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Article

The Metrization Problem in $[0,1]$-Topology

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Abstract: The paper discusses the classification of fuzzy metrics based on their continuity conditions, dividing them into Erceg, Deng, Yang-Shi, and Chen metrics. It explores the relationships between these types of fuzzy metrics, concluding that a Deng metric in $[0,1]$-topology must also be Erceg, Chen, and Yang-Shi metrics. The paper also proves that the product of countably many Deng pseudo-metric spaces remains a Deng pseudo-metric space, and demonstrates some σ-locally finite properties of Deng metric space. Additionally, the paper constructs two interrelated mappings based on normal space and concludes that if a $[0,1]$-topological space is T_1 and regular, and its topology has a σ-locally finite base, then it is Deng metrizable, and thus Erceg, Yang-Shi, and Chen metrizable as well.

Keywords: fuzzy point; $[0,1]$-topology; Deng pseudo-metric; σ-locally finite base; regular; T_1-space; distance; metrizable

MSC: 54A40; 03E72; 54E35

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1. Introduction

In general topology, given a topological space (X, δ), it is natural to ask whether there is a metric for X such that δ is the metric topology. Such a metric metricizes the topological space and the space is said to be metrizable. Around the 1950s, through the efforts of R.H. Bing [1], Y.M. Smirnov and C.H. Dowker[2], J. Nagata [3], M.H. Stone [4], this problem mentioned was satisfactorily solved, and eventually, their comprehensive conclusion is called Nagata-Smirnov metrization admittedly in general topology, unquestionably, which is the most important theorem of topology. By that time, the main theory of topology had been perfected. However, scholars engaged in academic research never stopped exploring the unknown areas and sought new ways to gain a breakthrough in topological theory. In 1968, C.L. Chang [5] introduced the fuzzy set theory of Zadeh [6] into topology for the first time, which declared the birth of $[0,1]$-topology. Soon after that, J.A. Goguen [7] further generalized L-fuzzy set to the proposed $[0,1]$-topology and his theory has been recognized as L-topology nowadays. From then on, this kind of lattice-valued topology formed another important branch of topology, and thereafter many creative results and original thoughts were presented (see [8–29], etc.).

Nevertheless, how to generalize classical metrics to the lattice-valued topology reasonably has always been a great challenge. So far, there are quite some fuzzy metrics introduced in the branch of learning (see [8,12,14,15,26,30–32], etc.). Considering the codomain is either an ordinary number or a fuzzy number, these metrics are roughly divided into two types.

One type is composed of these metrics, each of which is defined by such a function whose distance between objects is fuzzy, while the objects themselves are crisp. Additionally, each of them always induces a fuzzifying topology. In recent years, these metrics have been promoted by quite a few experts, such as I. Kramosil, J. Michalek, A. George, P. Veeramani, V. Gregori, S. Romaguera, J. Gutiérrez García, S. Morillas, F.G. Shi, etc. (see [15,16,25,33–44], etc.).

The other type consists of these metrics, each of which is defined by such a mapping $p : M \times M \to [0, +\infty)$, where M is the set of all standard fuzzy points of the underlying classical set X. In this case, every such fuzzy metric always induces a fuzzy topology (see [8,12–14,26,28], etc.).

About the latter, there are roughly three kinds of fuzzy metrics in history, with which the academic community has gradually been familiar. In addition, there is the fourth metric recently discovered. About the four fuzzy metrics, we will list them below one by one.

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The first is the Erceg metric, which was presented in 1979 by M.A. Erceg [14]. Since then, many scholars have been engaged in this research and have obtained many beautiful results. Among them, a typical conclusion is J.H. Liang [23] showed Urysohn's metrization theorem in 1984: an L-topological space is Erceg metrizable if it is T_1, regular, and C_{II}. In 1985, M.K. Luo [24] constructed an example of Erceg metric on I^X whose metric topology has no σ-locally finite base, which implies that the $[0,1]$-topological space of this example is not C_{II} of course, but then Liang's this conclusion is still the best one. In this paper, Liang's conclusion is only a corollary of our result in $[0,1]$-topology. Later on, based on Peng's simplification method [45], the Erceg metric was further simplified by P. Chen and F.G. Shi (see [11,46]) below:

(I) An *Erceg pseudo-metric* on L^X is a mapping $p : M \times M \to [0,+\infty)$ satisfying

(A1) If $a \geq b$, then $p(a,b) = 0$;
(A2) $p(a,c) \leq p(a,b) + p(b,c)$;
(B1) $p(a,b) = \bigvee_{c \ll b} p(a,c)$;
(A3) $\forall a,b \in M$, $\exists x \not\leq a'$ s.t. $p(b,x) < r \Leftrightarrow \exists y \not\leq b'$ s.t. $p(a,y) < r$.

An Erceg pseudo-metric p is called an *Erceg metric* if it further satisfies
(A4) If $p(a,b) = 0$, then $a \geq b$,

where " $\ll$ " is the way-below relation in Domain Theory and L^X is a completely distributive lattice [47,48].

The second is the Yang-Shi metric (or p.q. metric), which is proposed in 1988 by L.C. Yang [28]. It was proved by Yang that each topological molecular lattice with C_{II} property is p.q. metrizable (refer to [28,48] for details). After that, this kind of metric was studied in depth by F.G. Shi (see [11,26,46,49,50], etc.). Its definition is as follows:

(II) A *Yang-Shi pseudo-metric* (resp., *Yang-Shi metric*) on L^X is a mapping $p : M \times M \to [0,+\infty)$ satisfying (A1)-(A3) (resp., (A1)-(A4)) and the following

(B2) $p(a,b) = \bigwedge_{c \ll a} p(c,b)$.

Similarly, according to our later proofs in this paper, Yang's this conclusion is still a corollary of our result in $[0,1]$-topology.

The third is the Deng metric supplied in 1982 by Z.K. Deng [12], where Deng [13] showed that if a $[0,1]$-topological space is T_1, regular and C_{II} then it is Deng metrizable. In this paper, we will extend this result substantially. Incidentally, Y.Y. Lan and F. Long also provided a result about Deng pseudo-metrization problem [51]. However, the proof was not completely right after careful checking pointed out by us. It is worth mentioning that since Deng's research is only limited to this special lattice I^X, Deng pseudo-metric was later extended to L^X by P. Chen [52] below:

(III) An *extended Deng pseudo-metric* (resp., *extended Deng metric*) on L^X is a mapping $p : M \times M \to [0,+\infty)$ satisfying (A1)-(A3) (resp., (A1)-(A4)) and the following

(B3) $p(a,b) = \bigwedge_{b \ll c} p(a,c)$.

In a summary, the above three kinds of fuzzy metrics are defined by using (A1)-(A4) and different (B1), (B2), and (B3), respectively. Inspired by this, we conclude that there is another new metric defined as follows:

(IV) A *Chen pseudo-metric* (resp., *Chen metric*) on L^X is a mapping $p : M \times M \to [0,+\infty)$ satisfying (A1)-(A3) (resp., (A1)-(A4)) and the following

(B4) $p(a,b) = \bigvee_{a \ll c} p(c,b)$.

About it, some elementary properties related to it have been introduced [8].

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In this paper, we will focus on the latter mainly and study its metrization problem in $[0,1]$-topology. For this reason, we investigate the relationships between **(I)-(IV)** on I^X and fortunately acquire such a profound result: let $C = \{ p \mid p \text{ is a Chen metric}\}$, $E = \{ p \mid p \text{ is an Erceg metric} \}$, $D = \{ p \mid p \text{ is a Deng metric} \}$ and $Y = \{ p \mid p \text{ is a Yang-Shi metric} \}$ on I^X. Then $D = C \cap Y \cap E$.

Consequently, if a given $[0,1]$-topology is Deng metrizable, then it must be Erceg, Yang-Shi, and Chen metrizable. Thus this paper mainly will discuss the Deng metric and its metrization problem in $[0,1]$-topology.

To sum up, although so many scholars have been engaged in the study of fuzzy metrics, it is a little pity that the metrization problem in $[0,1]$-topology remains unsolved now. this paper aims to study the metrization problem in $[0,1]$-topology and will obtain the generalization of Nagata–Smirnov metrization theorem in $[0,1]$-topology.

2. Preliminaries

In this section, we cite the fundamental definitions that will be used in the sequel. The letter X always refers to a nonempty set throughout this paper, and I denotes the unit interval $[0,1]$.

A *fuzzy set* of X is a mapping $A : X \to I$, which forms the family I^X. The constant fuzzy set of X with the value 1 (resp., 0) is denoted by $\underline{1}$ (resp., $\underline{0}$). A *fuzzy point* (resp., *standard fuzzy point*) x_λ in X is a fuzzy set defined by $x_\lambda(x) = \lambda$ and $x_\lambda(y) = 0$ if $y \neq x$, where λ is a fixed number in $(0,1)$ (resp., $(0,1]$). The set of all fuzzy points (resp., all standard fuzzy points) of X is denoted by M_0 (resp., M). M_0 is a subfamily of M. Naturally, these properties of M_0 and M are also suitable for L-topology.

A subfamily δ of I^X is called a $[0,1]$-*topology* if it satisfies the following three conditions: (O1) $\underline{1}, \underline{0} \in \delta$; (O2) if $A, B \in \delta$, then $A \wedge B \in \delta$; (O3) if $\{A_\lambda \mid \lambda \in \Lambda\} \subseteq \delta$, then $\bigvee_{\lambda\in\Lambda} A_\lambda \in \delta$. The pair (X,δ) is called a $[0,1]$-*topological space* (a space for short). If $\delta \subseteq I^X$, then for each $A \in \delta$, A and A' are called a δ-*fuzzy open set* and a δ-*fuzzy closed set* (open set and closed set for short), respectively.

Two fuzzy sets A and B are called *quasi-coincident* if there exists x belonging to X such that $A(x) + B(x) > 1$ (see [53]). Let x_α be a fuzzy point and let A be a fuzzy set of X. The notation $x_\alpha \in A$ means $\alpha < A(x)$ [12]. The closure of a fuzzy set A of (X,δ) is the intersection of the members of the family of all closed sets containing A, denoted by $\overline{A}$ [12]. A fuzzy point x_α is called a *cluster point* of a fuzzy set U of (X,δ) if each open neighborhood of $x_{1-\alpha}$ is quasi-coincident with U. Consequently, $x_\alpha \leq \overline{A}$ if and only if x_α is a cluster point of A. Therefore, $\overline{A} = \bigvee\{y_\beta \mid y_\beta \text{ is a cluster point of } \overline{A}\}$ [12].

The space (X,δ) is called *regular* (resp., *normal*) if for any $x_\lambda \in M_0$ (resp., $\tau' \in \delta$), $\mu \in \delta$ with $x_\lambda \in \mu$ (resp., $\tau \leq \mu$), there exists v belonging to δ such that $x_\lambda \in v \leq \overline{v} \leq \mu$ (resp., $\tau \leq v \leq \overline{v} \leq \mu$) [20]. A $[0,1]$-topological space is T_1 if and only if x_λ is closed for each fuzzy point $x_\lambda \in M_0$. A family ψ of fuzzy sets is a *base* of δ if ψ is a subfamily of δ and for each fuzzy point x_λ and each open neighborhood μ of x_λ, there is a member v of δ such that $x_\lambda \in v \leq \mu$. A family κ of fuzzy sets is a *subbase* of δ if the family of finite intersections of members of κ is a base of δ [12,54]. The space (X,δ) is C_{II}, or called *second-countable* if the $[0,1]$-topology δ has a countable basis.

A family of fuzzy sets Ψ is called *locally finite* (resp., *discrete*) in a space (X,δ) if and only if each fuzzy point x_λ of the space has its an open neighborhood which is quasi-coincident with only finitely many members (resp., at most one member) of Ψ (see [48]). A family of fuzzy sets is called σ-*locally finite* (resp., σ-*discrete*) in a space (X,δ) if and only if it is the union of a countable number of locally finite (resp., discrete) subfamilies. A subfamily σ of I^X (resp., σ of δ) is called a (resp., an *open*) *cover* of a fuzzy set A in a space (X,δ) if for each $x_\alpha \in A$, there exists B belonging to σ such that $x_\alpha \in B$. Furthermore, if $A = \underline{1}$, then σ is called a *cover* of (X,δ). A cover $\mathcal{B}$ of a fuzzy set A is called a *refinement* of a cover $\mathcal{D}$ if each member of $\mathcal{B}$ is a subset of a member of $\mathcal{D}$ [48].

Let $\{X_t\}_{t\in T}$ be an indexed family of sets. The *cartesian product* of this indexed family, denoted by $\prod_{t\in T} X_t$, is the set of all functions $x : T \to \bigcup_{t\in T} X_t$ such that $x(t) \in X_t$ for each $t \in T$.

Let $X = \prod_{t \in T} X_t$. Then the t-th *projection* $J_t : I^X \to I^{X_t}$ is defined by $J_t(A)(y_t) = \sup\{A(x) \mid x_t = y_t\}$ for each $y_t \in X_t$ and let $J_t^{-1}(B) = \bigvee\{C \in I^X \mid J_t(C) \leq B\}$. The product space of $\{(X_t, \delta_t) \mid t \in T\}$ is defined by $\{J_t^{-1}(A_t) \mid A_t \in \delta_t, t \in T\}$ as a sub base [48].

Other unexplained terminologies and notations and further details can be found in [7,8,12,48,54].

Definition 1 ([8,12]). *A Deng pseudo-metric on I^X is a mapping $p : M_0 \times M_0 \to [0, +\infty)$ satisfying*

(D1) *If $\lambda_1 \geq \lambda_0$, then $p(x_{\lambda_1}, x_{\lambda_0}) = 0$;*
(D2) *$p(x_{\lambda_1}, z_{\lambda_3}) \leq p(x_{\lambda_1}, y_{\lambda_2}) + p(y_{\lambda_2}, z_{\lambda_3})$;*
(D3) *$p(x_{\lambda_1}, y_{\lambda_2}) = \bigwedge_{\lambda > \lambda_2} p(x_{\lambda_1}, y_\lambda)$;*
(D4) *$p(x_{\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{1-\lambda_1})$.*

A Deng pseudo-metric p is called a Deng metric if it further satisfies the following

(D5) *If $p(x_{\lambda_1}, y_{\lambda_2}) = 0$, then $x = y, \lambda_1 \geq \lambda_2$.*

Remark In [52] we have proved the following results: (1) a Deng metric p on I^X can be extended to an extended Deng metric p^*; (2) $p = p^* \mid M_0 \times M_0$; (3) p^* and p induce the same metric topology.

Based on the above (1)-(3), it is much easier to study Deng metric by using Definition 1 instead of (**III**) on I^X as its definition. A similar treatment to (**I**), (**II**), and (**VI**) on I^X is to restrict their domains to M_0 and use (D4) instead of (A3) while other conditions remain unchanged.

Theorem 1 ([12]). *Let p be a Deng pseudo-metric (resp., a Deng metric) on I^X. For each $r \in [0, 1)$ define $U_r(a) = \bigvee\{b \in M_0 \mid p(a, b) < r\}$. Then the family $\{U_r(a) \mid a \in M_0, r \in [0, +\infty)\}$ forms a base of δ_p, called the $[0, 1]$-topology induced by p. The space (X, δ_p) is called a Deng pseudo-metric space (resp., a Deng metric space).*

Theorem 2 ([12]). *If p is a Deng pseudo-metric on I^X, then (X, δ_p) is regular, normal.*

In [12], Deng has proved such a result: If (X, δ) is regular and C_{II}, then it is normal [12]. It is a special case of the following result:

Theorem 3 ([8,48]). *If (X, δ) is regular, and δ has a σ–locally finite base, then it is normal.*

Theorem 4 ([12,48]). *If $\{A_\lambda \mid \lambda \in \Gamma\}$ is locally finite in a space (X, δ), then $\overline{\bigvee_{\lambda \in \Gamma} A_\lambda} = \bigvee_{\lambda \in \Gamma} \overline{A_\lambda}$.*

Theorem 5 ([50]). *Suppose that (X, δ) is normal, and let a closed set A and an open set B satisfy $A \leq B$. Then there is a family $\{U_r \mid r \in Q_{[0,1]}\}$ such that each element is an open neighborhood of A and satisfies the following properties:*

(a) *$U_0 = A$, $U_1 = B$;*
(b) *If $r < s$, then $U_r \leq \overline{U}_r \leq U_s$.*

Theorem 6. *[50] Let p be a Yang-Shi pseudo-metric on L^X and define $P_r(b) = \bigvee\{c \in M \mid p(c, b) \geq r\}$. Then for $c, b \in M, c \leq P_r(b) \Leftrightarrow p(c, b) \geq r$.*

Theorem 7. *Let p be a Deng pseudo-metric on I^X. For any $a \in M_0$ and each $r \in [0, 1)$ define $B_r(a) = \bigvee\{b \in M_0 \mid p(a, b) \leq r\}$. Then*

(1) *$\overline{B_r(a)} = B_r(a)$;*
(2) *$b \leq B_r(a) \Leftrightarrow p(a, b) \leq r$.*

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Proof. Because of the following Theorem 9, Theorem 10 and the existing conclusion [11]: If p is an Erceg pseudo-metric on I^X, then it satisfies (1) and (2), this proposition holds as desired. □

3. The relationships between four kinds of fuzzy metrics on I^X

In this section, we will investigate the relationships between the four kinds of metrics: Erceg , Yang-Shi, Deng, and Chen metrics. First of all, we expose the main result as follows:

Theorem 8. *On I^X, let $C = \{ p \mid p$ is a Chen metric $\}$; $E = \{ p \mid p$ is an Erceg metric$\}$; $D = \{ p \mid p$ is a Deng metric $\}$; $Y = \{ p \mid p$ is a Yang-Shi metric$\}$. Then $D = C \cap Y \cap E$.*

Proof. It can be obtained from the following Theorem 9-12. □

Theorem 9. *If p is a Yang-Shi pseudo-metric on I^X, then it is an Erceg pseudo-metric.*

Proof. To prove that p is an Erceg pseudo-metric on I^X, we only need to prove that $p(x_\alpha, y_\beta) = \bigvee_{\gamma<\beta} p(x_\alpha, y_\gamma)$. The proof is as follows:

By (A1) and (A2), when $\gamma < \beta$, we have $p(x_\alpha, y_\gamma) \leq p(x_\alpha, y_\beta)$. Hence $p(x_\alpha, y_\beta) \geq \bigvee_{\gamma<\beta} p(x_\alpha, y_\gamma)$. If $p(x_\alpha, y_\beta) > \bigvee_{\gamma<\beta} p(x_\alpha, y_\gamma)$, then we may take two different numbers $s, r > 0$ such that

$$p(x_\alpha, y_\beta) > s > r \geq \bigvee_{\gamma<\beta} p(x_\alpha, y_\gamma).$$

In addition, for each $\gamma < \beta$, by triangle inequality $p(x_\alpha, y_\beta) \leq p(x_\alpha, y_\gamma) + p(y_\gamma, y_\beta)$, we have

$$s < p(x_\alpha, y_\beta) \leq p(y_\gamma, y_\beta) + r.$$

Therefore, $p(y_\gamma, y_\beta) > s - r$, so that $p(y_\beta, y_\beta) = \bigwedge_{\gamma<\beta} p(y_\gamma, y_\beta) \geq s - r > 0$. But this contradicts (A1), as desired. □

However, the converse is not true. Such a counterexample is given below.

Example 1. *Let $L = [0,1]$ and $X = x$. For convenience, we denote L^X and x_λ for L and λ respectively. Define a mapping $p : (0,1] \times (0,1] \to [0, +\infty)$ by:*

$$p(a,b) = \begin{cases} 0, & \text{if } a \geq b; \\ 1, & \text{if } a < b. \end{cases}$$

Firstly, let us prove that p is an Erceg pseudo-metric on $[0,1]$.

(A1) and (A2) are trivial.

(B1) if $a, b \in (0,1]$ and $a \geq b$, then $p(a,b) = 0$. Therefore, we have $\bigvee_{x<b} p(a,x) = 0$, so that in this case $p(a,b) = \bigvee_{x<b} p(a,x)$. Similarly, when $a < b$, we can prove $p(a,b) = \bigvee_{x<b} p(a,x) = 1$. Consequently, p satisfies (B1).

(A3) we only need to prove that $\bigwedge_{y>1-b} p(a,y) = \bigwedge_{x>1-a} p(b,x)$, which can be obtained from the following implications: $\bigwedge_{y>1-b} p(a,y) = 1 \Leftrightarrow y > 1-b$ implies $y > a \Leftrightarrow 1-b \geq a \Leftrightarrow x > 1-a$ implies $x > b \Leftrightarrow \bigwedge_{x>1-a} p(b,x) = 1$.

Secondly, we assert that p is not a Yang-Shi pseudo-metric. In fact, for any $a \in (0,1]$, we have $p(a,a) = 0$. But $\bigwedge_{c<a} p(c,a) = 1$. Thus $p(a,b) \neq \bigwedge_{c<a} p(c,b)$, as desired. □

Theorem 10. *If p is a Deng pseudo-metric on I^X, then it is a Yang-Shi pseudo-metric.*

Proof. For any two fuzzy points x_a and y_b, we only need to prove $p(x_a, y_b) = \bigwedge_{c<a} p(x_c, y_b)$. If $c < a$, then $p(x_a, y_b) \leq p(x_c, y_b)$. So $p(x_a, y_b) \leq \bigwedge_{c<a} p(x_c, y_b)$. If $p(x_a, y_b) = r < \bigwedge_{c<a} p(x_c, y_b) = t$, then by (D4) we have $p(y_{1-b}, x_{1-a}) = r < t$, so that by (D3) there exists a number $s > 1-a$ such that $p(y_{1-b}, x_s) < t$, i.e., $p(x_{1-s}, y_b) < t$. But that contradicts $\bigwedge_{c<a} p(x_c, y_b) = t$. Consequently, $p(x_a, y_b) = \bigwedge_{c<a} p(x_c, y_b)$, as desired. □

Conversely, we have the following conclusion:

Theorem 11. *If a Yang-Shi pseudo-metric p further is a Chen pseudo-metric on I^X, then p is a Deng pseudo-metric.*

To prove this, we first need to prove the following two Lemma 1 and Lemma 2.

Lemma 1. *Let p be a Yang-Shi pseudo-metric on I^X and for each $r \in [0,1)$ define $U_r(a) = \bigvee\{b \in I^X \mid p(a,b) < r\}$. Then $U_r(y_\lambda) = \bigvee_{\alpha>1-\lambda} P_r(y_\alpha)'$.*

Proof. Let $x_\beta \in \bigvee_{\alpha>1-\lambda} P_r(y_\alpha)'$ and take γ such that $x_\beta < x_\gamma \leq \bigvee_{\alpha>1-\lambda} P_r(y_\alpha)'$. Because $1-\gamma \geq \bigwedge_{\alpha>1-\lambda} P_r(y_\alpha)(x)$, there exists a number $\alpha > 1-\lambda$ such that $1-\gamma \geq P_r(y_\alpha)(x)$, and then for each $\delta > 1-\gamma$ we have $\delta > P_r(y_\alpha)(x)$. Therefore by Theorem 6, we can obtain $p(x_\delta, y_\alpha) < r$. Again by (A3) in (**I**) ((A3) on the special case I^X of L^X is for any $x_{\lambda_1}, y_{\lambda_2}$, $\exists t > 1-\lambda_1$ s.t. $p(y_{\lambda_2}, x_t) < r \Leftrightarrow \exists s > 1-\lambda_2$ s.t. $p(x_{\lambda_1}, y_s) < r$), there exists $x_\omega(x_\delta)$ (x_ω has something to do with x_δ) with $\omega > 1-\delta$ such that $p(y_\lambda, x_\omega) < r$. Let $x_q = \bigvee\{x_\omega(x_\delta) \mid \delta > 1-\gamma\}$. Then $x_\delta \not\leq x_{1-q}$, i.e., $x_\delta > x_{1-q}$. This implies that as long as $x_\delta > x_{1-\gamma}$, it must hold that $x_\delta > x_{1-q}$. Thus $x_\gamma \leq x_q$. Since $x_\beta < x_\gamma \leq x_q$, there exists $x_\omega(x_\delta)$ such that $x_\beta \leq x_\omega$, and so $p(y_\lambda, x_\beta) \leq p(y_\lambda, x_\omega) < r$. Hence $x_\beta \leq U_r(y_\lambda)$. Because x_β is arbitrary, we have $\bigvee_{\alpha>1-\lambda} P_r(y_\alpha)' \leq U_r(y_\lambda)$.

Conversely, let $x_\alpha \in U_r(y_\lambda)$. Then $p(y_\lambda, x_\alpha) < r$. For each $x_\beta > x_{1-\alpha}$, i.e., $\alpha > 1-\beta$, by (A3) there exists $\gamma > 1-\lambda$ such that $p(x_\beta, y_\gamma) < r$, and then by Theorem 6, $x_\beta \not\leq P_r(y_\gamma)$. Hence $x_\beta \not\leq \bigwedge_{\gamma>1-\lambda} P_r(y_\gamma)$. That is to say, as long as $x_\beta > x_{1-\alpha}$, i.e., $x_\beta \not\leq x_{1-\alpha}$, it is true that $x_\beta \not\leq \bigwedge_{\gamma>1-\lambda} P_r(y_\gamma)$. Consequently, $\bigwedge_{\gamma>1-\lambda} P_r(y_\gamma)(x) \leq x_{1-\alpha}$, i.e., $x_\alpha \leq \bigvee_{\gamma>1-\lambda} P_r(y_\gamma)'$. Because x_α is arbitrary, we have $U_r(y_\lambda) \leq \bigvee_{\gamma>1-\lambda} P_r(y_\gamma)'$, as desired. □

Lemma 2. *If p is a Yang-Shi pseudo-metric on I^X, then $\bigvee_{\alpha>1-\lambda_1} p(x_\alpha, y_{\lambda_2}) = \bigvee_{\beta>1-\lambda_2} p(y_{\lambda_\beta}, x_{\lambda_1})$.*

Proof. Denote $\bigvee_{\alpha>1-\lambda_1} p(x_\alpha, y_{\lambda_2}) = \bigvee_{\beta>1-\lambda_2} p(y_{\lambda_\beta}, x_{\lambda_1})$ as (H1). Then it is easy to check that (H1) is equivalent to the following property:

(H1)* $\exists\alpha > 1-\lambda_1$ s.t. $p(x_\alpha, y_{\lambda_2}) > r \Leftrightarrow \exists\beta > 1-\lambda_2$ s.t. $p(y_{\lambda_\beta}, x_{\lambda_1}) > r$.

Now let us prove (H1)*.

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Assume that there is α with $\alpha > 1-\lambda_1$ such that $p(x_\alpha, y_{\lambda_2}) > r$. Take a number s such that $p(x_\alpha, y_{\lambda_2}) > s > r$. By the process of proving of Theorem 7 and Theorem 9, we assert that $\lambda_2 > B_s(x_\alpha)(y)$. Therefore, by Lemma 1, we can obtain the following formula:

$$\lambda_2 > B_s(x_\alpha)(y) \geq U_s(x_\alpha)(y) = \bigvee_{\gamma > 1-\alpha} P_s(x_\gamma)'(y).$$

Thus, for every $\gamma > 1-\alpha$ it is true that $\lambda_2 > P_s(x_\gamma)'(y)$. That is to say, as long as $\alpha > 1-\lambda_1$, i.e., $x_{\lambda_1} \not\leq x_{1-\alpha}$ such that $p(x_\alpha, y_{\lambda_2}) > r$, it is true that $\lambda_2 > P_s(x_{\lambda_1})'(y)$, i.e., $1-\lambda_2 < P_s(x_{\lambda_1})(y)$. So there exists y_ω such that $y_{1-\lambda_2} < y_\omega \leq P_s(x_{\lambda_1})$, and then $p(y_\omega, x_{\lambda_1}) \geq s > r$ by Theorem 6. Similarly, so is the reverse, as desired. □

Proof. The proof of Theorem 11 is as follows:

Let p be a Yang-Shi pseudo-metric on I^X and it satisfies $p(x_{\lambda_2}, y_{\lambda_1}) = \bigvee_{s > \lambda_2} p(x_s, y_{\lambda_1})$. Then we only need to prove that p satisfies (D3) and (D4).

(D4). Given any $x_{\lambda_1}, y_{\lambda_2} \in M_0$. According to Lemma 2, we have

$$\bigvee_{\alpha > 1-\lambda_1} p(x_\alpha, y_{\lambda_2}) = \bigvee_{\beta > 1-\lambda_2} p(y_{\lambda_\beta}, x_{\lambda_1}),$$

and then $p(x_{1-\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{\lambda_1})$.

(D3). By (D1) and (D2), if $\lambda_3 > \lambda_1$, then $p(y_{\lambda_2}, x_{\lambda_1}) \leq p(y_{\lambda_2}, x_{\lambda_3})$. Thus, $p(y_{\lambda_2}, x_{\lambda_1}) \leq \bigwedge_{\lambda_3 > \lambda_1} p(y_{\lambda_2}, x_{\lambda_3})$.

Conversely, take any r with $r \in (0, +\infty)$ such that $p(y_{\lambda_2}, x_{\lambda_1}) < r$. Then by (D4) and (B1) we have

$$p(y_{\lambda_2}, x_{\lambda_1}) = p(x_{1-\lambda_1}, y_{1-\lambda_2}) = \bigwedge_{h < 1-\lambda_1} p(x_h, y_{1-\lambda_2}) < r.$$

Therefore, there at least exists h with $h < 1-\lambda_1$ such that $p(x_h, y_{1-\lambda_2}) < r$, i.e., $p(y_{\lambda_2}, x_{1-h}) < r$. Let $1-h = \lambda_3$. Then $h < 1-\lambda_1 \Leftrightarrow \lambda_1 < 1-h = \lambda_3$ and $p(y_{\lambda_2}, x_{\lambda_3}) < r$. Consequently, $p(y_{\lambda_2}, x_{\lambda_1}) \geq \bigwedge_{\lambda_3 > \lambda_1} p(y_{\lambda_2}, x_{\lambda_3})$, as desired. □

Theorem 12. *If p is a Deng pseudo-metric on I^X, then it is a Chen pseudo-metric.*

Proof. We only need to prove that $p(x_\alpha, y_\beta) = \bigvee_{\alpha < \gamma} p(x_\gamma, y_\beta)$. By (D1) and (D2) we have $p(x_\alpha, y_\beta) \geq \bigvee_{\alpha < \gamma} p(x_\gamma, y_\beta)$. If $p(x_\alpha, y_\beta) > \bigvee_{\alpha < \gamma} p(x_\gamma, y_\beta)$, then there exist two numbers s and r such that $p(x_\alpha, y_\beta) > s > r \geq \bigvee_{\alpha < \gamma} p(x_\gamma, y_\beta)$. Therefore, for any $\gamma > \alpha$ we have $s < p(x_\alpha, y_\beta) \leq p(x_\alpha, x_\gamma) + p(x_\gamma, y_\beta) \leq p(x_\alpha, x_\gamma) + r$, and then $s - r < p(x_\alpha, x_\gamma)$. Hence $0 < s - r \leq \bigwedge_{\alpha < \gamma} p(x_\alpha, x_\gamma) = p(x_\alpha, x_\alpha) = 0$. But this is a contradiction, and then it must hold $p(x_\alpha, y_\beta) = \bigvee_{\alpha < \gamma} p(x_\gamma, y_\beta)$, as desired. □

Theorem 13. *If p is a Chen pseudo-metric on I^X and satisfies the property $\bigvee_{s > 1-\lambda_1} p(x_s, y_{\lambda_2}) = \bigvee_{t > 1-\lambda_2} p(y_t, x_{\lambda_1})$, then p is an Erceg pseudo-metric.*

Proof. From $\bigvee_{s > 1-\lambda_1} p(x_s, y_{\lambda_2}) = \bigvee_{t > 1-\lambda_2} p(y_t, x_{\lambda_1})$ and $p(x_\alpha, y_\beta) = \bigvee_{\gamma > \alpha} p(x_\gamma, y_\beta)$, we can obtain $p(x_{1-\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{\lambda_1})$, and then

$$\bigvee_{s < \lambda_1} p(y_{\lambda_2}, x_s) = \bigvee_{s < \lambda_1} p(x_{1-s}, y_{1-\lambda_2})$$

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$$= \bigvee_{1-s>1-\lambda_1} p(x_{1-s}, y_{1-\lambda_2}) = p(x_{1-\lambda_1}, y_{1-\lambda_2}) = p(y_{\lambda_2}, x_{\lambda_1}).$$

Consequently, p is an Erceg pseudo-metric, as desired. □

Conversely, we have the following result:

Theorem 14. *If p is an Erceg pseudo-metric on I^X and satisfies the property $p(x_{1-\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{\lambda_1})$, then p is a Chen pseudo-metric.*

Proof. Since $p(x_{1-\lambda_1}, y_{\lambda_2}) = p(y_{1-\lambda_2}, x_{\lambda_1})$, we have the following equation:

$$\bigvee_{t>\lambda_2} p(y_t, x_{\lambda_1}) = \bigvee_{t>\lambda_2} p(x_{1-\lambda_1}, y_{1-t})$$

$$= \bigvee_{1-t<1-\lambda_2} p(x_{1-\lambda_1}, y_{1-t}) = p(x_{1-\lambda_1}, y_{1-\lambda_2}) = p(y_{\lambda_2}, x_{\lambda_1}).$$

Therefore, p is a Chen pseudo-metric, as desired. □

In summary, because of Theorem 8 in this section, we have asserted that if a given $[0,1]$-topology δ is Deng metrizable, then δ must be Erceg, Yang-Shi, and Chen metrizable. For this reason, next, we will mainly focus on the Deng metric and its metrization in $[0,1]$-topology.

4. The product of countable metric spaces

In this section, let $Q_{[0,1]}$ be the set of all rational numbers in $[0,1]$, let $\omega = \{1, 2, \cdots, n, \cdots\}$, and let $\mathcal{X} = \prod_{n\in\omega} X_n$, where X_n ($n \in \omega$) is a nonempty set. For clarity, the set of all fuzzy points in $\mathcal{X}$ is denoted by $M_0(\mathcal{X})$, and for $y \in \mathcal{X}$, $y = (y^1, y^2, \cdots, y^n, \cdots)$.

Theorem 15. *Let p be a Deng pseudo-metric on I^X and let $e(x_\alpha, y_\beta) = \min[1, p(x_\alpha, y_\beta)]$. Then (X, δ_e) is a Deng pseudo-metric space whose topology δ_e is identical to that of (X, δ_p).*

Consequently, each pseudo-metric space (X, δ_p) is homomorphic to a pseudo-metric space (X, δ_e), where the range of e is the unit interval $[0,1]$.

Proof. The proof is trivial and omitted. □

Theorem 16. *Let $\{(X_n, \delta_{p_n}) \mid n \in \omega\}$ be a sequence of Deng pseudo-metric spaces, and the range of p_n ($n \in \omega$) is the unit interval $[0,1]$. Define a mapping $p : M_0(\mathcal{X}) \times M_0(\mathcal{X}) \to [0,1]$ by:*

$$p(x_\alpha, y_\beta) = \sum_{n\in\omega} 2^{-n} p_n(J_n(x_\alpha), J_n(y_\beta)),$$

where $J_n : I^{\mathcal{X}} \to I^{X_n}$ is the n-th projection (see 2. Preliminaries on J_n). Then

(1) *For each $n \in \omega$, $p_n(J_n(x_\alpha), J_n(y_\beta)) = \bigwedge_{\tau>\beta} p_n(J_n(x_\alpha), J_n(y_\tau))$;*

(2) *The mapping p is a Deng pseudo-metric on $I^{\mathcal{X}}$;*

(3) *The space $(\mathcal{X}, \delta_p)$ is the product space of $\{(X_n, \delta_{p_n}) \mid n \in \omega\}$.*

Proof. (1) Since $J_n(y_\beta) = y^n_\beta$ ($n \in \omega$), we have

$$p_n(J_n(x_\alpha), J_n(y_\beta)) = p_n(J_n(x_\alpha), y^n_\beta) = \bigwedge_{\tau>\beta} p_n(J_n(x_\alpha), y^n_\tau) = \bigwedge_{\tau>\beta} p_n(J_n(x_\alpha), J_n(y_\tau)).$$

(2) Since p_n ($n \in \omega$) satisfies (D1) and (D2), it is easy to check that p also satisfies (D1) and (D2).

(D3) First, by the definition of p and (1), we have

$$\begin{aligned} p(x_\alpha, y_\beta) &= \sum_{n\in\omega} 2^{-n} p_n(J_n(x_\alpha), J_n(y_\beta)) = \sum_{n\in\omega} 2^{-n} \bigwedge_{\tau>\beta} p_n(J_n(x_\alpha), J_n(y_\tau)) \\ &\leq \bigwedge_{\tau>\beta} \sum_{n\in\omega} 2^{-n} p_n(J_n(x_\alpha), J_n(y_\tau)) \\ &= \bigwedge_{\tau>\beta} p(x_\alpha, y_\tau). \end{aligned}$$

Conversely, let $p(x_\alpha, y_\beta) = r$. Then for any $\varepsilon > 0$ we have

$$p(x_\alpha, y_\beta) = \sum_{n\in\omega} 2^{-n} p_n(J_n(x_\alpha, J_n(y_\beta)) = \sum_{n\in\omega} 2^{-n} \bigwedge_{\tau>\beta} p_n(J_n(x_\alpha), J_n(y_\tau)) < r + \varepsilon.$$

Let $r_n = 2^{-n} \bigwedge_{\tau>\beta} p_n(J_n(x_\alpha), J_n(y_\tau))$. Then $\bigwedge_{\tau>\beta} p_n(J_n(x_\alpha), J_n(y_\tau)) < 2^n \times r_n + \varepsilon$. Therefore, for each n there is a number τ_n with $\tau_n > \beta$ such that $p_n(J_n(x_\alpha), J_n(y_{\tau_n})) < 2^n \times r_n + \varepsilon$. Hence we have

$$\sum_{n\in\omega} 2^{-n} p_n J_n(x_\alpha), J_n(y_{\tau_n})) < \sum_{n\in\omega} r_n + \varepsilon \sum_{n\in\omega} 2^{-n} = r + \varepsilon.$$

Given every fixed natural number n, we can take a number μ_n with $\mu_n > \beta$ such that $\mu_n \leq \min\{\tau_1, \tau_2, \cdots, \tau_n\}$. Thus for any natural number m, we have

$$\begin{aligned} &\sum_{i=1}^{m} 2^{-i} p_i(J_i(x_\alpha), J_i(y_{\mu_m})) + \sum_{i=m+1}^{\infty} 2^{-i} p_i(J_i(x_\alpha), J_i(y_{\tau_i})) \\ &\quad \leq \sum_{i=1}^{\infty} 2^{-i} p_i(J_i(x_\alpha), J_i(y_{\tau_i})) < r + \varepsilon, \end{aligned}$$

which is equivalent to the following inequality:

$$\sum_{i=1}^{\infty} 2^{-i} p_i(J_i(x_\alpha), J_i(y_{\mu_m})) + \sum_{i=m+1}^{\infty} 2^{-i} [p_i(J_i(x_\alpha), J_i(y_{\tau_i})) - p_i(J_i(x_\alpha), J_i(y_{\mu_m})] < r + \varepsilon.$$

Consequently, for the fixed natural member m we have

$$\begin{aligned} &\sum_{i=1}^{\infty} 2^{-i} p_i(J_i(x_\alpha), J_i(y_{\mu_m})) \\ &< r + \varepsilon - \sum_{i=m+1}^{\infty} 2^{-i} [p_i(J_i(x_\alpha), J_i(y_{\tau_i})) - p_i(J_i(x_\alpha, J_i(y_{\mu_m}))] \\ &= r + \varepsilon + \sum_{i=m+1}^{\infty} 2^{-i} [p_i(J_i(x_\alpha), J_i(y_{\mu_m})) - p_i(J_i(x_\alpha), J_i(y_{\tau_i}))] \\ &\leq r + \varepsilon + \sum_{i=m+1}^{\infty} 2^{-i} p_i(J_i(y_{\tau_i}), J_i(y_{\mu_m})) \\ &\leq r + \varepsilon + \sum_{i=m+1}^{\infty} 2^{-i}. \end{aligned}$$

Hence

$$\bigwedge_{t>\beta} p(x_\alpha, y_t) \leq \bigwedge_{\mu_m>\beta} p(x_\alpha, y_{\mu_m}) = \bigwedge_{\mu_m>\beta} \sum_{i=1}^{\infty} 2^{-i} p_i(J_i(x_\alpha), J_i(y_{\mu_m})) \leq r + \varepsilon.$$

Because ε is arbitrary, we can obtain $\bigwedge_{\tau>\beta} p(x_\alpha, y_\tau) \leq r = p(x_\alpha, y_\beta)$. Therefore, p satisfies (D3).

(D4) Since it holds that $p_n(J_n(x_\alpha), J_n(y_\beta)) = p_n(J_n(y_{1-\beta}), J_n(x_{1-\alpha}))$ for each $n \in \omega$, we conclude that $p(x_\alpha, y_\beta) = p(y_{1-\beta}, x_{1-\alpha})$.

(3) First, let $(\mathcal{X}, \delta)$ be the product space of $\{(X_n, \delta_{p_n}), n \in \omega\}$. For any $V \in \delta_p$ and $x_\alpha \in V$, there is an open neighborhood $U_r(x_\alpha)$ of x_α such that $U_r(x_\alpha) \leq V$, where

$$U_r(x_\alpha) = \bigvee\{y_\beta \mid p(x_\alpha, y_\beta) = \sum_{i=1}^{\infty} 2^{-n} p_n(J_n(x_\alpha), J_n(y_\beta)) < r\}.$$

Taking a natural member q with $\frac{1}{2^q} < r$, we can get $U_{\frac{1}{2^q}}(x_\alpha) \leq V$. Thus, if we define

$$W = \bigvee\{z_\gamma \mid p_n(J_n(x_\alpha), J_n(z_\gamma)) < \frac{1}{2^{q+n+2}}, n \leq q+2\},$$

then $W \leq U_{\frac{1}{2^q}}(x_\alpha)$. This is because when $z_\gamma \in W$, we always have

$$p(x_\alpha, z_\gamma) < \sum_{n=1}^{n=q+2} \frac{1}{2^{q+n+2}} + \sum_{n=q+3}^{\infty} \frac{1}{2^n} < \frac{1}{2^{q+2}} + \frac{1}{2^{q+2}} = \frac{1}{2^{q+1}} < \frac{1}{2^q}.$$

Clearly, W is an open set in the product topology δ since it can be generated by the subbase of the product space $(\mathcal{X}, \delta)$. Therefore, we conclude that V is an open set in $(\mathcal{X}, \delta)$. Consequently, $\delta_p \subseteq \delta$.

Conversely, let $U = \bigvee\{x_\alpha \mid x_\alpha^n \in V\}$, where V is an open set of some δ_{p_n}. Then U is a member of the subbase of δ. If $x_\alpha \in \{x_\alpha \mid x_\alpha^n \in V\}$, then there is an open set $U_r(x_\alpha^n)$ (see Theorem 1 on U_r) belonging to δ_{p_n} such that $U_r(x_\alpha^n) \leq V$. Therefore, $J_n^{-1}(U_r(x_\alpha^n)) \leq U$. Since $p(x_\alpha, y_\beta) \geq 2^{-n} p_n(x_\alpha^n, y_\beta^n)$, the open set $U_{\frac{r}{2^n}}(x_\alpha)$ of δ_p is a subset of U. In fact, if $p(x_\alpha, y_\beta) < \frac{r}{2^n}$, then $p_n(x_\alpha^n, y_\beta^n) < r$. Hence $U_{\frac{r}{2^n}}(x_\alpha) \leq J_n^{-1}(U_r(x_\alpha^n))$, and then $U_{\frac{r}{2^n}}(x_\alpha) \leq U$. Therefore, U is the union of some open sets in δ_p. Consequently, $\delta \subseteq \delta_p$. To sum up, the proposition is proved. □

5. σ-locally finite property

In this section, some σ-locally finite properties of Deng pseudo-metric space will be examined based on a defined distance function between fuzzy sets.

Definition 2. *Let p be a Deng pseudo-metric on I^X. A distance function $d : I^X \times I^X \to [0, +\infty)$ is defined by:*

$$d[A, B] = \inf\{p(x_\alpha, y_\beta) \mid x_\alpha \in A, y_{1-\beta} \in B\}.$$

Let $A, B \in I^X$, and $x_\alpha \in M_0$. Then by definition, it is easy to prove that $d[A, x_{1-\alpha}] = \inf\{p(y_\beta, x_\alpha) \mid y_\beta \in A\}$, $d[x_\alpha, B] = \inf\{p(x_\alpha, y_\beta) \mid y_{1-\beta} \in B\}$ and $d[A, B] = d[B, A]$.

Theorem 17. *Let p be a Deng pseudo-metric on I^X. If fuzzy sets U and V are quasi-coincident, then $d[U, V] = d[V, U] = 0$.*

Proof. Because U and V are quasi-coincident, there is x belonging to X such that $U(x) + V(x) > 1$. Let $U(x) = \gamma$ and $V(x) = \beta$. Given α with $1 - \beta < \alpha < \gamma$, we have $x_\alpha \in U$. Take a number λ satisfying $\alpha > \lambda > 1 - \beta$. Since $x_{1-\lambda} \in V$, $d[U, V] \leq p(x_\alpha, x_\lambda) = 0$. Similarly, $d[V, U] = 0$. □

Theorem 18. *Let p be a Deng pseudo-metric on I^X. Suppose that $\mathbb{U}$ is an open cover of (X, δ_p), and for each U of $\mathbb{U}$ and each positive integer n, define $U_n = \bigvee\{x_\alpha \mid d[x_\alpha, U'] \geq \frac{1}{2^n}\}$. Then $d[U_n, U'_{n+1}] \geq \frac{1}{2^{n+1}}$.*

Proof. Let $x_\alpha \in U_n$ and $y_{1-\beta} \in U'_{n+1}$. For any $z_{1-\gamma} \in U'$ we can obtain $p(x_\alpha, y_\beta) + p(y_\beta, z_\gamma) \geq p(x_\alpha, z_\gamma) \geq d[x_\alpha, U']$. Hence it is clear that

$$p(x_\alpha, y_\beta) + d[y_\beta, U'] \geq d[x_\alpha, U'], \text{ i.e., } p(x_\alpha, y_\beta) \geq d[x_\alpha, U'] - d[y_\beta, U'].$$

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Because of $x_\alpha \in U_n$, there is x_η belonging to U such that $\eta > \alpha$ and $d[x_\alpha, U'] \geq d[x_\eta, U'] \geq \frac{1}{2^n}$. Because of $y_{1-\beta} \in U'_{n+1}$, we assert that $d[y_\beta, U'] < \frac{1}{2^{n+1}}$. Thus $p(x_\alpha, y_\beta) \geq d[x_\alpha, U'] - d[y_\beta, U'] > \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}}$. Consequently, $d[U_n, U'_{n+1}] \geq \frac{1}{2^{n+1}}$. □

Since $\mathbb{U}$ is a nonempty set, we can select a partial order on $\mathbb{U}$ such that $\mathbb{U}$ is well ordered, denoted by " $\prec$ " (see Theorem 25 in Chapter 0 of [54]: Every nonempty set can be well ordered).

Theorem 19. *Let p be a Deng pseudo-metric on I^X and let the family $\mathbb{U}$ be an open cover of (X, δ_p). Choose a relation $\prec$ which well orders the family $\mathbb{U}$ and for each $U \in \mathbb{U}$ and each $n \in \omega$ define $U_n^* = U_n \wedge (\vee\{V_{n+1} \mid V \in \mathbb{U} \text{ and } V \prec U\})'$. Then*

(1) *Either $V_n^* \leq U'_{n+1}$ or $U_n^* \leq V'_{n+1}$ is true, depending on whether U follows or precedes V in the ordering;*
(2) *In either case $d[U_n^*, V_n^*] \geq \frac{1}{2^{n+1}}$.*

Proof. (1) The proof is straightforward.

(2) It is easy to see that $V_n^* \leq V_n$. Furthermore, if $V \prec U$, then $U_n^* \leq V'_{n+1}$. Hence $d[U_n^*, V_n^*] \geq d[V'_{n+1}, V_n] = d[V_n, V'_{n+1}] \geq \frac{1}{2^{n+1}}$. Similarly, when $U \prec V$, $d[U_n^*, V_n^*] \geq \frac{1}{2^{n+1}}$. □

Theorem 20. *Let p be a Deng pseudo-metric on I^X and let the family $\mathbb{U}$ be an open cover of (X, δ_p). Given $U \in \mathbb{U}$ and given any $n \in \omega$, for each corresponding U_n^*, define $U_n^\sim = \bigvee\{x_\alpha \mid d[U_n^*, x_{1-\alpha}] < \frac{1}{2^{n+3}}\}$. Then*

(1) *$U_n^\sim$ is an open set;*
(2) *$d[U_n^\sim, V_n^\sim] \geq \frac{1}{2^{n+2}}$.*

Proof. (1) Take a fuzzy point x_α and a number s_α such that they satisfy $d[U_n^*, x_{1-\alpha}] = r < \frac{1}{2^{n+3}}$ and $0 < s_\alpha < \frac{1}{2^{n+3}} - r$, respectively. It follows that $d[U_n^*, z_{1-\gamma}] \leq d[U_n^*, x_{1-\alpha}] + p(x_\alpha, z_\gamma) < r + s_\alpha < \frac{1}{2^{n+3}}$ for any $z_\gamma \in U_{s_\alpha}(x_\alpha)$. Therefore, $U_{s_\alpha}(x_\alpha) \leq U_n^\sim$, so that $U_n^\sim$ is an open set. In addition, if $d[U_n^*, x_{1-\alpha}] < \frac{1}{2^{n+3}} < \frac{1}{2^{n+1}}$, then there is $y_\mu \in U_n^*$ such that $p(y_\mu, x_\alpha) < \frac{1}{2^{n+1}}$. But according to $d[U_n^*, U'_{n+1}] \geq d[U_n, U'_{n+1}] \geq \frac{1}{2^{n+1}}$, it must hold that $x_\alpha \leq U_{n+1}(x)$. Thus $U_n^\sim \leq U_{n+1} \leq U$.

(2) Taking $x_\alpha \in U_n^\sim$ and $y_\beta \in V_n^\sim$ such that $d[U_n^*, x_{1-\alpha}] < \frac{1}{2^{n+3}}$ and $d[V_n^*, y_{1-\beta}] < \frac{1}{2^{n+3}}$, we have $p(z_\gamma, x_\alpha) + p(x_\alpha, y_{1-\beta}) \geq p(z_\gamma, y_{1-\beta}) \geq d[U_n^*, y_\beta]$ for any $z_\gamma \in U_n^*$. Therefore, we have $d[U_n^*, x_\alpha] + p(x_\alpha, y_{1-\beta}) \geq d[U_n^*, y_\beta]$, so that

$$d[U_n^*, x_\alpha] + p(x_\alpha, y_{1-\beta}) + d[y_{1-\beta}, V^*] \geq d[U_n^*, y_\beta] + d[y_{1-\beta}, V_n^*].$$

Since for any $z_\gamma \in U_n^*$, $p(z_\gamma, y_{1-\beta}) + p(y_{1-\beta}, w_\alpha) \geq p(z_\gamma, w_\alpha) \geq d[z_\gamma, V_n^*]$ and $w_{1-\alpha} \in V_n^*$, we have $p(z_\gamma, y_{1-\beta}) + d[y_{1-\beta}, V_n^*] \geq d[z_\gamma, V_n^*]$. In addition, in view of $d[z_\gamma, V^*] \geq d[U_n^*, V^*]$, we can check that $p(z_\gamma, y_{1-\beta}) + d[y_{1-\beta}, V_n^*] \geq d[U_n^*, V_n^*]$. Therefore, we assert that $d[U_n^*, y_\beta] + d[y_{1-\beta}, V_n^*] \geq d[U_n^*, V_n^*]$, so that

$$d[U_n^*, x_{1-\alpha}] + p(x_\alpha, y_{1-\beta}) + d[y_{1-\beta}, V_n^*] \geq d[U_n^*, V_n^*].$$

Besides, from $x_\alpha \in U_n^\sim$ and $d[U_n^*, V_n^*] \geq \frac{1}{2^{n+1}}$, we have the following inequalities:

$$p(x_\alpha, y_{1-\beta}) \geq d[U_n^*, V_n^*] - d[U_n^*, x_{1-\alpha}] - d[y_{1-\beta}, V_n^*] > \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}} - d[y_{1-\beta}, V_n^*].$$

Note that $y_\beta \in V_n^\sim$ and $d[y_{1-\beta}, V_n^*] = \bigwedge_{x_\omega \in V_n^*} p(y_{1-\beta}, x_{1-\omega}) = \bigwedge_{x_\omega \in V_n^*} p(x_\omega, y_\beta) = d[V_n^*, y_{1-\beta}] \leq \frac{1}{2^{n+3}}$. Therefore, $p(x_\alpha, y_{1-\beta}) > \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}} - \frac{1}{2^{n+3}} = \frac{1}{2^{n+2}}$, so that $d[U_n^\sim, V_n^\sim] \geq \frac{1}{2^{n+2}}$, as desired. □

Theorem 21. *Let $\mathbb{V}_n$ be the family of all sets of the form $U_n^\sim (n \in \omega)$. Given fuzzy point x_α. Then*

(1) *If there is a fixed number $r > 0$ such that $d[x_\alpha, V_n^\sim] > r$ for each $V_n^\sim \in \mathbb{V}_n$, then $x_{1-\alpha} \not\leq \overline{\bigvee_{U_n^\sim \in \mathbb{V}_n} U_n^\sim}$;*

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(2) *If such a fixed number $r > 0$ is non-existent, then* $x_{1-\alpha} \leq \overline{\bigvee_{U_n^{\sim} \in \mathbb{V}_n} U_n^{\sim}}$;

(3) *If* $\alpha \leq \frac{1}{2}$, *then there at most exists a* $U_n^{\sim} \in \mathbb{V}_n$ *such that* $x_{1-\alpha} \leq \overline{U_n^{\sim}}$.

Proof. (1) Given $U_n^{\sim} \in \mathbb{V}_n$, we have $p(x_\alpha, y_{1-\beta}) > r$ for all $y_\beta \in U_n^{\sim}$. By Theorem 7, $y_{1-\beta} \not\leq B_r(x_\alpha)$, so that $y_{1-\beta} \not\leq U_r(x_\alpha)$, i.e., $U_r(x_\alpha)(y) + \beta < 1$. Therefore, $U_r(x_\alpha)(y) + U_n^{\sim}(y) \leq 1$ for all $y \in X$. Hence for each $V_n^{\sim} \in \mathbb{V}_n$, $U_r(x_\alpha)$ and $V_n^{\sim}$ are non-quasi-coincident, as desired.

(2) Because such a fixed number r is non-existent, for any $\varepsilon_k > 0$, there is $V_{n(\varepsilon_k)}^{\sim}$ such that $d[x_\alpha, V_{n(\varepsilon_k)}^{\sim}] < \varepsilon_k$. This means that each open neighborhood of x_α is quasi-coincident with $\bigvee_{U_n^{\sim} \in \mathbb{V}_n} U_n^{\sim}$. Therefore, $x_{1-\alpha} \leq \overline{\bigvee_{U_n^{\sim} \in \mathbb{V}_n} U_n^{\sim}}$, as desired.

(3) Bescause $\alpha \leq \frac{1}{2}$, we conclude that $p(x_{1-\alpha}, x_\alpha) = 0$. Assume that there exist two $U_n^{\sim}, V_n^{\sim} \in \mathbb{V}_n$ such that $x_{1-\alpha} \leq \overline{U_n^{\sim}}$ and $x_{1-\alpha} \leq \overline{V_n^{\sim}}$. Then by $d[x_\alpha, U_n^{\sim}] = 0$ and $d[x_\alpha, V_n^{\sim}] = 0$ we have

$$d[V_n^{\sim}, x_\alpha] + p(x_{1-\alpha}, x_\alpha) + d[x_\alpha, U_n^{\sim}] \geq d[V_n^{\sim}, U_n^{\sim}] \geq \frac{1}{2^{n+2}}.$$

Note that $d[V_n^{\sim}, x_\alpha] = d[x_\alpha, V_n^{\sim}]$. Therefore, we can obtain that $0 \geq \frac{1}{2^{n+2}}$. But this is a contradiction. □

6. Two interrelated mappings

To solve the metrization problem in $[0,1]$-topology in the next section, we shall construct two interrelated mappings in advance based on the normal spaces in this section.

Theorem 22. *Let (X, δ) be normal $[0,1]$-topological space and let $A \in \delta'$, $B \in \delta$ with $A \leq B$. Therefore, there exists a corresponding family $\{U_r \mid r \in Q_{[0,1]}\}$ relative to A and B satisfying* (a) and (b) in Theorem 5.

Define a mapping $f : M_0 \to [0,1]$ by

$$f(x_\alpha) = \begin{cases} \inf\{r \in Q_{[0,1]} \mid x_\alpha \in U_r\}, & x_\alpha \in B; \\ 1, & x_\alpha \notin B \end{cases}$$

and for all $x_\alpha, y_\beta \in M_0$ let $g(x_\alpha, y_\beta) = \max\{f(y_\beta) - f(x_\alpha), 0\}$.

(a) Let $V_r = U'_{1-r}$. Then the family $\{V_r \mid r \in Q_{[0,1]}\}$ satisfies the following properties: (1) $V_0 = B'$, $V_1 = A'$; (2) if $r < s$, then $V_r \leq V_s$;

(b) Define a mapping $f^* : M_0 \to [0,1]$ by

$$f^*(x_\alpha) = \begin{cases} \inf\{r \in Q_{[0,1]} \mid x_\alpha \leq V_r\}, & x_\alpha \leq A'; \\ 1, & x_\alpha \not\leq A' \end{cases}$$

and for all $x_\alpha, y_\beta \in M_0$ let $g^*(x_\alpha, y_\beta) = \max\{f^*(y_\beta) - f^*(x_\alpha), 0\}$. Then $g^*(y_{1-\beta}, x_{1-\alpha}) = g(x_\alpha, y_\beta)$;

(c) Both g and g^* satisfy the properties (D1)–(D3) in Definition 1.

Proof. (a) The proof is straightforward.

(b) Case 1. Assume that $x_\alpha \in B$. Then let us consider two subcases below.

Subcase 1. If $x_\alpha \in A$, then $f(x_\alpha) = \inf\{r \in Q_{[0,1]} \mid x_\alpha \in U_r\} = 0$.

(1) Assume that $y_\beta \in B$. (i) For the case of $y_\beta \in A$, we have $f(y_\beta) = 0$, and then $g(x_\alpha, y_\beta) = 0$. On the other hand, according to $x_\alpha \in A \Leftrightarrow x_{1-\alpha} \not\leq A'$ and $y_\beta \in A \Leftrightarrow y_{1-\beta} \not\leq A'$, we have $f^*(x_{1-\alpha}) = 1$ and $f^*(y_{1-\beta}) = 1$. Hence $g^*(y_{1-\beta}, x_{1-\alpha}) = 0$; (ii) For the case of $y_\beta \notin A$, let $f(y_\beta) = \inf\{r \in Q_{[0,1]} \mid y_\beta \in U_r\} = s$. If $s < 1$, then $g(x_\alpha, y_\beta) = s$ and there exists a monotonically decreasing sequence $S = \{s_n \mid s_n \geq s, y_\beta \in U_{s_n}, n \in \omega\} \subseteq Q_{[0,1]}$ such that $\lim_{n\to\infty} s_n = s$. Therefore, for each $\varepsilon \in Q_{[0,1]}$ with $\varepsilon > 0$ there is a natural number $N(\varepsilon)$ such that $y_{1-\beta} \in U'_{s_n-\varepsilon} = V_{1-s_n+\varepsilon}$ whenever $n > N(\varepsilon)$. Therefore,

$$f^*(y_{1-\beta}) = \inf\{r \in Q_{[0,1]} \mid y_{1-\beta} \leq V_r\} \leq \inf\{1 - s_n + \varepsilon \mid y_{1-\beta} \in V_{1-s_n+\varepsilon}\} \leq 1 - s + \varepsilon,$$

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so that $f^*(y_{1-\beta}) \le 1-s$ by the arbitrariness of ε. In addition, if $y_{1-\beta} \le V_r$, then from the equivalence $y_\beta \in U_{s_n} \Leftrightarrow 1-\beta > V_{1-s_n}(y)$ we have $V_r \ge V_{1-s_n}$, i.e., $r \ge 1-s_n$ for all $s_n \in S$. Thus $f^*(y_{1-\beta}) = \inf\{r \in Q_{[0,1]} \mid y_{1-\beta} \le V_r\} \ge 1-s$. Consequently $f^*(y_{1-\beta}) = 1-s$, and then $g^*(y_{1-\beta}, x_{1-\alpha}) = s$. If $s = 1$, i.e., $f(y_\beta) = 1$, then $g(x_\alpha, y_\beta) = 1$. In addition, by $f(y_\beta) = 1$, we assert that $y_\beta \in U_1$, but $y_\beta \notin U_r$ for all other $r \in Q_{[0,1]}$. Thus when $r \ne 0$, $y_{1-\beta} \le V_r$, and then $f^*(y_{1-\beta}) = 0$. Consequently $g^*(y_{1-\beta}, x_{1-\alpha}) = 1$.

(2) If $y_\beta \notin B$, then $f(y_\beta) = 1$, and thus $g(x_\alpha, y_\beta) = 1$. According to $y_\beta \notin B \Leftrightarrow y_{1-\beta} \le B'$, we assert that $y_{1-\beta} \le V_r$ for all $r \in Q_{[0,1]}$, so that $f^*(y_{1-\beta}) = 0$. Consequently, $g^*(y_{1-\beta}, x_{1-\alpha}) = 1$.

Subcase 2. Let $x_\alpha \notin A$ and let $f(x_\alpha) = t$.

(1) If $y_\beta \in B$, then (i) For the case of $y_\beta \in A$, we have $f(y_\beta) = 0$. So $g(x_\alpha, y_\beta) = 0$. Moreover, from $y_\beta \in A \Leftrightarrow y_{1-\beta} \not\le A'$, we know $f^*(y_{1-\beta}) = 1$, and then $g^*(y_{1-\beta}, x_{1-\alpha}) = 0$; (ii) For the case of $y_\beta \notin A$, let $f(y_\beta) = s$. Then $g(x_\alpha, y_\beta) = \max\{s-t, 0\}$. Similarly, it is true that $f^*(x_{1-\alpha}) = 1-t$ and $f^*(y_{1-\beta}) = 1-s$. Thus, $g^*(y_{1-\beta}, x_{1-\alpha}) = \max\{1-t-(1-s), 0\} = \max\{s-t, 0\}$.

(2) If $y_\beta \notin B$, then $f(y_\beta) = 1$ and $g(x_\alpha, y_\beta) = \max\{1-t, 0\} = 1-t$. In addition, by $f^*(y_{1-\beta}) = 0$ and $f^*(x_{1-\alpha}) = 1-t$, we have $g^*(y_{1-\beta}, x_{1-\alpha}) = 1-t$.

Case 2. Assume that $x_\alpha \notin B$. Then $f(x_\alpha) = 1$ and $f^*(x_{1-\alpha}) = 0$.

(1) Let $y_\beta \in B$. (i) If $y_\beta \in A$, then $f(y_\beta) = 0$, and thus $g(x_\alpha, y_\beta) = 0$. From $f^*(x_{1-\alpha}) = 0$ we have $g^*(y_{1-\beta}, x_{1-\alpha}) = 0$. (ii) If $y_\beta \notin A$, then from $f(x_\alpha) = 1$ we know $g(x_\alpha, y_\beta) = 0$. Meanwhile, because of $f^*(x_{1-\alpha}) = 0$, we can obtain $g^*(y_{1-\beta}, x_{1-\alpha}) = 0$.

(2) Let $y_\beta \notin B$. Then $f(y_\beta) = 1$. Note that $f(x_\alpha) = 1$. So $g(x_\alpha, y_\beta) = 0$. Owing to $f^*(x_{1-\alpha}) = 0$, we know $g^*(y_{1-\beta}, x_{1-\alpha}) = 0$.

(c) (D1) Let $x_\alpha \ge x_\beta$. If $x_\beta \notin B$, then $f(x_\alpha) = f(x_\beta) = 1$. So $g(x_\alpha, x_\beta) = 0$. If $x_\beta \in B$, then when $x_\alpha \notin B$, we have $f(x_\alpha) = 1$, and then $g(x_\alpha, x_\beta) = 0$; when $x_\alpha \in B$, we have $f(x_\alpha) = \inf\{r \in Q_{[0,1]} \mid x_\alpha \in U_r\} \ge \inf\{r \in Q_{[0,1]} \mid x_\beta \in U_r\} = f(x_\beta)$, and then $g(x_\alpha, x_\beta) = 0$. Besides, by $g^*(y_{1-\beta}, x_{1-\alpha}) = g(x_\alpha, y_\beta)$, it is easy to show that g^* also satisfies (D1).

(D2) To check $g(x_\alpha, y_\beta) + g(y_\beta, z_\gamma) \ge g(x_\alpha, z_\gamma)$ for any $x_\alpha, y_\beta, z_\gamma \in M_0$, we consider the following two cases: (a) when $g(x_\alpha, z_\gamma) = 0$, this conclusion is straightforward; (b) when $g(x_\alpha, z_\gamma) \ne 0$, we have $f(z_\gamma) > f(x_\alpha)$. In this case, (i) if $g(y_\beta, z_\gamma) = 0$, then $f(z_\gamma) \le f(y_\beta)$, and thus $g(x_\alpha, y_\beta) = f(y_\beta) - f(x_\alpha) \ge f(z_\gamma) - f(x_\alpha) = g(x_\alpha, z_\gamma)$; (ii) if $g(x_\alpha, y_\beta) = 0$, then $f(y_\beta) \le f(x_\alpha)$, and then $f(z_\gamma) - f(y_\beta) \ge f(z_\gamma) - f(x_\alpha)$. Therefore, $g(y_\beta, z_\gamma) \ge g(x_\alpha, z_\gamma)$; (iii) if $g(y_\beta, z_\gamma) \ne 0$ and $g(x_\alpha, y_\beta) \ne 0$, then $f(z_\gamma) > f(y_\beta)$ and $f(y_\beta) > f(x_\alpha)$, and then $g(x_\alpha, y_\beta) + g(y_\beta, z_\gamma) = [f(y_\beta) - f(x_\alpha)] + [f(z_\gamma) - f(y_\beta)] = f(z_\gamma) - f(x_\alpha) = g(x_\alpha, z_\gamma)$. Besides, by $g^*(y_{1-\beta}, x_{1-\alpha}) = g(x_\alpha, y_\beta)$, it is easy to see that g^* also satisfies (D2).

(D3) (1) Assume that $f(y_\beta) = 1$. (i) If $y_\beta \in B$, then besides $r = 1$, $y_\beta \notin U_r$ for all other $r \in Q_{[0,1]}$. Thus, for each η with $\beta < \eta < B(y)$ we assert that besides $r = 1$, $y_\eta \notin U_r$ for all other $r \in Q_{[0,1]}$. Consequently $f(y_\eta) = 1$. It follows that $g(x_\alpha, y_\eta) = \max\{1 - f(x_\alpha), 0\} = g(x_\alpha, y_\beta)$; (ii) if $y_\beta \notin B$, then for each γ with $\beta < \gamma < 1$ we have $y_\gamma \notin B$, and then $f(y_\gamma) = 1$. Therefore, $g(x_\alpha, y_\gamma) = \max\{1 - f(x_\alpha), 0\} = g(x_\alpha, y_\beta)$.

(2) Assume that $0 \le f(y_\beta) = p < 1$. (i) If there is a fixed number $h > 0$ such that $y_{\beta+h} \le U_q$ for each q with $q \in (p, 1] \cap Q_{[0,1]}$, and then $f(y_\gamma) = p$ for each γ with $\beta < \gamma < \beta + h$. Therefore, $g(x_\alpha, y_\gamma) = \max\{p - f(x_\alpha), 0\} = g(x_\alpha, y_\beta)$. (ii) If such a fixed h is non-existent, then by $f(y_\beta) = \inf\{r \in Q_{[0,1]} \mid y_\beta \in U_r\} = p$ we assert that for any $\varepsilon > 0$ there exists a r belonging to $Q_{[0,1]}$ such that $y_\beta \in U_r$ and $r - p < \varepsilon$. Take a number γ satisfying $\beta < \gamma < \beta + \frac{r-p}{2}$. Then $f(y_\gamma) < p + \varepsilon$. Let $\bigwedge_{\gamma > \beta} f(y_\gamma) = q$. Clearly, $p \le q$. Thus for any $\varepsilon > 0$ we have $q = \bigwedge_{\gamma > \beta} f(y_\gamma) < p + \varepsilon$. Because ε is arbitrary, we can obtain $q \le p$. Thus $f(y_\beta) = \bigwedge_{\gamma > \beta} f(y_\gamma)$. Consequently, $g(x_\alpha, y_\beta) = \max\{p - f(x_\alpha), 0\} = \max\{\bigwedge_{\gamma > \beta} f(y_\gamma) - f(x_\alpha), 0\}$. (i) If $p - f(x_\alpha) < 0$, then $g(x_\alpha, y_\beta) = 0$ and there exists a γ satisfying $\gamma > \beta$ such that $f(y_\gamma) - f(x_\alpha) < 0$. Meanwhile, $\bigwedge_{\gamma > \beta} g(x_\alpha, y_\gamma) = \bigwedge_{\gamma > \beta} \{\max\{f(y_\gamma) - f(x_\alpha), 0\}\} = 0$.

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Hence $g(x_\alpha, y_\beta) = \bigwedge_{\gamma>\beta} g(x_\alpha, y_\gamma)$. (ii) If $p - f(x_\alpha) = 0$, then this means that for any $\varepsilon > 0$ there exists a γ satisfying $\gamma > \beta$ such that $f(y_\gamma) - f(x_\alpha) < \varepsilon$. Therefore, $\bigwedge_{\gamma>\beta} g(x_\alpha, y_\gamma) = \bigwedge_{\gamma>\beta} \{\max\{f(y_\gamma) - f(x_\alpha), 0\}\} < \varepsilon$, so that $\bigwedge_{\gamma>\beta} g(x_\alpha, y_\gamma) = 0$. (iii) If $p - f(x_\alpha) > 0$, then $f(y_\gamma) - f(x_\alpha) > 0$ for each $\gamma > \beta$. It follows that $\bigwedge_{\gamma>\beta} g(x_\alpha, y_\gamma) = \bigwedge_{\gamma>\beta} \{\max\{f(y_\gamma) - f(x_\alpha), 0\}\} = \bigwedge_{\gamma>\beta} \{f(y_\gamma) - f(x_\alpha)\} = p - f(x_\alpha)$. In summary, g satisfies (D3). Similarly, so does g^*. □

7. Metrization theorem

8. Metrization theorem

For a $[0,1]$-topological space (X, δ), if there is a Deng pseudo-metric (resp., Deng metric) p on I^X such that $\delta = \delta_p$, where δ_p is the pseudo-metric topology, then the space is said to be *Deng pseudo-metrizable* (resp., *Deng metrizable*). Similar treatment to Erceg metric, Chen metric, and Yang-Shi metric.

Theorem 23. *If a $[0,1]$-topological space is regular, and the topology has a σ-locally finite base, then it is Deng pseudo-metrizable.*

Proof. The sketch of proof is: Firstly, by Theorem 22 and the property of σ-locally finite base, we will generate a countable family of Deng pseudo-metric spaces $\{(X_n, \delta_{p_n}) \mid n \in \omega\}$. Secondly, by Theorem 16, we will construct a Deng pseudo-metric on $I^{\mathcal{X}}$ and prove that the space $(\mathcal{X}, \delta_p)$ is exactly the product space of the family $\{(X_n, \delta_{p_n}) \mid n \in \omega\}$. Finally, we will deduce that (X, δ) can be embedded into $(\mathcal{X}, \delta_p)$.

Now let us complete the proof step by step. First, for each $n \in \omega$ let $\sigma^n = \{A_i^n \mid i \in \Gamma_n\}$ be locally finite in (X, δ) and let the union $\sigma = \bigcup\{\sigma^n \mid n \in \omega\}$ be a base of (X, δ).

Arbitrarily select a pair of positive integers m, n. Let $A_i^n \in \sigma^n$ and let it be fixed for the moment. We consider the following open set

$$A_i^{m,n} = \bigvee\{A^m \mid A^m \in \sigma^m, \overline{A^m} \leq A_i^n\}.$$

For convenience, we denote $A_i^{m,n}$ as A_i. Because δ_m is locally finite, by Theorem 4 we have $\overline{A_i} \leq A_i^n$. Next, the proof shall be divided into the following five steps.

Step 1. By Theorem 22 there exists a family $\{U_r \mid r \in Q_{[0,1]}\}$ corresponding to $\overline{A_i}$ and A_i^n. Therefore, we can define a mapping $f_i : M_0 \to [0,1]$ by

$$f_i(x_\alpha) = \begin{cases} \inf\{r \in Q_{[0,1]} \mid x_\alpha \in U_r\}, & x_\alpha \in A_i^n; \\ 1, & x_\alpha \notin A_i^n. \end{cases}$$

For any $x_\alpha, y_\beta \in M_0$ let $g_i(x_\alpha, y_\beta) = \max\{f_i(y_\beta) - f_i(x_\alpha), 0\}$ and for each $r \in Q_{[0,1]}$ let $V_r = U'_{1-r}$. Then there exists a family $\{V_r \mid r \in Q_{[0,1]}\}$. Therefore, we can define a mapping f_i^* by

$$f_i^*(x_\alpha) = \begin{cases} \inf\{r \in Q_{[0,1]} \mid x_\alpha \leq V_r\}, & x_\alpha \leq (\overline{A_i})'; \\ 1, & x_\alpha \not\leq (\overline{A_i})'. \end{cases}$$

Let $g_i^*(x_\alpha, y_\beta) = \max\{f_i^*(y_\beta) - f_i^*(x_\alpha), 0\}$. Then by Theorem 22, we have $g^*(y_{1-\beta}, x_{1-\alpha}) = g(x_\alpha, y_\beta)$.

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Step 2. Let $p_i(x_\alpha, y_\beta) = [g_i(x_\alpha, y_\beta) + g_i^*(x_\alpha, y_\beta)]/2$. Then p_i is a Deng pseudo-metric on I^X. This is because both g_i and g_i^* satisfy the properties (D1)-(D3), and so does p_i. Besides, by Theorem 22, we have the following two equalities:

$$p_i(x_\alpha, y_\beta) = \frac{g_i(x_\alpha, y_\beta) + g_i^*(x_\alpha, y_\beta)}{2} = \frac{g_i(x_\alpha, y_\beta) + g_i(y_{1-\beta}, x_{1-\alpha})}{2};$$

$$p_i(y_{1-\beta}, x_{1-\alpha}) = \frac{g_i(y_{1-\beta}, x_{1-\alpha}) + g_i^*(y_{1-\beta}, x_{1-\alpha})}{2} = \frac{g_i(y_{1-\beta}, x_{1-\alpha}) + g_i(x_\alpha, y_\beta)}{2}.$$

Therefore, p_i satisfies (D4).

Step 3. Since σ^n is locally finite, for any $x_{1-\alpha}$, $y_{1-\beta} \in M_0$ there are two open sets: an open neighborhood $U_{x_{1-\alpha}}$ of $x_{1-\alpha}$ and an open neighborhood $U_{y_{1-\beta}}$ of $y_{1-\beta}$ such that they are quasi-coincident with only a finite family $\{C_p \mid p = p_1, p_2, \cdots, p_l\} \subseteq \sigma^n$ and a finite set $\{B_j \mid j = j_1, j_2, \cdots, j_m\} \subseteq \sigma^n$, respectively. Let

$$\{A_i^n \mid i = k_1, k_2, \cdots, k_q\} = \{C_p \mid p = p_1, p_2, \cdots, p_l\} \cup \{B_j \mid j = j_1, j_2, \cdots, j_m\}.$$

Therefore, either $U_{x_{1-\alpha}}(z_i) + A_i^n(z_i) > 1$ or $U_{y_{1-\beta}}(z_i) + A_i^n(z_i) > 1$ is true for each i ($i = k_1, k_2, \cdots, k_q$) and corresponding $z_i \in X$. This implies that there exists at most a finite family $\{A_i^n \mid i = k_1, k_2, \cdots, k_q\}$ such that $x_\alpha \in A_i^n$ or $y_\beta \in A_i^n$. It follows that $f_i(x_\alpha) \leq 1$ or $f_i(y_\beta) \leq 1$ ($i = k_1, k_2, \cdots, k_q$). On the other hand, when $k \in \Gamma_n$ and $k \neq i$ ($i = k_1, k_2, \cdots, k_q$), it is easy to show that $x_\alpha \notin A_k^n$ and $y_\beta \notin A_k^n$, and then $f_k(x_\alpha) = 1$ and $f_k(y_\beta) = 1$. Hence $g_k(x_\alpha, y_\beta) = 0$. In other words, when $i \in \Gamma_n$, it may be correct that $g_i(x_\alpha, y_\beta) \neq 0$ only if i belongs to the finite index set $\{k_1, k_2, \cdots, k_q\}$.

Similarly, for any $x_\alpha, y_\beta \in M_0$, there exists at most a finite family $\{A_i^n \mid i = q_1, q_2, \cdots, q_m\}$ such that $x_\alpha \not\leq (A_i^n)'$ or $y_\beta \not\leq (A_i^n)'$. It follows that $f_i^*(x_\alpha) \neq 0$ or $f_i^*(y_\beta) \neq 0$ holds ($i = q_1, q_2, \cdots, q_m$). When $k \in \Gamma_n$ and $k \neq i$ ($i = q_1, q_2, \cdots, q_m$), we have $x_\alpha \leq (A_i^n)'$ and $y_\beta \leq (A_i^n)'$, and then $f_k^*(x_\alpha) = 0$ and $f_k^*(y_\beta) = 0$. Therefore, $g_k^*(x_\alpha, y_\beta) = 0$. In other words, when $i \in \Gamma_n$, it may be correct that $g_i^*(x_\alpha, y_\beta) \neq 0$ only if i belongs to the finite index set $\{q_1, q_2, \cdots, q_m\}$.

Let $J = \{k_1, k_2, \cdots, k_q\} \cup \{q_1, q_2, \cdots, q_m\}$. Then for any $x_\alpha, y_\beta \in M_0$, there exists at most a finite index set J such that when $i \in J$, it may be correct that $p_i(x_\alpha, y_\beta) \neq 0$. Therefore, for the two positive integers m, n, we can define a mapping $p_{m,n} : M_0 \times M_0 \to [0, +\infty)$ by

$$p_{m,n}(x_\alpha, y_\beta) = \sum\{p_i(x_\alpha, y_\beta) \mid A_i^n \in \sigma^n, i \in J\}.$$

Next, we will prove that each $p_{m,n}$ is also a Deng pseudo-metric on I^X. The proof is as follows:
(D1) Because each $p_i (i \in J)$ satisfies (D1), when $x_\alpha \geq y_\beta$, we have

$$p_{m,n}(x_\alpha, y_\beta) = \sum\{p_i(x_\alpha, y_\beta) \mid A_i^n \in \sigma^n, i \in J\} = 0.$$

(D2) Let $x_\alpha, y_\beta, z_\gamma \in M_0$. Because p_i satisfies (D2) for each i, we have

$$\begin{aligned}
&p_{m,n}(x_\alpha, y_\beta) + p_{m,n}(y_\beta, z_\gamma)\\
&= \sum\{p_i(x_\alpha, y_\beta) \mid A_i^n \in \sigma^n, i \in J\} + \sum\{p_i(y_\beta, z_\gamma) \mid A_i^n \in \sigma^n, i \in J\}\\
&= \sum\{p_i(x_\alpha, y_\beta) + p_i(y_\beta, z_\gamma) \mid A_i^n \in \sigma^n, i \in J\}\\
&\geq \sum\{p_i(x_\alpha, z_\gamma) \mid A_i^n \in \sigma^n, i \in J\}\\
&= p_{m,n}(x_\alpha, z_\gamma).
\end{aligned}$$

(D3) Because g_i and g_i^* satisfy (D3), i.e., we have the following two equalities:

$$\sum\{g_i(x_\alpha, y_\beta) \mid A_i^n \in \sigma^n, i \in J\} = \sum\{\bigwedge_{\gamma > \beta} g_i(x_\alpha, y_\gamma) \mid A_i^n \in \sigma^n, i \in J\},$$

$$\sum\{g_i^*(x_\alpha, y_\beta) \mid A_i^n \in \sigma^n, i \in J\} = \sum\{\bigwedge_{\gamma > \beta} g_i^*(x_\alpha, y_\gamma) \mid A_i^n \in \sigma^n, i \in J\}.$$

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Note that the above two formulas are finite sums. Therefore,

$$
\begin{aligned}
& p_{m,n}(x_\alpha, y_\beta) \\
&= \sum\{p_i(x_\alpha, y_\beta) \mid A_i^n \in \sigma^n, i \in J\} \\
&= \sum\{[g_i(x_\alpha, y_\beta) + g_i^*(x_\alpha, y_\beta)]/2 \mid A_i^n \in \sigma^n, i \in J\} \\
&= \sum\{g_i(x_\alpha, y_\beta)/2 \mid A_i^n \in \sigma^n, i \in J\} + \sum\{g_i^*(x_\alpha, y_\beta)/2 \mid A_i^n \in \sigma^n, i \in J\} \\
&= \sum\{\bigwedge_{\gamma>\beta} g_i(x_\alpha, y_\gamma)/2 \mid A_i^n \in \sigma^n, i \in J\} + \sum\{\bigwedge_{\gamma>\beta} g_i^*(x_\alpha, y_\gamma))/2 \mid A_i^n \in \sigma^n, i \in J\} \\
&= \sum\{(\bigwedge_{\gamma>\beta} g_i(x_\alpha, y_\gamma) + g_i^*(x_\alpha, y_\gamma))/2 \mid A_i^n \in \sigma^n, i \in J\} \\
&= \bigwedge_{\gamma>\beta} \sum\{(g_i(x_\alpha, y_\gamma) + g_i^*(x_\alpha, y_\gamma))/2 \mid A_i^n \in \sigma^n, i \in J\} \\
&= \bigwedge_{\gamma>\beta} p_{m,n}(x_\alpha, y_\gamma).
\end{aligned}
$$

(D4) Because p_i satisfies (D4) for each $i \in J$, we have the following equalities:

$$
\begin{aligned}
& p_{m,n}(x_\alpha, y_\beta) \\
&= \sum\{p_i(x_\alpha, y_\beta) \mid A_i^n \in \sigma^n, i \in J\} \\
&= \sum\{p_i(y_{1-\beta}, x_{1-\alpha}) \mid A_i^n \in \sigma^n, i \in J\} \\
&= p_{m,n}(y_{1-\beta}, x_{1-\alpha}).
\end{aligned}
$$

Therefore, $\{p_{m,n} \mid m \in \omega, n \in \omega\}$ is a countable family of Deng pseudo-metrics. Meanwhile, we denote the topology generated by $p_{m,n}$ as $\delta_{p_{m,n}}$.

Step 4. We will prove that $\bigcup \delta_{p_{m,n}}$ is a base of (X, δ). For this purpose, we only need to prove the following (a) and (b).

(a) $\delta_{p_{m,n}} \subseteq \delta \ (m, n \in \omega)$.

By Theorem 1, it is sufficient to find an open set $V_{x_\alpha} \in \delta$ such that $x_\alpha \in V_{x_\alpha} \le U_\varepsilon(x_\alpha)$ for any open set $U_\varepsilon(x_\alpha) \in \delta_{p_{m,n}}$.

Since σ^n is locally finite, there is an open neighborhood $U_{x_{1-\alpha}}$ of $x_{1-\alpha}$ which is only quasi-coincident with finitely many members: $\{A_{i_l}^n \mid l = 1, 2, \cdots, k\} \subseteq \sigma^n$. Therefore,

$$f_{i_1}(x_\alpha) \le 1, \cdots, f_{i_k}(x_\alpha) \le 1 \tag{1}$$

Since $f_{i_l}(x_\alpha) = \inf\{r_{i_l} \in Q_{[0,1]} \mid x_\alpha \in U_{r_{i_l}}\} = t_{i_l}$ $(l = 1, 2, \cdots, k)$, we may select an open set $U_{r_{i_l}}$ with $x_\alpha \in U_{r_{i_l}}$ such that $r_{i_l} - t_{i_l} < \frac{\varepsilon}{2k}$. Therefore, when $y_\beta \in U_{r_{i_l}}$, we have $f_{i_l}(y_\beta) \le r_{i_l}$ $(l = 1, 2, \cdots k)$. Thus

$$g_{i_l}(x_\alpha, y_\beta) = \max\{f_{i_l}(y_\beta) - f_{i_l}(x_\alpha), 0\} \le \max\{r_{i_l} - t_{i_l}, 0\} < \frac{\varepsilon}{2k}, l = 1, \cdots, k.$$

When $A_{i_m}^n \in \sigma^n$ with $m \ne 1, 2, \cdots, k$, it must hold $f_{i_m}(x_\alpha) = 1$, and then $g_{i_m}(x_\alpha, y_\beta) = \max\{f_{i_m}(y_\beta) - f_{i_m}(x_\alpha), 0\} = 0$.

Similarly, there is an open neighborhood U_{x_α} of x_α which is only quasi-coincident with finitely many members: $\{A_{j_t}^n \mid t = 1, 2, \cdots, p\} \subseteq \sigma^n$. This implies that $x_\alpha \not\le (A_{j_t}^n)'$ for each j_t $(t = 1, 2, \cdots, p)$, so that

$$f_{j_1}^*(x_\alpha) > 0, \cdots, f_{j_p}^*(x_\alpha) > 0 \tag{2}$$

Let $f_{j_t}^*(x_\alpha) = \inf\{r_{jt} \in Q_{[0,1]} \mid x_\alpha \le V_{j_t}\} = s_{j_t}$ $(t = 1, 2, \cdots, p)$. Then we can select an open set $V_{r_{j_t}}$ with $x_\alpha \le V_{r_{j_t}}$ $(t = 1, 2, \cdots, p)$ such that $r_{j_t} - s_{j_t} < \frac{\varepsilon}{4p}$. Take a number h_{j_t} of $Q_{[0,1]}$ satisfying $r_{j_t} < h_{j_t} < r_{j_t} + \frac{\varepsilon}{4p}$ such that $x_\alpha \le V_{h_{j_t}}$ $(t = 1, 2, \cdots, p)$. Hence $h_{j_t} - s_{j_t} < \frac{\varepsilon}{2p}$ $(t = 1, 2, \cdots, p)$. Because $r_{j_t} < h_{jt}$, it is true that $U_{1-h_{j_t}} \le U_{1-r_{j_t}}$. Therefore, by the property of $\{U_r \mid r \in Q_{[0,1]}\}$

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we have $\overline{U}_{1-h_{jt}} \leq U_{1-r_{jt}}$, i.e., $U'_{1-r_{jt}} \leq (\overline{U}_{1-h_{jt}})'$. Furthermore, because $U_{1-h_{jt}} \leq \overline{U}_{1-h_{jt}}$, it is true that $(\overline{U}_{1-h_{jt}})' \leq U'_{1-h_{jt}}$. Hence $x_\alpha \leq V_{r_{jt}} \leq (\overline{U}_{1-h_{jt}})' \leq V_{h_{jt}}$. Let $O_{h_{j_t}} = V^\circ_{h_{j_t}}, t = 1, 2, \cdots, p$. Then $x_\alpha \leq (\overline{U}_{1-h_{jt}})' \leq O_{h_{j_t}}$. If $z_\gamma \in O_{h_{j_t}} \leq V_{h_{j_t}}, t = 1, 2, \cdots p$, then $f^*_{j_t}(z_\gamma) \leq h_{j_t}$. Hence

$$g^*_{j_t}(x_\alpha, z_\gamma) = \max\{f^*_{j_t}(z_\gamma) - f^*_{j_t}(x_\alpha), 0\} \leq \max\{h_{j_t} - s_{j_t}, 0\} < \frac{\varepsilon}{2p}, t = 1, \cdots, p.$$

Now, let

$$V_{x_\alpha} = U_{x_\alpha} \wedge U_{r_{i_1}} \wedge \cdots \wedge U_{r_{i_k}} \wedge O_{h_{j_1}} \wedge \cdots \wedge O_{h_{jp}} \tag{3}$$

Since U_{x_α} and $A^n_{j_s}$ are not quasi-coincident for each j_s ($s \neq 1, 2, \cdots, p$), when $y_\beta \in V_{x_\alpha}$, we have $y_\beta \leq (A^n_{j_s})'$ and then $f^*_{j_s}(y_\beta) = 0$ ($s \neq 1, 2, \cdots, p$). And consequently $g^*_{j_s}(x_\alpha, y_\beta) = 0$ ($s \neq 1, 2, \cdots, p$). If $k \leq p$, then

$$p_{m,n}(x_\alpha, y_\beta) \leq [\frac{(\frac{\varepsilon}{2k} + \frac{\varepsilon}{2p})}{2}] \times k + \frac{0 + \frac{\varepsilon}{2p}}{2} \times (p - k) = \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$

If $k > p$, then

$$p_{m,n}(x_\alpha, y_\beta) \leq [\frac{(\frac{\varepsilon}{2k} + \frac{\varepsilon}{2p})}{2}] \times p + \frac{\varepsilon}{2k} \times (k - p) = \frac{\varepsilon}{4} + \frac{\varepsilon}{4k}(2k - p) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon.$$

In either case $x_\alpha \in V_{x_\alpha} \leq U_\varepsilon(x_\alpha)$. Therefore, (a) is proved.

Incidentally, to make the above proof more perfect, we add the following two points. If $f_{i_l}(x_\alpha) = 1$ for all i_l ($l = 1, \cdots, k$), then $g_{i_l}(x_\alpha, y_\beta) = 0$. Meanwhile, let us consider two more special cases below.

(i) If there exists a nonempty set $\Lambda = \{j_{w_1}, \cdots, j_{w_q}\} \subseteq \{j_1, \cdots, j_p\}$ such that each element $j_{w_i} \in \Lambda$ satisfies $0 < f^*_{j_w}(x_\alpha) < 1$, then

$$V_{x_\alpha} = U_{x_\alpha} \wedge O_{j_{w_1}} \wedge \cdots \wedge O_{j_{w_q}}.$$

(ii) If $f^*_{j_t}(x_\alpha) = 1$ for all j_t ($t = 1, \cdots, p$), then $g^*_{j_s}(x_\alpha, y_\beta) = 0$. Let $V_{x_\alpha} = U_{x_\alpha}$. Then for any $y_\beta \in U_{x_\alpha}$ we have $p_{m,n}(x_\alpha, y_\beta) = 0$, and thus $y_\beta \in U_\varepsilon(x_\alpha)$, i.e., $x_\alpha \in V_{x_\alpha} \leq U_\varepsilon(x_\alpha)$.

(b) Each member in δ is the union of some members in $\cup\delta_{p_{m,n}}$.

Let $x_\alpha \in B \in \delta$. Because (X, δ) is regular, there exists an open set v belonging to σ such that $x_\alpha \in v \leq \overline{v} \leq B$. Therefore, it is easy to show that there is a natural member n such that $v \in \sigma^n \subseteq \sigma$. For convenience, we denote v as A^n_i. Similarly, for $x_\alpha \in A^n_i$, there are another natural member m and an open set A^m_j belonging to σ^m such that

$$x_\alpha \in A^m_j \leq \overline{A^m_j} \leq A^n_i \leq \overline{A^n_i} \leq B.$$

Let $A_i = \bigvee\{A^m \mid A^m \in \sigma^m, \overline{A^m} \leq A^n_i\}$. Clearly, $x_\alpha \in A^m_j \leq A_i \leq \overline{A_i} \leq A^n_i \leq \overline{A^n_i} \leq B$. Therefore, by Theorem 22 there exists a corresponding family $\{U_r \mid r \in Q_{[0,1]}\}$ relative to $\overline{A_i}$ and B such that $x_\alpha \in U_r$ for all U_r ($r \in Q_{[0,1]}$). And consequently $f_i(x_\alpha) = 0$. If $y_\beta \notin B$, then $y_\beta \notin A^n_i$. Thus $f_i(y_\beta) = 1$, and then $g_i(x_\alpha, y_\beta) = \max\{f_i(y_\beta) - f_i(x_\alpha), 0\} = 1$. Therefore, we assert that $p_{m,n}(x_\alpha, y_\beta) \geq \frac{1}{2}$. In other words, as long as $p_{m,n}(x_\alpha, y_\beta) < \frac{1}{2}$, it must hold that $y_\beta \in B$. This implies that for each $x_\alpha \in B$ there exists $U_{\frac{1}{2}}(x_\alpha)$ belonging to $\delta_{p_{m,n}}$ such that $U_{\frac{1}{2}}(x_\alpha) \leq B$. Thus $B = \bigvee_{x_\alpha \in B} U_{\frac{1}{2}}(x_\alpha)$. That is to say, if $B \in \delta$, then there is $D \subseteq \bigcup_{n,m \in \omega} \delta_{p_{m,n}}$ such that $B = \bigvee D$. Therefore, (b) is proved.

Step 5. Based on the discussions above, we renumber the countable set $\{p_{m,n} \mid m = 1, 2, \cdots, n = 1, 2, \cdots\}$ as $\{p_n \mid n \in \omega\}$. Let $\mathcal{X} = \prod_{n \in \omega, X_n = X} X_n$. By Theorem 4.2, we define a mapping $p : M_0 \times M_0 \to [0, 1]$ by

$$p(x_\alpha, y_\beta) = \sum_{n\in\omega} 2^{-n} p_n(J_n(x_\alpha), J_n(y_\beta)),$$

where $J_n : I^{\mathcal{X}} \to I^X$ is the n-th projection, and affirm that p is a Deng pseudo-metric on $I^{\mathcal{X}}$ and $(\mathcal{X}, \delta_p)$ is the product space of $\{(X, \delta_{p_n}) \mid n \in \omega\}$, where $(\mathcal{X}, \delta_p)$ is generated by $\Gamma_p = \{J_n^{-1}(U) \mid U \in \delta_{p_n}, n \in \omega\}$ as a subbase. Now let us prove that (X, δ) can be embedded into $(\mathcal{X}, \delta_p)$.

Let $x^\omega = (x, x, \cdots, x, \cdots)$ and denote x^ω_α as the fuzzy point whose support and value are x^ω and $\alpha \in (0, 1)$, respectively. All these fuzzy points are denoted by $M_1 = \{x^\omega_\alpha \mid x \in X, \alpha \in (0, 1)\}$. Let $\mathbb{X} = \{x^\omega \mid x \in X\}$. A mapping $e : M_1 \to M_0$ is defined by $e(x^\omega_\alpha) = x_\alpha$. Obviously, e is a bijection and its inverse mapping e^{-1} embeds M_0 into $I^{\mathcal{X}}$. Let $p_e = p \mid M_1 \times M_1$. Consequently, we regenerate a new mapping $p_e : M_1 \times M_1 \to [0, 1]$. It is easy to prove that p_e is a Deng pseudo-metric on $I^{\mathbb{X}}$, and $(\mathbb{X}, \delta_{p_e})$ is a subspace of $(\mathcal{X}, \delta_p)$. Because Γ_p is a subbase of $(\mathcal{X}, \delta_p)$, $\Gamma_p \mid \mathbb{X}$ is certainly a subbase of $(\mathbb{X}, \delta_{p_e})$. Moreover, because of Step 4, $\Gamma_p \mid \mathbb{X}$ is exactly a base of $(\mathbb{X}, \delta_{p_e})$. Hence (X, δ) and $(\mathbb{X}, \delta_{p_e})$ are homeomorphic. In fact, let $p_{e_1}(x_\alpha, y_\beta) = p_e(x^\omega_\alpha, y^\omega_\beta)$ for any $x^\omega_\alpha, y^\omega_\beta \in M_1$. Then (X, δ) can be embedded into $(\mathcal{X}, \delta_p)$ and the mapping p_{e_1} is a Deng pseudo-metric on I^X which metricizes the $[0, 1]$-topological space (X, δ). Consequently, $\delta = \delta_{p_{e_1}}$. In summary, the proof has been completed. □

Theorem 24. *A $[0, 1]$-topological space is Deng metrizable if and only if it is T_1 and Deng pseudo-metrizable.*

Proof. (*Sufficiency*). Let p be a Deng metric on I^X and let $y_{\lambda_2} \le \bigwedge_{r>0} B_r(x_{\lambda_1})$ (see Theorem 7 on B_r). Then for any $r > 0$ we have $y_{\lambda_2} \le B_r(x_{\lambda_1})$ and then $p(x_{\lambda_1}, y_{\lambda_2}) \le r$, so that $p(x_{\lambda_1}, y_{\lambda_2}) = 0$. By (D5) we can obtain $x = y$ and $\lambda_1 \ge \lambda_2$. Hence $\bigwedge_{r>0} B_r(x_{\lambda_1}) = x_{\lambda_1}$. Besides, by Theorem 7 we can assert that $B_r(x_{\lambda_1})$ $(r > 0)$ is a closed set. Thus, $\overline{\bigwedge_{r>0} B_r(x_{\lambda_1})} = \bigwedge_{r>0} B_r(x_{\lambda_1})$. Consequently, $\overline{x}_{\lambda_1} = x_{\lambda_1}$, as desired.

(*Necessity*). If $p(x_{\lambda_1}, y_{\lambda_2}) = 0$, then $p(y_{1-\lambda_2}, x_{1-\lambda_1}) = 0$. For any $r > 0$, by (D3) we can take a number $1 - \lambda_r > 1 - \lambda_1$ such that $p(y_{1-\lambda_2}, x_{1-\lambda_r}) < r$, and then $x_{1-\lambda_1} < x_{1-\lambda_r} \le U_r(y_{1-\lambda_2})$, i.e., $x_{\lambda_1} + U_r(y_{1-\lambda_2}) > 1$. Consequently $y_{\lambda_2} \le \overline{x}_{\lambda_1} = x_{\lambda_1}$. Therefore, $x = y$ and $\lambda_1 \ge \lambda_2$, so that p satisfies (D5). □

Because of the conclusion $D = C \bigcap Y \bigcap E$ in Theorem 8, Theorem 22 and Theorem 23, we can obtain the main result in this paper as follow

Corollary 1. *If a $[0, 1]$-topological space (X, δ) is T_1 and regular, and δ has a σ-locally finite base, then it is Deng, Erceg, Chen, and Yang-Shi metrizable.*

9. Conclusions

In this paper, we study the metrization problem: whether there is a metric such that a given $[0, 1]$-topology coincides with the metric topology. Eventually, we obtain a desired result:

Metrization Theorem. *If a $[0, 1]$-topological space (X, δ) is T_1 and regular, and δ has a σ-locally finite base, then it is Deng, Erceg, Chen, and Yang-Shi metrizable.*

Based on the result, we can conclude that Deng's, Liang's, and Yang's metric results appeared in Introduction (refer to [12,23,28] for details) are all special cases of our conclusion. This is because if (X, δ) is C_{II}, then δ must have a σ-locally finite base, but the converse is not true. Therefore, Corollary 1 proved by us is the most satisfactory solution to metrization problem in $[0, 1]$-topology so far.

In the future, we will consider whether or not our results can be generalized to L-topology [8,14]. In addition, we will further investigate the Erceg metric, the Yang-Shi metric, the Deng metric, and the Chen metric. Besides, we will continue to research the kind of lattice-valued topological spaces each of whose topologies has a σ-locally finite base. Beyond that, we will also intend to inquire into some questions on the fuzzifying metric topology (see [15,31,37,43]).

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