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Number of Components in the Graph Exponential with Complete Graph Exponents

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Keywords: graph products; graph exponentiation; graph exponential; graph product components

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Number of Components in the Graph Exponential with Complete Graph Exponents

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Abstract: Although Lovász introduced the graph exponential G^K in 1967, this product has received little attention. This paper focuses on the number of components in G^K when exponent K is a complete graph K_n. For a connected bipartite G, when K is K_2, G^{K_2} has three components; and when K is K_n with $n > 2$, G^{K_n} produces $\lceil \frac{n+1}{2} \rceil$ components. A connected G with an odd cycle produces one component in G^{K_n}.

Keywords: graph products; graph exponentiation; graph exponential; graph product components

MSC: 05C76; 05C75

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1. Introduction

The graph exponentiation product that produces the graph exponential G^K was introduced by Lovász in 1967 [7]; yet this interesting graph product has received little attention. In 1969, mention is given to G^K in [4]. G^K is further developed in [8] (2001) where focus is on Hedetniemi's conjecture. With primary attention given to neighborhood multisets, G^K is addressed in the 2021 article [2]. The G^K adjacency matrix is discussed in [5] (2024).

This note provides the number of components in G^K when K is a complete graph. Assume G is finite, undirected and connected as discussed in [5] that gives the construction of the G^K adjacency matrix A_{G^K}. Due to its bipartite nature, exponent K_2 is examined separately from K_n where $n > 2$. Proposition 2 states that given a connected bipartite G with K_2 as K, then G^{K_2} has three components. Theorem 2 states that for a connected bipartite G and for exponent K_n ($n > 2$), G^{K_n} has $\lceil \frac{n_K+1}{2} \rceil$ components where n_K is the order of K_n. Corollary 1 examines G^{K_n} when G contains an odd cycle for the K_n exponent producing a single component.

Section 2 discusses notation and briefly discusses matrices and the direct product. Section 3 gives information concerning the form of G^K discussed in this note. Section 4 covers the number of components in G^{K_2} while Section 5 gives the number of components for G^{K_n} with $n > 2$.

2. Background

Basic graph theory knowledge as found in [1] is assumed. The vertex set of graph G is denoted by $V(G)$ while $E(G)$ indicates the edge set. It is assumed that all graphs have the same vertex labeling scheme of $\{1, 2, \cdots, n\}$ where n is graph order with n_G as the order for a particular G. Graph order is also indicated with $|G|$. For any vertex $v \in V(G)$, $N(v)$ reflects the open neighborhood of v, and $N[v]$ is used for the closed neighborhood. Vertex adjacency is given by $v_1 \sim v_2$. Two graphs being isomorphic is indicated using $G \cong H$, and $G + H$ is used for the disjoint union of G and H. The disconnected graph of x copies of G is noted as xG and the automorphism group of G is $\text{Aut}(G)$.

Complete graphs are K_n and complete graphs with a loop at each vertex are K_n^*. C_n is the cycle graph with order n. Notation D_n is the disjoint union of n number of K_1 subgraphs and is called the *empty graph*. The *null graph* $\mathbb{O}$ has empty vertex and edge sets. We assume that all bipartite G with partite sets V_1 and V_2 have vertices labeled so V_1 contains only even labeled vertices while V_2 contains only odd vertices. TCall this bipartite labeling scheme the *parity labeling*.

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2.1. Matrices

In this paper, loops in an adjacency matrix are given a 1 along the diagonal. The direct power $G^x = G \times G \times \cdots$ is the direct product of G with itself x times. As is shown in [5], G^x has connections to G^K as seen in the adjacency matrices of both products. Let the adjacency matrix of graph G be A_G. Denoted A_{G^x}, the adjacency matrix for the direct power G^x is the Kronecker product $A_G \otimes A_G \otimes \cdots$ of G with itself x times. The adjacency matrix for G^K is given by A_{G^K} as explained in [5] and in Section 3. Denote an all zero matrix as Z while J is a matrix of all ones.

2.2. Direct Product

The direct product $G \times H$ has vertex set that is the Cartesian product $V(G) \times V(H)$ producing ordered pair vertices (g, h) where $g \in V(G)$ and $h \in V(H)$. An edge $\big((g, h)(g', h')\big)$ in $E(G \times H)$ is defined when $(g, g') \in E(G)$ and $(h, h') \in E(H)$. Thus, the edge structure of $G \times H$ is defined on homomorphisms that preserve the adjacency structures of G and H. Additional information on the direct product can be found in [3].

3. Graph Exponentiation

There exist multiple definitions for graph exponentiation (see [6] Figure 1 for an alternate definition). This section provides an overview of the graph exponentiation product G^K for this paper, with a more detailed explanation given in [5] that discusses the adjacency matrix A_{G^K}, and in [2], [4], [7] plus [8]. Although K_n indicates a complete graph, in this paper K with no subscript exclusively refers to the graph that is the *exponent* in the graph exponentiation product, G^K.

Graph exponentiation is a graph product operation that produces the *graph exponential* G^K as follows. Set $V(G^K)$ is the set of all functions $f : V(K) \to V(G)$ where two functions f_1 and f_2 are adjacent in G^K if $(f_1(k)f_2(k')) \in E(G)$ for all $(k, k') \in E(K)$. Thus the order of $V(G^K)$ is $|G|^{|K|}$. Since only undirected G and K are considered here, all functions are symmetric and G^K is undirected as discussed in [8].

In addition to the notation $|K|$ as graph order, let n_K indicate the order of K. If $V(K) = \{k_1, k_2, \cdots, k_{n_K}\}$ then each function f can be represented by n_K-tuple $(g_1, g_2, \cdots, g_{n_K})$ where $g_i \in V(G)$, reflecting that $f(k_i) = g_i$.

As this product can generate loops in G^K even though G and K may be loopless (see Figure 1), G and K are considered to be finite undirected and connected graphs possibly with loops but without multiple edges as also assumed in [5]. Hence, G^K is over the set of G that might have loops but no mulitple edges. Trivial graphs with $n = 1$ are not considered.

Suppose G is K_2. Figure 1 displays G plus G^2 and G^{K_2} along with their respective adjacency matrices.

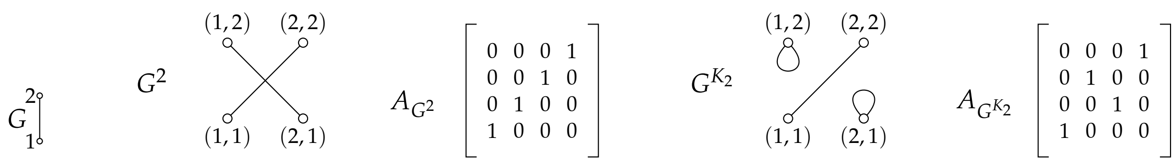

Figure 1. For $G := K_2$, graphs G^2 and G^{K_2} with their respective adjacency matrices.

3.1. The Set of "Edge-Generating" f_i Function Combinations

Depending on the edge structures of G and K, not all f function combinations generate edges in G^K. As an example, let K be K_3 with $V(G) = \{v_1, v_2, v_3\}$ and let G be K_2. Let σ be a f combination. Although the set of σ is $\mathrm{Aut}(K_3)$ that is the dihedral group on 3 (excluding the identity, $\mathrm{Aut}(K_3)$ is: $(v_2v_3v_1)$, $(v_3v_1v_2)$, $(v_2v_1v_3)$, $(v_1v_3v_2)$, $(v_3v_2v_1)$), any σ function combination containing a fixed vertex is disregarded as these combinations of f cannot generate an edge in G^K. Thus, the f combinations in this case that *generate edges in* G^K is set $\{(v_2v_3v_1), (v_3v_1v_2)\}$; and $K_2^{K_3}$ is the disjoint union $C_6 + K_2$ [4].

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The edge-generating function combinations are also referred to in this note as *the f combinations that apply to G^K*.

3.2. Known Properties of G^K

As K_1^* has one function that maps vertex v to v then $G^{K_1^*} = G$. Thus K_1^* is the identity for this product.

Given graphs G, H and K plus direct product $G \times H$, the following hold as proved in [7].

- $G^{K_1^*} = G$.
- $G^{x(K_1^*)} = G \times G \times \cdots \times G$ x number of times is G^x,
- $G^{\mathbb{O}} = K_1^*$
- $G^H \times G^K \cong G^{H+K}$,
- $(G \times H)^K \cong G^K \times H^K$,
- $(G^H)^K \cong G^{H \times K}$.

Any G has a *neighborhood multiset* $\mathcal{N}(G)$ of the open neighborhoods $N(g)$ for $g \in V(G)$. For K_2 with $V(K_2) = \{0,1\}$, as $N(0) = 1$ and $N(1) = 0$, then $\mathcal{N}(K_2) = \{\{1\},\{0\}\}$. It is not always true that if $\mathcal{N}(G) = \mathcal{N}(H)$ then $G \cong H$. See [2] for a couple of examples. Graph G is said to be *neighborhood reconstructible* if $\mathcal{N}(G) = \mathcal{N}(H)$ implies that $G \cong H$ [2]. In [2] it is shown that G is neighborhood reconstructible if and only if $G^{K_2} \cong H^{K_2}$ implies that $G \cong H$ for all H. This paper appears to be the only instance in which cancellation in G^K is explored.

3.3. Adjacency Matrix for the Exponential

Most of the following material in this subsection is given in [5] and repeated here for completeness. As an overview, based on $|G^K| = |G|^{|K|}$, adjacency matrix A_{G^K} is the Kronecker product matrix $A_{G\otimes} = \prod_{i=1}^{(n_K)-1} A_G \otimes (A_G)_i$ of A_G over $|K|$ that is row permuted by the f_i function cmbinations that generate edges in the exponential.

Thus, when K is K_2, the exponential adjacency matrix $A_{G^{K_2}}$ is a single permutation of the adjacency matrix for G^2 as given by the following proposition from [5].

Proposition 1. *Let G be any graph without multiple edges and let K in G^K be K_2. Construct a block matrix $A^*_{G^K} = A_G \otimes A_G$ using the colexicographic ordering of $V(G^K)$ as row index labels while column indices are lexicographically labeled. Matrix $A^*_{G^K}$ produces A_{G^K} by row permuting to have both row and column indices in lexicographic order.* ■

Examination of the adjacency matrices for the two graphs in Figure 1 gives an example for Proposition 1. Let $(g,g') \in V(G^{K_2})$ and π be the permutation of A_{G^2} to $A_{G^{K_2}}$. When $g = g'$, as shown by the (g,g') row in A_{G^2}, π fixes $N(g,g')$. When $g \neq g'$, $N(g,g') \mapsto N(g',g)$ due to the transposition action of π on A_{G^2} resulting in $A_{G^{K_2}}$.

Following are some definitions from [5]. Imagine set $\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$ of n_K-tuple f_j functions combinations where $f_j : V(K) \to V(G)$ and k is the number of σ_i combinations of f_j that apply to G^K. Let $\pi_i \in \Omega$ be a $|G|^{|K|} \times |G|^{|K|}$ permutation matrix for each σ_i. Thus, based on the structures of G and K, Ω_G contains only the k number of π_i where the σ_i produce edges in G^K.

Let $A_{G\otimes}$ be the Kronecker product of A_G over $|K|$ where the Z blocks are maximized by having A_G as the first multiplicand:

$$A_{G\otimes} = \prod_{i=1}^{n_K-1} A_G \otimes (A_G)_i \tag{1}$$

Define $\Omega_{A_\wedge}$ as a collection of $|\Omega_G|$ number of $(A_\wedge)_i = \pi_i * A_{G\otimes}$. Thus, each $(A_\wedge)_i$ is associated with a distinct member π_i of Ω_G; and each $(A_\wedge)_i$ member of $\Omega_{A_\wedge}$ is a submatrix of A_{G^K} based on a specific π_i, with set $\Omega_{A_\wedge}$ over all π_i in Ω_G.

Define $\sum_\Delta$ as a matrix sum of $(A_\wedge)_i$ such that a_{ij} is changed to 1 for all a_{ij} where $a_{ij} > 1$. This eliminates the miscounting of redundantly generated neighbors in G^K.

Figure 2 contains a general A_{G^K} construction algorithm while Theorem 1 discusses this matrix.

CONSTRUCTION ALGORITHM FOR A_{G^K}

1. Determine the π_i members of Ω_G based on the σ_i that apply to G^K.
2. Utilizing A_G and Kronecker product, produce $A_{G\otimes}$ using equation 1.
3. For $|\Omega_G|$ number of $A_{G\otimes}$, build the set $\Omega_{A_\wedge}$ where each $(A_\wedge)_i$ member matrix is $(A_\wedge)_i = \pi_i * A_{G\otimes}$.
4. Apply $\sum_\Delta$ to the set of $(A_\wedge)_i$ in $\Omega_{A_\wedge}$ to generate A_{G^K} of G^K.

Figure 2. The construction algorithm of A_{G^K}.

The proof of Theorem 1 is found in [5].

Theorem 1. *Given G^K where G and K are graphs without multiple edges, A_{G^K} is the $\sum_\Delta$ sum of the members of $\Omega_{A_\wedge}$.* ∎

4. Number of Components for K_2 as K

Complete graph K_2 is also a bipartite path graph; thus, this exponent is examined separately from the general K_n case. Note that when K is K_2, then $V(G^K) = V(G^2)$ and edges $(g_1, g_1') \sim (g_2, g_2')$ in G^{K_2} if and only if edges (g_1, g_2') and (g_1', g_2) are in $E(G)$. We begin when G is a connected bipartite graph.

4.1. Bipartite Connected G

Weishcel's Theorem (WT) [3] states that given any nontrivial connected G and H, if either G or H has an odd cycle, then $G \times H$ is connected. However, if both G and H are bipartite, WT states that $G \times H$ has exactly two components.

In addition to WT stating that bipartite G and H create exactly two components, it is also known that both of the two generated components are bipartite. Thus, the time-lapsed product $G_1 \times G_2 \times \cdots \times G_k = \times_{i=1}^{k} G_i$ produces 2^{x-1} number of components where x is the number of bipartite factors.

In our case, as G^{n_k} and G^K are intimately linked, conisder when G is bipartite and G^{n_k} produces $2^{(n_k)-1}$ bipartite graphs. In bipartite graphs, the two partite sets generate partitions of even and odd walks between vertex pairs based on the partite set members. As the edges in $G \times H$ are defined on the edges of G and H, the adjacency structures of both G and H are preserved in the product via homomorphisms that define the product edges. Let $(g,h), (g',h') \in V(G \times H)$. The proof of WT is based on the fact that if a g,g'-walk of length n exists in G and a h,h'-walk of length n is in H, then there is a length n $(g,h), (g',h')$-walk in $G \times H$. If no such length n walk exists for two vertices in $G \times H$, then the distance between the vertex pair is infinite and the two vertices must be in different components.

As G is bipartite in this subsection, then G has no loops. Let the two copies of G in G^2 be G and G'. Let the partite sets of G be V_1 and V_2 with the parity vertex labeling scheme, and give the same assignment to the two partite sets of G'. Based on the definition of the direct product, then one component C_s has vertices (g, g') where g and g' have the same parity and there exists an even length walk between (g, g') and (g', g) for all same parity g and g' pairs in C_s. Note that C_s includes when $g = g'$. Component C_d consists of (g, g') where g and g' have different parities resulting in $(g, g') \sim (g', g)$ for all g and g' in C_d.

Reiterated from earlier and as shown in [5], when K is K_2, the exponential adjacency matrix is found by colexicographically labeling the rows of A_{G^2} to give $A^*_{G^2}$, followed by row permuting $A^*_{G^2}$ to lexicographic row order. Let π be the permutation matrix that row permutes $A^*_{G^2}$ to lexicographic row order. The π permutation fixes rows where $g = g'$ but maps all other rows to their transpose. In other words, for all g and g', if $g \neq g'$ then (g, g') in $A^*_{G^2}$ maps to row (g', g) in $A^*_{G^2}$ resulting in the (g', g) row of $A_{G^{K_2}}$. Component C_s in G^2 has an isomorphic component C^*_s in G^{K_2} while C_d becomes two components in the exponential as discussed in Proposition 2. First we give a general lemma that applies to all G^K and is utilized in the proof of Proposition 2.

Lemma 1. *For any connected bipartite G and any K both with order $n > 1$, graph exponential G^K cannot have fewer than two components.*

Proof. Suppose that the partite sets of connected biparitite G have parity vertex labeling.The edges in the direct product are defined on homomorphisms that preserve the adjacency structure of G in G^{n_K}. When G is a connected bipartite graph, it is a given from WT that direct power G^{n_K} has $2^{(n_K)-1}$ bipartite components, one of which contains vertices whose elements have the same parity while all other components contain vertices of mixed parity. The fewest possible number of components in G^{n_K} is when $n_K = 2$; so consider this case. Since the construction of A_{G^K} is the row permutation of $A_{G^{n_K}}$, it is a given that row permuting G^{n_K} does not change the adjacency structure of G^{n_K}; nor does it change the parity of the vertex elements. Thus, this particular G^K maintains at least two components.

Now ponder a K with $n_K > 2$ resulting in G^{n_K} having $2^{(n_K)-1}$ components for a connected bipartite G. As before, one of these components contains vertices whose elements have the same parity while the other components contain mixed parity vertices. It is later shown that the permutations of $A_{G^{n_K}}$ to A_{G^K} can indeed reduce the number of components. However, it is still a fact that the row permutations cannot alter the parity of the elements in the vertices. Thus, it is not possible to have fewer than two components in G^K when G is connected and bipartite. □

Proposition 2. *Given a connected bipartite G, exponential G^{K_2} has three components.*

Proof. Let G have bipartite vertex sets V_1 and V_2 where V_1 has only even labeled vertices and V_2 has only odd vertices. It is a given that due to the bipartite nature of G, G^{K_2} cannot have fewer than two components from Lemma 1; and that the components of G^{K_2} are connected based on the parity of g and g' in $(g, g') \in V(G^{K_2})$ due to the parity labeling of the two vertex partitions in G. Define components C_s and C_d in G^2 as given previously and have C^*_s be the counterpart of C_s in G^{K_2}. Thus component C_s and C^*_s have vertex elements with the same parity. It is a given that a transpose of g and g' in $(g, g') \in C_s$ does not alter the parity of either g or g'. First we want to show that both C_s components are isomorphic.

Let $(g, g') \in V(C_s)$ and have $(g^{*'}, g^*) \in V(C^*_s)$. Because π does not change the parity of g and g', then C^*_s contains only $(g^{*'}, g^*)$ where $g^{*'}$ has the same parity as g^*. Thus the even walk between (g, g) and (g', g) in G^2 is maintained in G^{K_2}. Permutation matrix π that applies a transpose to (g, g') in G^2 generating $(g^{*'}, g^*)$ in G^{K_2} where $(g^{*'}, g^*)$ is merely the transpose of (g, g'). In other words, matrix π maps $N(g, g')$ in G^2 to $N(g^{*'}, g^*)$ in G^{K_2} preserving the vertex pair walks. Thus there exists a bijection between the members of C_s and C^*_s resulting in $C_s \cong C^*_s$.

Now let C_d be the component of G^2 such that for all $(g, g') \in C_d$ there exists a parity difference between g and g'. Then, for every (g, g'), transpose $(g', g) \in N(g, g')$ and vice versa. Consider the impact of π on the vertices in C_d. Similar to $C_s \rightarrow C^*_s$, π maps $N(g, g')$ in G^2 to $N(g^{*'}, g^*)$ in G^{K_2}; but now $(g^{*'}, g^*) \in N(g^{*'}, g^*)$ creates a loop in G^{K_2} and eliminates edge $((g, g'), (g', g))$ in C_d. As $N(g, g') \mapsto N(g^{*'}, g^*)$, the rest of $N(g, g')$ maps directly without change. Thus, for all (g, g') and all (g', g) in G^2, π generates two components, C^*_{d1} and C^*_{d2} where $(g, g') \in C^*_{d1}$ and $(g', g) \in C^*_{d2}$. Therefore, G^{K_2} has three components when G is a connected bipartite graph. □

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4.2. Connected G with Odd Cycle

WT tells us that if connected G contains an odd cycle, then G^2 is connected. Thus, for any vertex pair in G containing an odd cycle there is a walk of some length n. As G^2 is connected, then there also exists a n length walk between vertex pairs in G^2. As π merely row permutes, A_{G^2} to give $A_{G^{K_2}}$, then the number of neighbors in G^2 for any (g, g') becomes the number of neighbors of (g', g). In particular, we refer to two specific situations: (1) $(g', g) \in N(g, g')$ in G^2 and (2) (g, g') is a pendant in G^2.

For (1), let (g, g') be such a vertex in G^2 so there exists edge $(g, g') \in E(G)$. As G has an odd cycle, then the degree of both (g, g') and (g', g) in G^2 is at least 3. Then π maps $N(g, g')$ to the neighborhood of (g', g) in G^{K_2}; and edge $((g, g'), (g', g))$ is eliminated in G^{K_2} and both vertices in G^{K_2} gain loops. However, both neighborhoods in G^2 have degree at least 3, so (g, g') and (g', g) remain connected in G^{K_2}.

Now ponder situation (2). If G^2 contains a pendant vertex, then G must contain a pendant; so let the pendant in G be v. As G contains an odd cycle, then a neighbor of v must have degree at least 2. Given an arbitrary vertex x in G, then vx and xv both have degrees greater than 1 in G^2; and the only vertex with degree 1 in G^2 (and in G^{K_2}) is vv which is fixed under π so its degree is that of v. Hence, G^{K_2} remains connected even if G^2 contains a pendant. Corollary 1 in the next section states that for all K_n exponents, when G is connected and has an odd cycle, then G^K has one component.

5. Number of Components for K_n $(n > 2)$ as K

In this section, the order of K is assumed to be greater than 2. For $n_K > 2$, $\mathrm{Aut}(K_n)$ is the symmetric group on the set of n; thus $|\mathrm{Aut}(K_n)| = n!$ and the number of Hamiltonian cycles is $\frac{(n-1)!}{2}$ generating $(n-1)!$ directed Hamiltonian cycles (dH). Prior to focusing on the general case, we give a brief discussion regarding K_3 as it is also cycle graph C_3.

As K_3 is an odd cycle, the automorphism group for K_3 is also the dihedral group on three elements. Thus, for K_3 with $V(G) = \{1, 2, 3\}$, excluding the identity, the f_j combinations as n_K-tuples are $(2, 3, 1)$, $(3, 1, 2)$, $(2, 1, 3)$, $(1, 3, 2)$, $(3, 2, 1)$. Focusing on edge generation in G^K, any function combination containing a fixed vertex is disregarded and the set of f_j combinations that apply to G^K is $\{(2, 3, 1), (3, 1, 2)\}$.

As with $K := K_2$, we begin with a connected bipartite G. We also assume that G has the parity labeling scheme so V_1 has vertices with only even labels while V_2 has only odd labeled vertices.

5.1. Bipartite Connected G

From WT it is a given that for any bipartite G, G^{n_k} generates $2^{(n_k)-1}$ bipartite components. Based on the proof of WT and the parity vertex labeling scheme, one of the $2^{(n_k)-1}$ bipartite factors has vertex tuple elements that share the same parity. In this component, there are even length walks between the all-even vertex pairs and between the all-odd vertex pairs.

The other components of G^{n_K} all have vertex elements with mixed parity. Denote an even element as e and let o indicate an odd element. Using shorthand notation and example G^3, since these components are bipartite, vertex (eeo) $\sim$ (ooe); but vertex pair (oeo)$\sim$(eoe) is in a separate component from that of (eeo) and (ooe). In other words, given parity vertex labeling, the components of G^{n_K} are constructed based on both the ratio of odd to even elements and their ordered arrangement in the vertex tuples. As the choices are binary and the tuple size is n_K, then there are 2^{n_k} ordered combinations of e and o over the $2^{(n_K)-1}$ components so each component has $\frac{2^{n_K}}{2^{(n_K)-1}} = 2$ e-o ordered combinations.

Lemma 2. *Given a parity labeling of G, t he formation of the $2^{(n_k)-1}$ components of G^{n_K} is based on the 2^{n_K} even and odd ordered combinations of vertex element pairs where pairing is based on opposite parity.* ■

A bipartite G implies that G is loopless; thus, the focus here is on the $(n_K - 1)!$ directed Hamiltonian cycles (dH) of K_n as K $(n > 2)$. Denote the tuple vertices by the number of even

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and odd elements so that $(n)e$ indicates that there are n number of even tuple elements in a particular tuple; and similarly for odd elements. Every set of G^{n_K} components has a subset of components with tuples $((n_K-1)\mathrm{e}, 1\mathrm{o}) \sim (1\mathrm{e}, (n_K-1)\mathrm{o})$. For these components, in the set of dH, any repetition of a digit in a particular tuple position creates possibly redundant edges in G^{K_n}. As the regular degree for any K_n is $n_K - 1$, there are $(n_K-2)!$ listings of each digit d in a particular tuple position not indexed by d in the set of dH for K_n. This is based on the edge choices in each dH at each vertex that follow the initial vertex. Thus, excluding the component with same parity tuples, $\frac{(n_K-1)!}{(n_K-2)!} = n_K - 1$ number of dH that generate distinct edges in G^{K_n} for *all* of the mixed parity components. It needs to be noted that some subsets of components, such as those where $|\mathrm{e}| = |\mathrm{o}|$ in the tuples, have additional dH that create distinct edges in the exponential; but focus here is on the minimum number of dH that generate distinct exponential edges as these edges join components of G^{n_K} to form the components of G^{K_n}.

To see the redundant dH, suppose that K is K_4 with $V(K_4) = \{1,2,3,4\}$ resulting in distinct dH of (2341), (4123), (4312), (3142), (2413), and (3421) excluding the identity. Imagine K_2 as G with $V(K_2) = \{\mathrm{e},\mathrm{o}\}$ so (eoee) is a vertex in $(K_2)^4$ and in $K_2^{K_4}$. Denote $\pi_1 = (4123)$ and $\pi_2 = (3142)$. Then $\pi_1 \cdot (\mathrm{eoee}) = (\mathrm{eooo}) = \pi_2 \cdot (\mathrm{eoee})$ as $2 \mapsto 1$ in both π_1 and π_2.

Lemma 3. *When $n > 2$, for a connected bipartite G with the vertex parity labeling scheme, and excluding G^{K_n} vertices where all vertex tuple elements have the same parity, each K_n has a minimum of $n_K - 1$ number of directed Hamiltonian cycles that generate distinct edges in G^{K_n} for its vertices.*■

When $G := K_2$ has $V(K_2) = \{\mathrm{e},\mathrm{o}\}$, in the $2^{(n_K)-1}$ components, there are 2^{n_k} e-o ordered combinations that are also the vertices of G^{n_K} since $|G| = 2$. However, if G has $|G| > 2$ such as $G := C_4$ with $V(C_4) = \{1,2,3,4\}$ where $V_1 = \{2,4\}$ while $V_2 = \{1,3\}$, then e represents $\{2,4\}$ and o reflects $\{1,3\}$. In this case, there are still 2^{n_K} e-o ordered combinations but these combinations are over 4^{n_K} vertices of $C_4^{n_k}$ in $2^{(n_K)-1}$ components.

Again suppose K_2 is G with $V(K_2) = \{\mathrm{e},\mathrm{o}\}$, but now let K be K_3 with $V(K_3) = \{1,2,3\}$. Then the permutations that apply to G are $\pi_1 = (2,3,1)$ and $\pi_2 = (3,1,2)$. Thus, $\pi_1 \cdot (\mathrm{ooo}) = (\mathrm{eee}) = \pi_2 \cdot (\mathrm{ooo})$ and $\pi_1 \cdot (\mathrm{oeo}) = (\mathrm{eeo})$ and $\pi_2 \cdot (\mathrm{oeo}) = (\mathrm{oee})$ as shown in Figure 3. As noted in the figure, the components in $(K_2)^3$ are derived from the e to o ratio plus the ordering of the e and o elements; however, the $K_2^{K_3}$ components are formed exclusively on the e to o ratio. In other words, the e and o element ordering does not matter in the $K_2^{K_3}$ components; only the e-o ratio is used in component formation. $K_2^{K_3}$ is $K_2 + C_6$ [4].

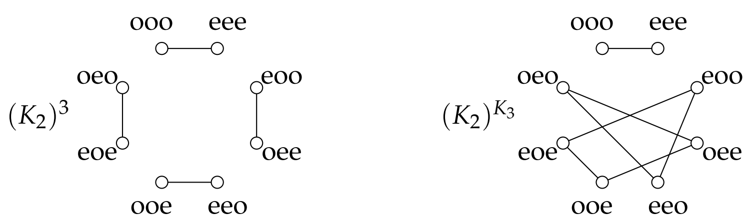

Figure 3. Vertex parity and connectivity for $(K_2)^3$ and $K_2^{K_3}$.

Theorem 2. *With a connected bipartite G and K_n as K where $n > 2$, then the $2^{(n_K)-1}$ number of G^{n_K} components reduces to $\lceil \frac{n_K+1}{2} \rceil$ components in G^{K_n}.*

Proof. Suppose connected bipartite G has parity vertex labeling for V_1 and V_2. It is a given that G^{n_K} generates $2^{(n_K)-1}$ bipartite components based on ordered combinations of even and odd tuple elements by Lemma 2. However, these $2^{(n_K)-1}$ ordered combination pairs can be partitioned into $n_K + 1$ unordered combinations based on the e to o ratio. Within these unordered combinations, the combination that contains all-even tuple elements plus the combination with all-odd elements, can be eliminated as it was shown in the proof of Proposition 2 that this component (call it F_1), consisting of

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two combinations in G^{n_K}, presents an isomorphic component in G^{K_n}; and the rationale is the same for any n.

Our focus is now on the $n_k - 1$ unordered combinations that have mixed parity in their tuples. Lemma 3 tells us that the minimum number of directed Hamiltonian cycles in K_n ($n > 2$ here) is $n_k - 1$; so apply these directed cycles to the $n_K - 1$ unordered even-odd combinations to get $\lceil \frac{n_K - 1}{2} \rceil$ components with mixed parity tuples. Adding the component isomorphic to F_1 to this component set generates $\lceil \frac{n_K + 1}{2} \rceil$ components in G^{K_n} when $n > 2$. $\square$

5.2. Connected G with Odd Cycle

When connected G has an odd cycle, then G^{n_K} has a single component. Suppose K is K_n. Then the set of dH contains no reason that the single component in G^{n_K} is partitioned. Hence, the rationale discussed for K_2 as K also applies to all K_n as K when $n > 2$ and G is connected with an odd cycle.

Corollary 1. *Assume that G is a connected graph with an odd cycle. Then G^{K_n} has one connected component.* ■

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