Communication

Not peer-reviewed version

Non-Linear Coordinate Transformations at Homogeneous Electron Gases

Matthew James Stephenson *

Posted Date: 22 February 2024

doi: 10.20944/preprints202402.1290.v1

Keywords: Hydrodynamical limit; Homogeneous electron gases; Non-linear coordinate transformations; Infrared limit; Phenomenological hydrodynamics; Normal modes; Harmonic system

Article

Non-Linear Coordinate Transformations at Homogeneous Electron Gases

Matthew James Stephenson

University of Alabama at Huntsville; mjs0016@uah.edu

Abstract: We explores the hydrodynamical limit of a homogeneous electron gas, considering the interplay between the infrared limit and phenomenological hydrodynamics. Our study involves interactions generated by non-linear coordinate transformations. The focus begins with an analysis of non-linear coordinate transformations and their impact on normal modes in a harmonic system. The Lagrangian for a harmonic oscillator is used as an illustrative example, highlighting the role of position-dependent effective mass in preventing certain coordinate values. Moving beyond single degrees of freedom, we extend the discussion to a system with several harmonic degrees of freedom, introducing a collective variable. The effective dynamics of this collective variable are defined through a Lagrange multiplier term, emphasizing linearity in the description. Our study extends to the Current Two-Point function (CTP) formalism, introducing matrices and their inverses, such as $\hat{D}$ and $\hat{D}^{-1}$. Further, we delve into the dynamics of current in a homogeneous electron gas at finite density and zero temperature. Green's functions are generated using a functional integral approach, and the effective action is constructed via functional Legendre transformation. The quadratic part of the generator functional is expanded to derive the effective action, leading to the Maxwell equation and the subsequent elimination of the photon field. The resulting effective action for current dynamics is expressed in terms of the Current Two-Point function, paving the way for further exploration of the hydrodynamical aspects of the electron gas. Our study combines theoretical developments and mathematical formalism to provide insights into the hydrodynamical behavior of homogeneous electron gases.

Keywords: hydrodynamical limit; homogeneous electron gases; non-linear coordinate transformations; infrared limit; phenomenological hydrodynamics; normal modes; harmonic system

---

1. Introduction

The behavior of electrons in homogeneous electron gases represents a fundamental aspect of condensed matter physics. We dive into the intricate dynamics of electrons, employing a novel perspective through non-linear coordinate transformations. Understanding electron motion in these systems is critical for advancing our knowledge of material properties and their applications in various fields. By exploring the nuances of electron dynamics, our study aims to provide valuable insights that may pave the way for enhanced electronic devices and innovative technological solutions. The subsequent sections will unfold the methodology, results, and implications of our investigation into the non-linear coordinate transformations, shedding light on the fascinating world of electron behavior.

2. IR Limit vs. Phenomenological Hydrodynamics: Insights from Non-Linear Coordinate Transformations

In the realm of electron dynamics, the Infrared (IR) limit plays a pivotal role in scrutinizing the behavior of electrons at low energies. Phenomenological hydrodynamics, as a macroscopic framework, facilitates the description of collective electron motion in condensed matter systems. This section delves into the intricate interplay between the IR limit and phenomenological hydrodynamics, with a specific focus on non-linear coordinate transformations.

 doi:10.20944/preprints202402.1290.v1

2.1. IR Limit: Low-Energy Electron Interactions

Consider a homogeneous electron gas, and let k represent the wave vector associated with the electron dynamics. In the IR limit, where $k \to 0$, we probe the low-energy behavior of the system. Examining electron interactions in this limit unveils insights into the long-wavelength aspects of the system, crucial for understanding collective electron motion. Non-linear coordinate transformations, expressed through differential geometry or suitable mathematical formalisms, become indispensable tools in elucidating the nuanced details of electron interactions at low energies.

2.2. Phenomenological Hydrodynamics: Macroscopic Dynamics

Phenomenological hydrodynamics describes the macroscopic evolution of electron densities and currents. Let $\rho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ denote the charge density and current density, respectively. Macroscopic equations, such as the continuity equation $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$, encapsulate the emergent behavior of electrons. However, establishing a direct link between these macroscopic equations and the microscopic origins of electron interactions remains challenging.

2.3. Non-Linear Coordinate Transformations: Bridging Microscopic and Macroscopic Realms

Non-linear coordinate transformations provide a mathematical bridge between the microscopic and macroscopic descriptions of electron dynamics. Let $\mathbf{X}(\mathbf{r}, t)$ represent the transformed coordinates. By leveraging differential geometry and advanced mathematical techniques, we unravel the intricate connection between the IR limit and phenomenological hydrodynamics. The transformation equations become essential in expressing how the microscopic electron interactions manifest in the macroscopic observables described by hydrodynamics.

The subsequent sections will delve into the mathematical methodology and present results, showcasing the pivotal role of non-linear coordinate transformations in gaining a deeper understanding of electron dynamics in condensed matter systems.

3. Intricacies of Non-linear Coordinate Transformations and Interactions

The foundational notion of normal modes as the non-interacting degrees of freedom in a harmonic system undergoes a profound transformation when confronted with the intricacies arising from the dynamics governed by a non-linear combination of coordinates. This mathematical nuance is distinctly elucidated through the lens of Lagrangian formalism.

Consider a harmonic oscillator described by the Lagrangian

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2, \tag{1}$$

where m is the mass, ω is the angular frequency, and q is the coordinate. The transition to a non-linear coordinate $Q = q^2/2$ induces a metamorphosis in the Lagrangian, assuming the form

$$L = \frac{m}{4Q}\dot{Q}^2 - m\omega^2 Q, \tag{2}$$

where the position-dependent effective mass, $m(Q) = m/(2Q)$, becomes a pivotal player, accelerating oscillations and imposing constraints to prevent Q from traversing the point $Q = 0$, particularly when the system commences with an initial value $Q_{\text{in}} > 0$.

Extending this concept to a system with multiple harmonic degrees of freedom x_n, where $n = 1, \ldots, N$, governed by a quadratic action $S_0[x]$, the effective dynamics of a collective variable $y = F(x_1, \ldots, x_N)$ is delineated through the augmented action $S[y, x] = S_0[x] + S_i[y, x]$. The Lagrange multiplier term is expressed as

$$S_i[y, x] = -K \int dt \, \{y(t) - F[x_1(t), \ldots, x_N(t)]\}^2, \tag{3}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 22 February 2024 doi:10.20944/preprints202402.1290.v1

where K stands as an arbitrary constant. The effective action, denoted as $S_{\text{eff}}[y]$, emerges by eliminating the original set of variables $\{x_1, \ldots, x_N\}$ via their equations of motion for any fixed trajectory $y(t)$. Remarkably, this algorithm yields a quadratic, harmonic effective action for y exclusively when the function $F[x_1, \ldots, x_N]$ manifests linearity in x.

4. Chiral Transformation Properties (CTP)

The Chiral Transformation Properties (CTP) are characterized by the matrices $\hat{D}$ and $\hat{D}^{-1}$, given by

$$\hat{D} = \begin{pmatrix} D_n + iD_i & D_f - iD_i \\ D_f + iD_i & D_n - iD_i \end{pmatrix}, \quad \hat{D}^{-1} = \hat{\sigma} \begin{pmatrix} \Delta_n + i\Delta_i & \Delta_f - i\Delta_i \\ \Delta_f + i\Delta_i & \Delta_n - i\Delta_i \end{pmatrix} \hat{\sigma}, \tag{4}$$

where $D_{\substack{r\\a}} = D_n \pm D_f$ and $D^{-1}_{\substack{r\\a}} = \Delta_n \pm \Delta_f$. These matrices encapsulate the essential transformation properties inherent in the Chiral Symmetry breaking phenomenon.

5. Current Dynamics in a Homogeneous Electron Gas

Consideration will be given here to electrons at finite density and vanishing temperature. The Green's functions are generated by the functional

$$e^{\frac{i}{\hbar}W[\hat{a},\hat{j}]} = \int D[\hat{\psi}]D[\hat{\bar{\psi}}]D[\hat{A}]e^{\frac{i}{\hbar}\hat{\bar{\psi}}[\hat{F}^{-1}+\not{\hat{a}}-\frac{e}{c}\hat{\sigma}\not{\hat{A}}]\hat{\psi}+\frac{i}{2\hbar}\hat{A}\hat{D}_0^{-1}\hat{A}+\frac{i}{\hbar}\hat{j}\hat{A}}, \tag{5}$$

where the scalar products denote space-time integrations,

$$fg = \frac{1}{c}\int d^4x f(x)g(x) = c\int \frac{d^4q}{(2\pi)^4} f(-q)g(q), \tag{6}$$

and fields with hats represent CTP doublets, e.g.,

$$\hat{\psi} = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} A^+ \\ A^- \end{pmatrix}, \quad \hat{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

The free photon and electron propagators are given by

$$\hat{D}_0^{\mu\nu}(p) = -\hat{D}_+(p;0)T^{\mu\nu}, \tag{8}$$

with

$$T^{\mu\nu} = g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}, \tag{9}$$

and

$$\hat{F}(p) = (\not{p} + m)\left[\hat{D}_-(p;m) + 2\pi i\delta(p^2 - m^2)n(p)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right], \tag{10}$$

where

$$n(p) = \frac{\Theta(p^0)}{e^{\beta(\epsilon_p - \mu)} + 1} + \frac{\Theta(-p^0)}{e^{\beta(\epsilon_p + \mu)} + 1} \tag{11}$$

and $\epsilon_{\boldsymbol{p}} = c\sqrt{m^2c^2 + \hbar^2\boldsymbol{p}^2}$. The removal of UV divergences is achieved as in the usual single-time axis formalism, [19], and the counterterms will be suppressed hereafter.

The generator functional $W[\hat{a},\hat{j}]$ constructs the effective action through functional Legendre transformation,

$$W[\hat{a},\hat{j}] = \Gamma[\hat{J},\hat{A}] + \hat{a}\hat{J} + \hat{j}\hat{A}, \tag{12}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 22 February 2024 doi:10.20944/preprints202402.1290.v1

where the new variables

$$\hat{J} = \frac{\delta W[\hat{a},\hat{j}]}{\delta \hat{a}}, \quad \hat{A} = \frac{\delta W[\hat{a},\hat{j}]}{\delta \hat{j}} \tag{13}$$

are the expectation values, $J^\mu(x) = \mathrm{Tr}[\rho\bar{\psi}(x)\gamma^\mu\psi(x)] = (c\rho, \boldsymbol{j})$, $A_\mu(x) = \mathrm{Tr}[\rho A_\mu(x)]$ for $\hat{a} = \hat{j} = 0$. The Legendre transformation of a real, convex function can be defined geometrically or algebraically. We follow the latter route and use Eqs. (13) to define the effective action, a complex functional in an algebraic manner. The inverse transformation is based on the variable

$$\frac{\delta\Gamma[\hat{J},\hat{A}]}{\delta\hat{J}} = -\hat{a}, \quad \frac{\delta\Gamma[\hat{J},\hat{A}]}{\delta\hat{A}} = -\hat{j}, \tag{14}$$

therefore the variational equation of the effective action is satisfied by the expectation values, obtained for vanishing external source. We are interested in the effective current dynamics, and the corresponding effective action, $\Gamma[\hat{J}]$, will be obtained by eliminating $\hat{A}$ from $\Gamma[\hat{J},\hat{A}]$ using the second equation in (14). The effective action is real in the physical case, $j_+ = -j_-$, $a_+ = -a_-$; therefore, it is sufficient to retain the real part, $\Re\Gamma$, in the equation of motion.

The quadratic part of the generator functional W and the effective action Γ will be calculated by expanding in $\hbar$ and retaining the first two orders. We start with W by integrating out the electron field,

$$e^{\frac{i}{\hbar}W[\hat{a},\hat{j}]} = \int D[\hat{A}]e^{\mathrm{Tr}[\hat{F}^{-1}+\not{a}-\frac{e}{c}\hat{\sigma}\not{\hat{A}}]+\frac{i}{2\hbar}\hat{A}\hat{D}^{-1}\hat{A}+\frac{i}{\hbar}\hat{j}\hat{A}}. \tag{15}$$

Keeping the $\mathcal{O}\left(\hbar^0\hat{a}^2\right)$ part of the exponent, we obtain

$$e^{\frac{i}{\hbar}W[\hat{a},\hat{j}]} = \int D[\hat{A}]e^{-\frac{i}{2}\hat{a}\hat{G}\hat{a}+\frac{i}{\hbar}\hat{k}\hat{A}+\frac{i}{2\hbar}\hat{A}\hat{D}^{-1}\hat{A}}, \tag{16}$$

where

$$\hat{D}^{-1} = \hat{D}_0^{-1} - \frac{e^2}{c^2}\hat{\sigma}\hat{G}\sigma, \tag{17}$$

and the source

$$\hat{k} = \hat{j} + \frac{e}{c}\hat{\sigma}\hat{G}\hat{a}, \tag{18}$$

are given in terms of the one-loop current-current Green function

$$G_{(\sigma_1\mu_1)(\sigma_2\mu_2)}(x_1,x_2) = -i\hbar\mathrm{tr}[\hat{F}_{\sigma_1\sigma_2}(x_1,x_2)\gamma_{\mu_2}\hat{F}_{\sigma_2\sigma_1}(x_2,x_1)\gamma_{\mu_1}]. \tag{19}$$

The presence of a neutralizing, homogeneous background charge was assumed in Eq. (18). The Gaussian integral, (16), yields

$$W[\hat{a},\hat{j}] = -\frac{\hbar}{2}\hat{a}\hat{G}\hat{a} - \frac{e}{c}\hat{j}\hat{D}_0\sigma\hat{G}\hat{a} - \frac{1}{2}\hat{j}\left(\hat{D}_0 + \frac{e^2}{c^2}\hbar\hat{D}_0\hat{\sigma}\hat{G}\hat{\sigma}\hat{D}_0\right)\hat{j} \tag{20}$$

in the desired order.

Although the orders of the expansion of $W[\hat{a},\hat{j}]$ in $\hbar$ and in the number of loops correspond to each other in the usual manner, this is not the case anymore when considering the effective action. The reason is that the variables of the effective action have different orders in $\hbar$ ($\hat{J} = \mathcal{O}(\hbar)$, $\hat{A} = \mathcal{O}\left(\hbar^0\right)$) compared to the variables of W ($\hat{a} = \mathcal{O}\left(\hbar^0\right)$, $\hat{j} = \mathcal{O}\left(\hbar^0\right)$). The Legendre transformation (12) gives

$$\Gamma[\hat{J},\hat{A}] = \frac{1}{2\hbar}\hat{J}\hat{G}^{-1}\hat{J} + \frac{1}{2}\hat{A}\hat{D}_0^{-1}\hat{A} - e\hat{A}\hat{\sigma}\hat{J} \tag{21}$$

 doi:10.20944/preprints202402.1290.v1

for the $\mathcal{O}(\hbar)$ effective action. After the effective action has been derived in the desired accuracy, we set $\hbar = 1$ for the remainder of this work.

The Maxwell equation,

$$\hat{A} = e\hat{D}_0\sigma\hat{J}, \tag{22}$$

can be used to eliminate the photon field and arrive at the effective action,

$$\Gamma[\hat{J}] = \frac{1}{2}\hat{J}\left(\hat{G}_0^{-1} - \frac{e^2}{c^2}\hat{\sigma}\hat{D}_0\hat{\sigma}\right)\hat{J}. \tag{23}$$

As any bosonic two-point function, $\hat{G}$, too, has the CTP block structure of Eq. (4), allowing us to define the retarded and advanced parts,

$$\left(\hat{G}^{-1} - \frac{e^2}{c^2}\hat{\sigma}\hat{D}_0\hat{\sigma}\right)_{\substack{r\\a}} = (\hat{G}_{\substack{r\\a}})^{-1} - \frac{e^2}{c^2}\hat{D}_{0\substack{r\\a}}. \tag{24}$$

6. Equations of Motion

It is advantageous to use the parametrization $a^{\pm} = \bar{a}/2 \pm a$ for the source, where a stands for a physical source and $\bar{a}$ is a book-keeping device. The effective action for the current is then defined by

$$W[\bar{a}, a] = \Gamma[J, J_d] + \bar{a}J + aJ_d, \tag{25}$$

where

$$J = \frac{\delta W[\bar{a}, a]}{\delta\bar{a}}, \quad J_d = \frac{\delta W[\bar{a}, a]}{\delta a}. \tag{26}$$

When the book-keeping variable is set to zero, $\bar{a} = 0$, then

$$J = \frac{\delta W[\hat{a}]}{\delta a^+}_{|a^+=-a_-=a} = \frac{\delta W[\hat{a}]}{\delta a^-}_{|a^+=-a_-=a} \tag{27}$$

becomes the expectation value of the current in the presence of a physical external source a, and the auxiliary field J_d is vanishing.

The effective action, defined by Eqs. (25)-(27), yields the equation of motion,

$$a = -G^{r-1}J, \tag{28}$$

for the true expectation value, obtained for $\bar{a} = 0$ when $J_d = 0$.

7. Conclusion

In conclusion, our exploration into the hydrodynamical intricacies of a homogeneous electron gas has been underpinned by a rigorous mathematical framework. Employing advanced tools such as non-linear coordinate transformations, we have delved into the profound interplay between the infrared (IR) limit and phenomenological hydrodynamics within condensed matter systems. Initiating our analysis with a meticulous examination of normal modes in a harmonic system, we underscored the consequential impact of position-dependent effective mass introduced through non-linear coordinate transformations. Expanding our purview beyond single degrees of freedom, the incorporation of a collective variable in a multi-harmonic system revealed the subtle dynamics encapsulated by a Lagrange multiplier term, highlighting the inherent linearity in our model. A significant stride in our investigation involved the integration of the Current Two-Point function (CTP) formalism, introducing matrices such as $\hat{D}$ and $\hat{D}^{-1}$ to unravel the intricate symmetries associated with the Chiral Symmetry breaking phenomenon. Extending our mathematical scrutiny to the dynamics of current in a homogeneous electron gas at finite density and zero temperature, we harnessed the power of functional integral techniques. The resultant effective action for current dynamics, elegantly expressed

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 22 February 2024 doi:10.20944/preprints202402.1290.v1

in terms of the Current Two-Point function, provides a nuanced foundation for delving into the hydrodynamical aspects of electron behavior. In essence, our study fuses sophisticated theoretical developments with a rigorous mathematical apparatus. Non-linear coordinate transformations emerge as a key analytical tool, bridging the microscopic and macroscopic facets of electron dynamics. This work not only advances our understanding of condensed matter physics but also lays the groundwork for future mathematical explorations in this intriguing domain.

Appendix A. Current Two-Point Function at Finite Density in the Non-Relativistic Limit

The result of the calculation of the current-current Green function, given by Eq. (19),

$$G^{\mu\nu}_{\sigma\tau}(q) = -i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}[\gamma^\mu F_{\sigma\tau}(q+p)\gamma^\nu F_{\tau\sigma}(p)], \tag{A1}$$

at finite density and vanishing temperature in the non-relativistic limit, $c\to\infty$, is briefly summarized in this Appendix.

Appendix A.1. Lorentz Structure

The bilinear form is symmetric, $G_{(\sigma\mu)(\sigma'\nu)}(p) = G_{(\sigma'\nu)(\sigma\mu)}(-p)$, and transverse, $p^\mu G_{(\sigma\mu)(\sigma'\nu)}(p) = 0$. Moreover, it is covariant and involves two four-vectors, p^μ and β^μ, defined as $\beta^\mu = (1,\mathbf{0})$ in the inertial frame where the electron gas is at rest. These vectors yield two independent kinematic scalar combinations, $\mathbf{q}^2 = -[q-u(uq)]^2$ and $\xi = uq/|\mathbf{q}| = \omega/c|\mathbf{q}|$, where the notation $q^\mu = (\omega/c,\mathbf{q})$ is employed. This tensor can be parametrized by two Lorentz scalars as

$$G^{\mu\nu} = G_\ell P^{\mu\nu}_\ell + D_t P^{\mu\nu}_t, \tag{A2}$$

where P_t and P_ℓ are projectors onto the three-dimensional transverse and longitudinal subspaces,

$$P^{\mu\nu}_t = -\begin{pmatrix}0 & 0\\ 0 & \mathbf{T}\end{pmatrix},$$
$$P^{\mu\nu}_\ell = \frac{1}{1-\xi^2}\begin{pmatrix}1 & \mathbf{n}\xi\\ \mathbf{n}\xi & \xi^2\mathbf{L}\end{pmatrix}, \tag{A3}$$

respectively, with $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$, $\mathbf{L} = \mathbf{n}\otimes\mathbf{n}$, and $\mathbf{T} = \mathbb{I}-\mathbf{L}$. The inverse, defined by $G\mu{}_\rho G^{-1\rho\nu} = T^{\mu\nu}$, is given by

$$G^{-1} = \frac{1}{G_\ell}P_\ell + \frac{1}{G_t}P_t. \tag{A4}$$

Appendix A.2. Vacuum Contribution

The bilinear form is the sum of vacuum and finite density contributions, $\hat{G} = \hat{G}_{\mathrm{vac}} + \hat{G}_{\mathrm{gas}}$, and both the vacuum and finite density contributions are of the form (**??**). The vacuum contributions, $\hat{G}^{\mu\nu}_{\mathrm{vac}} = \hat{G}_{\mathrm{vac}}T^{\mu\nu}$, are straightforward to find. The CTP diagonal block, $G^{++}_{\ell\,\mathrm{vac}} = G^{++}_{t\,\mathrm{vac}} = G^{++}_{\mathrm{vac}}$, is the standard one,

$$\begin{aligned}G^{++}_{\mathrm{vac}}(q) &= \frac{1}{3\pi}q^2\left\{\frac{1}{3}+2\left(1+\frac{2m^2c^2}{q^2}\right)\left[\sqrt{\frac{4m^2c^2}{q^2}-1}\,\mathrm{arccot}\sqrt{\frac{4m^2c^2}{q^2}-1}-1\right]\right\}\\ &= \frac{q^2}{15\pi}\left[\frac{q^2}{m^2c^2}+\mathcal{O}\left(\frac{q^2}{m^2c^2}\right)^2\right].\end{aligned} \tag{A5}$$

 doi:10.20944/preprints202402.1290.v1

The CTP non-diagonal block, calculated using the free propagator (**??**), is $G^{+-}_{\ell\,\mathrm{vac}} = G^{+-}_{t\,\mathrm{vac}} = G^{+-}_{\mathrm{vac}}$, with

$$G^{+-}_{\mathrm{vac}}(q) = \frac{i}{3}\int \frac{d^4p}{(2\pi)^4} 2\pi\delta((p+q)^2 - m^2)\Theta(-p^0 - q^0)2\pi\delta(p^2 - m^2)\Theta(p^0)\mathrm{Tr}\, N(p+q,q), \tag{A6}$$

where the trace is over the Lorentz indices of the trace formula,

$$N^{\mu\nu}(p,q) = \mathrm{Tr}\left(\gamma^\mu\left[(p_\alpha\gamma^\alpha + mc)\gamma^\nu(q_\beta\gamma^\beta + mc)\right]\right) = 4(m^2c^2 - pq)g^{\mu\nu} + 4p^\mu q^\nu + 4p^\nu q^\mu. \tag{A7}$$

The expression simplifies to

$$G^{+-}_{\mathrm{vac}} = \frac{i(c^2m^2 + \frac{q^2}{2})}{3\pi c|\mathbf{q}|}\Theta(-q^0 - m)\int_0^{\sqrt{q^{02} - m^2c^2}} \frac{dpp}{\omega_p}\Theta(2p|\mathbf{q}| - |q^2 + 2\omega_p q^0|), \tag{A8}$$

an expression negligible in the non-relativistic limit.

Appendix A.3. Fermi-Sphere Contribution

To determine $\hat{G}_\ell$ and $\hat{G}_t$ for the electron gas, we need $\hat{T} = \hat{G}^{00}_{\mathrm{gas}}$ and the spatial trace, $\hat{S} = \hat{G}^{jj}_{\mathrm{gas}}$. For this purpose, we use the trace factors

$$\begin{aligned} t &= N^{00}(p+q,p)_{|p^2=m^2c^2} = t' - 2(q^2 + 2pq), \\ s &= N^{jj}(p+q,p)_{|p^2=m^2c^2} = s' + 2(q^2 + 2pq), \end{aligned} \tag{A9}$$

where $t' = 8(p^{02} + p^0q^0) + 2q^2$ and $s' = 8(p^0q^0 + \mathbf{p}^2) - 2q^2$ in the loop integrals

$$\begin{aligned} \binom{T}{S}_{++}(q) &= i\int_p \binom{t}{s}\Bigg[2\pi\delta((q+p)^2 - m^2c^2)n_{q+p}2\pi\delta(p^2 - m^2c^2)n_p \\ &\quad - i\frac{2\pi\delta(p^2 - m^2c^2)n_p}{(p+q)^2 - m^2c^2 + i\epsilon} - i\frac{2\pi\delta((p+q)^2 - m^2c^2)n_{p+q}}{p^2 - m^2c^2 + i\epsilon}\Bigg], \\ \binom{T}{S}_{+-}(q) &= -i\int_p \binom{t'}{s'}2\pi\delta((q+p)^2 - m^2c^2)2\pi\delta(p^2 - m^2c^2)[\Theta(-p^0 - q^0)n_p \\ &\quad + \Theta(p^0)n_{p+q} - n_{q+p}n_p]. \end{aligned} \tag{A10}$$

It is advantageous to introduce the integrals

$$\begin{aligned} I^{(1)}[q;f] &= \int \frac{d^4p}{(2\pi)^4} f(p,q)2\pi\delta(p^2 - m^2c^2), \\ I^{(2)}[q;f] &= \int \frac{d^4p}{(2\pi)^4} f(p,q)2\pi\delta(p^2 - m^2c^2)2\pi\delta(q^2 + 2pq), \end{aligned} \tag{A11}$$

and write

$$\begin{aligned} \binom{T}{S}_{++}(q) &= I^{(1)}\left[q; \frac{\binom{t}{s}n_p}{q^2 + 2pq + i\epsilon}\right] + \frac{i}{2}I^{(2)}[q; \tbinom{t'}{s'}n_p n_{q+p}] + (q \to -q), \\ \binom{T}{S}_{+-}(q) &= -iI^{(2)}[q; \tbinom{t'}{s'}(\Theta(-p^0 - q^0)n_p + \Theta(p^0)n_{p+q} - n_{q+p}n_p)]. \end{aligned} \tag{A12}$$

The integrals $I^{(1)}$ and $I^{(2)}$ are evaluated by assuming that the integrands are spherically symmetric and nonvanishing for $p^0 > 0$ and $|\mathbf{p}| \ll mc$. One finds

$$\begin{aligned}
I^{(1)}[q; f(p^0,\mathbf{p})] &= \frac{1}{4\pi^2 mc}\int dpp^2 f(mc^2,\mathbf{p}),\\
I^{(1)}[q; \frac{g(p^0,\mathbf{p})}{q^2+2pq+i\epsilon}] &= \frac{1}{16\pi^2|\mathbf{q}|mc}\int_0^\infty dppg(mc^2,\mathbf{p})\ln\frac{k+p+i\epsilon}{k-p+i\epsilon},\\
I^{(2)}[q; h((p^0,\mathbf{p}),(\frac{\omega}{c},\mathbf{q}))] &= \frac{1}{16\pi^2|\mathbf{q}|mc}\int d^3ph\left((mc^2,\mathbf{p}),(\frac{\omega}{c},\mathbf{q})\right)\delta(k-p_z),
\end{aligned} \tag{A13}$$

where $k = \frac{q^2+2m\omega}{2|\mathbf{q}|}$.

We need the Fourier transform of the real part of the CTP diagonal block in the leading order in $1/c$,

$$\begin{aligned}
\Re\binom{T}{S}_{++}(q) &= \frac{1}{c}\left[\binom{T}{S}_{++}(q) + \binom{T}{S}^*_{++}(-q)\right]\\
&= \Re I^{(1)}\left[\frac{\binom{t}{s}n_p}{q^2+2pq+i\epsilon}\right] + (\omega\to-\omega)\\
&= \frac{1}{16\pi^2|\mathbf{q}|mc}\int_0^{k_F} dppn_p\left[\binom{8m^2c^2}{8(m\omega+p^2)+2\mathbf{q}^2}\ln\left|\frac{k+p}{k-p}\right| + 8|\mathbf{q}|p\binom{-1}{1}\right] + (\omega\to-\omega)
\end{aligned} \tag{A14}$$

The momentum integral can be easily carried out at vanishing temperature, $n_p = \Theta(p^0)\Theta(k_F - |\mathbf{p}|)$, yielding

$$\begin{aligned}
\Re T_{++}(q) &= \frac{k_F^2 mc}{2\pi^2|\mathbf{q}|}L^{(1)}(r) + (\omega\to-\omega),\\
\Re S_{++}(q) &= \frac{k_F^2}{2\pi^2 mc|\mathbf{q}|}\left[k_F^2 L^{(3)}(r) + \left(m\omega + \frac{\mathbf{q}^2}{4}\right)L^{(1)}(r)\right] + \frac{k_F^3 c}{6\pi^2 m} + (\omega\to-\omega),
\end{aligned} \tag{A15}$$

where $r = \frac{q^2+2m\omega}{2|\mathbf{q}|k_F}$ and

$$\begin{aligned}
L^{(1)}(r) &= \int_0^1 dkk\ln\left|\frac{a+k}{a-k}\right| = r + \frac{1}{2}(1-r^2)\ln\left|\frac{r+1}{r-1}\right|,\\
L^{(3)}(r) &= \int_0^1 dkk^3\ln\left|\frac{r+k}{r-k}\right| = \frac{r}{6} + \frac{r^3}{2} + \frac{1}{4}(1-r^4)\ln\left|\frac{r+1}{r-1}\right|.
\end{aligned} \tag{A16}$$

The leading order $1/c$ contribution to the off-diagonal CTP block is

$$\binom{T}{S}_{+-}(q) = -iI^{(2)}[\binom{t'}{s'}) \tag{A17}$$

References

1. Zhenjie Yan, Parth B. Patel, Biswaroop Mukherjee, Chris J. Vale, Richard J. Fletcher, Martin W. Zwierlein, *Thermography of the superfluid transition in a strongly interacting Fermi gas*, *Science*, **383**(6683), 629–633, 2024. DOI: 10.1126/science.adg3430. https://www.science.org/doi/abs/10.1126/science.adg3430. PDF.
2. Alvise Bastianello, *Sine-Gordon model from coupled condensates: A generalized hydrodynamics viewpoint*, *Phys. Rev. B*, **109**(3), 035118, 2024. DOI: 10.1103/PhysRevB.109.035118. https://link.aps.org/doi/10.1103/PhysRevB.109.035118.
3. Thibault Bonnemain and Benjamin Doyon, *Soliton gas of the integrable Boussinesq equation and its generalised hydrodynamics*, *arXiv:2402.08669*, 2024. https://arxiv.org/abs/2402.08669.
4. Michael Strickland, *Hydrodynamization and resummed viscous hydrodynamics*, *arXiv:2402.09571*, 2024. https://arxiv.org/abs/2402.09571.
5. H. D. Zeh, Found. Phys. **1**, 69 (1970).

 doi:10.20944/preprints202402.1290.v1

6. O. Kübler, H. D. Zeh, Ann. Phys. (NY) **76**, 405 (1973).
7. W. H. Zurek, Phys. Rev. D **24**, 1516 (1981).
8. W. H. Zurek, Phys. Rev. D **26**, 1862 (1982).
9. E. Joos, H. D. Zeh, Z. Phys. B. **59**, 223 (1985).
10. E. M. Lifsitz, L. P. Pitaevskii, *Course of Theoretical Physics, Vol 10., Physical Kinetics* Pergamon Press, Oxford, 1981.
11. J. Schwinger, J. Math. Phys. **2**, 407 (1961); *Particles and Sources*, vol. I., II., and III.,Addison-Wesley, Cambridge, Mass. 1970-73.
12. A. O. Caldeira, A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981).
13. L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) (Sov. Phys. JETP **20**, 1018 (1965)).
14. R. P. Feynman, F. L. Vernon, Ann. Phys. **24**, 118 (1963).
15. H. F. Dowker, J. J. Halliwell, Phys. Rev. **D46**, 1580 (1992).
16. C. Anastopoulos, Phys. Rev. **D63**, 125024 (2001).
17. J. J. Halliwell, Phys. Rev. **D58**, 105015 (1998).
18. Y. Aharonov, P. G. Bergmann, J. L. Lebowitz, Phys. Rev. **B134**, 1410 (1964).
19. J. Polonyi, Phys. Rev. **D74**, 065014 (2006).
20. U. Weiss, *Quantum Dissipative Systems*, World Scientific, 1993, Singapore.
21. H. D. Zeh, *The Direction of Time*, Springer-Verlag, Berlin, 1989.
22. P. Ehrenfest, T. Ehrenfest, *Conceptual Foundations of the Statistical Approach in Mechanics*, Cornell University Press, Ithaca, 1959.
23. E. Jaynes, Phys. Rev. **106**, 620 (1957).
24. J. Gemmer, M. Michel, G. Mahler, *Quantum Thermodyanmics*, Springer Verlag, Berlin, 2004.
25. P. Bergmann, J. Lebowitz, Phys. Rev. **99**, 578 (1955).
26. R. B. Griffith, J. Stat. Phys. **36**, 219 (1984).
27. R. Omnès, J. Stat. Phys. **53**, 893 (1988).
28. M. Gell-Mann, J. B. Hartle, Phys. Rev. **D47**, 3345 (1993).
29. J. Polonyi, Phys. Rev. D **77**, 125018 (2008).
30. P. C. Clemmow, J. P. Dougherty, *Electrodynamics of Particles and Plasmas* Addison-Wesley, 1969.
31. J. A. Wheeler, in *Quantum Theory and Measurement*, ed. J. A. Wheeler, W. H. Zuker, Princeton University Press, Princeton, 1983.
32. J. A. Wheeler, R. Feynman, Rev. Mod. Phys. **17**, 157 (1945).
33. Matthew Stephenson, *Hydrodynamical Limit of a Homogeneous Electron Gas* , *viXra:2304.0152*, 2023. https://vixra.org/abs/2304.0152.