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Article

Weakly Singular Wendroff-Type Integral Inequalities of Multiple Variables with Multiple Nonlinear Terms and Their Applications

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Abstract

We establish the estimation of solutions of two classes of weakly singular Wendroff-type integral inequalities of multiple variables with multiple nonlinear terms, apply them to fractional partial differential equations, and give the proof procedures for uniqueness, boundedness and continuous dependence.

Keywords: inequality of integration; Wendroff-type; weak singularity

1. Introduction

The research significance of singular kernel integral inequality is that it provides a theoretical framework for dealing with integral equations with singular kernels, which is widely used in mathematical analysis, physics, engineering and other fields. Such inequalities can help to analyze and solve systems with singular properties, especially in describing nonlinear and non-smooth problems. Its practical value is reflected in several aspects: in signal processing, image reconstruction, quantum mechanics, fluid dynamics and other fields, singular kernel integral inequalities can be used to describe complex physical phenomena, such as electromagnetic fields, particle propagation, etc. In addition, it also plays an important role in numerical analysis and optimization algorithms, helping to improve the solution efficiency and accuracy. Through in-depth study of these inequalities, we can promote the modeling, analysis and solution of practical problems. Singular integral inequality is a very important part in the theory of differential equations, and a series of results have been obtained in this field.

For example, in 1981, Henry [1] proposed a linear integral inequality with a singular kernel, shown as follows:

$$u(t) \leq a + b \int_0^t (t-s)^{\beta-1} u(s) ds.$$

The results can be used to study the qualitative property equations of parabolic differentials.

In 1997, Medved [2] gave a new method to study linear and nonlinear singular integral Gronwall-Bellman-Bihari-type inequality and Henry-type inequality, which is shown as follows:

$$u(t) \leq a(t) + \int_0^t (x-s)^{\beta-1} s^{\gamma-1} F(s) u(s) ds,$$

$$u(t) \leq a(t) + \int_0^t (x-s)^{\beta-1} F(s) w(u(s)) ds,$$

and the estimates of the solutions are given separately.

In 2002, Ma and Yang [3] studied more generalized weakly singular Volterra-type integral inequalities using the improved *Medved* method. It looks like this:

$$u(t) \leq a(t) + b(t) \int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} g(s) u(s) ds + c(t) \int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} g(s) \omega(u(s)) ds.$$

In 2008, Wu [4] studied a class of weakly odd Volterra-type integral inequalities with n nonlinear terms, which are shown as follows:

$$u(t) \leq a(t) + \sum_{i=1}^n \int_{b_i(0)}^{b_i(t)} (t^{\alpha_i} - s^{\alpha_i})^{\beta_i-1} s^{\gamma_i-1} f_i(t, s) w_i(u(s)) ds.$$

Thereafter, in 2008, Cheung et al. [5] respectively studied the following binary weakly singular Wendroff-type integral inequalities by applying the method in reference [3], which are shown as follows:

$$u^p(x, y) \leq a(x, y) + b(x, y) \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} \times (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} f(s, t) u^q(s, t) dt ds.$$

On this basis, in 2010, Wang and Zheng [6] discussed a more general form of nonlinear Wendroff-type weakly singular integral inequality for a class of binary functions, which is shown as follows:

$$u(x, y) \leq a(x, y) + \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} \times (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} f(x, y, s, t) w(u(s, t)) ds dt.$$

In 2013, Wu et al. [7] studied a new class of weakly singular Wendroff-type integral inequalities, which are shown as follows:

$$u(x, y) \leq a(x, y) + \int_0^x \int_0^y (x^{\alpha_1} - s^{\alpha_1})^{\beta_1-1} s^{\gamma_1-1} (y^{\alpha_2} - t^{\alpha_2})^{\beta_2-1} \times t^{\gamma_2-1} f_1(x, y, s, t) w(u(s, t)) ds dt + \int_0^x \int_0^y (x^{\alpha_3} - s^{\alpha_3})^{\beta_3-1} s^{\gamma_3-1} (y^{\alpha_4} - t^{\alpha_4})^{\beta_4-1} \times t^{\gamma_4-1} f_2(x, y, s, t) u(s, t) ds dt.$$

Subsequently, in 2014, the weakly singular Wendroff-type integral inequalities of two nonlinear terms were studied in [8], which are shown as follows:

$$u(x, y) \leq a(x, y) + \sum_{i=1}^2 \int_0^x \int_0^y (x^{\alpha_i} - s^{\alpha_i})^{\beta_i-1} s^{\gamma_i-1} (y^{\sigma_i} - t^{\sigma_i})^{\mu_i-1} \times t^{\tau_i-1} f_i(x, y, s, t) w_i(u(s, t)) ds dt.$$

On the basis of Zheng [6] and Wu [8], in 2017, Liu [9] studied a class of weakly singular Wendroff-type integral inequalities with n nonlinear terms, which are shown as follows:

$$u(x, y) \leq a(x, y) + \sum_{i=1}^n \int_0^x \int_0^y (x^{\alpha_i} - s^{\alpha_i})^{\beta_i-1} s^{\gamma_i-1} (y^{\sigma_i} - t^{\sigma_i})^{\mu_i-1} \times t^{\tau_i-1} f_i(x, y, s, t) w_i(u(s, t)) ds dt.$$

In 2024, Yang [10] studied a class of generalized Wendroff-type weakly singular integral inequalities, which are shown as follows:

$$u(x, y) \leq a(x, y) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(x_0)}^{c_i(x)} \alpha_i(x, y, s, t) f_i(x, y, s, t) w_i(u(s, t)) ds dt.$$

More work on singular integral inequalities is emerging, see ref. [14–16]. It can be seen that with the development of integral and differential equations, some new integral inequalities need to be considered in order to meet the wider application of differential and integral equations. In this paper we focus on Wendroff-type integral inequalities and their application to the study of well-posed solutions of differential and integral equations.

On the basis of reference [3–10], this paper considers the weakly singular Wendroff-type integral inequality of multiple variables with n nonlinear terms as follows.

$$\begin{aligned} u(x, y, z) &\leq a(x, y, z) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} (x^{\alpha_i} - s^{\alpha_i})^{\beta_i-1} s^{\gamma_i-1} \\ &\quad \times (y^{\alpha_i} - t^{\alpha_i})^{\beta_i-1} t^{\gamma_i-1} (z^{\alpha_i} - \tau^{\alpha_i})^{\beta_i-1} \tau^{\gamma_i-1} f_i(x, y, z, s, t, \tau) \\ &\quad \times W_i(u(s, t, \tau)) ds dt, \end{aligned}$$

$$\begin{aligned} u(x_1, x_2, \dots, x_n) &\leq a(x_1, x_2, \dots, x_n) + \sum_{i=1}^n \int_{b_{1,i}(x_{1,0})}^{b_{1,i}(x_1)} \int_{b_{2,i}(x_{2,0})}^{b_{2,i}(x_2)} \dots \int_{b_{n,i}(x_{n,0})}^{b_{n,i}(x_n)} \\ &\quad \times (x_1^{\alpha_i} - s_1^{\alpha_i})^{\beta_i-1} s_1^{\gamma_i-1} (x_2^{\alpha_i} - s_2^{\alpha_i})^{\beta_i-1} s_2^{\gamma_i-1} \dots (x_n^{\alpha_i} - s_n^{\alpha_i})^{\beta_i-1} s_n^{\gamma_i-1} \\ &\quad \times f_i(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n) W_i(u(s_1, s_2, \dots, s_n)) ds_1 ds_2 \dots ds_n. \end{aligned}$$

Before this, we need to study the estimation of the solution of a class of Wendroff-type integral inequalities as a theorem that can be used to refer to. Secondly, combined with the modified Medved method, we can obtain the estimation of unknown functions of Wendroff-type weakly singular integral inequalities. Finally, it is applied to study the boundedness, uniqueness and continuous dependence of solutions of fractional partial differential equations.

2. Definitions and Lemmas

Definition 1 (see [10]). (Nondecreasing and expanding large monotonicity) For a given number of functional f , such as fruit set and conformity on f of reduction function $mon(f) \geq f$ is set up, says $mon(f)$ is the reduction of f amplification drab.

Definition 2 The Riemann-Liouville integral of order α of a function $f(t)$ with respect to t is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t \geq 0.$$

Definition 3 The derivative of a function $f(t)$ with respect to t of order γ is defined as

$$D_t^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t (t-s)^{-\gamma} f(s) dt, \quad t \geq 0.$$

Lemma 1 (see [3]). Let α, β, γ and p be positive constants. Then,

$$\int_0^t (t^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\gamma-1)} ds = \frac{t^\theta}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \geq 0.$$

Where $B[\xi, \eta] = \int_0^1 s^{\xi-1} (1-s)^{\eta-1} ds$ ($\operatorname{Re} \xi > 0, \operatorname{Re} \eta > 0$) is well-known B-function and $\theta = p[\alpha(\beta-1) + \gamma-1] + 1$.

Lemma 2 (see [3]). (Discrete Jensen's inequality) Let $A_1, A_2, \dots, A_n$ be nonnegative real numbers, $r > 1$ be real numbers and n be natural numbers, then

$$(A_1, A_1, \dots, A_n)^r \leq n^{r-1} (A_1^r, A_2^r, \dots, A_n^r).$$

Lemma 3 (see [12]). (Holder integral inequality) Suppose the function $f(x) \in L^p([t_0, +\infty], \mathbb{R})$, the function $g(x) \in L^q([t_0, +\infty], \mathbb{R})$, the constant $p, q \geq 1$ and $1/p + 1/q = 1$. Then

$$\int_{t_0}^t |f(s)g(s)| ds \leq \left(\int_{t_0}^t |f(s)|^p ds \right)^{\frac{1}{p}} \left(\int_{t_0}^t |g(s)|^q ds \right)^{\frac{1}{q}}.$$

Lemma 4 (see [12]). Suppose the function $f(x) \in L^p([t_0, +\infty], \mathbb{R})$, the function $g(x) \in L^q([t_0, +\infty], \mathbb{R})$, the constant $p, q \geq 1$ and $1/p + 1/q = 1$, $\alpha(t)$ is a continuously differentiable nondecreasing function, $\alpha(t) \leq t, \alpha(t_0) = t_0$. Then

$$\int_{\alpha(t_0)}^{\alpha(t)} |f(s)g(s)| ds \leq \left(\int_{\alpha(t_0)}^{\alpha(t)} |f(s)|^p ds \right)^{\frac{1}{p}} \left(\int_{\alpha(t_0)}^{\alpha(t)} |g(s)|^q ds \right)^{\frac{1}{q}}.$$

3. Main Results

In this section, we will prove estimates of solutions of weakly singular integral inequalities of Wendroff-type for multiple variables with n nonlinear terms. In this chapter, let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ denotes the set of nonnegative real numbers. x_0, y_0 is a constant real number, $I := (x_0, +\infty), J := (y_0, +\infty), T := (z_0, +\infty), \Lambda = I \times J \times T \subset \mathbb{R}^3$, define the following set of functions:

$$\Omega(\mathbb{R}_+, \mathbb{R}_+) := \{ \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ | \varphi \text{ is a nondecreasing function} \},$$

$$C^1(I, I) := \{ \varphi : I \rightarrow I | \varphi \text{ is a continuously differentiable function} \},$$

$$C(\Lambda, \mathbb{R}_+) := \{ \varphi : \Lambda \rightarrow \mathbb{R}_+ | \varphi \text{ is a continuous function} \},$$

$$C_\Omega(\Lambda, \mathbb{R}_+) := \left\{ \varphi : \Lambda \rightarrow \mathbb{R}_+ \left| \begin{array}{l} \varphi \text{ is continuous and nondecreasing} \\ \text{with respect to each variable} \end{array} \right. \right\}.$$

Assume that

$$(K_1) \quad \alpha_i \in (0, 1], \beta_i \in (0, 1), p > 1 \text{ and satisfy } p(\beta_i - 1) + 1 > 0, \gamma_i > 1 - \frac{1}{p} \text{ and } \frac{1}{p} + \alpha_i(\beta_i - 1) + \gamma_i - 1 \geq 0, \text{ where } i = 1, 2, \dots, n.$$

$$(K_2) \quad a \in C_\Omega(\Lambda, \mathbb{R}_+).$$

$$(K_3) \quad b_i \in C^1(I, I) \cap \Omega(I, I), c_i \in C^1(J, J) \cap \Omega(J, J), d_i \in C^1(T, T) \cap \Omega(T, T), \text{ and for all } x \in I, y \in J, z \in T \text{ there exists } 0 \leq b_i(x) \leq x, 0 \leq c_i(y) \leq y, 0 \leq d_i(z) \leq z.$$

$$(K_4) \quad f_i(\cdot, \cdot, \cdot, s, t, \tau) \in C_\Omega(\Lambda, \mathbb{R}_+), f_i(x, y, z, \cdot, \cdot, \cdot) \in C(\Lambda, \mathbb{R}_+).$$

$$(K_5) \quad W_i(\eta) \in \Omega(\mathbb{R}_+, \mathbb{R}_+) \text{ is monotonically nondecreasing, when } \eta > 0 \text{ has } W_i(\eta) > 0, \text{ and } \frac{W_2(\eta)}{W_1(\eta)}, \frac{W_3(\eta)}{W_2(\eta)}, \dots, \frac{W_n(\eta)}{W_{n-1}(\eta)} \text{ is monotonically nondecreasing, where } i = 1, 2, \dots, n.$$

$$(K_6) \quad I_i := (x_i, +\infty), \Lambda^1 = \prod_{i=1}^n I_i \subset \mathbb{R}^n, a \in C_\Omega(\Lambda^1, \mathbb{R}_+), b_i \in C^1(I_i, I_i) \cap \Omega(I_i, I_i),$$

and for all $x_i \in I_i$, there exists $0 \leq b_{i,j}(x_i) \leq x_i$. $f_i(\cdot, \cdot, \dots, \cdot, s_1, s_2, \dots, s_n) \in C_\Omega(\Lambda^1, \mathbb{R}_+), f_i(x_1, x_2, \dots, x_n, \cdot, \cdot, \dots, \cdot) \in C(\Lambda^1, \mathbb{R}_+).$

Theorem 1 Under the condition that Assumption (K_2) – (K_5) holds, if $u \in C(\Lambda, \mathbb{R}_+)$, and $u(x, y, z)$ satisfies the weakly singular integral inequality

$$u(x, y, z) \leq a(x, y, z) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \times f_i(x, y, z, s, t, \tau) W_i(u(s, t, \tau)) ds dt d\tau, \quad (1)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$. Then

$$u(x, y, z) \leq G_n^{-1} \left[G_n \left(G_{n-1}^{-1}(a_n(x, y, z)) \right) + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \hat{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right], \quad (2)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$. Where

$$a_1(x, y, z) = \max_{0 \leq \sigma \leq x, 0 \leq \mu \leq y, 0 \leq \zeta \leq z} a(\sigma, \mu, \zeta), \quad (3)$$

$$a_i(x, y, z) = G_{i-1} \left(G_{i-2}^{-1}(a_{i-1}(x, y, z)) \right) + \int_{b_{i-1}(x_0)}^{b_{i-1}(x)} \int_{c_{i-1}(y_0)}^{c_{i-1}(y)} \int_{d_{i-1}(z_0)}^{d_{i-1}(z)} \hat{f}_{i-1}(x, y, z, s, t, \tau) ds dt d\tau, \quad (4)$$

for $i = 2, 3, \dots, n$.

$$\hat{f}_i(x, y, z, s, t, \tau) = \max_{0 \leq \sigma \leq x, 0 \leq \mu \leq y, 0 \leq \zeta \leq z} f_i(\sigma, \mu, \zeta, s, t, \tau), \quad (5)$$

$G_i^{-1}(v)$ is the inverse function of $G_i(v)$,

$$G_i(v) = \int_{v_i}^v \frac{d\zeta}{W_i(\zeta)}, \quad v \geq v_i > 0, \quad i = 1, 2, \dots, n. \quad (6)$$

And $X, Y, Z \in \mathbb{R}_+$ are chosen such that

$$G_i \left(G_{i-1}^{-1}(a_i(X, Y, Z)) \right) + \int_{b_i(x_0)}^{b_i(X)} \int_{c_i(y_0)}^{c_i(Y)} \int_{d_i(z_0)}^{d_i(Z)} \times \hat{f}_i(X, Y, Z, s, t, \tau) ds dt d\tau \in \text{Dom}(G_i^{-1}). \quad (7)$$

Remark From Definition 1, we know that, when assuming that Condition (K_2) , (K_4) , (K_5) does not hold, we can solve the auxiliary inequality of (1) as follows

$$\begin{aligned} u(x,y,z) &\leq a(x,y,z) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} f_i(x,y,z,s,t,\tau) \\ &\quad \times W_i(u(s,t,\tau)) ds dt d\tau \\ &\leq mon(a)(x,y,z) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} mon(f_i)(x,y,z,s,t,\tau) \\ &\quad \times W_i(u(s,t,\tau)) ds dt d\tau. \end{aligned}$$

The induction hypothesis will be used next to prove Theorem 1.
Proof. Mathematical induction is used.

Let the right-hand side of inequality (1) be the function $v(x,y,z)$,

$$\begin{aligned} v(x,y,z) &= a(x,y,z) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \\ &\quad \times f_i(x,y,z,s,t,\tau) W_i(u(s,t,\tau)) ds dt d\tau, \end{aligned} \tag{8}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$. Which is

$$u(x,y,z) \leq v(x,y,z). \tag{9}$$

(i) When $n = 1$, for any definite $\tilde{x} \in [x_0, X], \tilde{y} \in [y_0, Y], \tilde{z} \in [z_0, Z]$, and the definition of (3) and (5), then follows from Equation (8)

$$\begin{aligned} v(x,y,z) &= a(\tilde{x}, \tilde{y}, \tilde{z}) + \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \\ &\quad \times f_1(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) W_1(u(s,t,\tau)) ds dt d\tau, \end{aligned} \tag{10}$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

Take the partial derivative of the left and right sides of (10) with respect to x , using the function $W_i(v) > 0 \quad (i = 1, 2, \dots, n)$ and monotonically nondecreasing with respect to u can be obtained

$$\begin{aligned} \frac{dv(x,y,z)}{dx} &= b'_1(x) \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, x, t, \tau) W_1(u(x,t,\tau)) dt d\tau \\ &\leq b'_1(x) W_1(v(x,y,z)) \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, x, t, \tau) dt d\tau, \end{aligned} \tag{11}$$

Equation (11) shows that,

$$\frac{dv(x,y,z)}{W_1(v(x,y,z))} \leq \left[b'_1(x) \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, x, t, \tau) dt d\tau \right] dx, \tag{12}$$

it follows from $W_1(v) > 0$, ntegrating both sides of (12) from x_0 to x , and having the definition of $G_1(v)$ in Equation (6)

$$\begin{aligned} G_1(v(x,y,z)) - G_1(v(x_0,y,z)) \\ \leq \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) ds dt d\tau, \end{aligned} \tag{13}$$

since $v(x_0, y, z) = a_1(\tilde{x}, \tilde{y}, \tilde{z})$, the arrangement of Equation (13) can be obtained

$$v(x, y, z) \leq G_1^{-1} \left[G_1(a_1(\tilde{x}, \tilde{y}, \tilde{z})) + \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) ds dt d\tau \right],$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

Since $\tilde{x}, \tilde{y}$, and $\tilde{z}$, are arbitrary, we replace $\tilde{x}, \tilde{y}$, and $\tilde{z}$ by x, y, z , respectively. And it can be obtained from Equation (9):

$$u(x, y, z) \leq G_1^{-1} \left[G_1(a_1(x, y, z)) + \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(x, y, z, s, t, \tau) ds dt d\tau \right],$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

So when $n = 1$, Equation (2) holds.

(ii) Suppose that when $n = k$, Equation (2) holds. The proof is that when $n = k + 1$, Equation (2) also holds. For any definite $\tilde{x} \in [x_0, X], \tilde{y} \in [y_0, Y], \tilde{z} \in [z_0, Z]$, and the definition of (3) and (5), then follows from Equation (8)

$$\begin{aligned} v(x, y, z) &= a_1(\tilde{x}, \tilde{y}, \tilde{z}) + \sum_{i=1}^{k+1} \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \\ &\quad \times \hat{f}_i(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) W_i(u(s, t, \tau)) ds dt d\tau, \end{aligned} \quad (14)$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

Take the partial derivative of the left and right sides of (14) with respect to x , using the function $W_i(v) > 0$ ($i = 1, 2, \dots, n$) and monotonically nondecreasing with respect to v can be obtained

$$\begin{aligned} \frac{dv(x, y, z)}{dx} &= \sum_{i=1}^{k+1} b'_i(x) \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \hat{f}_i(\tilde{x}, \tilde{y}, \tilde{z}, x, t, \tau) W_i(v(x, t, \tau)) dt d\tau \\ &\leq W_1(v(x, y, z)) b'_1(x) \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, x, t, \tau) dt d\tau \\ &\quad + \sum_{i=2}^{k+1} b'_i(x) \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \hat{f}_i(\tilde{x}, \tilde{y}, \tilde{z}, x, t, \tau) W_i(v(x, t, \tau)) dt d\tau, \end{aligned}$$

we have

$$\begin{aligned} \frac{dv(x, y, z)}{W_1(v(x, y, z))} &\leq \left[b'_1(x) \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, x, t, \tau) dt d\tau \right] dx \\ &\quad + \sum_{i=1}^k \left[b'_{i+1}(x) \int_{c_{i+1}(y_0)}^{c_{i+1}(y)} \int_{d_{i+1}(z_0)}^{d_{i+1}(z)} \hat{f}_{i+1}(\tilde{x}, \tilde{y}, \tilde{z}, x, t, \tau) \right. \\ &\quad \times \left. \frac{W_{i+1}(v(x, t, \tau))}{W_1(v(x, t, \tau))} dt d\tau \right] dx. \end{aligned} \quad (15)$$

Let $M_{i+1}(u) = \frac{W_{i+1}(v)}{W_1(v)}$ ($i = 1, 2, \dots, k$), it follows from the definition of $W_1(v)$ that $M_{i+1}(v)$ is a monotone and nonnegative function, and $\frac{M_3(v)}{M_2(v)}, \frac{M_4(v)}{M_3(v)}, \frac{M_{k+1}(v)}{M_k(v)}$ is monotone and nondecreasing, integrating from x_0 to x on both sides of Equation (15), and it follows from the definition of $G_1(v)$ that:

$$\begin{aligned} & G_1(v(x, y, z)) - G_1(v(x_0, y, z)) \\ & \leq \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) ds dt d\tau \\ & + \sum_{i=1}^k \int_{b_{i+1}(x_0)}^{b_{i+1}(x)} \int_{c_{i+1}(y_0)}^{c_{i+1}(y)} \int_{d_{i+1}(z_0)}^{d_{i+1}(z)} \hat{f}_{i+1}(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) \\ & \times M_{i+1}[v(s, t, \tau)] ds dt d\tau, \end{aligned} \quad (16)$$

since $v(x_0, y, z) = a_1(\tilde{x}, \tilde{y}, \tilde{z})$, the arrangement of Equation (16) can be obtained

$$\begin{aligned} G_1(v(x, y, z)) & \leq G_1(a_1(\tilde{x}, \tilde{y}, \tilde{z})) + \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \\ & \times \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) ds dt d\tau \\ & + \sum_{i=1}^k \int_{b_{i+1}(x_0)}^{b_{i+1}(x)} \int_{c_{i+1}(y_0)}^{c_{i+1}(y)} \int_{d_{i+1}(z_0)}^{d_{i+1}(z)} \hat{f}_{i+1}(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) \\ & \times M_{i+1}[v(s, t, \tau)] ds dt d\tau. \end{aligned} \quad (17)$$

According to Equation (4), the definition of $a_2(x, y, z)$ is as follows.

$$a_2(\tilde{x}, \tilde{y}, \tilde{z}) = G_1(a_1(\tilde{x}, \tilde{y}, \tilde{z})) + \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} \int_{d_1(z_0)}^{d_1(z)} \hat{f}_1(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) ds dt d\tau.$$

Then Equation (17) can be reduced to

$$\begin{aligned} G_1(v(x, y, z)) & \leq a_2(\tilde{x}, \tilde{y}, \tilde{z}) + \sum_{i=1}^k \int_{b_{i+1}(x_0)}^{b_{i+1}(x)} \int_{c_{i+1}(y_0)}^{c_{i+1}(y)} \int_{d_{i+1}(z_0)}^{d_{i+1}(z)} \\ & \times \hat{f}_{i+1}(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) M_{i+1}[v(s, t, \tau)] ds dt d\tau, \end{aligned} \quad (18)$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

Let

$$G_1(v(x, y, z)) = g(x, y, z), \quad (19)$$

then

$$v(x, y, z) = G_1^{-1}(g(x, y, z)). \quad (20)$$

Using Equations (19) and (20), Equation (18) can be transformed into

$$\begin{aligned} g(x, y, z) & \leq a_2(\tilde{x}, \tilde{y}, \tilde{z}) + \sum_{i=1}^k \int_{b_{i+1}(x_0)}^{b_{i+1}(x)} \int_{c_{i+1}(y_0)}^{c_{i+1}(y)} \int_{d_{i+1}(z_0)}^{d_{i+1}(z)} \\ & \times \hat{f}_{i+1}(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) M_{i+1}[G_1^{-1}(g(s, t, \tau))] ds dt d\tau, \end{aligned} \quad (21)$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

Equation (21) has the same form as Equation (1). From functions $a_2(\tilde{x}, \tilde{y}, \tilde{z})$ and $M_{i+1}(v)$ satisfying

the conditions of functions $a(x, y, z)$ and $W_i(v)$ in Equation (1) respectively, and from the result of induction hypothesis $n = k$, we obtain

$$g(x, y, z) \leq \Phi_{k+1}^{-1} \left[\Phi_{k+1} \left(\Phi_k^{-1} (a_{k+1}(x, y, z)) \right) + \int_{b_{k+1}(x_0)}^{b_{k+1}(x)} \int_{c_{k+1}(y_0)}^{c_{k+1}(y)} \int_{d_{k+1}(z_0)}^{d_{k+1}(z)} \hat{f}_{k+1}(x, y, z, s, t, \tau) ds dt d\tau \right], \tag{22}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$. Where

$$\Phi_{k+1}(v) = \int_{\tilde{v}_{k+1}}^v \frac{d\tilde{\xi}}{M_{k+1}(G_1^{-1}(\tilde{\xi}))},$$

Equation (19) shows that

$$\tilde{v}_{k+1} = G_1(v_{k+1}).$$

$\Phi_{k+1}^{-1}(v)$ is the inverse function of $\Phi_{k+1}(v)$, and notice that

$$\begin{aligned} \Phi_{k+1}(v) &= \int_{\tilde{v}_{k+1}}^v \frac{d\tilde{\xi}}{M_{k+1}(G_1^{-1}(\tilde{\xi}))} = \int_{G_1(v_{k+1})}^v \frac{W_1(G_1^{-1}(\tilde{\xi}))}{W_{k+1}(G_1^{-1}(\tilde{\xi}))} d\tilde{\xi} \\ &= \int_{v_{k+1}}^{G_1^{-1}(v)} \frac{d\tilde{\xi}}{W_{k+1}(\tilde{\xi})} = G_{k+1}(G_1^{-1}(v)). \end{aligned}$$

So

$$\Phi_{k+1}^{-1}(v) = G_1(G_{k+1}^{-1}(v)), \tag{23}$$

$$\Phi_k(v) = G_k(G_1^{-1}(v)). \tag{24}$$

Then Equation (22) can be arranged by Equation (23) and Equation (24) as follows.

$$g(x, y, z) \leq G_1 \left\{ G_{k+1}^{-1} \left[G_{k+1} \left(G_k^{-1} (a_{k+1}(x, y, z)) \right) + \int_{b_{k+1}(x_0)}^{b_{k+1}(x)} \int_{c_{k+1}(y_0)}^{c_{k+1}(y)} \int_{d_{k+1}(z_0)}^{d_{k+1}(z)} \hat{f}_{k+1}(x, y, z, s, t, \tau) ds dt d\tau \right] \right\}, \tag{25}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Furthermore, Equation (25) can be transformed from Equation (19) and (9) to

$$u(x, y, z) \leq G_{k+1}^{-1} \left[G_{k+1} \left(G_k^{-1} (a_{k+1}(x, y, z)) \right) + \int_{b_{k+1}(x_0)}^{b_{k+1}(x)} \int_{c_{k+1}(y_0)}^{c_{k+1}(y)} \int_{d_{k+1}(z_0)}^{d_{k+1}(z)} \hat{f}_{k+1}(x, y, z, s, t, \tau) ds dt d\tau \right],$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Thus when $n = k + 1$, the proposition also holds.

Finally, Theorem 1 is proved by (i) and (ii).

Judging the boundedness of the solutions of integral inequalities leads to the boundedness of the solutions of the corresponding integral or differential equations. Next we can discuss the boundedness of Equation (1). Judging the boundedness of Equation (1) with recursive form requires the

following assumption:

(K₇) $a \in C_\Omega(\Lambda, \mathbb{R}_+)$ is bounded.

(K₈) For all $\hat{f}_i, b_i, c_i, d_i$ such that:

$$\lim_{(x,y,z) \rightarrow (+\infty, +\infty, +\infty)} \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \hat{f}_i(x, y, z, s, t, \tau) ds dt d\tau < +\infty.$$

From the assumption condition (K₇), we know that function $a_1(x, y, z)$ is bounded, and combining with Equations (4) and (6) in Theorem 1, we can show that function $a_i(x, y, z)$ ($i = 1, 2, \dots, n$) is bounded. It is also known from the assumption condition (K₈) that the function $u(x, y, z)$ is bounded with respect to $(x, y, z) \in \Lambda$ in Inequality (1).

Corollary 1 Under the condition that Assumption (K₂)-(K₄) holds, if $u \in C(\Lambda, \mathbb{R}_+)$, and $u(x, y, z)$ satisfies the weakly singular integral inequality

$$u(x, y, z) \leq a(x, y, z) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \hat{f}_i(x, y, z, s, t, \tau) (u(s, t, \tau))^k ds dt d\tau, \quad (26)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$. Then

(i) If $k = 1$,

$$u(x, y, z) \leq \tilde{G}_{n-1}^{-1}(a_n(x, y, z)) \exp \left[\int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \hat{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right],$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Where $\tilde{G}_i^{-1}(v)$ is the inverse function of $\tilde{G}_i(v)$,

$$\tilde{G}_i(v) = \int_{v_i}^v \frac{d\tilde{\xi}}{\tilde{\xi}} = \ln \frac{v}{v_i}, \quad v \geq v_i > 0, \quad i = 1, 2, \dots, n. \quad (27)$$

$$\tilde{G}_i^{-1}(v) = v_i e^v, \quad \text{Dom}(\tilde{G}_i^{-1}) = [0, \infty), \quad i = 1, 2, \dots, n. \quad (28)$$

And $X, Y, Z \in \mathbb{R}_+$ are chosen such that

$$\tilde{G}_i \left(\tilde{G}_{i-1}^{-1}(a_i(X, Y, Z)) \right) + \int_{b_i(x_0)}^{b_i(X)} \int_{c_i(y_0)}^{c_i(Y)} \int_{d_i(z_0)}^{d_i(Z)} \hat{f}_i(X, Y, Z, s, t, \tau) ds dt d\tau \in \text{Dom}(\tilde{G}_i^{-1}).$$

(ii) If $0 < k < 1$,

$$u(x, y, z) \leq \left[\left(\tilde{G}_{n-1}^{-1}(a_n(x, y, z)) \right)^{1-k} + (1-k) \times \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \hat{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right]^{1/(1-k)},$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Where $\bar{G}_i^{-1}(v)$ is the inverse function of $\bar{G}_i(v)$,

$$\bar{G}_i(v) = \int_{v_i}^v \frac{d\bar{\xi}}{\bar{\xi}^k} = \frac{1}{1-k} (v^{1-k} - v_i^{1-k}), \quad v \geq v_i > 0, \quad i = 1, 2, \dots, n. \quad (29)$$

$$\begin{aligned}\overline{G}_i^{-1}(v) &= \left(v_i^{1-k} + (1-k)v\right)^{1/(1-k)}, \\ \text{Dom}(\overline{G}_i^{-1}) &= [0, \infty), \quad i = 1, 2, \dots, n.\end{aligned}\tag{30}$$

And $X, Y, Z \in \mathbb{R}_+$ are chosen such that

$$\begin{aligned}\overline{G}_i\left(\overline{G}_{i-1}^{-1}(a_i(X, Y, Z))\right) &+ \int_{b_i(x_0)}^{b_i(X)} \int_{c_i(y_0)}^{c_i(Y)} \int_{d_i(z_0)}^{d_i(Z)} \\ &\times \hat{f}_i(X, Y, Z, s, t, \tau) ds dt d\tau \in \text{Dom}(\overline{G}_i^{-1}).\end{aligned}$$

The functions $a_i(x, y, z)$, $\hat{f}_i(x, y, z, s, t, \tau)$ are all defined in Theorem 1.

Proof. It is not difficult to see that inequality (26) is a special case of the inequality (1) in Theorem 1, and taking $W[\eta] = \eta^k$ yields inequality (26). When applying Theorem 1 to inequality (26), since (i) If $k = 1$, from Equation (2) combined with Equation (27), we can obtain the following.

$$\begin{aligned}u(x, y, z) &\leq \tilde{G}_n^{-1}\left[\ln\left(\frac{1}{v_n}\left(\tilde{G}_{n-1}^{-1}(a_n(x, y, z))\right)\right)\right. \\ &\quad \left.+ \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \hat{f}_n(x, y, z, s, t, \tau) ds dt d\tau\right],\end{aligned}\tag{31}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

The Equation (31) is combined with Equation (28) to obtain

$$u(x, y, z) \leq \tilde{G}_{n-1}^{-1}(a_n(x, y, z)) \exp\left[\int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \hat{f}_n(x, y, z, s, t, \tau) ds dt d\tau\right],$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

(ii) If $0 < k < 1$, from inequality (2) combined with Equations (29), we can obtain the following.

$$\begin{aligned}u(x, y, z) &\leq \overline{G}_n^{-1}\left[\frac{1}{1-k}\left(\left(\overline{G}_{n-1}^{-1}(a_n(x, y, z))\right)^{1-k} - v_n^{1-k}\right)\right. \\ &\quad \left.+ \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \hat{f}_n(x, y, z, s, t, \tau) ds dt d\tau\right],\end{aligned}\tag{32}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

The inequality (32) is combined with equation (30) to obtain

$$\begin{aligned}u(x, y, z) &\leq \left[\left(\overline{G}_{n-1}^{-1}(a_n(x, y, z))\right)^{1-k} + (1-k)\right. \\ &\quad \left.\times \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \hat{f}_n(x, y, z, s, t, \tau) ds dt d\tau\right]^{1/(1-k)},\end{aligned}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

It follows from (i) and (ii) that Corollary 1 is proved.

Theorem 2 Under the condition that Assumption (K_1) – (K_5) holds, if $u \in C(\Lambda, \mathbb{R}_+)$, and $u(x, y, z)$ satisfies the weakly singular integral inequality

$$\begin{aligned} u(x, y, z) \leq & a(x, y, z) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} (x^{\alpha_i} - s^{\alpha_i})^{\beta_i-1} s^{\gamma_i-1} \\ & \times (y^{\alpha_i} - t^{\alpha_i})^{\beta_i-1} t^{\gamma_i-1} (z^{\alpha_i} - \tau^{\alpha_i})^{\beta_i-1} \tau^{\gamma_i-1} \\ & \times f_i(x, y, z, s, t, \tau) W_i(u(s, t, \tau)) ds dt d\tau, \end{aligned} \tag{33}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$. Then

$$\begin{aligned} u(x, y, z) \leq & \left\{ \hat{G}_n^{-1} \left[\hat{G}_n \left(\hat{G}_{n-1}^{-1} \left(A_n(x, y, z) \right) \right) \right. \right. \\ & \left. \left. + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right] \right\}^{1/q}, \end{aligned} \tag{34}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$. Where

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} &= 1, \\ \tilde{a}(x, y, z) &= \max_{0 \leq \sigma \leq x, 0 \leq \mu \leq y, 0 \leq \zeta \leq z} a(\sigma, \mu, \zeta), \end{aligned} \tag{35}$$

$$A_1(x, y, z) = (n + 1)^{q-1} \tilde{a}^q(x, y, z), \tag{36}$$

$$\begin{aligned} A_i(x, y, z) &= \hat{G}_{i-1} \left(\hat{G}_{i-2}^{-1} (A_{i-1}(x, y, z)) \right) \\ &+ \int_{b_{i-1}(x_0)}^{b_{i-1}(x)} \int_{c_{i-1}(y_0)}^{c_{i-1}(y)} \int_{d_{i-1}(z_0)}^{d_{i-1}(z)} \tilde{f}_{i-1}(x, y, z, s, t, \tau) ds dt d\tau, \end{aligned} \tag{37}$$

for $i = 1, 2, \dots, n$.

$$\hat{f}_i(x, y, z, s, t, \tau) = \max_{0 \leq \sigma \leq x, 0 \leq \mu \leq y, 0 \leq \zeta \leq z} f_i(\sigma, \mu, \zeta, s, t, \tau), \tag{38}$$

$$\tilde{f}_i(x, y, z, s, t, \tau) = (n + 1)^{q-1} H_i^q \hat{f}_i^q(x, y, z, s, t, \tau), \tag{39}$$

for $i = 1, 2, \dots, n$.

$$H_i = \alpha_i^{-3/p} (xyz)^{(1/p) + \alpha_i(\beta_i-1) + (\gamma_i-1)} B \left[\frac{p(\gamma_i-1) + 1}{\alpha_i}, p(\beta_i-1) + 1 \right]^{3/p},$$

for $i = 1, 2, \dots, n$.

$\hat{G}_i^{-1}(v)$ is the inverse function of $\hat{G}_i(v)$,

$$\hat{G}_i(v) = \int_{v_i}^v \frac{d\zeta}{\tilde{W}_i(\zeta)}, \quad v \geq v_i > 0, \quad i = 1, 2, \dots, n. \tag{40}$$

Where

$$\tilde{W}_i(u) = W_i^q(u^{1/q}). \tag{41}$$

And $X, Y, Z \in \mathbb{R}_+$ are chosen such that

$$\begin{aligned} & \hat{G}_i \left(\hat{G}_{i-1}^{-1}(a_i(X, Y, Z)) \right) + \int_{b_i(x_0)}^{b_i(X)} \int_{c_i(y_0)}^{c_i(Y)} \int_{d_i(z_0)}^{d_i(Z)} \\ & \times \tilde{f}_i(X, Y, Z, s, t, \tau) ds dt d\tau \in \text{Dom}(\hat{G}_i^{-1}), \end{aligned}$$

for $i = 1, 2, \dots, n$.

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$, and $p > 1, q > 0$, apply Lemma 4 to Equation (33), and it follows

$$\begin{aligned} u(x, y, z) & \leq a(x, y, z) + \sum_{i=1}^n \left(\int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} (x^{\alpha_i} - s^{\alpha_i})^{p(\beta_i-1)} s^{p(\gamma_i-1)} \right. \\ & \times (y^{\alpha_i} - t^{\alpha_i})^{p(\beta_i-1)} t^{p(\gamma_i-1)} (z^{\alpha_i} - \tau^{\alpha_i})^{p(\beta_i-1)} \tau^{p(\gamma_i-1)} \Big)^{1/p} \\ & \times \left(\int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} f_i^q(x, y, z, s, t, \tau) W_i^q(u(s, t, \tau)) ds dt d\tau \right)^{1/q}, \end{aligned}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Since $0 \leq b_i(x) \leq x, 0 \leq c_i(y) \leq y, 0 \leq d_i(z) \leq z$ can be obtained

$$\begin{aligned} u(x, y, z) & \leq a(x, y, z) + \sum_{i=1}^n \left(\int_0^x \int_0^y \int_0^z (x^{\alpha_i} - s^{\alpha_i})^{p(\beta_i-1)} s^{p(\gamma_i-1)} \right. \\ & \times (y^{\alpha_i} - t^{\alpha_i})^{p(\beta_i-1)} t^{p(\gamma_i-1)} (z^{\alpha_i} - \tau^{\alpha_i})^{p(\beta_i-1)} \tau^{p(\gamma_i-1)} \Big)^{1/p} \\ & \times \left(\int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} f_i^q(x, y, z, s, t, \tau) W_i^q(u(s, t, \tau)) ds dt d\tau \right)^{1/q}, \end{aligned} \quad (42)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Defining the function N_1, N_2, N_3 , and using Lemma 1 can be obtained

$$N_1 = \int_0^x (x^{\alpha_i} - s^{\alpha_i})^{p(\beta_i-1)} s^{p(\gamma_i-1)} ds = \frac{x^{\theta_i}}{\alpha_i} B \left[\frac{p(\gamma_i-1)+1}{\alpha_i}, p(\beta_i-1)+1 \right],$$

$$N_2 = \int_0^y (y^{\alpha_i} - t^{\alpha_i})^{p(\beta_i-1)} t^{p(\gamma_i-1)} dt = \frac{y^{\theta_i}}{\alpha_i} B \left[\frac{p(\gamma_i-1)+1}{\alpha_i}, p(\beta_i-1)+1 \right],$$

$$N_3 = \int_0^z (z^{\alpha_i} - \tau^{\alpha_i})^{p(\beta_i-1)} \tau^{p(\gamma_i-1)} d\tau = \frac{z^{\theta_i}}{\alpha_i} B \left[\frac{p(\gamma_i-1)+1}{\alpha_i}, p(\beta_i-1)+1 \right],$$

where $\theta_i = p(\alpha_i(\beta_i-1) + \gamma_i-1) + 1, i = 1, 2, \dots, n$.

Define the function H_i as

$$\begin{aligned} H_i & = (N_1 N_2 N_3)^{1/p} \\ & = \alpha_i^{-3/p} (xyz)^{(1/p)+\alpha_i(\beta_i-1)+(\gamma_i-1)} B \left[\frac{p(\gamma_i-1)+1}{\alpha_i}, p(\beta_i-1)+1 \right]^{3/p}, \end{aligned} \quad (43)$$

for $i = 1, 2, \dots, n$.

From Equation (43), Equation (42) is changed to

$$u(x, y, z) \leq a(x, y, z) + \sum_{i=1}^n H_i \left(\int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} f_i^q(x, y, z, s, t, \tau) \right. \\ \left. \times W_i^q(u(s, t, \tau)) ds dt d\tau \right)^{1/q}, \quad (44)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Taking a constant q to the second power on both sides of Equation (44) and applying Lemma 2, we obtain the following result.

$$u^q(x, y, z) \leq (n+1)^{q-1} \left[a^q(x, y, z) + \sum_{i=1}^n H_i^q \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \right. \\ \left. \times f_i^q(x, y, z, s, t, \tau) W_i^q(u(s, t, \tau)) ds dt d\tau \right], \quad (45)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

It follows from Equations (35) and (38) that $a(x, y, z) \leq \tilde{a}(x, y, z)$, $f_i(x, y, z, s, t, \tau) \leq \hat{f}_i(x, y, z, s, t, \tau)$, and $\tilde{a}(x, y, z)$ are monotonically nondecreasing with respect to x, y, z , and $\hat{f}_i(x, y, z, s, t, \tau)$ is continuous and monotonically nondecreasing with respect to x, y, z . Equation (45) can then be transformed into

$$u^q(x, y, z) \leq (n+1)^{q-1} \left[\tilde{a}^q(x, y, z) + \sum_{i=1}^n H_i^q \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \right. \\ \left. \times \hat{f}_i^q(x, y, z, s, t, \tau) W_i^q(u(s, t, \tau)) ds dt d\tau \right], \quad (46)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

For any fixed value of $\tilde{x} \in [x_0, X]$, $\tilde{y} \in [y_0, Y]$, $\tilde{z} \in [z_0, Z]$, and Equation (36) and Equation (39) make Equation (46) become

$$u^q(x, y, z) \leq A_1(\tilde{x}, \tilde{y}, \tilde{z}) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \tilde{f}_i^q(x, y, z, s, t, \tau) \\ \times W_i^q(u(s, t, \tau)) ds dt d\tau,$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

Let the function $V_i(x, y, z)$ be

$$V_i(x, y, z) = A_1(\tilde{x}, \tilde{y}, \tilde{z}) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \tilde{f}_i^q(x, y, z, s, t, \tau) \\ \times W_i^q(u(s, t, \tau)) ds dt d\tau, \quad (47)$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

And from Equation (47) we can get

$$V_i(x_0, y, z) = A_1(\tilde{x}, \tilde{y}, \tilde{z}), \quad (48)$$

$$u^q(x, y, z) \leq V_i(x, y, z) \Rightarrow u(x, y, z) \leq V_i^{1/q}(x, y, z), \quad (49)$$

from Equation (49), Equation (48) can be changed into

$$V_i(x, y, z) = A_1(\tilde{x}, \tilde{y}, \tilde{z}) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \tilde{f}_i^q(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) \times W_i^q(V_i^{1/q}(s, t, \tau)) ds dt d\tau, \quad (50)$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

Define $\tilde{G}_i(u)$ in Equation (40), using Equation (41) can make Equation (50) become

$$V_i(x, y, z) = A_1(\tilde{x}, \tilde{y}, \tilde{z}) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} \tilde{f}_i^q(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) \times \tilde{W}_i(V(s, t, \tau)) ds dt d\tau, \quad (51)$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

From the definitions of $A_1(\tilde{x}, \tilde{y}, \tilde{z})$, $\tilde{f}_i(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau)$, and $\tilde{W}_i(V(s, t, \tau))$, we know that $A_1(\tilde{x}, \tilde{y}, \tilde{z})$, $\tilde{f}_i(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau)$, and $\tilde{W}_i(V(s, t, \tau))$ satisfy the conditions of functions $a(x, y, z)$, $f_i(x, y, z, s, t, \tau)$, and $W_i(v)$ in Theorem 1, respectively, then Equation (51) can be obtained by applying Theorem 1 as follows.

$$V_i(x, y, z) \leq \hat{G}_n^{-1} \left[\hat{G}_n \left(\hat{G}_{n-1}^{-1} (A_n(\tilde{x}, \tilde{y}, \tilde{z})) \right) + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \tilde{f}_n(\tilde{x}, \tilde{y}, \tilde{z}, s, t, \tau) ds dt d\tau \right],$$

for $x_0 \leq x \leq \tilde{x} \leq X, y_0 \leq y \leq \tilde{y} \leq Y, z_0 \leq z \leq \tilde{z} \leq Z$.

Since $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are arbitrary, we replace $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ by x , y and z , respectively.

$$V_i(x, y, z) \leq \hat{G}_n^{-1} \left[\hat{G}_n \left(\hat{G}_{n-1}^{-1} (A_n(x, y, z)) \right) + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right], \quad (52)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Finally, Equation (49) makes Equation (52) become

$$V_i(x, y, z) \leq \left\{ \hat{G}_n^{-1} \left[\hat{G}_n \left(\hat{G}_{n-1}^{-1} (A_n(x, y, z)) \right) + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right] \right\}^{1/q},$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Thus Theorem 2 is proved.

Corollary 2 Under the condition that Assumption (K_1) – (K_4) holds, if $u \in C(\Lambda, \mathbb{R}_+)$, and $u(x, y, z)$ satisfies the weakly singular integral inequality

$$u(x, y, z) \leq a(x, y, z) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} \int_{d_i(z_0)}^{d_i(z)} (x^{\alpha_i} - s^{\alpha_i})^{\beta_i-1} s^{\gamma_i-1} \times (y^{\alpha_i} - t^{\alpha_i})^{\beta_i-1} t^{\gamma_i-1} (z^{\alpha_i} - \tau^{\alpha_i})^{\beta_i-1} \tau^{\gamma_i-1} \times f_i(x, y, z, s, t, \tau) (u(s, t, \tau))^k ds dt d\tau, \quad (53)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$. Then
(i) If $k = 1$,

$$u(x,y,z) \leq \left(\widetilde{G}_{n-1}^{-1}(A_n(x,y,z))\right)^{1/q} \exp \left[\frac{1}{q} \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \widetilde{f}_n(x,y,z,s,t,\tau) ds dt d\tau \right],$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.
Where $\widetilde{G}_i^{-1}(u)$ is the inverse function of $\widetilde{G}_i(u)$, $\widetilde{G}_i(u)$ is defined in Equation (27), $A_i(x,y,z)$ in Equation (37), and $\widetilde{f}_i(x,y,z,s,t,\tau)$ in Equation (39).
And $X,Y,Z \in \mathbb{R}_+$ are chosen such that

$$\widetilde{G}_i\left(\widetilde{G}_{i-1}^{-1}(A_i(X,Y,Z))\right) + \int_{b_i(x_0)}^{b_i(X)} \int_{c_i(y_0)}^{c_i(Y)} \int_{d_i(z_0)}^{d_i(Z)} \widetilde{f}_i(X,Y,Z,s,t,\tau) ds dt d\tau \in Dom(\widetilde{G}_i^{-1}),$$

for $i = 1, 2, \dots, n$.
(ii) If $0 < k < 1$,

$$u(x,y,z) \leq \left[\left(\overline{G}_{n-1}^{-1}(A_n(x,y,z))\right)^{1-k} + (1-k) \right. \\ \left. \times \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \widetilde{f}_n(x,y,z,s,t,\tau) ds dt d\tau \right]^{1/[q(1-k)]},$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.
Where $\overline{G}_i^{-1}(u)$ is the inverse function of $\overline{G}_i(u)$, $\overline{G}_i(u)$ is defined in Equation (29), $A_i(x,y,z)$ in Equation (37), and $\widetilde{f}_i(x,y,z,s,t,\tau)$ in Equation (39).
And $X,Y,Z \in \mathbb{R}_+$ are chosen such that

$$\overline{G}_i\left(\overline{G}_{i-1}^{-1}(A_i(X,Y,Z))\right) + \int_{b_i(x_0)}^{b_i(X)} \int_{c_i(y_0)}^{c_i(Y)} \int_{d_i(z_0)}^{d_i(Z)} \widetilde{f}_i(X,Y,Z,s,t,\tau) ds dt d\tau \in Dom(\overline{G}_i^{-1}),$$

for $i = 1, 2, \dots, n$.

Proof. It is not difficult to see that inequality (53) is a special case of the inequality (33) in Theorem 2, and taking $W[\eta] = \eta^k$ yields inequality (53). When applying Theorem 2 to inequality (53), since
(i) If $k = 1$, by replacing $\hat{G}_i(u)$ by $\widetilde{G}_i(u)$ in Equation (34) according to the change of proposition, it can be obtained

$$u(x,y,z) \leq \left\{ \widetilde{G}_n^{-1} \left[\widetilde{G}_n \left(\widetilde{G}_{n-1}^{-1} \left(A_n(x,y,z) \right) \right) \right. \right. \\ \left. \left. + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \widetilde{f}_n(x,y,z,s,t,\tau) ds dt d\tau \right] \right\}^{1/q}, \tag{54}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Using Equations (27) and (28) in Corollary 1, Equation (54) can be obtained

$$\begin{aligned} u(x, y, z) &\leq \left\{ \tilde{G}_n^{-1} \left[\tilde{G}_n \left(\tilde{G}_{n-1}^{-1} \left(A_n(x, y, z) \right) \right) \right. \right. \\ &\quad \left. \left. + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right] \right\}^{1/q} \\ &= \left\{ \tilde{G}_n^{-1} \left[\ln \left(\frac{1}{u_n} \left(\tilde{G}_{n-1}^{-1} \left(A_n(x, y, z) \right) \right) \right) \right. \right. \\ &\quad \left. \left. + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right] \right\}^{1/q} \\ &= \left(\tilde{G}_{n-1}^{-1} \left(A_n(x, y, z) \right) \right)^{1/q} \exp \left[\frac{1}{q} \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \right. \\ &\quad \left. \times \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right]. \end{aligned}$$

(ii) If $0 < k < 1$, by replacing $\hat{G}_i(u)$ by $\bar{G}_i(u)$ in Equation (34) according to the change of proposition, it can be obtained

$$\begin{aligned} u(x, y, z) &\leq \left\{ \bar{G}_n^{-1} \left[\bar{G}_n \left(\bar{G}_{n-1}^{-1} \left(A_n(x, y, z) \right) \right) \right. \right. \\ &\quad \left. \left. + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right] \right\}^{1/q}, \end{aligned} \quad (55)$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y, z_0 \leq z \leq Z$.

Using Equations (29) and (30) in Corollary 1, Equation (55) can be obtained

$$\begin{aligned} u(x, y, z) &\leq \left\{ \bar{G}_n^{-1} \left[\bar{G}_n \left(\bar{G}_{n-1}^{-1} \left(A_n(x, y, z) \right) \right) \right. \right. \\ &\quad \left. \left. + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right] \right\}^{1/q} \\ &= \left\{ \bar{G}_n^{-1} \left[\frac{1}{1-k} \left(\bar{G}_{n-1}^{-1} \left(A_n(x, y, z) \right) \right)^{1-k} - u_n^{1-k} \right] \right. \\ &\quad \left. + \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right\}^{1/q} \\ &= \left[\left(\bar{G}_{n-1}^{-1} \left(A_n(x, y, z) \right) \right)^{1-k} + (1-k) \int_{b_n(x_0)}^{b_n(x)} \int_{c_n(y_0)}^{c_n(y)} \int_{d_n(z_0)}^{d_n(z)} \right. \\ &\quad \left. \times \tilde{f}_n(x, y, z, s, t, \tau) ds dt d\tau \right]^{1/[q(1-k)]}. \end{aligned}$$

Thus by (i) and (ii), Corollary 2 is proved.

Theorem 3 Under the condition that Assumption (K_1) , (K_5) and (K_6) holds, if $u \in C(\Lambda^1, \mathbb{R}_+)$, and $u(x_1, x_2, \dots, x_n)$ satisfies the weakly singular integral inequality

$$\begin{aligned} u(x_1, x_2, \dots, x_n) &\leq a(x_1, x_2, \dots, x_n) + \sum_{i=1}^n \int_{b_{1,i}(x_{1,0})}^{b_{1,i}(x_1)} \int_{b_{2,i}(x_{2,0})}^{b_{2,i}(x_2)} \dots \int_{b_{n,i}(x_{n,0})}^{b_{n,i}(x_n)} \\ &\quad \times (x_1^{\alpha_i} - s_1^{\alpha_i})^{\beta_i-1} s_1^{\gamma_i-1} (x_2^{\alpha_i} - s_2^{\alpha_i})^{\beta_i-1} s_2^{\gamma_i-1} \dots (x_n^{\alpha_i} - s_n^{\alpha_i})^{\beta_i-1} s_n^{\gamma_i-1} \\ &\quad \times f_i(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n) W_i(u(s_1, s_2, \dots, s_n)) ds_1 ds_2 \dots ds_n, \end{aligned}$$

for $x_{1,0} \leq x_1 \leq X_1, x_{2,0} \leq x_2 \leq X_2, \dots, x_{n,0} \leq x_n \leq X_n$. Then

$$u(x_1, x_2, \dots, x_n) \leq \left\{ \hat{G}_n^{-1} \left[\hat{G}_n \left(\hat{G}_{n-1}^{-1} \left(\tilde{A}_n(x_1, x_2, \dots, x_n) \right) \right) + \int_{b_{1,n}(x_{1,0})}^{b_{1,n}(x_1)} \int_{b_{2,n}(x_{2,0})}^{b_{2,n}(x_2)} \dots \int_{b_{n,n}(x_{n,0})}^{b_{n,n}(x_n)} \tilde{f}_n(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n) \times ds_1 ds_2 \dots ds_n \right] \right\}^{1/q},$$

for $x_{1,0} \leq x_1 \leq X_1, x_{2,0} \leq x_2 \leq X_2, \dots, x_{n,0} \leq x_n \leq X_n$. Where

$$\frac{1}{p} + \frac{1}{q} = 1,$$

$$\bar{a}(x_1, x_2, \dots, x_n) = \max_{x_{1,0} \leq \sigma_1 \leq x_1, x_{2,0} \leq \sigma_2 \leq x_2, \dots, x_{n,0} \leq \sigma_n \leq x_n} a(\sigma_1, \sigma_2, \dots, \sigma_n),$$

$$\tilde{A}_1(x_1, x_2, \dots, x_n) = (n+1)^{q-1} \bar{a}^q(x_1, x_2, \dots, x_n),$$

$$\begin{aligned} \tilde{A}_i(x_1, x_2, \dots, x_n) &\leq \hat{G}_{i-1} \left(\hat{G}_{i-2}^{-1} \left(\tilde{A}_{i-1}(x_1, x_2, \dots, x_n) \right) \right) \\ &+ \int_{b_{1,i-1}(x_{1,0})}^{b_{1,i-1}(x_1)} \int_{b_{2,i-1}(x_{2,0})}^{b_{2,i-1}(x_2)} \dots \int_{b_{n,i-1}(x_{n,0})}^{b_{n,i-1}(x_n)} \\ &\times \tilde{f}_{i-1}(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n) ds_1 ds_2 \dots ds_n, \end{aligned} \quad (56)$$

for $i = 1, 2, \dots, n$.

$$\hat{f}_i(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n) = \max_{0 \leq \sigma \leq x, 0 \leq \mu \leq y, 0 \leq \zeta \leq z} f_i(\sigma_1, \sigma_2, \dots, \sigma_n, s_1, s_2, \dots, s_n),$$

$$\begin{aligned} \tilde{f}_i(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n) &= (n+1)^{q-1} \tilde{H}_i^q \\ &\times \hat{f}_i^q(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n), \end{aligned} \quad (57)$$

for $i = 1, 2, \dots, n$.

$$\begin{aligned} \tilde{H}_i &= \alpha_i^{-n/p} \left(\prod_{i=1}^n x_i \right)^{(1/p) + \alpha_i(\beta_i - 1) + (\gamma_i - 1)} \\ &\times B \left[\frac{p(\gamma_i - 1) + 1}{\alpha_i}, p_i(\beta_i - 1) + 1 \right]^{n/p}, \end{aligned} \quad (58)$$

for $i = 1, 2, \dots, n$.

$\hat{G}_i^{-1}(u)$ is the inverse function of $\hat{G}_i(u)$, $\hat{G}_i(u)$ is defined in Equation (40) of Theorem 2.

And $X_1, X_2, \dots, X_n \in \mathbb{R}_+$ are chosen such that

$$\begin{aligned} &\hat{G}_i \left(\hat{G}_{i-1}^{-1} \left(A_i(X_1, X_2, \dots, X_n) \right) \right) + \int_{b_{1,n}(x_{1,0})}^{b_{1,n}(X_1)} \int_{b_{2,n}(x_{2,0})}^{b_{2,n}(X_2)} \dots \int_{b_{n,n}(x_{n,0})}^{b_{n,n}(X_n)} \\ &\times \tilde{f}_i(X_1, X_2, \dots, X_n, s_1, s_2, \dots, s_n) ds_1 ds_2 \dots ds_n \in \text{Dom}(\hat{G}_i^{-1}), \end{aligned}$$

for $i = 1, 2, \dots, n$.

Corollary 3 Under the condition that Assumption (K_1) and (K_6) holds, if $u \in C(\Lambda^1, \mathbb{R}_+)$, and $u(x_1, x_2, \dots, x_n)$ satisfies the weakly singular integral inequality

$$\begin{aligned} u(x_1, x_2, \dots, x_n) &\leq a(x_1, x_2, \dots, x_n) + \sum_{i=1}^n \int_{b_{1,i}(x_{1,0})}^{b_{1,i}(x_1)} \int_{b_{2,i}(x_{2,0})}^{b_{2,i}(x_2)} \dots \int_{b_{n,i}(x_{n,0})}^{b_{n,i}(x_n)} \\ &\quad \times (x_1^{\alpha_i} - s_1^{\alpha_i})^{\beta_i-1} s_1^{\gamma_i-1} (x_2^{\alpha_i} - s_2^{\alpha_i})^{\beta_i-1} s_2^{\gamma_i-1} \dots (x_n^{\alpha_i} - s_n^{\alpha_i})^{\beta_i-1} s_n^{\gamma_i-1} \\ &\quad \times f_i(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n) (u(s_1, s_2, \dots, s_n))^k ds_1 ds_2 \dots ds_n, \end{aligned}$$

for $x_{1,0} \leq x_1 \leq X_1, x_{2,0} \leq x_2 \leq X_2, \dots, x_{n,0} \leq x_n \leq X_n$. Then

(i) If $k = 1$,

$$\begin{aligned} u(x_1, x_2, \dots, x_n) &\leq \left(\tilde{G}_{n-1}^{-1} \left(\tilde{A}_n(x_1, x_2, \dots, x_n) \right) \right)^{1/q} \\ &\quad \times \exp \left[\frac{1}{q} \int_{b_{1,n}(x_{1,0})}^{b_{1,n}(x_1)} \int_{b_{2,n}(x_{2,0})}^{b_{2,n}(x_2)} \dots \int_{b_{n,n}(x_{n,0})}^{b_{n,n}(x_n)} \right. \\ &\quad \left. \times \tilde{f}_n(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n) ds_1 ds_2 \dots ds_n \right], \end{aligned}$$

for $x_{1,0} \leq x_1 \leq X_1, x_{2,0} \leq x_2 \leq X_2, \dots, x_{n,0} \leq x_n \leq X_n$. Where $\tilde{G}_i^{-1}(u)$ is the inverse function of $\tilde{G}_i(u)$, $\tilde{G}_i(u)$ is defined in Equation (27), $\tilde{A}_i(x_1, x_2, \dots, x_n)$ in Equation (56), and $\tilde{f}_i(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n)$ in Equation (57).

And $X_1, X_2, \dots, X_n \in \mathbb{R}_+$ are chosen such that

$$\begin{aligned} \tilde{G}_i \left(\tilde{G}_{i-1}^{-1} \left(\tilde{A}_i(X_1, X_2, \dots, X_n) \right) \right) + \int_{b_{1,n}(x_{1,0})}^{b_{1,n}(X_1)} \int_{b_{2,n}(x_{2,0})}^{b_{2,n}(X_2)} \dots \int_{b_{n,n}(x_{n,0})}^{b_{n,n}(X_n)} \\ \times \tilde{f}_i(X_1, X_2, \dots, X_n, s_1, s_2, \dots, s_n) ds_1 ds_2 \dots ds_n \in \text{Dom}(\tilde{G}_i^{-1}), \end{aligned}$$

for $i = 1, 2, \dots, n$.

(ii) If $0 < k < 1$,

$$\begin{aligned} u(x_1, x_2, \dots, x_n) &\leq \left[\left(\bar{G}_{n-1}^{-1} \left(\tilde{A}_n(x_1, x_2, \dots, x_n) \right) \right)^{1-k} \right. \\ &\quad + (1-k) \int_{b_{1,n}(x_{1,0})}^{b_{1,n}(x_1)} \int_{b_{2,n}(x_{2,0})}^{b_{2,n}(x_2)} \dots \int_{b_{n,n}(x_{n,0})}^{b_{n,n}(x_n)} \\ &\quad \left. \times \tilde{f}_n(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n) ds_1 ds_2 \dots ds_n \right]^{1/[q(1-k)]}, \end{aligned}$$

for $x_{1,0} \leq x_1 \leq X_1, x_{2,0} \leq x_2 \leq X_2, \dots, x_{n,0} \leq x_n \leq X_n$.

Where $\bar{G}_i^{-1}(u)$ is the inverse function of $\bar{G}_i(u)$, $\bar{G}_i(u)$ is defined in Equation (29), $\tilde{A}_i(x_1, x_2, \dots, x_n)$ in Equation (56), and $\tilde{f}_i(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n)$ in Equation (57).

And $X_1, X_2, \dots, X_n \in \mathbb{R}_+$ are chosen such that

$$\begin{aligned} \bar{G}_i \left(\bar{G}_{i-1}^{-1} \left(\tilde{A}_i(X_1, X_2, \dots, X_n) \right) \right) + \int_{b_{1,n}(x_{1,0})}^{b_{1,n}(X_1)} \int_{b_{2,n}(x_{2,0})}^{b_{2,n}(X_2)} \dots \int_{b_{n,n}(x_{n,0})}^{b_{n,n}(X_n)} \\ \times \tilde{f}_i(X_1, X_2, \dots, X_n, s_1, s_2, \dots, s_n) ds_1 ds_2 \dots ds_n \in \text{Dom}(\bar{G}_i^{-1}), \end{aligned}$$

for $i = 1, 2, \dots, n$.

4. Application

[i] In this section, we consider the following boundary value problem (BVP) for delayed fractional partial differential equations:

$$\begin{cases} ({}^c D_{\chi_0}^\beta u)(x, y, z) = U(x, y, z, u(x, y, z), u(b(x), c(y), d(z))), \\ u(x, y_0, z_0) = \varphi(x), u(x_0, y, z_0) = \psi(y), u(x_0, y_0, z) = \phi(z), \end{cases} \quad (59)$$

for $x, y, z \in \Lambda$. Where $\chi_0 = (x_0, y_0, z_0)$, $U = C(\Lambda \times \mathbb{R}^3, \mathbb{R})$, the functions $\varphi = C_\Omega(I, \mathbb{R})$, $\psi = C_\Omega(J, \mathbb{R})$, $\phi = C_\Omega(T, \mathbb{R})$ and satisfy the condition $\varphi(x_0) = \psi(y_0) = \phi(z_0)$, $b = C_\Omega^1(I, I)$, $c = C_\Omega^1(J, J)$, $d = C_\Omega^1(T, T)$, $0 \leq b(x) \leq x$, $0 \leq c(y) \leq y$, $0 \leq d(z) \leq z$.

The operator ${}^c D_{\chi_0}^\beta$ denotes the Caputo fractional function and $\beta := (\beta_1, \beta_2, \beta_3) \in (0, 1]^3$ is its order. With Definitions 2 and 3, we know that

$$({}^c D_{\chi_0}^\beta u)(x, y, z) := I_{\chi_0}^{1-\beta} \frac{\partial^3 u(x, y, z)}{\partial x \partial y \partial z},$$

$$\begin{aligned} (I_{\chi_0}^\beta u)(x, y, z) &:= \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)} \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z (x-s)^{\beta_1-1} \\ &\quad \times (y-t)^{\beta_2-1} (z-\tau)^{\beta_3-1} u(s, t, \tau) ds dt d\tau. \end{aligned}$$

In the following, we will discuss the boundedness, uniqueness and continuity of the solution of Equation (1) in turn. To obtain the conditions for boundedness, we first present the estimated solution of Equation (1).

Considering that the functions φ , ψ and ϕ in equation BVP (59) are all bounded, suppose that

$$\begin{aligned} &|U(x, y, z, u(x, y, z), u(b(x), c(y), d(z)))| \\ &\leq h_1(x, y, z)W_1(|u(x, y, z)|) + h_2(x, y, z)W_2(|u(b(x), c(y), d(z))|), \end{aligned} \quad (60)$$

where $h_i \in C(\Lambda, R_+)$, W_i satisfies assumption condition (K_5) ($i = 1, 2$). The estimation of $u(x, y, z)$ in equation BVP(59) can be obtained as follows.

$$\begin{aligned} |u(x, y, z)| &\leq G_2^{-1} \left[G_2 \left(G_1^{-1}(a_2(x, y, z)) \right) \right. \\ &\quad \left. + \int_{b(x_0)}^{b(x)} \int_{c(y_0)}^{c(y)} \int_{d(z_0)}^{d(z)} F_2(x, y, z, s, t, \tau) ds dt d\tau \right], \end{aligned} \quad (61)$$

where

$$G_i(v) := \int_{v_i}^v \frac{d\tilde{\zeta}}{W_i(\tilde{\zeta})}, \quad v > v_i > 0, \quad i = 1, 2. \quad (62)$$

$$a_1(x, y, z) := \varphi(x) + \psi(y) + \phi(z) - \varphi(x_0), \quad (63)$$

$$a_2(x, y, z) := G_1(a_1(x, y, z)) + \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z F_1(x, y, z, s, t, \tau) ds dt d\tau, \quad (64)$$

$$F_1(x, y, z, s, t, \tau) := \alpha(x, y, z, s, t, \tau)h_1(x, y, z), \quad (65)$$

$$F_2(x, y, z, s, t, \tau) := \alpha(x, y, z, s, t, \tau) \left[\frac{h_2(b^{-1}(s), c^{-1}(t), d^{-1}(\tau))}{|b'(b^{-1}(s))c'(c^{-1}(t))d'(d^{-1}(\tau))|} \right], \quad (66)$$

$$\alpha(x, y, z, s, t, \tau) := \frac{(x-s)^{\beta_1-1}(y-t)^{\beta_2-1}(z-\tau)^{\beta_3-1}}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}, \quad (67)$$

$$\lim_{(x,y,z) \rightarrow (+\infty, +\infty, +\infty)} \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z F_1(x, y, z, s, t, \tau) ds dt d\tau < +\infty, \quad (68)$$

$$\lim_{(x,y,z) \rightarrow (+\infty, +\infty, +\infty)} \int_{b(x_0)}^{b(x)} \int_{c(y_0)}^{c(y)} \int_{d(z_0)}^{d(z)} F_2(x, y, z, s, t, \tau) ds dt d\tau < +\infty. \quad (69)$$

Proof. According to the definition of Caputo fractional partial derivative in reference [13], equation BVP (59) can be transformed into the following equivalent integral equation.

$$u(x, y, z) \leq a_1(x, y, z) + \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) \times U(s, t, \tau, u(s, t, \tau), u(b(s), c(t), d(\tau))) ds dt d\tau, \quad (70)$$

for $x, y, z \in \Lambda$. Where function $a_1(x, y, z)$ is defined in Equation (63) and function $\alpha(x, y, z, s, t, \tau)$ is defined in Equation (67). Substituting Equation (60) into Equation (70) and setting $s_1 = b(s)$, $t_1 = c(t)$, $\tau_1 = d(\tau)$, we can deduce

$$\begin{aligned} |u(x, y, z)| &\leq |a_1(x, y, z)| + \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) h_1(s, t, \tau) \\ &\quad \times W_1(|u(s, t, \tau)|) ds dt d\tau + \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) \\ &\quad \times h_2(s, t, \tau) W_2(|u(b(s), c(t), d(\tau))|) ds dt d\tau \\ &\leq |a_1(x, y, z)| + \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) h_1(s, t, \tau) \\ &\quad \times W_1(|u(s, t, \tau)|) ds dt d\tau + \int_{b(x_0)}^{b(x)} \int_{c(y_0)}^{c(y)} \int_{d(z_0)}^{d(z)} \\ &\quad \times \alpha(x, y, z, b^{-1}(s_1), c^{-1}(t_1), d^{-1}(\tau_1)) \\ &\quad \times \frac{h_2(b^{-1}(s_1), c^{-1}(t_1), d^{-1}(\tau_1))}{|b'(b^{-1}(s_1))c'(c^{-1}(t_1))d'(d^{-1}(\tau_1))|} W_2(|u(s_1, t_1, \tau_1)|) ds_1 dt_1 d\tau_1. \end{aligned} \quad (71)$$

Clearly, Inequality (71) satisfies the form of Equation (1) in Theorem 1, and the conditions (K_2) - (K_5) are all assumed to hold. Therefore, the estimated equation (61) for $u(x, y, z)$ can be obtained from Theorem 1. Note that since the functions φ , ψ , ϕ are all bounded functions, the holding of equations (68) and (69) implies that the conditions (K_7) , (K_8) are assumed to be satisfied, thus reusing the boundedness proof of the estimated solution $u(x, y, z)$ in Theorem 1, we can show that $u(x, y, z)$ is bounded.

Next, we will discuss the uniqueness of the solution of Equation BVP (59). For all $(x, y, z) \in \Lambda$, suppose

$$\begin{aligned} &|U(x, y, z, u(x, y, z), u(b(x), c(y), d(z))) - U(x, y, z, \tilde{u}(x, y, z), \tilde{u}(b(x), c(y), d(z)))| \\ &\leq h_1(x, y, z) W_1(|u(x, y, z) - \tilde{u}(x, y, z)|) \\ &\quad + h_2(x, y, z) W_2(|u(b(x), c(y), d(z)) - \tilde{u}(b(x), c(y), d(z))|), \end{aligned} \quad (72)$$

where $h_i \in C(\Lambda, R_+)$, W_i satisfies assumption condition (K_5) ($i = 1, 2$). Suppose that both $u(x, y, z)$ and $\tilde{u}(x, y, z)$ are solutions of Equation BVP (59), then there is at most one solution of equation BVP (59) on Λ .

Proof. By equivalent integral equation (70) and setting $s_1 = b(s)$, $t_1 = c(t)$, $\tau_1 = d(\tau)$, we can deduce

$$\begin{aligned}
 & |u(x, y, z) - \tilde{u}(x, y, z)| \\
 & \leq \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) h_1(s, t, \tau) W_1(|u(s, t, \tau) - \tilde{u}(s, t, \tau)|) ds dt d\tau \\
 & \quad + \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) h_2(s, t, \tau) \\
 & \quad \times W_2(|u(b(s), c(t), d(\tau)) - \tilde{u}(b(s), c(t), d(\tau))|) ds dt d\tau \\
 & \leq \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) h_1(s, t, \tau) W_1(|u(s, t, \tau) - \tilde{u}(s, t, \tau)|) ds dt d\tau \\
 & \quad + \int_{b(x_0)}^{b(x)} \int_{c(y_0)}^{c(y)} \int_{d(z_0)}^{d(z)} \alpha(x, y, z, b^{-1}(s_1), c^{-1}(t_1), d^{-1}(\tau_1)) \\
 & \quad \times \frac{h_2(b^{-1}(s_1), c^{-1}(t_1), d^{-1}(\tau_1))}{|b'(b^{-1}(s_1))c'(c^{-1}(t_1))d'(d^{-1}(\tau_1))|} W_2(|u(s_1, t_1, \tau_1) - \tilde{u}(s_1, t_1, \tau_1)|) ds_1 dt_1 d\tau_1.
 \end{aligned} \tag{73}$$

Clearly, Inequality (73) satisfies the form of equation (1) in Theorem 1, similar to the proof of the boundedness of solutions, it can be verified that the discovery hypothesis conditions (K_2) – (K_5) are all true.

Using Theorem 1 for Equation (70), it can be obtained

$$\begin{aligned}
 |u(x, y, z) - \tilde{u}(x, y, z)| & \leq G_2^{-1} \left[G_2 \left(G_1^{-1}(a_2(x, y, z)) \right) \right. \\
 & \quad \left. + \int_{b(x_0)}^{b(x)} \int_{c(y_0)}^{c(y)} \int_{d(z_0)}^{d(z)} F_2(x, y, z, s, t, \tau) ds dt d\tau \right],
 \end{aligned} \tag{74}$$

where G_i , F_i ($i = 1, 2$), $a_2(x, y, z)$ are respectively defined in Equation (62), Equation (65), Equation (66), and Equation (67). It can be found that $a(x, y, z) = 0$, such that $a_1(x, y, z) = 0$. Combining the definition of G_i and the properties of W_i , can be obtained

$$\lim_{v \rightarrow 0} G_i(v) = -\infty, \quad \lim_{v \rightarrow -\infty} G_i^{-1}(v) = 0, \quad i = 1, 2. \tag{75}$$

Since Equations (67) and (68) both hold, then we have

$$a_2(x, y, z) := G_1(a_1(x, y, z)) + \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z F_1(x, y, z, s, t, \tau) ds dt d\tau = -\infty,$$

$$G_1^{-1}(a_2(x, y, z)) = 0,$$

$$G_2 \left(G_1^{-1}(a_2(x, y, z)) \right) + \int_{b(x_0)}^{b(x)} \int_{c(y_0)}^{c(y)} \int_{d(z_0)}^{d(z)} F_2(x, y, z, s, t, \tau) ds dt d\tau = -\infty,$$

$$G_2^{-1} \left[G_2 \left(G_1^{-1}(a_2(x, y, z)) \right) + \int_{b(x_0)}^{b(x)} \int_{c(y_0)}^{c(y)} \int_{d(z_0)}^{d(z)} F_2(x, y, z, s, t, \tau) ds dt d\tau \right] = 0.$$

Therefore, the conclusion $|u(x, y, z) - \tilde{u}(x, y, z)| \leq 0$ is obtained from Equation (74), that is, there is $u(x, y, z) = \tilde{u}(x, y, z)$ on $(x, y, z) \in \Lambda$. Therefore, the uniqueness of the solution to equation BVP (59) is proved.

Finally, we will discuss the continuous dependence of the solution of equation BVP (59) on the given functions U, φ, ψ, ϕ . Consider another delayed fractional partial differential equation BVP:

$$\begin{cases} ({}^c D_{\chi_0}^\beta u)(x, y, z) = \tilde{U}(x, y, z, \tilde{u}(x, y, z), \tilde{u}(b(x), c(y), d(z))), \\ \tilde{u}(x, y_0, z_0) = \tilde{\varphi}(x), \tilde{u}(x_0, y, z_0) = \tilde{\psi}(y), \tilde{u}(x_0, y_0, z) = \tilde{\phi}(z), \\ \tilde{\varphi}(x_0) = \tilde{\psi}(y_0) = \tilde{\phi}(z_0), \end{cases} \quad (76)$$

for $x, y, z \in \Lambda$. Where $U = C(\Lambda \times \mathbb{R}^3, \mathbb{R})$, $\varphi = C_\Omega(I, \mathbb{R})$, $\psi = C_\Omega(J, \mathbb{R})$, $\phi = C_\Omega(T, \mathbb{R})$. Suppose that the function U satisfies Equation (68) on a finite interval $\hat{I} := [x_0, x_1]$, $\hat{J} := [y_0, y_1]$, $\hat{T} := [z_0, z_1]$, and $\tilde{\Lambda} = \hat{I} \times \hat{J} \times \hat{T} \subset \mathbb{R}^3$, $\left(\left| \tilde{U} - U \right|, \left| \tilde{\varphi} - \varphi \right|, \left| \tilde{\psi} - \psi \right|, \left| \tilde{\phi} - \phi \right| \right) \rightarrow 0$. Then the solution $\tilde{u}(x, y, z)$ in BVP (72) is sufficiently close to the solution $u(x, y, z)$ in BVP (59).

Proof. Using the uniqueness of the estimated solution $u(x, y, z)$, for arbitrarily small constant χ_0 , assume

$$\max_{x \in \hat{I}} \left| \tilde{\varphi} - \varphi \right| \leq \chi_0, \quad \max_{y \in \hat{J}} \left| \tilde{\psi} - \psi \right| \leq \chi_0, \quad \max_{z \in \hat{T}} \left| \tilde{\phi} - \phi \right| \leq \chi_0,$$

$$\max_{(x, y, z, s, t, \tau) \in \tilde{\Lambda}} \left| \tilde{U} - U \right| \leq \chi_0.$$

By the equivalent integral equation (70) as well as Equations (63) and (67), and setting $s_1 = b(s)$, $t_1 = c(t)$, $\tau_1 = d(\tau)$, we can deduce

$$\begin{aligned} & \left| \tilde{u}(x, y, z) - u(x, y, z) \right| \\ & \leq \left| \tilde{\varphi}(x) - \varphi(x) + \tilde{\psi}(y) - \psi(y) + \tilde{\phi}(z) - \phi(z) + \varphi(x_0) - \tilde{\varphi}(x_0) \right| \\ & \quad + \left| \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) \left[\tilde{U}(s, t, \tau, \tilde{u}(s, t, \tau), \tilde{u}(b(s), c(t), d(\tau))) \right. \right. \\ & \quad \left. \left. - U(s, t, \tau, u(s, t, \tau), u(b(s), c(t), d(\tau))) \right] ds dt d\tau \right| \\ & \leq 4\chi_0 + \left| \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) \left[\tilde{U}(s, t, \tau, \tilde{u}(s, t, \tau), \tilde{u}(b(s), c(t), d(\tau))) \right. \right. \\ & \quad \left. \left. - U(s, t, \tau, \tilde{u}(s, t, \tau), \tilde{u}(b(s), c(t), d(\tau))) \right] ds dt d\tau \right| \\ & \quad + \left| \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) \left[U(s, t, \tau, \tilde{u}(s, t, \tau), \tilde{u}(b(s), c(t), d(\tau))) \right. \right. \\ & \quad \left. \left. - U(s, t, \tau, u(s, t, \tau), u(b(s), c(t), d(\tau))) \right] ds dt d\tau \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left[4 + \frac{(x_1 - x_0)^{\beta_1} (y_1 - y_0)^{\beta_2} (z_1 - z_0)^{\beta_3}}{\Gamma(\beta_1 + 1) \Gamma(\beta_2 + 1) \Gamma(\beta_3 + 1)} \right] \chi_0 \\
&\quad + \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \alpha(x, y, z, s, t, \tau) h_1(s, t, \tau) W_1(|\tilde{u}(s, t, \tau) - u(s, t, \tau)|) ds dt d\tau \\
&\quad + \int_{b(x_0)}^{b(x)} \int_{c(y_0)}^{c(y)} \int_{d(z_0)}^{d(z)} \alpha(x, y, z, b^{-1}(s_1), c^{-1}(t_1), d^{-1}(\tau_1)) \\
&\quad \times \frac{h_2(b^{-1}(s_1), c^{-1}(t_1), d^{-1}(\tau_1))}{|b'(b^{-1}(s_1))c'(c^{-1}(t_1))d'(d^{-1}(\tau_1))|} W_2(|\tilde{u}(s_1, t_1, \tau_1) - u(s_1, t_1, \tau_1)|) ds_1 dt_1 d\tau_1.
\end{aligned} \tag{77}$$

Clearly, Inequality (77) satisfies the form of equation (1) in Theorem 1, applying Equation (77), we can obtain that

$$\begin{aligned}
|\tilde{u}(x, y, z) - u(x, y, z)| &\leq G_2^{-1} \left[G_1^{-1} (a_2(x, y, z)) \right] \\
&\quad + \int_{b(x_0)}^{b(x)} \int_{c(y_0)}^{c(y)} \int_{d(z_0)}^{d(z)} F_2(x, y, z, s, t, \tau) ds dt d\tau,
\end{aligned} \tag{78}$$

where

$$a_1(x, y, z) = \left[4 + \frac{(x_1 - x_0)^{\beta_1} (y_1 - y_0)^{\beta_2} (z_1 - z_0)^{\beta_3}}{\Gamma(\beta_1 + 1) \Gamma(\beta_2 + 1) \Gamma(\beta_3 + 1)} \right] \chi_0,$$

G_i, F_i ($i = 1, 2$), $a_2(x, y, z)$ are respectively defined in Equation (62), Equation (65), Equation (66), and Equation (67). It can be concluded that when $\chi_0 \rightarrow 0$, $a_i(x, y, z) \rightarrow 0$ ($i = 1, 2$). Finally, through equations (75) and (78), we can obtain:

$$\lim_{\chi_0 \rightarrow 0} |\tilde{u}(x, y, z) - u(x, y, z)| = 0.$$

Thus, the continuity of the solution of the equation has been proven.

[ii] Consider the following inequality

$$\begin{aligned}
u(x, y, z) &\leq \frac{1}{x + y + z + 1} + \int_0^{\sqrt{x}} \int_0^{\sqrt{y}} \int_0^{\sqrt{z}} (x - s)^{-1/5} s^{-1/2} \\
&\quad \times (y - t)^{-1/5} t^{-1/2} (z - \tau)^{-1/5} \tau^{-1/2} x y z e^{-s-2t-3\tau} \\
&\quad \times \sqrt[3]{u(s, t, \tau)} ds dt d\tau + \int_0^x \int_0^y \int_0^z (x - s)^{-1/5} s^{-1/2} \\
&\quad \times (y - t)^{-1/5} t^{-1/2} (z - \tau)^{-1/5} \tau^{-1/2} \sqrt{u(s, t, \tau)} ds dt d\tau,
\end{aligned} \tag{79}$$

for $x \geq 0, y \geq 0, z \geq 0$. Let $\alpha_i = 1$ ($i = 1, 2$), $\beta_1 = \frac{4}{5}$, $\gamma_1 = \frac{1}{2}$, $\beta_2 = \frac{2}{3}$, $\gamma_2 = \frac{2}{3}$. From $\gamma_i > 1 - \frac{1}{p}$ and $p > 1$, we get $1 < p < 2$; And then from $\frac{1}{p} + \alpha_i(\beta_i - 1) + \gamma_i - 1 \geq 0$, we can take $p = \frac{4}{3}$. It's easy to compute $q = 4$ from $\frac{1}{p} + \frac{1}{q} = 1$. Let $a(x, y, z) = \frac{1}{x+y+z+1}$, $f_1(x, y, z, s, t, \tau) = x y z e^{-s-2t-3\tau}$, $f_2(x, y, z, s, t, \tau) = 1$, $W_1(u) = \sqrt[3]{u}$, $W_2(u) = \sqrt{u}$ again, it follows that $\frac{W_2(u)}{W_1(u)} = u^{1/6}$ is monotonically nondecreasing.

Obviously, Equation (75) satisfies the conditions in Theorem 2, so it is easy to calculate from Theorem 2 as follows.

$$\tilde{a}(x, y, z) = \max_{0 \leq \sigma \leq x, 0 \leq \mu \leq y, 0 \leq \zeta \leq z} a(\sigma, \mu, \zeta) = 1,$$

$$A_1(x, y, z) = (2 + 1)^{4-1} \tilde{a}^4(x, y, z) = 27,$$

$$\begin{aligned} H_1 &= \alpha_1^{-3/p}(xyz)^{(1/p)+\alpha_1(\beta_1-1)+(\gamma_1-1)}\mathbf{B}\left[\frac{p(\gamma_1-1)+1}{\alpha_1},p(\beta_1-1)+1\right]^{3/p} \\ &= (xyz)^{1/20}\mathbf{B}\left[\frac{1}{3},\frac{11}{15}\right]^{9/4}, \end{aligned}$$

$$\begin{aligned} H_2 &= \alpha_2^{-3/p}(xyz)^{(1/p)+\alpha_2(\beta_2-1)+(\gamma_2-1)}\mathbf{B}\left[\frac{p(\gamma_2-1)+1}{\alpha_2},p(\beta_2-1)+1\right]^{3/p} \\ &= (xyz)^{1/2}\mathbf{B}\left[\frac{5}{9},\frac{5}{9}\right]^{9/4}, \end{aligned}$$

$$\hat{f}_1(x,y,z,s,t,\tau)=\max_{0\leq\sigma\leq x,0\leq\mu\leq y,0\leq\zeta\leq z}f_1(\sigma,\mu,\zeta,s,t,\tau)=xyze^{-s-2t-3\tau},$$

$$\hat{f}_2(x,y,z,s,t,\tau)=\max_{0\leq\sigma\leq x,0\leq\mu\leq y,0\leq\zeta\leq z}f_2(\sigma,\mu,\zeta,s,t,\tau)=1,$$

$$\begin{aligned} \tilde{f}_1(x,y,z,s,t,\tau) &= (2+1)^{4-1}H_1^4\hat{f}_1^4(x,y,z,s,t,\tau) \\ &= 27(xyz)^{21/5}\mathbf{B}\left[\frac{1}{3},\frac{11}{15}\right]^9e^{-4s-8t-12\tau}, \end{aligned}$$

$$\begin{aligned} \tilde{f}_2(x,y,z,s,t,\tau) &= (2+1)^{4-1}H_2^4\hat{f}_2^4(x,y,z,s,t,\tau) \\ &= 27(xyz)^{1/3}\mathbf{B}\left[\frac{5}{9},\frac{5}{9}\right]^9, \end{aligned}$$

$$\hat{G}_1(v)=\int_{v_1}^v\frac{1}{\sqrt[3]{\xi}}d\xi=\frac{3}{2}\Big(v^{2/3}-v_1^{2/3}\Big),\quad v\geq v_1>0.$$

$$\hat{G}_1^{-1}(v)=\left(\frac{2}{3}v-v_1^{2/3}\right)^{3/2}.$$

$$\hat{G}_2(v)=\int_{v_2}^v\frac{1}{\sqrt{\xi}}d\xi=2(\sqrt{v}-\sqrt{v_2}),\quad v\geq v_2>0.$$

$$\hat{G}_2^{-1}(v)=\left(\frac{v}{2}+\sqrt{v_2}\right)^2.$$

$$\begin{aligned} A_2(x,y,z) &= \hat{G}_1(A_1(x,y,z))+\int_0^{\sqrt{x}}\int_0^{\sqrt{y}}\int_0^{\sqrt{z}}\tilde{f}_1(x,y,z,s,t,\tau)dsdtd\tau \\ &= \frac{3}{2}\Big(9-v_1^{2/3}\Big)-\frac{9}{128}(xyz)^{21/5}\mathbf{B}\left[\frac{1}{3},\frac{11}{15}\right]^9 \\ &\quad\times\Big(e^{-4\sqrt{x}}-1\Big)\Big(e^{-8\sqrt{y}}-1\Big)\Big(e^{-12\sqrt{z}}-1\Big). \end{aligned}$$

Since $\int_{v_1}^{\infty} \frac{1}{\sqrt[3]{\xi}} d\xi = \infty$, $\int_{v_2}^{\infty} \frac{1}{\sqrt[5]{\xi}} d\xi = \infty$, so $X = \infty$, $Y = \infty$, $Z = \infty$. Therefore, Theorem 2 of this paper can be obtained

$$\begin{aligned} & u(x, y, z) \\ & \leq \left\{ \hat{G}_2^{-1} \left[\hat{G}_2 \left(\hat{G}_1^{-1} (A_2(x, y, z)) \right) + \int_0^x \int_0^y \int_0^z \tilde{f}_2(x, y, z, s, t, \tau) ds dt d\tau \right] \right\}^{1/4} \\ & = \left\{ \left[9 - \frac{3}{63} (xyz)^{21/5} B \left[\frac{1}{3}, \frac{11}{15} \right]^9 \left(e^{-4\sqrt{x}} - 1 \right) \left(e^{-8\sqrt{y}} - 1 \right) \left(e^{-12\sqrt{z}} - 1 \right) \right]^{3/4} \right. \\ & \quad \left. + \frac{27}{2} (xyz)^{4/3} B \left[\frac{5}{9}, \frac{5}{9} \right]^9 \right\}^{1/2}, \end{aligned}$$

for $x \geq 0, y \geq 0, z \geq 0$. Which implies that $u(x, y, z)$ in Equation (79) is bounded.

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