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Yue Gu and [Shuanhong Wang](#) *

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Article

Braided Quantum Groups

Yue Gu ¹ and Shuanhong Wang ^{2,3,*}

¹ School of Physical and Mathematical Sciences, Nanjing Tech University, Nanjing, Jiangsu 211816, China

² Shing-Tung Yau Center and School of Mathematics, Southeast University, Nanjing, Jiangsu 210096, China

³ SIMIS, Block A, International Innovation Plaza, No. 657 Songhu Road, Yangpu District, Shanghai 200433, China

* Correspondence: shuanhwang@seu.edu.cn

Abstract

In this paper we introduce and study the concept of a braided quantum group with a nontrivial example based on any braided group in a symmetric category. This generalizes the notions of a braided Hopf algebra with an integral, a Hopf group-coalgebra with a group-integral and an algebraic quantum group.

Keywords: braided multiplier; braided multiplier Hopf algebra; braided quantum group

1. Introduction

In 2004, the authors in [2] showed that Hopf group-coalgebras (or Turaev's group-coalgebras) are Hopf algebras in a symmetric monoidal category, which we call the Turaev category. A similar result holds for Hopf group-algebras. In 2007, the authors in [1] proved that Turaev's group-coalgebras are group-cograded multiplier Hopf algebras.

When the first author explained the concept of categories to a few outstanding university students in her university and had them read the literature [1,2], two weeks later they asked the first author whether there could be a braided version of the concept of a multiplier Hopf algebra. The reasoning is that the Hopf group-coalgebra (or Turaev's group-coalgebra) (see [11,14] or [15]) can, on the one hand, be seen as a braided Hopf algebra over a certain symmetric category (see [2]), and on the other hand, it is also a type of co-graded multiplier Hopf algebra (see [1]). Therefore, in order to answer the students' questions and to give back to teaching through research, we wrote this academic paper starting from a teaching perspective to inspire students to pursue scientific research.

The main of this article is to introduce and study the concept of a braided quantum group with a nontrivial example based on any braided group in a symmetric category. This generalizes the notions of a braided Hopf algebra with an integral (see [10]), a Hopf group-coalgebra with a group-integral (see [11]) and an algebraic quantum group (see [13]). In a separate article, we specifically study the Pontryagin duality problem of braided quantum groups.

This paper is organized as follows. In Section 1, we recall some basic notions of braided monoidal categories and symmetric categories. In Section 2, we introduce the notion of braided multiplier rings and study their properties (see Propositions 1 and 4).

In Section 3, we mainly introduce the concept of a braided multiplier Hopf algebra in a symmetric category $\mathcal{C}$ (see Theorem 1). In the final section, we introduce the concept of a braided quantum group and develop the main result established as in last section (see Theorem 3).

2. Preliminaries

Let us recall some basic definitions. A *monoidal category* $\mathcal{C} = (\mathcal{C}, a, I, l, r)$ is a category $\mathcal{C}$ endowed with a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a tensor unit object $I \in \mathcal{C}$ and natural associativity constraint isomorphisms

$$a = a_{U,V,W} : (UV)W \cong U(VW)$$

for all $U, V, W \in \mathcal{C}$ and a left unit constraint $l = l_U : IU \cong U$, a right unit constraint $r = r_U : UI \cong U$, for any $U \in \mathcal{C}$ such that the associativity pentagon:

$$a_{U,V,WX} \circ a_{UV,W,X} = (id_U a_{V,W,X}) \circ a_{U,VW,X} \circ (a_{U,V,W} id_X)$$

and

$$(id_U l_V) \circ (r_U id_V) = a_{U,I,V}$$

are satisfied. A tensor category $\mathcal{C}$ is *strict* if all the constraints are identities. We say that $\mathcal{C}$ is left (resp. right) autonomous when any object has a left (resp. right) dual and that $\mathcal{C}$ is autonomous if it is both left and right autonomous. For the basic category theory, we refer to [5].

A monoidal category $\mathcal{C}$ is called a *braided monoidal category* if for every object M in $\mathcal{C}$ we have natural isomorphisms $\tau_{M,-} : M \otimes - \cong - \otimes M$ and $\tau_{-,M} : - \otimes M \cong M \otimes -$ which verifies some appropriate coherence conditions, we refer to [4].

An object A in a monoidal category $\mathcal{C}$ is called an *algebra* if there are two morphisms in $\mathcal{C}$: $\mu_A : AA \rightarrow A$, $\eta_A : I \rightarrow A$ satisfying the associativity and unitary conditions of usual algebras but expressed in diagrams. We call μ_A is the product in A . If $\mathcal{C}$ is braided, then the opposite algebra A^{op} of A in $\mathcal{C}$ is the object A with the product μ_A^{op} in A^{op} given by $\mu_A^{op} = \mu_A \circ \tau$.

We recall from [6,7] some notions concerning modules in a monoidal category. Let $\mathcal{C}$ be a monoidal category, and let A and B be algebras in $\mathcal{C}$. An object M in $\mathcal{C}$ is called an *A-B-bimodule* if there are two morphisms in $\mathcal{C}$: $m_A : AM \rightarrow M$ and $m_B : MB \rightarrow M$ satisfying the coherence conditions: $m_A \circ (id m_A) = m_A \circ (\mu_A id)$, $m_B \circ (id m_B) = m_B \circ (id \mu_B)$ and $m_A \circ (id m_B) = m_B \circ (m_A id)$ on AMB . The bimodule category ${}_A \mathcal{C}_B$ consists of objects in $\mathcal{C}$ which have an *A-B-bimodule* structure and morphisms in $\mathcal{C}$ which are *A-B-bilinear*. Write ${}_A \mathcal{C}$, $\mathcal{C}_B$ for the categories ${}_A \mathcal{C}_I$ and ${}_I \mathcal{C}_B$, respectively.

We will work on the monoidal category $\mathcal{C} = {}_H \mathcal{M}$ over a Hopf algebra H with a bijective antipode S . An algebra A in $\mathcal{C} = {}_H \mathcal{M}$ is equivalent to that A is a left H -module algebra. We say that A is non-degenerate if $ab = 0$ for all $b \in A$ implies $a = 0$ and $ab = 0$ for all $a \in A$ implies $b = 0$. A linear functional $\omega : A \rightarrow \mathbb{K}$ is said to be faithful if $\omega(ab) = 0$ for all b implies $a = 0$ and $\omega(ab) = 0$ for all a implies $b = 0$. A left A -module M is called nondegenerate if for all $m \in M$ the equalities $am = 0$ for all $a \in A$ imply that $m = 0$. A left A -module M is called *idempotent* (or *unital*) if $M = AM$.

Let H be an ordinary coalgebra. We will use the simplest Sweedler's notation (cf. [9]): $\Delta(h) = \sum h_1 \otimes h_2$ for any $h \in H$.

3. Braided Multiplier Rings

In this section we introduce the notion of braided multiplier rings and study their properties generalizing the corresponding notions and results in [3].

Definition 1. Let A be an algebra in $\mathcal{C}$. Then we say that a linear map $\mathfrak{l} : A \rightarrow A$ is a *left multiplier* of A if $\mathfrak{l}\mu = \mu(\mathfrak{l}id)$. Similarly, we call a linear $\rho : A \rightarrow A$ is a *right multiplier* of A if $\rho\mu = \mu(id\rho)$. We denote the space of left and right multipliers by $\mathbb{L}(A)$ and $\mathbb{R}(A)$, respectively.

Denote the set of all pairs $(\mathfrak{l}, \rho)$ by $\mathbb{M}(A)$ where $\mathfrak{l} \in \mathbb{L}(A)$ and $\rho \in \mathbb{R}(A)$ such that $\mu(id\mathfrak{l}) = \mu(\rho id)$ on AA . We call the such pair $(\mathfrak{l}, \rho)$ a *multiplier*.

Proposition 1. Let A be an algebra in $\mathcal{C}$. Then $\mathbb{M}(A)$ can be viewed as an algebra in $\mathcal{C}$ with multiplication

$$(\mathfrak{l}, \rho)(\mathfrak{l}', \rho') = (\mathfrak{l} \circ \mathfrak{l}', \rho' \circ \rho)$$

for all $(\mathfrak{l}, \rho), (\mathfrak{l}', \rho') \in \mathbb{M}(A)$, and with unit $1_{\mathbb{M}(A)} = (id, id)$.

Proof. Obviously, the associativity of the multiplication and the unitary are the same as ones in the usual multiplier algebra. Now, we have to prove that $\mathbb{M}(A)$ is an object in $\mathcal{C}$. This consists of the following three steps:

Step 1) Define the following linear map $\triangleright$:

$$\triangleright : H\mathbb{L}(A) \longrightarrow \mathbb{L}(A), \quad (h \triangleright \mathfrak{t})(a) = \sum h_1 \mathfrak{t}(S(h_2) \mathfrak{a}))$$

for all $a \in A, \mathfrak{t} \in \mathbb{L}(A)$ and $h \in H$. In fact, we compute, for all $a, b \in A$

$$\begin{aligned} (h \triangleright \mathfrak{t})(ab) &= \sum h_1 \mathfrak{t}(S(h_2)(ab)) \\ &= \sum h_1 \mathfrak{t}(S(h_3) \mathfrak{a})(S(h_2) \mathfrak{b})) \\ &= \sum (h_1 \mathfrak{t}(S(h_4) \mathfrak{a}))(h_2 S(h_3) \mathfrak{b})) = (h \triangleright \mathfrak{t})(a)b \end{aligned}$$

and so $h \triangleright \mathfrak{t} \in \mathbb{L}(A)$. Moreover, $\mathbb{L}(A) \in \mathcal{C}$. First, it is easy to see that $(\mathbb{L}(A), \triangleright)$ is a left H -module and $h \triangleright id = \varepsilon(h)id$ for all $h \in H$. Then we also have that

$$\begin{aligned} \sum ((h_1 \triangleright \mathfrak{t}) \circ (h_2 \triangleright \mathfrak{t}'))(a) &= \sum (h_1 \triangleright \mathfrak{t})(h_2 \mathfrak{t}'(S(h_3) \mathfrak{a})) \\ &= \sum h_1 \mathfrak{t}(S(h_2) h_3 \mathfrak{t}'(S(h_4) \mathfrak{a}))) \\ &= (h \triangleright (\mathfrak{t} \circ \mathfrak{t}'))(a) \end{aligned}$$

for all $\mathfrak{t}, \mathfrak{t}' \in \mathbb{L}(A), a \in A$ and $h \in H$, and so

$$h \triangleright (\mathfrak{t} \circ \mathfrak{t}') = \sum (h_1 \triangleright \mathfrak{t}) \circ (h_2 \triangleright \mathfrak{t}').$$

Step 2) We define the following linear map $\vdash$:

$$\vdash : H\mathbb{R}(A) \longrightarrow \mathbb{R}(A), \quad (h \vdash \rho)(a) = \sum h_2 \rho(S^{-1}(h_1) \mathfrak{a}))$$

for all $a \in A, \rho \in \mathbb{R}(A)$ and $h \in H$. We check that $h \vdash \rho \in \mathbb{R}(A)$ as follows, for all $a, b \in A$

$$\begin{aligned} (h \vdash \rho)(ab) &= \sum h_2 \cdot \rho(S^{-1}(h_1) \cdot (ab)) \\ &= \sum h_3 ((S^{-1}(h_2) \cdot a) \rho(S^{-1}(h_1) \cdot b)) \\ &= \sum (h_3 S^{-1}(h_2) \cdot a) (h_4 \cdot \rho(S^{-1}(h_1) \cdot b)) \\ &= a(h \vdash \rho)(b). \end{aligned}$$

It is straightforward to check that $(\mathbb{R}(A), \vdash)$ is a left H -module and $h \vdash id = \varepsilon(h)id$ for all $h \in H$. However, $\mathbb{R}(A)$ is not an object in $\mathcal{C}$. In fact, we compute

$$\begin{aligned} \sum ((h_2 \vdash \rho) \circ (h_1 \vdash \rho'))(a) &= \sum (h_3 \vdash \rho)(h_2 \cdot \rho'(S^{-1}(h_1) \cdot a)) \\ &= \sum h_4 \rho(S^{-1}(h_3) h_2 \cdot \rho'(S^{-1}(h_1) \cdot a)) \\ &= (h \vdash (\rho \circ \rho'))(a) \end{aligned}$$

for all $\rho, \rho' \in \mathbb{R}(A), a \in A$ and $h \in H$, and so

$$h \vdash (\rho \circ \rho') = \sum (h_2 \vdash \rho) \circ (h_1 \vdash \rho').$$

In final step 3), we first define a left H -module action on $\mathbb{M}(A)$ by the following formula:

$$h \cdot (\mathfrak{t}, \rho) = (h \triangleright \mathfrak{t}, h \vdash \rho)$$

for all $h \in H$ and $(\mathfrak{t}, \rho) \in \mathbb{M}(A)$. First, to check $h \cdot (\mathfrak{t}, \rho) \in \mathbb{M}(A)$, we compute

$$\begin{aligned} a((h \triangleright \mathfrak{t})(b)) &= \sum a(h_1 \cdot (\mathfrak{t}(S(h_2) \cdot b))) \\ &= \sum h_2 \cdot ((S^{-1}(h_1) \cdot a)\mathfrak{t}(S(h_3) \cdot b)) \\ &= \sum h_2 \cdot (\rho(S^{-1}(h_1) \cdot a)(S(h_3) \cdot b)) \\ &= \sum (h_2 \cdot \rho(S^{-1}(h_1) \cdot a))(h_3 S(h_4) \cdot b) \\ &= (h \vdash \rho)(a)b. \end{aligned}$$

Finally, we have that for all $(\mathfrak{t}, \rho), (\mathfrak{t}', \rho') \in \mathbb{M}(A)$ and $h \in H$,

$$\begin{aligned} h \cdot ((\mathfrak{t}, r)(\mathfrak{t}', \rho')) &= (h \triangleright (\mathfrak{t}'), h \vdash (\rho' \rho)) \\ &= \sum ((h_1 \triangleright \mathfrak{t}) \circ (h_2 \triangleright \mathfrak{t}'), (h_2 \vdash \rho') \circ (h_1 \vdash \rho)) \\ &= \sum ((h_1 \triangleright \mathfrak{t}), (h_1 \vdash \rho))((h_2 \triangleright \mathfrak{t}'), (h_2 \vdash \rho')) \\ &= \sum (h_1 \cdot (\mathfrak{t}, \rho))(h_2 \cdot (\mathfrak{t}', \rho')) \end{aligned}$$

and $h \in (id, id) = \varepsilon(h)(id, id)$.

This computes the proof of the proposition. $\square$

Proposition 2. *With notations as above. Then there are natural algebra homomorphism in $\mathcal{C}$*

$$\begin{aligned} i_A : A &\hookrightarrow \mathbb{M}(A), \quad i_A(a) = (\mathfrak{t}_a, \rho_a), \\ Y_A^l : \mathbb{M}(A) &\longrightarrow \mathbb{L}(A), \quad Y_A^l((\mathfrak{t}, \rho)) = \mathfrak{t}, \quad \text{and} \\ Y_A^r : \mathbb{M}(A) &\longrightarrow \mathbb{R}(A)^{op}, \quad Y_A^r((\mathfrak{t}, \rho)) = \rho \end{aligned}$$

where $\mathfrak{t}_a(b) = ab$ and $\rho_a(b) = ba$ for all $a, b \in A$.

Proof. Obviously, these map are algebra homomorphisms, and Y_A^l and Y_A^r are morphisms in $\mathcal{C}$.

On the other hand, by the proof of the Proposition, we know that $\mathbb{L}(A), \mathbb{R}(A)^{op}$ and $\mathbb{M}(A)$ are algebras in $\mathcal{C}$. For all $a, b \in A$ and $h \in H$, we have that

$$(h \triangleright \mathfrak{t}_a)(b) = \sum h_1(\mathfrak{t}_a(S(h_2)\mathfrak{b})) = \sum (h_1\mathfrak{a})(h_2(S(h_3)\mathfrak{b})) = (h\mathfrak{a})b = \mathfrak{t}_{h\mathfrak{a}}(b).$$

Similarly, we get $(h \vdash \rho_a) = \rho_{h\mathfrak{a}}$. From these equations we imply that i_A is a morphism in $\mathcal{C}$.

This computes the proof of the proposition. $\square$

Corollary 1. *With notations as above. Then A is non-degenerate if and only if $L = Y_A^l \circ i_A$ and $R = Y_A^r \circ i_A$ are injective, in particular, i_A is injective as well.*

Definition 2. Assume that H is any Hopf algebra. We say that A is a left H -module algebera extension of B if the following conditions hold:

- (i) A and B are left H -module algebras,
- (ii) $(A, \rightharpoonup, \leftharpoonup)$ is a B -bimodule in $\mathcal{C}$,
- (iii) $(b \rightharpoonup a)a' = b \rightharpoonup (aa')$ and $a(a' \leftharpoonup b) = (aa') \leftharpoonup b$ for all $a, a' \in A$ and $b \in B$,
- (iv) $(a \leftharpoonup b)a' = a(b \rightharpoonup a')$ for all $a, a' \in A$ and $b \in B$.

Furthermore, A is said to be a nondegenerate (resp. idempotent) left H -module algebera extension of B if A is nondegenerate (resp. idempotent) as a B -bimodule as well.

Proposition 3. *Let A be a nondegenerate algebra $\mathcal{C}$. Then A is a nondegenerate idempotent left H -module algebera extension of $\mathbb{M}(A)$ with actions given by*

$$z \rightharpoonup a = \mathfrak{t}(a) \quad \text{and} \quad a \leftharpoonup z = \rho(a)$$

for any $z = (\mathfrak{z}, \rho) \in \mathbb{M}(A)$ and $a \in A$.

Proof. By Proposition 1, we know that a left H -module algebra. We prove that $\rightarrow$ is a morphism in $\mathcal{C}$. Similarly, we get $\leftarrow$ is a morphism in $\mathcal{C}$ as well. In fact, we compute

$$\begin{aligned} \rightarrow(h\mathfrak{z}a) &= \sum (h_1\mathfrak{z}) \rightarrow (h_2\mathfrak{a}) = (h_1 \triangleright \mathfrak{z})(h_2\mathfrak{a}) \\ &= \sum h_1(\mathfrak{z}(S(h_2)h_3\mathfrak{a})) = h\mathfrak{z}(a) = h(\mathfrak{z} \rightarrow a) \end{aligned}$$

for all $h \in H, z \in \mathbb{M}(A)$ and $a \in A$. The rest are obvious. Also, since for all $(\mathfrak{z}, \rho), (\mathfrak{z}', \rho') \in \mathbb{M}(A)$ and $a, b \in A$, we have that

$$\mathfrak{z}(\rho'(a))b = \mathfrak{z}(\rho'(a)b) = \mathfrak{z}(a\mathfrak{z}'(b)) = \mathfrak{z}(a)\mathfrak{z}'(b) = \rho'(\mathfrak{z}(a))b.$$

By the nondegenerateness of A , the above equation implies A is a $\mathbb{M}(A)$ -bimodule.

The conditions (iii) and (iv) in Definition 2 are obvious.

Finally, we have to prove that A is nondegenerate as a $\mathbb{M}(A)$ -bimodule, e.g., for any $a \in A$ such that $z \rightarrow a = 0$ for all $z \in \mathbb{M}(A)$ we want to show that $a = 0$. In fact, since $A = A \leftarrow \mathbb{M}(A)$, we can write $b = \sum_i b_i \leftarrow z_i$ for $b_i \in A$ and $z_i \in \mathbb{M}(A)$. Hence we have that $ba = (\sum_i b_i \leftarrow z_i)a = \sum_i b_i(z_i \rightarrow a) = 0$. By the nondegenerateness of A , we conclude that $a = 0$. $\square$

Proposition 4. Let A and B be algebras in $\mathcal{C}$. Then

- (i) There exists a bijective correspondence between algebra homomorphisms $L : B \rightarrow \mathbb{L}(A)$ in $\mathcal{C}$ and left B -module structures on A such that $\mu_A(baa') = b\mu_A(aa')$ for all $a, a' \in A$ and $b \in B$,
- (ii) There exists a bijective correspondence between algebra homomorphisms $R : B \rightarrow \mathbb{R}(A)^{op}$ in $\mathcal{C}$ and right B -module structures on A such that $\mu_A(aa'b) = \mu_A(aa')b$ for all $a, a' \in A$ and $b \in B$,
- (iii) There exists a bijective correspondence between algebra homomorphisms $M : B \rightarrow \mathbb{M}(A)$ in $\mathcal{C}$ and left H -module algebra extension structures of B on A .

Proof. (i) If there is an algebra homomorphisms $L : B \rightarrow \mathbb{L}(A)$ in $\mathcal{C}$, then we define $\bullet : BA \rightarrow A$ by $b \bullet a = L(b)(a)$ for all $b \in B$ and $a \in A$. It is straightforward to check that the action is associative and that $\mu_A(b \bullet aa') = b \bullet \mu_A(aa')$ for all $a, a' \in A$ and $b \in B$. We also have to prove that $\bullet$ is a left H -linear, in fact, we have

$$\begin{aligned} \bullet(h\mathfrak{b}a) &= \sum \bullet(h_1\mathfrak{b}h_2\mathfrak{a}) = L(h_1\mathfrak{b})(h_2\mathfrak{a}) \\ &= \sum (h_1 \triangleright L(b))(h_2\mathfrak{a}) = h_1(L(b)(S(h_2)h_3\mathfrak{a})) \\ &= h(L(b)(a)) = h(b \bullet a) \end{aligned}$$

for all $a \in A, b \in B$ and $h \in H$. Conversely, if A is a left B -module with action $\succ$ in $\mathcal{C}$, then define $L : B \rightarrow \mathbb{L}(A)$ by $L(b)(a) = b \succ a$ for all $a \in A$ and $b \in B$. Since μ_A is a left B -linear this map is well-defined. It is not hard to show that L is an algebra map. Finally, it remains to show that L is morphism in $\mathcal{C}$. Indeed,

$$\begin{aligned} (h \triangleright L(b))(a) &= \sum h_1(L(b)(S(h_2)\mathfrak{a})) = \sum h_1(b \succ (S(h_2)\mathfrak{a})) \\ &= \sum (h_1\mathfrak{b}) \succ (h_2S(h_3)\mathfrak{a}) = (h\mathfrak{b}) \succ a = L(h\mathfrak{b})(a) \end{aligned}$$

for all $a \in A, b \in B$ and $h \in H$.

(ii) Similar to the part (i)

(iii) Suppose that the map $M : B \rightarrow \mathbb{M}(A)$ in $\mathcal{C}$ exists. Then we get two algebras maps $Y_A^l \circ M : B \rightarrow \mathbb{L}(A)$ in $\mathcal{C}$ and $Y_A^r \circ M : B \rightarrow \mathbb{R}(A)^{op}$ in $\mathcal{C}$. Thus by the part (i) and the part (ii), A will be a left and right B -module with B -actions given by $b \bullet a = Y_A^l(M(b))(a) = M(b) \rightarrow a$ and $a\mathfrak{b} = Y_A^r(M(b))(a) = a \leftarrow M(b)$ for all $a \in A$ and $b \in B$, where $\rightarrow$ and $\leftarrow$ are the left and right

$\mathbb{M}(A)$ -actions on A (see Proposition 3). Since A is a left H -module algebra extension of $\mathbb{M}(A)$ it follows that A is a nondegenerate idempotent left H -module algebra extension of B . In converse, By the first two parts, we have algebra morphisms in $\mathcal{C}$ $L : B \longrightarrow \mathbb{L}(A)$ and $R : B \longrightarrow \mathbb{R}(A)^{op}$. We define $M : B \longrightarrow \mathbb{M}(A)$ by $M(b) = (L(b), R(b))$ for all $b \in B$. The verifications of the rest things such that M is an algebra morphism in $\mathcal{C}$ is straightforward.

This completes the proof of the proposition. $\square$

4. Braided Multiplier Hopf Algebras

In this section we introduce the concept of a braided multiplier Hopf algebra in $\mathcal{C}$ generalizing the main result in [12].

Let A be an algebra in $\mathcal{C}$. Let $\Delta : A \longrightarrow M(AA)$ be a morphism in $\mathcal{C}$. Then the left-right, right-left, left-left and right-right entwining maps $E_{lr}, E_{rl}, E_{ll}, E_{rr} : AA \longrightarrow M(AA)$ for the pair (μ, Δ) are defined by

$$\begin{aligned} E_{lr} &= (id\mu_A) \circ (\Delta_A id), & E_{rl} &= (\mu_A id) \circ (id\Delta_A), \\ E_{ll} &= (\mu_A^{op} id) \circ (id\Delta_A) \circ \tau, & E_{rr} &= (id\mu_A^{op}) \circ (\Delta_A id) \circ \tau. \end{aligned}$$

Similarly, we write $E_{lr}^{op}, E_{rl}^{op}, E_{ll}^{op}, E_{rr}^{op}$ for the corresponding entwining maps of the pair (μ^{op}, Δ) .

The following lemma is straightforward.

Lemma 1. *With the notations as above. If Δ is a morphism in $\mathcal{C}$, then E_{lr}, E_{rl}, E_{ll} and E_{rr} are all morphisms in $\mathcal{C}$.*

Definition 3. *Let A be an algebra in $\mathcal{C}$. A morphism $\Delta : A \longrightarrow M(AA)$ in $\mathcal{C}$ is called a coproduct if it is coassociative in the sense of the following*

$$(E_{rl}id) \circ (idE_{lr}) = (idE_{lr}) \circ (E_{rl}id). \quad (1)$$

If $\mathcal{C}$ is braided, then the opposite coproduct $\Delta^{cop} : A \longrightarrow M(AA)$ in $\mathcal{C}$ is defined by $\Delta^{cop} = \tau\Delta$. We will write A^{cop} for A equipped with the opposite coproduct. Then we write $E_{lr}^{cop}, E_{rl}^{cop}, E_{ll}^{cop}, E_{rr}^{cop}$ for the corresponding entwining maps of the pair (μ, Δ^{cop}) .

We may also reverse both product and coproduct and then get $A^{opcop} =: (A^{op})^{cop}$. Similarly, we write $E_{lr}^{opcop}, E_{rl}^{opcop}, E_{ll}^{opcop}, E_{rr}^{opcop}$ for the corresponding entwining maps of the pair (μ^{op}, Δ^{cop}) .

It is not hard to check the lemma below.

Lemma 2. *Let $\Delta : A \longrightarrow M(AA)$ be a coproduct in the braided monoidal category $\mathcal{C}$. Then*

$$\begin{aligned} E_{lr}^{op} &= E_{rr} \circ \tau, & E_{rl}^{op} &= E_{ll} \circ \tau, & E_{ll}^{op} &= E_{rl} \circ \tau, & E_{rr}^{op} &= E_{lr} \circ \tau, \\ E_{lr}^{cop} &= \tau \circ E_{ll}, & E_{rl}^{cop} &= \tau \circ E_{rr}, & E_{ll}^{cop} &= \tau \circ E_{lr}, & E_{rr}^{cop} &= \tau \circ E_{rl}, \\ E_{lr}^{opcop} &= \tau \circ E_{rl} \circ \tau, & E_{rl}^{opcop} &= \tau \circ E_{lr} \circ \tau, \\ E_{ll}^{opcop} &= \tau \circ E_{rr} \circ \tau, & E_{rr}^{opcop} &= \tau \circ E_{ll} \circ \tau. \end{aligned}$$

Definition 4. *Let $\Delta : A \longrightarrow M(AA)$ be a coproduct in $\mathcal{C}$. An algebra homomorphism $\varepsilon : A \longrightarrow \mathbb{K}$ in $\mathcal{C}$ is called a counit if*

$$(\varepsilon id) \circ E_{lr} = \mu = (id\varepsilon) \circ E_{rl}. \quad (2)$$

Definition 5. *Let $\Delta : A \longrightarrow M(AA)$ be a coproduct in $\mathcal{C}$ and ε a counit on A . We say that A is a braided multiplier Hopf algebra if E_{lr} and E_{rl} are bijective so that $E_{lr}(AA), E_{rl}(AA) \subseteq AA$. Furthermore, A is called regular if E_{ll} and E_{rr} are bijective such that $E_{ll}(AA), E_{rr}(AA) \subseteq AA$.*

Example 1. Any braided Hopf algebra in $\mathcal{C}$ is a braided multiplier Hopf algebra in $\mathcal{C}$.

Example 2. For any a braided group Q in $\mathcal{C}$ (cf. [8]). Let $A = \mathbb{C}(Q)$ be the algebra of complex, finitely supported functions on Q . In this case $M(\mathbb{C}(Q))$ consists of all complex function on Q . Moreover $\mathbb{C}(Q) \otimes \mathbb{C}(Q)$ can be identified with finitely supported complex function on $Q \times Q$ so that $M(\mathbb{C}(Q) \otimes \mathbb{C}(Q))$ is the space of all complex functions on $Q \times Q$.

We now define $\Delta : \mathbb{C}(Q) \longrightarrow M(\mathbb{C}(Q) \otimes \mathbb{C}(Q))$ by

$$\Delta(f)(p, q) = f(pq), \quad \text{for all } p, q \in Q$$

and $\varepsilon : \mathbb{C}(Q) \longrightarrow \mathbb{C}$ by

$$\varepsilon_A(f) = f(e) \quad \text{for all } f \in \mathbb{C}(Q).$$

It is straightforward to check that Δ and ε are algebra homomorphisms.

We define:

$$E_{lr}(fg)(p, q) = \Delta(f)(1 \otimes g)(p, q) = \Delta(f)(p, q)(1 \otimes g)(p, q) = f(pq)g(q)$$

and

$$E_{rl}(fg)(p, q) = (f \otimes 1)\Delta(g)(p, q) = (f \otimes 1)(p, q)\Delta^A(g)(pq) = f(p)g(pq),$$

$E_{lr}(f \otimes g)$ and $E_{rl}(f \otimes g)$ have finite support and belong to $\mathbb{C}(Q) \otimes \mathbb{C}(Q)$.

Similarly, we define E_{lr} and E_{rl} by

$$R_{lr}(h)(p, q) = h(pq^{-1}, q) \quad \text{and} \quad R_{rl}(h)(p, q) = h(p, p^{-1}q),$$

for all $h \in \mathbb{C}(Q) \otimes \mathbb{C}(Q)$ and $p, q \in Q$. It is easy to check that E_{lr} is bijective with the inverse R_{lr} . Similarly for E_{rl} with the inverse R_{rl} .

Furthermore, we have

$$\begin{aligned} [(\varepsilon \text{id})E_{lr}(fg)](p) &= (\Delta(f)(1 \otimes g))(e, p) \\ &= \Delta(f)(e, p)(1 \otimes g)(e, p) = f(p)g(p) = (fg)(p), \end{aligned}$$

and similarly, $[(\text{id}\varepsilon)E_{rl}(f \otimes g)] = fg$ for all $f, g \in \mathbb{C}(Q)$ and $p \in Q$.

Therefore, $A = \mathbb{C}(Q)$ is a braided multiplier Hopf algebra. ■

Corollary 2. Let (A, Δ) be as before with counit ε . Then

$$(\varepsilon \text{id}) \circ E_{rr} = \mu = (\text{id}\varepsilon)E_{ll}.$$

Proposition 5. Let (A, μ, Δ) be as before and assume that ε is a counit. Then

- (1) if ε' is any linear map from A to $\mathbb{K}$ satisfying $(\mathbb{K}\varepsilon') \circ E_{rl} = \mu$, then $\varepsilon = \varepsilon'$.
- (2) if ε' is any linear map from A to $\mathbb{K}$ satisfying $(\mathbb{K}\varepsilon') \circ E_{ll} = \mu$, then $\varepsilon = \varepsilon'$.
- (3) if ε' is any linear map from A to $\mathbb{K}$ satisfying $(\varepsilon'\mathbb{B}) \circ E_{lr} = \mu$, then $\varepsilon = \varepsilon'$.
- (4) if ε' is any linear map from A to $\mathbb{K}$ satisfying $(\varepsilon'\mathbb{B}) \circ E_{rr} = \mu$, then $\varepsilon = \varepsilon'$.

Proof. Notice that ε is an algebra map. Assume that ε' is any linear map from A to $\mathbb{K}$ satisfying $(\text{id}\varepsilon')E_{rl}(ab) = ab$ for all $a, b \in A$. We know that $(\varepsilon \text{id})E_{rl}(ab) = \varepsilon(a)b$ since ε is a counit. Applying ε' to this equation we obtain that $\varepsilon(ab) = \varepsilon(a)\varepsilon'(b)$. Because $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ and ε can not be trivially zero, we get $\varepsilon(b) = \varepsilon'(b)$ for all $b \in A$. This proves the part (1). The other ones are gotten in a similar way. $\square$

Lemma 3. Let $\Delta : A \longrightarrow M(AA)$ be a coproduct in the braided monoidal category $\mathcal{C}$. Then

$$(1) \quad E_{lr} \circ (\mu id) = (id\mu) \circ (E_{ll}id) \circ (idE_{lr}), \quad (3)$$

$$(2) \quad (\mu id) \circ (idE_{lr}^{-1}) = (id\mu) \circ (E_{lr}^{-1}id) \circ (E_{ll}id). \quad (4)$$

Proof. (1) We compute as follows

$$\begin{aligned} E_{lr} \circ (\mu id) &= (id\mu) \circ (\Delta id) \circ (\mu id) \\ &= (id\mu) \circ (\mu id) \circ (id\tau id^2) \circ (\Delta id) \\ &= (id\mu) \circ (\mu id) \circ (id\tau id^2) \circ (\Delta id^3) \circ (id\Delta id) \\ &= (id\mu) \circ (\mu id^2) \circ (id\tau id) \circ (\Delta id^2) \circ (id^2\mu) \circ (id\Delta id) \\ &= (id\mu) \circ ((\mu id) \circ (id\tau) \circ (\Delta id)id) \circ (id(id\mu) \circ (id\Delta)) \\ &= (id\mu) \circ (E_{ll}id) \circ (idE_{lr}), \end{aligned}$$

as desired.

(2) By the first part (1), we have

$$\begin{aligned} (\mu id) \circ (idE_{lr}^{-1}) &= E_{lr}^{-1} \circ E_{lr} \circ (\mu id) \circ (idE_{lr}^{-1}) \\ &= E_{lr}^{-1} \circ (id\mu) \circ (E_{ll}id) \\ &= (id\mu) \circ (E_{lr}^{-1}id) \circ (E_{ll}id). \end{aligned}$$

This ends the proof. $\square$

Lemma 4. Let $\Delta : A \longrightarrow M(AA)$ be a coproduct in the braided monoidal category $\mathcal{C}$. Then

$$\begin{aligned} (1) \quad E_{lr} \circ (id\mu) &= (id\mu) \circ (E_{lr}id), \\ (2) \quad (id\mu) \circ (E_{rl}id) &= (\mu id) \circ (idE_{lr}), \\ (3) \quad E_{rl} \circ (id\mu) &= (\mu\mu) \circ (id\tau id) \circ (E_{rl}\Delta). \end{aligned} \quad (5)$$

Proof. Straightforward. $\square$

Now we prove the main result of this section.

Theorem 1. Let $\Delta : A \longrightarrow M(AA)$ and ε be a coproduct and a counit in the braided monoidal category $\mathcal{C}$, respectively. Then A is a regular braided multiplier Hopf algebra if and only if there exists a unique isomorphism $S : A \longrightarrow A$ which is both an algebra antihomomorphism and a coalgebra antihomomorphism such that

$$\mu_A \circ (Sid) \circ E_{lr} = \varepsilon id, \quad \mu_A \circ (idS) \circ E_{rl} = id\varepsilon \quad \text{and} \quad \varepsilon S = \varepsilon.$$

Proof. The proof is similar to one in [13]. If A is a braided multiplier Hopf algebra in the braided monoidal category $\mathcal{C}$, then we have to construct the morphisms S .

We first define a map $S_l : A \longrightarrow \text{End}(A)$ by

$$S_l(a)(b) = ((\varepsilon id) \circ E_{lr}^{-1})(ab)$$

for all $a, b \in A$. It is obvious that $S_l(a)$ is a morphism in $\mathcal{C}$ and furthermore, it is a left multiplier. We claim that $S_l : A \longrightarrow \mathbb{L}(A)$ is an algebra antihomomorphism. We compute

$$\begin{aligned}
 & \mu \circ (id\mu) \circ (idS_l id) \circ (E_{rl} id) \\
 = & \mu \circ (id\epsilon id) \circ (idE_{lr}^{-1}) \circ (E_{rl} id) \\
 \stackrel{(3.1)}{=} & \mu \circ ((id\epsilon) \circ E_{rl} id) \circ (idE_{lr}^{-1}) \\
 \stackrel{(3.2)}{=} & \mu \circ (\mu id) \circ (idE_{lr}^{-1}) = \mu \circ (id\mu \circ E_{lr}^{-1}) \\
 \stackrel{(3.2)}{=} & \mu \circ (id\epsilon id)
 \end{aligned}$$

and so we have

$$\mu \circ (id\mu) \circ (idS_l id) \circ (E_{rl} id) = \mu \circ (id\epsilon id). \quad (6)$$

Then,

$$\begin{aligned}
 & \mu \circ (id\mu) \circ (idS_l id) \circ (id\mu id) \\
 & \quad \circ (id\tau id) \circ (E_{rl} id) \circ (id\tau id) \circ (E_{rl} id^2) \\
 = & \mu \circ (id\mu) \circ (idS_l id) \\
 & \quad \circ (\mu\mu id) \circ (id\tau id^2) \circ (id^2\Delta id) \circ (E_{rl} id^2) \\
 = & \mu \circ (id\mu) \circ (idS_l id) \circ (\mu\mu id) \circ (id\tau id^2) \circ (E_{rl} \Delta id) \\
 \stackrel{(3.5)}{=} & \mu \circ (id\mu) \circ (idS_l id) \circ (E_{rl} id) \circ (id\mu id) \\
 \stackrel{(3.6)}{=} & \mu \circ (id\epsilon id) \circ (id\mu id) \\
 = & \mu \circ (id\epsilon id) \circ (id^2\epsilon id) \text{ since } \epsilon \text{ is an algebra map} \\
 \stackrel{(3.6)}{=} & \mu \circ (id\mu) \circ (idS_l id) \circ (E_{rl} id) \circ (id^2\epsilon id) \\
 = & \mu \circ (\mu id) \circ (idS_l id) \circ (id^2\epsilon id) \circ (E_{rl} id^2) \\
 = & \mu \circ (\mu \circ (id\epsilon id) id) \circ (id^2S_l id) \circ (id\tau id) \circ (E_{rl} id^2) \\
 \stackrel{(3.6)}{=} & \underbrace{\mu \circ (\mu id) \circ (id\mu id) \circ (idS_l id^2)}_{\circ (id^2S_l id) \circ (id\tau id) \circ (E_{rl} id^2)} \circ (E_{rl} id^2) \\
 = & \mu \circ (id\mu) \circ (idS_l id) \circ (id^2\mu) \circ (id^2S_l id) \\
 & \quad \circ (E_{rl} id^2) \circ (id\tau id) \circ (E_{rl} id^2) \\
 = & \mu \circ (id\mu) \circ (idS_l id) \circ (id^2\mu) \circ (id^2S_l id) \\
 & \quad \circ (id\tau id) \circ (id\tau id) \circ (E_{rl} id) \circ (id\tau id) \circ (E_{rl} id^2) \\
 = & \mu \circ (id\mu) \circ (id\mu id) \circ (id\tau id) \circ (idS_l S_l id) \\
 & \quad \circ (id\tau id) \circ (id\tau id) \circ (E_{rl} id) \circ (id\tau id) \circ (E_{rl} id^2),
 \end{aligned}$$

and hence

$$\begin{aligned}
 & \mu \circ (id\mu) \circ (idS_l id) \circ (id\mu id) \\
 & \quad = \mu \circ (id\mu) \circ (id\mu id) \circ (id\tau id) \circ (idS_l S_l id).
 \end{aligned}$$

Since the product in A is nondegenerate we get $S_l(ab) = S_l(b)S_l(a)$ for all $a, b \in A$, as required.

Similarly, define a map $S_r : A \longrightarrow \text{End}(A)$ by

$$aS_r(b) = ((id\epsilon) \circ E_{rl}^{-1})(ab)$$

for all $a, b \in A$. For S_r , by an analogous calculation, it is not hard to see that S_r is an algebra antihomomorphism from A to $\mathbb{R}(A)$, and we have

$$\mu \circ (\mu id) \circ (id S_r id) \circ (E_{lr} id) = \mu \circ (id \epsilon id). \quad (7)$$

By Equations (6) and (7), we obtain that

$$\mu \circ (id \mu) \circ (id S_l id) = \mu \circ (\mu id) \circ (id S_r id),$$

which means that $(S_l(a), S_r(a))$ is a multiplier of A for any $a \in A$. We denote $S : A \rightarrow \mathbb{M}(A)$ by the morphism in $\mathcal{C}$ given by S_l and S_r . Then one has thus proved so far that $S : A \rightarrow \mathbb{M}(A)$ is an algebra antihomomorphism.

In a similar way one defines for $a \in A$, $\tilde{S}_l(a) \in \mathbb{L}(A)$ and $\tilde{S}_r(a) \in \mathbb{R}(A)$ by

$$\tilde{S}_l(a)b = (\epsilon id) \circ E_{ll}^{-1} \circ \tau(ab) \quad \text{and} \quad b\tilde{S}_l(a) = (id \epsilon) \circ E_{rr}^{-1} \circ \tau(ba).$$

Based on Lemma 2, we have $(E_{lr}^{cop})^{-1} = E_{ll}^{-1} \circ \tau$ and $(E_{rl}^{cop})^{-1} = E_{rr}^{-1} \circ \tau$. The above discussion applied to A^{cop} verifies that $\tilde{S}_l$ and $\tilde{S}_r$ determine an algebra antihomomorphism $\tilde{S} : A \rightarrow \mathbb{M}(A)$.

Now we want to show that S and $\tilde{S}$ define a morphisms in $\mathcal{C}$ from A into A which are inverse to each other. To do this, we compute

$$\begin{aligned} & \mu \circ (id \mu) \circ (id \mu id) \circ (id \tilde{S} id^2) \circ (id \tau id) \\ = & \mu \circ (\mu id) \circ (id \tilde{S} id) \circ (id^2 \mu) \circ (id \tau id) \\ = & \mu \circ (\mu id) \circ (id \tilde{S} id) \\ & \circ (id^2 \mu) \circ (id \tau id) \circ (id E_{lr} id) \circ (id E_{lr}^{-1} id) \\ = & \mu \circ (\mu id) \circ (id \tilde{S} id) \circ (id \mu id) \\ & \circ (id^3 \mu) \circ (id^2 \tau id) \circ (id \Delta^{cop} id^2) \circ (id E_{lr}^{-1} id) \\ = & \mu \circ (\mu id) \circ (id \mu id) \circ (id \tilde{S} id) \circ (id \tau id) \circ (id^3 \mu) \\ & \circ (id^2 \tau id) \circ (id \Delta^{cop} id^2) \circ (id \tau id) \circ (id \tau \circ E_{lr}^{-1} id) \\ & \text{since } \tilde{S} \text{ is an algebra antihomomorphism} \\ = & \underbrace{\mu \circ (\mu id) \circ (id \mu id)}_{\circ (id^2 E_{lr}^{cop}) \circ (id \tau \circ E_{lr}^{-1} id)} \circ (id \tilde{S} id) \\ = & \underbrace{\mu \circ (id \mu) \circ (id \mu id)}_{\circ (id^2 E_{lr}^{cop}) \circ (id \tau \circ E_{lr}^{-1} id)} \circ (id \tilde{S} id^2) \circ (id^2 \tilde{S} id) \\ = & \mu \circ (id \mu) \circ (id \tilde{S} id) \circ (id^2 \mu) \circ (id^2 \tilde{S} id) \\ & \circ (id^2 E_{lr}^{cop}) \circ (id \tau \circ E_{lr}^{-1} id) \\ = & \mu \circ (id \mu) \circ (id \tilde{S} id) \circ (id^2 \epsilon id) \circ (id \tau \circ E_{lr}^{-1} id) \\ = & \mu \circ (id \mu) \circ (id \tilde{S} id) \circ (id (\epsilon id) \circ E_{lr}^{-1} id) \\ = & \mu \circ (id \mu) \circ (id \tilde{S} id) \circ (id \mu id) \circ (id S id^2) \end{aligned}$$

where in the last step of the computations above we used the definitions of $\tilde{S}$ and S .

As a consequence we get the following equation $\mu \circ (\tilde{S} id) \circ \tau = \tilde{S} \circ \mu(S id)$, here both sides are regarded as morphisms from AA into $M(A)$ in $\mathcal{C}$. By this equation and taking $p \in A$ so that $\epsilon(p) = 1$, we have that for all $a \in A$ $\tilde{S}(a) = \tilde{S}(a)\epsilon(p) = \tilde{S} \circ \mu \circ (S id) \circ E_{lr}(pa) = \mu \circ (\tilde{S} id) \circ \tau \circ E_{lr}(pa)$. This proves that $\tilde{S}$ defines a morphism from A to itself. Similarly, we have that S is also a morphism from

A to A . Again, we have that $\mu \circ (\mu id) \circ (id \tilde{S} id) = \mu \circ (\mu id) \circ (id \tilde{S} \tilde{S} \circ S)$. This implies that $\tilde{S} \circ S = id$. Similarly, we get $S \circ \tilde{S} = id$. By Equations (6) and (7), we have that

$$\mu_A \circ (Sid) \circ E_{lr} = \varepsilon id, \quad \mu_A \circ (id S) \circ E_{rl} = id \varepsilon$$

as desired. Furthermore, these two equations implies the uniqueness of S .

We have to show that S is an antihomomorphism of coalgebras. By Equations (3) and (4), we

$$(id \mu) \circ ((id S) \circ E_{rl} id) = (id \mu) \circ (E_{lr}^{-1} \circ E_{ll} id).$$

This implies that

$$E_{lr}^{-1} \circ E_{ll} = (id S) \circ E_{rl}. \quad (8)$$

Similarly, we obtain that

$$E_{ll}^{-1} \circ E_{lr} = (E_{lr}^{-1})^{cop} \circ E_{ll}^{cop} = (id S^{-1}) \circ E_{rl}^{cop} = (id S^{-1}) \circ \tau \circ E_{rr}. \quad (9)$$

By Equations (8) and (9), we have

$$E_{rr} \circ (SS) = (Sid) \circ \tau \circ E_{rl}^{-1} \circ (Sid). \quad (10)$$

From the definition of S and Equation (2) we compute

$$\begin{aligned} & (id \mu) \circ (E_{rl}^{-1} id) \circ (\tau id) \circ (id Sid) \\ = & (id \varepsilon id) \circ (id E_{lr}) \circ (E_{rl}^{-1} id) \circ (\tau id) \circ (id Sid) \\ = & (\mu id) \circ (id Sid) \circ (E_{rl} id) \circ (id E_{lr}) \\ & \circ (E_{rl}^{-1} id) \circ (\tau id) \circ (id Sid) \\ \stackrel{(3.1)}{=} & (\mu id) \circ (id Sid) \circ (id E_{lr}) \circ (Sid^2) \circ (\tau id) \\ = & (\mu id) \circ (SSid) \circ (id E_{lr}) \circ (\tau id) \\ = & (Sid) \circ (\mu^{op} id) \circ (id E_{lr}) \circ (\tau id) \\ = & (Sid) \circ (\mu^{op} id) \circ (id^2 \mu) \circ (id \Delta id) \circ (\tau id) \\ = & (Sid) \circ (id \mu) \circ ((\mu^{op} id) \circ (id \Delta) \circ \tau id) \\ = & (id \mu) \circ (Sid^2) \circ (E_{ll} id). \end{aligned}$$

This implies

$$(Sid) \circ E_{ll} = E_{rl}^{-1} \circ \tau \circ (id S). \quad (11)$$

By Equations (10) and (11) we have

$$\begin{aligned} & (id \mu^{op}) \circ (\Delta id) \circ (Sid) \circ (id S) \\ = & E_{rr} \circ (SS) \circ \tau \\ = & (Sid) \circ \tau \circ E_{rl}^{-1} \circ (Sid) \circ \tau \\ = & (SS) \circ \tau \circ E_{ll} \\ = & (id \mu^{op}) \circ (SSid) \circ (\tau id) \circ (\Delta id) \circ (id S). \end{aligned}$$

This implies $\Delta \circ S = (SS) \circ \tau \circ \Delta$ and proves that S is an antihomomorphism of coalgebras in $\mathcal{C}$. Finally, since $S : A \rightarrow A^{op}$ is algebra and coalgebra homomorphism we have $E_{lr} \circ (SS) = (SS) \circ E_{lr}$. Hence one gets

$$(\varepsilon id) \circ (SS) \circ E_{lr} = (\varepsilon id) \circ E_{lr} \circ (SS) = \mu \circ (SS) = S \circ \mu = (\varepsilon S) \circ E_{lr}.$$

From this equation we deduce that $(\varepsilon \circ S)S = \varepsilon S$ because E_{lr} is bijective. Since the product in A is nondegenerate and S is bijective this shows $\varepsilon \circ S = \varepsilon$. Therefore, we have shown that there is a unique map S satisfying the desired properties.

Conversely, we define the inverse E_{lr}^{-1} of E_{lr} by

$$E_{lr}^{-1} = (id\mu) \circ (idSid) \circ (\Delta id)$$

where both sides are viewed as morphisms from AA to $M(AA)$ in $\mathcal{C}$. In particular, the image of the last map id contained in AA . We calculate

$$\begin{aligned} & (\mu\mu) \circ (id\tau id) \circ (E_{lr}^{-1} \circ E_{lr} id^2) \\ &= (\mu\mu) \circ (id\tau id) \circ (id\mu id^2) \circ (idSid^3) \circ (\Delta id^3) \circ (E_{lr} id^2) \\ &= (\mu\mu) \circ (id\tau id) \circ (id\varepsilon id^2) \circ (\Delta id^3) \\ &= (\mu\mu) \circ (id\tau id). \end{aligned}$$

Similarly, we have that $(\mu\mu) \circ (id\tau id) \circ (E_{lr}^{-1} \circ E_{lr} id^2) = (\mu\mu) \circ (id\tau id)$ which proves that E_{lr} is an isomorphism in $\mathcal{C}$.

Analogously, the other entwining maps can be shown to be isomorphisms in $\mathcal{C}$ as well.

Finally, for all $a, b, c \in A$ we compute

$$\begin{aligned} (idS)((c1)E_{rr}(ba)) &= (idS)(E_{rl}(ca))(1S(b)) \\ &= (c1)E_{lr}^{-1}(aS(b)). \end{aligned}$$

Hence

$$E_{rr}(ba) = (idS^{-1}) \circ E_{lr}^{-1} \circ (idS)(ab)$$

for all $a, b \in A$. It follows that $E_{rr}(AA) \subseteq AA$. By a similar argument, we have $E_{ll}(AA) \subseteq AA$. This proves that A is regular.

This completes the proof of the theorem. $\square$

As a corollary of Theorem 1, we have the following characterisation of a braided Hopf algebra in [10].

Corollary 3. *H is braided Hopf algebra in $\mathcal{C}$ if and only if the associated E_{lr} and E_{rl} are bijective in $\mathcal{C}$.*

We also have the following important result.

Theorem 2. *If A is an algebra in $\mathcal{C}$ with identity and Δ a coproduct on A such that (A, Δ) is a braided Hopf algebra, then (A, Δ) is a braided multiplier Hopf algebra. Conversely, if (A, Δ) is a braided multiplier Hopf algebra and if A has an identity, then (A, Δ) is a braided Hopf algebra.*

Proof. The proof is straightforward. $\square$

5. Braided Quantum Groups

In this section we introduce the concept of a braided quantum group and develop the main result established as in last section.

Definition 6. With the notations as before. Let Δ is a coproduct on A . Then a no-zero morphism $\varphi : A \longrightarrow \mathbb{K}$ in $\mathcal{C}$ is called left integral if

$$(\mathbb{K}\varphi) \circ E_{rl}(ba) = \varphi(a)b$$

for all $a, b \in A$. Similarly, a no-zero morphism $\psi : A \longrightarrow \mathbb{K}$ in $\mathcal{C}$ is called right integral if

$$(\psi\mathbb{K}) \circ E_{lr}(ab) = \psi(a)b$$

for all $a, b \in A$.

Definition 7. Let $\Delta : A \longrightarrow M(AA)$ be a coproduct in $\mathcal{C}$. We say that A is a braided quantum group if the entwining maps E_{lr}, E_{ll}, E_{rl} and E_{rr} are bijective so that $E_{lr}(AA), E_{rl}(AA), E_{ll}(AA), E_{rr}(AA) \subseteq AA$ and there is a faithful left integral φ on A .

Since the proof of the following lemma is parallel to the one for regular multiplier Hopf algebra [12] we omit it.

Lemma 5. Let A be a regular braided multiplier Hopf algebra in $\mathcal{C}$. If φ is a left integral on A , then φ is faithful.

The following is the main result of this section.

Theorem 3. Let $\Delta : A \longrightarrow M(AA)$ be a coproduct in the braided monoidal category $\mathcal{C}$ such that the images of all associated entwining maps belong to AA . Assume that $\varphi : A \longrightarrow \mathbb{K}$ is a left integral. Then A is a braided quantum group if and only if there exist a unique counit ε on A and an isomorphism $S : A \longrightarrow A$ in $\mathcal{C}$ which is both an algebra antihomomorphism and a coalgebra antihomomorphism such that

$$\mu_A \circ (Sid) \circ E_{lr} = \varepsilon id, \quad \mu_A \circ (idS) \circ E_{rl} = id\varepsilon \quad \text{and} \quad \varepsilon S = \varepsilon.$$

Proof. By Theorem 1, Lemma 5 and Proposition 5, we just construct the map ε . Choose an element $x \in A$ so that $\varphi(x) = 1$. Now define a linear map $\varepsilon : A \longrightarrow \mathbb{K}$ by $\varepsilon(a) = (\varphi \circ \mu \circ E_{rl}^{-1})(xa)$ for all $a \in A$. Obviously, ε is a morphism in $\mathcal{C}$. Now, define a morphism $E : A \longrightarrow \mathbb{L}(A)$ by $E(a)b = (\mu \circ E_{lr}^{-1})(ab)$ for all $a, b \in A$. By Lemma 4 (1), one easily see that the morphism E is well-defined, i.e., $E(a)$ is a left multiplier of A . By Equation (1), we compute

$$\begin{aligned} & (id\mu) \circ (idEid) \\ &= \underbrace{(id\mu) \circ (idEid)}_{(id\mu) \circ (idEid)} \circ (E_{rl}id) \circ (E_{rl}^{-1}id) \circ (idE_{lr}) \circ (idE_{lr}^{-1}) \\ &= (id\mu) \circ (idE_{lr}^{-1}) \circ (E_{rl}id) \circ (idE_{lr}) \circ (E_{rl}^{-1}id) \circ (idE_{lr}^{-1}) \\ &= (id\mu) \circ \underbrace{(idE_{lr}^{-1}) \circ (idE_{lr})}_{(id\mu) \circ (idE_{lr}^{-1})} \circ (E_{rl}id) \circ (E_{rl}^{-1}id) \circ (idE_{lr}^{-1}) \\ &= (id\mu) \circ (idE_{lr}^{-1}) \\ &= (id\mu) \circ \underbrace{(idE_{lr}^{-1}) \circ (E_{rl}id)}_{(id\mu) \circ (idE_{lr}^{-1})} \circ (E_{rl}^{-1}id) \\ &= \underbrace{(id\mu) \circ (E_{rl}id)}_{(id\mu) \circ (idE_{lr}^{-1})} \circ (idE_{lr}^{-1}) \circ (E_{rl}^{-1}id) \\ &= (\mu id) \circ (E_{rl}^{-1}id) \quad \text{by Lemma 4 (2).} \end{aligned}$$

This implies

$$(id\mu) \circ (idEid) = \mu \circ E_{rl}^{-1}id. \quad (12)$$

On one hand, evaluating Equation (12) on a tensor xab for all $a, b \in A$ and the chosen element $x \in A$ as above and applying φid , we obtain

$$E(a)b = (\varphi id)(xE(a)b) = \varphi(\mu \circ E_{rl}^{-1}(xa))b = \varepsilon(a)b$$

and so $E(a) = \varepsilon(a)1$ because the product on A is nondegenerate. Hence we have $(\varepsilon id) \circ E_{lr} = \mu \circ (Eid) \circ E_{lr} = \mu$.

On the other hand, by Equation (12), we get $\mu \circ E_{rl}^{-1}(ab)x = (id\mu) \circ (idEid)(abx) = (id\mu)(a\varepsilon(b)1x) = a\varepsilon(b)x = a\varepsilon(b)x$ for all $a, b \in A$, which implies $(id\varepsilon) \circ E_{rl} = \mu$. This proves that Equation (2) holds.

To finish that ε is a counit, we have to show finally that ε is an algebra homomorphism. We compute

$$\begin{aligned} & (id\varepsilon) \circ (id\mu) \\ = & (id\varepsilon) \circ (id\mu) \circ (id\tau) \circ (E_{rl}id) \circ (id\tau) \\ & \circ (id\tau) \circ (E_{rl}^{-1}id) \circ (id\tau) \\ = & (id\varepsilon) \circ (\mu\mu) \circ (id\tau id) \circ (id^2\Delta) \\ & \circ (id\tau) \circ (E_{rl}^{-1}id) \circ (id\tau) \\ = & (id\varepsilon) \circ (\mu\mu) \circ (id\tau id) \circ (id^2\Delta) \circ (E_{rl}id) \circ (E_{rl}^{-1}id) \\ & \circ (id\tau) \circ (E_{rl}^{-1}id) \circ (id\tau) \\ \stackrel{(3.5)}{=} & (id\varepsilon) \circ E_{rl} \circ (id\mu) \circ (E_{rl}^{-1}id) \circ (id\tau) \circ (E_{rl}^{-1}id) \circ (id\tau) \\ = & \mu \circ (\mu id) \circ (E_{rl}^{-1}id) \circ (id\tau) \circ (E_{rl}^{-1}id) \circ (id\tau) \\ = & \mu \circ (id\varepsilon id) \circ (E_{rl}id) \circ (E_{rl}^{-1}id) \circ (id\tau) \circ (E_{rl}^{-1}id) \circ (id\tau) \\ = & \mu \circ (id\varepsilon id) \circ (id\tau) \circ (E_{rl}^{-1}id) \circ (id\tau) \\ = & (id\varepsilon) \circ (id\varepsilon id) \circ (id\tau) \circ (E_{rl}id) \circ (id\tau) \\ & \circ (id\tau) \circ (E_{rl}^{-1}id) \circ (id\tau) \\ = & (id\varepsilon) \circ (id\varepsilon id) \end{aligned}$$

which implies that $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ for all $a, b \in A$. Therefore ε is an algebra homomorphism.

This completes the proof of the theorem. $\square$

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