kg}$	0.018	7226 km/s	7650 km/s	$\approx 5.5\%$
S4711	$2.15 \times 10^{13} \text{m}$	$1.482 \times 10^{37} \text{kg}$	0.012	6880 km/s	6700 km/s	$\approx 2.7\%$
S4716	$1.5 \times 10^{13} \text{m}$	$1.255 \times 10^{37} \text{kg}$	0.029	7687 km/s	8000 km/s	$\approx 3.9\%$

Conclusion: The a priori calculation (prediction) for the three high-velocity stars, requiring only the distance r , yields theoretical velocities that are in nearly perfect agreement with actual observations (with a maximum error of only $\sim 5\%$). This not only reaffirms the validity of our

framework, but also provides theoretical velocity support for the falsifiable predictions of periastron orbital precession. Furthermore, our method stands in stark contrast to traditional approaches that rely on orbit fitting: for the study of the S2 orbit alone, as many as 145 astrometric measurements and 44 radial velocity measurements were integrated, and both datasets were input into the theoretical model to fit the periastron velocity.

D.7.5. Summary and Unified Expression

In summary, we obtain the complete unified expression for the effective coupling of nonlocal entanglement strength. For orbital precession observations, its form is:

$$k_{eff}(r, \mathcal{O}) = \begin{cases} \frac{k}{1 + (r/r_{sf})^\beta} \cdot \frac{1}{1 + \lambda_{nl}(\Delta r_{orbit}/r)^n} & (\text{strong field} \rightarrow \text{quasi} - \text{strong field}) \\ \frac{k}{1 + (r/r_{qsf})^\alpha} \cdot \frac{1}{1 + \lambda_{nl}(\Delta r_{orbit}/r)^n} & (\text{quasi} - \text{strong field} \rightarrow \text{weak field}) \end{cases} \quad (42)$$

where $k = M_{SgrA^*}/M$.

- The bare coupling that runs with scale:

$$k_{bare}(r) = \begin{cases} \frac{k}{1 + (r/r_{sf})^\beta} & (\text{strong field} \rightarrow \text{quasi} - \text{strong field}) \\ \frac{k}{1 + (r/r_{qsf})^\alpha} & (\text{quasi} - \text{strong field} \rightarrow \text{weak field}) \end{cases}$$

- The orbital averaging suppression term to the bare coupling:

$$\mathcal{R}_{orbit}(r) = \frac{1}{1 + \lambda_{nl}(\Delta r_{orbit}/r)^n}$$

When $r = r_p$ (let $X = \Delta r_{orbit}/r_p$, where $\Delta r_{orbit} = r_a - r_p = 2ae$ and $r_p = a(1 - e)$):

$$\mathcal{R}_{orbit}(r_p) = \frac{1}{1 + \lambda_{nl}[2e/(1 - e)]^n}$$

Thus, the effective coupling $k_{eff}(r, \mathcal{O})$ can also be written as:

$$k_{eff}(r, \mathcal{O}) = k_{bare}(r)\mathcal{R}_{orbit}(r)$$

This formula clearly integrates three levels of physics:

1. The system bare entanglement relative strength factor (M_{SgrA^*}/M): realizes unified scaling for gravitational systems with different central black hole masses.
2. The scale running term $[1 + (r/r_{sf})^\beta]^{-1}$ or $[1 + (r/r_{qsf})^\alpha]^{-1}$: describes the renormalization group-like running behavior from black hole strong field to galactic scale (exactly corresponding to the renormalization group flow of the nested duality we adopt). Its transition scales r_{sf} , r_{qsf} and power law β are solved simultaneously by the "closest to local saturation" high-velocity star near the central black hole and the minimum precession star (S4714 and S2 for Sgr A*).
3. The orbital suppression term $[1 + \lambda_{nl}X^n]^{-1}$: characterizes the weakening of effective coupling by nonlocal orbital averaging, whose intensity is determined by orbital eccentricity e , parameters λ_{nl} and n . And λ_{nl} and n are solved simultaneously by the two high-velocity stars close to local saturation near the central black hole (S4714 and S62 for Sgr A*).

D.7.6. Core Conclusions

In this section, we complete the empirical constraint and establishment of the unified running framework through specific mathematical models and observational data, with the main conclusions as follows:

1. The orbital precession observations in the strong-field region at the center of the Milky Way can solve for the nonlocal suppression parameters: $\lambda_{nl} \approx 6.28 \times 10^{-13}$, $n \approx 6$. This leads to the conclusion that these parameters can be solved simultaneously by two fast stars near the central black hole that are closest to local saturation (only the orbital eccentricity and periastron precession of the stars need to be observed). In addition, the introduction of the orbital suppression term $\mathcal{R}_{orbit}(r)$ is not purely empirical parameterization, but stems from the essential difference between local and nonlocal observations in holographic duality: local observations correspond to the single-point response of local operators, while orbital precession corresponds to the nonlocal correlation function averaged over the complete orbit, which is thus diluted by the average effect of holographic entanglement entropy. Therefore, the power n can be interpreted as the dimension of the compactified phase space corresponding to the nonlocal orbital average correlation function, which is further constrained by the simultaneous solution of S62 and S4714 to be $n \approx 6$. This result is also consistent with the sum of the phase space dimensions of the gauge group $(U(1)_Y \times SU(2)_L \times SU(3)_C)$ (see Appendix B.1), thus providing observational data support for the previous motivational discussion.

2. After introducing the bare entanglement relative strength factor M_{SgrA^*}/M with Sgr A* as the reference, the theory can uniformly describe gravitational systems with different central black hole masses.

3. The scale running parameters in the strong-field region at the center of the Milky Way can also be solved: $r_{sf} \cong 3.4 \times 10^{12} \text{m}$, $\beta \cong 2.4$, $r_{qsf} \cong 4.5 \times 10^{14} \text{m}$. And draw the conclusion: they are solved simultaneously by the "closest to local saturation" high-velocity star and the minimum precession star near the central black hole (observe the periastron distances of the two stars, the periastron velocity of the "closest to local saturation" star (which can invert its $k_{bare}(r)$), and then use the solved λ_{nl} and n to invert the $k_{bare}(r)$ of the minimum precession star).

4. Formula (42), as a compact expression, uniformly describes gravitational phenomena from stellar-scale orbital motion to galactic-scale rotation curves.

- As described in Appendix D.7.1, starting from the quasi-strong field, the orbit-averaged coupling (k_{eff}^{orbit}) approximately coincides with the bare coupling ($k_{bare}(r_p)$): $\mathcal{R}_{orbit}(r) = \frac{1}{1+\lambda_{nl}(\Delta r_{orbit}/r)^n} \cong 1$, with orbital average corrections becoming secondary. When transitioning to the galactic scale described in Section 5 of the main text:

$$k_{eff}(r, \mathcal{O}) = k_{bare}(r) \mathcal{R}_{orbit}(r) \cong k_{bare}(r) = \frac{k}{1 + (r/r_{qsf})^\alpha} \Rightarrow k_{eff}(r, \mathcal{O}) \cong \frac{k}{1 + (r/r_{qsf})^\alpha}$$

Upon entering the galactic scale, the previous quasi-strong field scale r_{qsf} is no longer applicable and naturally switches to the rotation curve scale r_{peak} used in Section 5 (i.e., $r_{peak} = \xi_0 r_{qsf}$, $\xi_0 \gg 1$). The scale transformation in the weak field causes $k \rightarrow k_0$ (i.e., $k_0 = \epsilon_0 k$, $\epsilon_0 \ll 1$) (due to the rapid decay of bare entanglement with increasing distance):

$$k_{eff}^{galaxy}(r, \mathcal{O}) \cong \frac{\epsilon_0 k}{1 + (r/\xi_0 r_{qsf})^\alpha} = \frac{k_0}{1 + (r/r_{peak})^\alpha} \sim k_0 \left(\frac{r_{peak}}{r} \right)^\alpha$$

This recovers $k(r) = k_0 \left(\frac{r_{peak}}{r} \right)^\alpha$.

- On the other hand, $M(r) = M_{SgrA^*} \cdot e^{r/r_{ss}}$ (which transitions to the strong field as $r \rightarrow 0$: $M(r) \cong M$ ($M = M_{SgrA^*}$ for the Milky Way), suitable for predicting black hole shadows via the metric) is equally applicable to the galactic scale (weak field):

$$M(r) = M_{baryon,topo} (1 - e^{-r/r_0}) = M_{SgrA^*} \cdot e^{r_{eq}/r_{ss}}$$

This implies that any radial distance r on the galactic scale can be uniquely mapped to an equivalent radial distance r_{eq} in the quasi-strong field such that $M(r)$ is equal, with the solution:

$$r_{eq} = r_{ss} \ln \frac{M_{baryon,topo}(1 - e^{-r/r_0})}{M_{SgrA*}}$$

i.e., there exists a one-to-one correspondence between radial distances: $r \Leftrightarrow r_{eq}$.

- Taking the distance between the Solar System and Sgr A* ($r \approx 8$ kpc) as an example.

As described in Section 5.2.1 of the main text, this distance lies in the mid-disk region of the rotation curve:

$$M_{baryon,topo} \approx 1.79 \times 10^{41} \text{kg}, r_0 \approx 6 \text{kpc} \Rightarrow r_{eq} \approx 3.77 \times 10^{14} \text{m}$$

i.e.,

$$r \approx 8 \text{kpc} \Leftrightarrow r_{eq} \approx 3.77 \times 10^{14} \text{m}$$

- This description, spanning black hole shadows in the strong field $\rightarrow$ stellar orbital motion in the quasi-strong field $\rightarrow$ rotation curves in the weak field, is all based on the same modified gravitational potential framework. This not only demonstrates the powerful cross-scale adaptability of the effective framework but also reflects the renormalization group flow characteristics under nested duality through the consistency between a priori calculations and observational data.

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