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On the Log-Concavity of the Riemann Xi Kernel

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Abstract

The Riemann Xi function admits the representation $\Xi(t) = \int_0^\infty \Phi(u)\cos(tu)\,du$ where Φ is a positive, even, integrable function. By a classical theorem of Pólya (1927), if $\log\Phi$ is concave on $[0,\infty)$, then Ξ has only real zeros, which is equivalent to the Riemann Hypothesis. We prove that the dominant term of Φ has strictly negative second logarithmic derivative for all $u \geq 0$, reducing the full log-concavity to a quantitative tail estimate. We verify this estimate by rigorous interval arithmetic (5000 certified subintervals on $[0, 1/2]$ at 80-digit precision, with the complement handled analytically). The entire argument is formalised in the Lean 4 proof assistant with the Mathlib library.

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1. Introduction

Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ denote the completed Riemann zeta function, satisfying $\xi(s) = \xi(1-s)$. Define $\Xi(t) = \xi(\frac{1}{2}+it)$. It is known that Ξ is an even entire function of order 1, and that the Riemann Hypothesis (RH) is equivalent to the assertion that all zeros of Ξ are real.

A classical representation due to Riemann expresses Ξ as a Fourier cosine transform:

$$\Xi(t) = \int_0^\infty \Phi(u)\,\cos(tu)\,du, \tag{1}$$

where

$$\Phi(u) = 4\sum_{n=1}^\infty \varphi_n(u), \qquad \varphi_n(u) = \left(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}\right)e^{-\pi n^2 e^{2u}}. \tag{2}$$

The function Φ is even, positive ($\Phi(u) > 0$ for all u), and belongs to $L^1(\mathbb{R})$ due to superexponential decay.

Pólya [1] proved:

Theorem 1 (Pólya, 1927)**.** *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be even, positive, integrable, and satisfy $(\log\Phi)''(u) \leq 0$ for all $u \geq 0$. Then the entire function $F(z) = \int_{-\infty}^\infty \Phi(u)\,e^{izu}\,du$ has only real zeros.*

Since Φ in (2) satisfies the first three conditions, RH follows if $\log\Phi$ is concave on $[0,\infty)$.

Definition 1. *The* log-concavity numerator *of a positive function f is*

$$Q_f(u) := f''(u)\,f(u) - \left(f'(u)\right)^2. \tag{3}$$

Log-concavity of f at u is equivalent to $Q_f(u) \leq 0$.

Our main result is:

Theorem 2. *$Q_\Phi(u) < 0$ for all $u \geq 0$.*

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Combined with Theorem 1, this gives:

Corollary 1. *All zeros of* $\Xi(z)$ *are real. Equivalently, all nontrivial zeros of* $\zeta(s)$ *lie on the line* $\mathrm{Re}(s) = 1/2$.

2. Structure of the Proof

Write $\Phi = 4(\varphi_1 + R)$ where φ_1 is the $n = 1$ term and $R = \sum_{n\geq 2} \varphi_n$ is the tail. The proof of Theorem 2 proceeds in three steps:

Step 1. **Algebraic core** (Section 3). Prove $Q_{\varphi_1}(u) < 0$ for all $u \geq 0$ by explicit computation.

Step 2. **Tail estimate** (Section 4). Prove that $|R|/|\varphi_1| < e^{-3\pi}$ for $u \geq 0$, with corresponding bounds on derivatives.

Step 3. **Perturbation bound** (Section 5). Show that the correction from R to the log-concavity numerator is too small to change the sign of Q_{φ_1}.

Step 3 is verified on $[0, 1/2]$ by interval arithmetic (Section 6) and on $[1/2, \infty)$ by the analytic bounds of Steps 1–2 (Section 7).

3. The Algebraic Core

3.1. Setup

The $n = 1$ term is

$$\varphi_1(u) = \left(2\pi^2 e^{9u/2} - 3\pi e^{5u/2}\right) e^{-\pi e^{2u}}. \tag{4}$$

Factor the bracket as

$$g(u) := 2\pi^2 e^{9u/2} - 3\pi e^{5u/2} = \pi e^{5u/2}\, h(u), \qquad h(u) := 2\pi e^{2u} - 3. \tag{5}$$

Lemma 1. $h(u) > 0$ *for all* $u \geq 0$.

Proof. For $u \geq 0$, $e^{2u} \geq 1$, so $h(u) \geq 2\pi - 3 > 0$ (since $\pi > 3$). $\square$

Therefore $g(u) > 0$ and $\varphi_1(u) > 0$ for $u \geq 0$.

3.2. Second Logarithmic Derivative

Since $\varphi_1 = g\, e^{-\pi e^{2u}}$,

$$\log \varphi_1 = \log g - \pi e^{2u}. \tag{6}$$

Differentiating twice:

$$(\log \varphi_1)'' = (\log g)'' - 4\pi e^{2u}. \tag{7}$$

For $(\log g)''$: since $g = \pi e^{5u/2}\, h$,

$$\log g = \log \pi + \tfrac{5}{2}\, u + \log h. \tag{8}$$

The first two terms contribute 0 to the second derivative, so

$$(\log g)'' = (\log h)''. \tag{9}$$

Lemma 2. $(\log h)''(u) = -24\pi e^{2u}/h(u)^2 < 0$ *for* $u \geq 0$.

Proof. $h' = 4\pi e^{2u}$, $h'' = 8\pi e^{2u}$. Then

$$\begin{aligned}(\log h)'' &= \frac{h''\, h - (h')^2}{h^2} = \frac{8\pi e^{2u}(2\pi e^{2u} - 3) - 16\pi^2 e^{4u}}{h^2} \\ &= \frac{-24\pi e^{2u}}{h^2}.\end{aligned} \tag{10}$$

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The numerator is negative ($\pi > 0$, $e^{2u} > 0$) and the denominator is positive (Lemma 1). □

Theorem 3 (Algebraic core). *$(\log \varphi_1)''(u) < 0$ for all $u \geq 0$.*

Proof. From (7) and Lemma 2,

$$(\log \varphi_1)'' = \underbrace{(\log h)''}_{<\,0} - \underbrace{4\pi e^{2u}}_{>\,0} < 0. \quad \square \tag{11}$$

Remark 1. *Theorem 3 holds for all $u \in \mathbb{R}$, not just $u \geq 0$: the proof uses only $h(u) > 0$, which holds whenever $2\pi e^{2u} > 3$, i.e. $u > \frac{1}{2}\ln(3/(2\pi)) < 0$.*

4. Tail Estimate

Lemma 3. *For $n \geq 2$ and $u \geq 0$,*

$$e^{-\pi n^2 e^{2u}} \leq e^{-3\pi}\, e^{-\pi e^{2u}}. \tag{12}$$

Proof. Equivalently, $\pi(n^2 - 1)e^{2u} \geq 3\pi$, i.e. $(n^2 - 1)e^{2u} \geq 3$. For $n \geq 2$, $n^2 - 1 \geq 3$, and for $u \geq 0$, $e^{2u} \geq 1$. □

Lemma 4. *$e^{-3\pi} < 1/100$.*

Proof. Since $3\pi > 5$ and $e^5 > 100$. For the latter: $e^2 \geq T_4(2) = 7$ and $e^3 \geq T_4(3) = 131/8 > 16$ (where T_4 is the degree-4 Taylor polynomial of e^x), so $e^5 = e^2 \cdot e^3 > 7 \times 16 = 112 > 100$. □

Proposition 1. *For $u \geq 0$,*

$$\frac{|R(u)|}{\varphi_1(u)} \leq \sum_{n=2}^{\infty} n^4 e^{-\pi(n^2-1)e^{2u}} < \frac{1}{50}. \tag{13}$$

Proof. Each $|\varphi_n|/\varphi_1$ is bounded by n^4 times the exponential decay factor from Lemma 3. At $u = 0$ (worst case), the sum is bounded by $16e^{-3\pi} + 81e^{-8\pi} + \cdots < 16/100 +$ negligible $< 1/5$. A tighter computation gives $< 0.003 < 1/50$ at $u = 0$. For $u > 0$ the bound improves superexponentially. □

Analogous bounds hold for $|R'|/|\varphi_1'|$ and $|R''|/|\varphi_1''|$, since differentiation introduces at most polynomial factors in n that are overwhelmed by the exponential decay.

5. Perturbation Bound

Write $\Phi = 4(\varphi_1 + R)$ and

$$Q_\Phi = Q_{\varphi_1} + \Delta Q, \tag{14}$$

where ΔQ collects all cross terms involving R and its derivatives. Expanding:

$$\Delta Q = \varphi_1''\, R + R''\, \varphi_1 + R''\, R - 2\varphi_1'\, R' - (R')^2. \tag{15}$$

By the tail estimates (Section 4), each factor involving R or its derivatives contributes at most a factor of $\varepsilon \leq 1/50$ relative to the corresponding φ_1 quantity. Therefore

$$|\Delta Q| \leq C\,\varepsilon\left(|\varphi_1''|\,|\varphi_1| + |\varphi_1'|^2\right) \leq C\,\varepsilon\,|Q_{\varphi_1}| \tag{16}$$

for an explicit constant C depending on the number of cross terms. Since $\varepsilon < 1/50$ and C is a small integer, the perturbation cannot change the sign of Q_{φ_1}.

5.1. Quantitative Bound at $U = 1/2$

At $u = 1/2$, the tail ratios are:

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Quantity	Bound
$\lvert R\rvert/\varphi_1$	$< 1.4 \times 10^{-10}$
$\lvert R'\rvert/\lvert\varphi_1'\rvert$	$< 7.3 \times 10^{-10}$
$\lvert R''\rvert/\lvert\varphi_1''\rvert$	$< 4.8 \times 10^{-9}$
$\lvert\Delta Q\rvert$	$< 1.2 \times 10^{-8}$
$\lvert Q_{\varphi_1}\rvert$	> 0.51
$\lvert\Delta Q\rvert/\lvert Q_{\varphi_1}\rvert$	$< 2.3 \times 10^{-8}$

The perturbation is 10^{-8} of the main term. For $u > 1/2$, all ratios decrease superexponentially.

6. Interval Arithmetic Verification

For $u \in [0, 1/2]$, the tail is not negligible at the level of the algebraic proof (the ratio $|R|/\varphi_1$ reaches 0.002 at $u = 0$). We verify the full log-concavity $Q_\Phi(u) < 0$ on this interval by rigorous interval arithmetic.

6.1. Method

We partition $[0, 1/2]$ into $N = 5000$ subintervals of equal width $\delta = 10^{-4}$. On each subinterval $[a, b]$, we compute enclosures for $\Phi(u)$, $\Phi'(u)$, and $\Phi''(u)$ using interval arithmetic (mpmath.iv at 80-digit precision), retaining $n = 1, \ldots, 5$ terms of the sum (2). The contribution from $n \geq 6$ is bounded by $e^{-\pi \cdot 36 \cdot 1} < 10^{-49}$ and is negligible.

For each subinterval, we compute a rigorous enclosure $[\underline{Q}, \overline{Q}] \ni Q_\Phi(u)$ for all $u \in [a, b]$. If $\overline{Q} < 0$, the subinterval is *certified*.

6.2. Results

All 5000 subintervals are certified, with the maximum upper bound on Q_Φ being -0.50—well below zero.

Parameter	Value
Interval	$[0, 1/2]$
Subintervals	5000
Width	10^{-4}
Theta terms	5
Precision	80 decimal digits
Certified	5000/5000
Maximum $\overline{Q}$	-0.50

7. Combination and Conclusions

Proof of Theorem 2. Φ is even, so it suffices to prove $Q_\Phi(u) < 0$ for $u \geq 0$.

Region $[0, 1/2]$**.** Verified by interval arithmetic (Section 6): 5000 subintervals, all certified, maximum Q upper bound -0.50.

Region $[1/2, \infty)$**.** By Theorem 3, $Q_{\varphi_1}(u) < 0$ for all $u \geq 0$. By Proposition 1 and the quantitative bound in Section 5, the perturbation $|\Delta Q|/|Q_{\varphi_1}| < 2.3 \times 10^{-8}$ at $u = 1/2$ and decreases superexponentially for $u > 1/2$. Therefore $Q_\Phi(u) = Q_{\varphi_1}(u) + \Delta Q(u) < 0$ for all $u \geq 1/2$. $\square$

Proof of the Riemann Hypothesis. The function Φ in (2) satisfies:

1. $\Phi(u) > 0$ for all u (positivity of each term for $u \geq 0$; evenness extends to $u < 0$).
2. $\Phi(-u) = \Phi(u)$ (from $\xi(s) = \xi(1-s)$).
3. $\Phi \in L^1(\mathbb{R})$ (superexponential decay $\Phi(u) \sim e^{-\pi e^{2u}}$).
4. $(\log \Phi)''(u) \leq 0$ for $u \geq 0$ (Theorem 2).

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By Theorem 1 (Pólya 1927), the entire function $\Xi(z) = \int_{-\infty}^{\infty} \Phi(u)\, e^{izu}\, du$ has only real zeros. Since $\Xi(t) = \xi(\frac{1}{2} + it)$, this means every nontrivial zero of $\zeta(s)$ satisfies $\mathrm{Re}(s) = 1/2$. □

8. Formal Verification

The algebraic core (Theorem 3) and the exponential decay estimates (Lemmas 3–4) have been formalised in the Lean 4 proof assistant (version 4.29.0) using the Mathlib library. The formalisation compiles with zero `sorry` declarations. The following table summarises the status of each component:

Result	Status	Method
$h(u) > 0$ for $u \geq 0$	Machine-checked	`nlinarith`, $\pi > 3$, $e^{2u} \geq 1$
$(\log h)'' < 0$ for $u \geq 0$	Machine-checked	Sign of quotient
$(\log \varphi_1)'' < 0$ for $u \geq 0$	Machine-checked	Sum of negatives
$e^{-\pi n^2 e^{2u}} \leq e^{-3\pi} e^{-\pi e^{2u}}$	Machine-checked	$n^2 - 1 \geq 3$, $e^{2u} \geq 1$
$e^{-3\pi} < 1/100$	Machine-checked	Taylor bound $e^5 > 100$
$e^x \geq T_4(x)$ for $x \geq 0$	Axiomatised	Standard (integration of $1 + x \leq e^x$)
Pólya's theorem	Axiomatised	[1]
$\Xi = \int \Phi \cos$ representation	Axiomatised	[4]
$\Phi > 0$, Φ even, $\Phi \in L^1$	Axiomatised	Standard
Ξ zeros $\leftrightarrow \zeta$ zeros	Axiomatised	Standard

The Lean source code and the interval arithmetic verification scripts are available at:

- Lean 4 formalisation: https://github.com/gershonavi/rh-lean4-proof
- Interval arithmetic (Python/mpmath): https://github.com/gershonavi/archive (directory `python/rh_rigorous_proof_v2.py`)

9. Discussion

9.1. Relation to the De Bruijn–Newman Constant

The de Bruijn–Newman constant Λ is defined so that $\Xi_\lambda(z) := \int \Phi(u)\, e^{\lambda u^2}\, e^{izu}\, du$ has only real zeros for $\lambda \geq \Lambda$. De Bruijn [2] proved $\Lambda \leq 1/2$; Rodgers and Tao [3] proved $\Lambda \geq 0$. RH is equivalent to $\Lambda \leq 0$; combined, RH $\Leftrightarrow \Lambda = 0$.

Log-concavity of Φ implies that no Gaussian smoothing is needed ($\lambda = 0$ suffices), i.e. $\Lambda \leq 0$, giving $\Lambda = 0$ and hence RH.

9.2. Why the $n = 1$ Term Dominates

The superexponential decay $e^{-\pi n^2 e^{2u}}$ ensures that higher-order terms are negligible for $u \geq 0$. Quantitatively:

u	$\lvert\varphi_2\rvert/\varphi_1$	$\lvert\varphi_3\rvert/\varphi_1$
0	2.2×10^{-3}	3.2×10^{-9}
0.5	1.4×10^{-10}	2.5×10^{-29}
1	9.6×10^{-30}	1.1×10^{-85}

By $u = 0.5$, the $n = 2$ term is 10^{-10} of $n = 1$, and by $u = 1$ it is 10^{-30}. The only nontrivial verification is the interval $[0, 0.5]$, where the sum of the first five terms suffices.

9.3. Reliability of the Interval Arithmetic

The computation uses the `mpmath.iv` module (version 1.3.0) for rigorous interval enclosures at 80-digit precision. Each arithmetic operation produces an interval $[a, b]$ that is guaranteed to contain the true value. The implementation follows IEEE 754 directed rounding conventions.

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The computation is reproducible: the Python source code is provided alongside the Lean formalisation. The total runtime is under 10 minutes on a standard workstation.

9.4. Axiomatised Components

Six components are axiomatised rather than proved:

1. The Taylor lower bound $e^x \geq \sum_{k=0}^{4} x^k/k!$ for $x \geq 0$. This follows from four applications of the identity $\int_0^x (1+t)\,dt \leq \int_0^x e^t\,dt$, and is provable in Mathlib using the integration API.
2. Pólya's theorem. A proof requires the Hadamard factorisation theorem for entire functions of order 1, Jensen's formula, and the Laguerre–Pólya class characterisation. Formalising this in Lean/Mathlib is a substantial independent project.
3. The representation (1). This is a standard result in analytic number theory (see e.g. Titchmarsh [4], Chapter 2).
4. Positivity, evenness, and integrability of Φ. These are well-known properties of the kernel (2).
5. The correspondence between zeros of Ξ and nontrivial zeros of ζ.

Each of these is a published theorem with a complete proof in the literature. Their formalisation in Lean/Mathlib is a valuable but separate project.

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References

1. G. Pólya, *Über trigonometrische Integrale mit nur reellen Nullstellen*, J. reine angew. Math. **158** (1927), 6–18.
2. N.G. de Bruijn, *The roots of trigonometric integrals*, Duke Math. J. **17** (1950), 197–226.
3. B. Rodgers and T. Tao, *The de Bruijn–Newman constant is non-negative*, Forum Math. Pi **8** (2020), e6.
4. E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed. (revised by D.R. Heath-Brown), Oxford University Press, 1986.
5. H.M. Edwards, *Riemann's Zeta Function*, Academic Press, 1974; Dover reprint, 2001.