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Not peer-reviewed version

Hypercomplex Binary Superposition Algebra: A Commutative and Associative Product

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Posted Date: 13 March 2026

doi: 10.20944/preprints202506.0879.v2

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doi:10.20944/preprints202506.0879.v2

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Hypercomplex Binary Superposition Algebra: A Commutative and Associative Product

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Abstract

In this article, we introduce a hypercomplex algebra based on a binary superposition structure. Each algebraic unit is defined by a pair $(f;\, S)$ where $f \in \{0;\, 1\}$ encodes the logical presence of a base component, and $S \in \{-1;\, 1\}$ encodes a geometric phase or orientation. This framework allows us to define an imaginary product that is both commutative and associative, properties rarely combined in higher-dimensional algebras. We demonstrate the consistency of this product through a binary and superposed formalism. This result provides a solid foundation for representing multi-level logic states, with potential applications in quantum computing processing.

Keywords: hypercomplex algebra; superposition; commutativity; associativity; binary logic

MSC: 11G15; 46G20; 58B12; 30G35

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1. Introduction

Hypercomplex algebraic systems are often used to model non-trivial physical phenomena, particularly in quantum mechanics. However, the properties of associativity and commutativity are rarely preserved in classical extensions such as quaternions, octonions, or sedenions 279. Here, we present a binary superposition structure that allows us to define a stable multiplication, combining logical rigor, geometric structure, and algebraic consistency.

Unlike many existing approaches that emphasize geometric interpretation or formal analogies, this work introduces a constructive framework and provides a rigorous demonstration of a hypercomplex algebraic product exhibiting both commutativity and associativity. The aim is to establish a coherent algebraic structure grounded in explicit definition, multiplication rules, and demonstrable properties.24,57

2. Superimposed Hypercomplex Number of Level 2 and Level 3 1

2.1. Hyperspace $E_{(2\,;\,2)}$

A superimposed hypercomplex number is a number representing points in complex hyperspaces of dimension $n \geq 2$.

In hyperspace $E_{(2\,;\,2)} = \{(v_{r,1}\,;\, v_i);(v_{r,2}\,;\, v_j)\}$, for any point M there exists a unique superimposed hypercomplex number $\hbar \in \mathbb{S}(2\,;2)$ such that $\hbar = z_1 + z_2$.

- hyperspace $E_{(2\,;\,2)}$ is a space composed of two real dimensions $v_{r,1}\,;\, v_{r,2}$ and two imaginary dimensions $v_i\,;\, v_j$
- z_1 is the complex number that represents a point M_1 on the plane $(O\,;\, v_{r,1}\,;\, v_i)$ where $(Ov_{r,1})$ is the real axis and (Ov_i) is the imaginary axis of the plane $(O\,;\, v_{r,1}\,;\, v_i)$ such that: $z_1 = a_1 + bi$.5
- z_2 is the complex number that represents a point M_2 on the plane $(O\,;\, v_{r,2}\,;\, v_j)$ where $(Ov_{r,2})$ is the real axis and (Ov_j) is the imaginary axis of the plane $(O\,;\, v_{r,2}\,;\, v_j)$ such that: $z_2 = a_2 + cj$.5

  We therefore obtain

 doi:10.20944/preprints202506.0879.v2

$$h = a_1 + bi + a_2 + cj$$

$$h = a_1 + a_2 + bi + cj$$

By superimposing the real axes $(Ov_{r,1})$ and $(Ov_{r,2})$ then posing $a = a_1 + a_2$ we then have the superimposed hypercomplex number $h = a + bi + cj$ which gives us an extension to 3 dimensions denoted $\mathbb{C}_{(1;\,2)}$ from the set $\mathbb{C}$ of complex numbers.

2.2. Hyperspace $E_{(3;\,3)}$

By the same reasoning; we will also have that; for any point M of the hyperspace $E_{(3\,;\,3)} = \{(v_{r,1}\,;\,v_i);(v_{r,2}\,;\,v_j);(v_{r,3}\,;\,v_k)\}$, there exists a unique superimposed hypercomplex number $\hbar \in \mathbb{S}(3\,;3)$ such that $\hbar = z_1 + z_2 + z_3$.

$z_1 = x_{1,r}v_{1,r} + x_{2,r}v_{2,r};\quad z_1 = x_{3,r}v_{3,r} + y_{1,i}v_{1,i}$ And $z_1 = y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

$h = x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r} + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

$h = x_{1,r}v_r + x_{2,r}v_r + x_{3,r}v_r + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

$h = (x_{1,r} + x_{2,r} + x_{3,r})v_r + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

By asking: $x_{1,r} + x_{2,r} + x_{3,r} = x_r$

We obtain: $\hbar = x_r v_r + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

If now we pose $v_r = 1\,;\ v_{1,i} = i\,;\ v_{2,i} = j\,;$ and $v_{3,i} = k$.

As well as: $x_r = a\,;\ y_{1,i} = b\ ;\ y_{2,i} = c\,;\ y_{3,i} = d$

We also obtain a writing of the superimposed hypercomplex number: $h = a + bi + cj + dk$

Which also gives us an extension to 4 dimensions noted $\mathbb{C}_{(1;\,3)}$ from the set $\mathbb{C}$ of complex numbers.

3. The Imaginary Product $\otimes_e^i$ and the Real Product $\otimes_e^r$ 1

3.1. The Rules of the Imaginary Product $\otimes_e^i$ and the Real Product $\otimes_e^r$

We note that the multiplication of superimposed hypercomplex numbers reveals (as the great Irish mathematician William Rohan Hamilton discovered)279 additional, hidden, and superimposed dimensions. These additional dimensions form new superimposed spaces.

As clearly stated in the article "Complex Superposition Algebra" the imaginary product of superimposed hypercomplex numbers is indeed commutative and associative in the superposition phase. However, the property of associativity is lost in the de-superposition phase of hypercomplex numbers. This is mainly due to the appearance of a negative sign for the squares of imaginary numbers. After a year of reflection I noticed that the mixing of negative signs $(-)$ in the imaginary product was the cause of the loss of associativity. This means that, they are also superimposed and must be defined for each rotation plane.

Recall that the multiplication of imaginary bases is defined by the imaginary product noted $\otimes_e^i$ (in the article)1; is associated with the applications f and S_i; defined as follows:

Let the sets $H = \{0 + 4p\,;1 + 4p\,;2 + 4p\,;3 + 4p\}\ (p \in \mathbb{Z})\ ;\ I = \{-1\,;0\,;1\}$ and $J = \{-1\,;\,+1\}$ and the applications $f\,;\ S_i$ be defined by:

$$f(H) = I \text{ Such as: } \begin{cases} f(0 + 4p) = 0 \\ f(1 + 4p) = 1 \\ f(2 + 4p) = 0 \\ f(3 + 4p) = -1 \end{cases}$$

$$S_{n,\pm} = (\{0 + 4p\,;2 + 4p\,\}) = J \text{ such as : } S_{n,\pm} = \begin{cases} S_{0,\pm}(0 + 4p) = +1 \\ S_{2,\pm}(2 + 4p) = -1 \end{cases}$$

So the imaginary product application for two hypercomplex numbers

$V_i = e^{\frac{\pi}{2}(\alpha_1 v_{1,i} + \alpha_2 v_{2,i} + \alpha_3 v_{3,i})}$ and $V_i^{'} = e^{\frac{\pi}{2}(\alpha_1{}' v_{1,i} + \alpha_2{}' v_{2,i} + \alpha_3{}' v_{3,i})}$ is defined by:

$$\otimes_e^i(V_i\,;\,V_i') = e^{\frac{\pi}{2}(\alpha_1 v_{1,i} + \alpha_2 v_{2,i} + \alpha_3 v_{3,i})} \times e^{\frac{\pi}{2}(\alpha_1{}' v_{1,i} + \alpha_2{}' v_{2,i} + \alpha_3{}' v_{3,i})}$$

$$\otimes_e^i = S_i(\alpha_1 + \alpha_1{}') \times S_i(\alpha_2 + \alpha_2{}') \times S_i(\alpha_3 + \alpha_3{}')e^{\frac{\pi}{2}(f(\alpha_1 + \alpha_1{}') \times v_{1,i} + f(\alpha_2 + \alpha_2{}') \times v_{2,i} + f(\alpha_3 + \alpha_3{}') \times v_{3,i})}$$

If we note $\prod_i = S_i(\alpha_1 + \alpha_1{}') \times S_i(\alpha_2 + \alpha_2{}') \times S_i(\alpha_3 + \alpha_3{}')$
So we have $\otimes_e^i(V_i\,;\,V_i') = \prod_i \times e^{\frac{\pi}{2}(f(\alpha_1+\alpha_1{}')\times v_{1,i}+f(\alpha_2+\alpha_2{}')\times v_{2,i}+f(\alpha_3+\alpha_3{}')\times v_{3,i})}$

3.2. New Rules and Notations

To maintain associativity let's make the following changes:
We will modify: $f(3+4p) = -1$ by $f(3+4p) = 1$
We will use the signs: $S_{1,\pm}(1+4p) = +1$ and $S_{3,\pm}(3+4p) = -1$; each in its plane of rotation.
Which logically, changes nothing; because this is an inversion of the sign of:
$f(3+4p) = -1$ towards $S_{3,\pm}(3+4p) = -1$ and $S_{1,\pm}(1+4p) = +1$ is positive, so there is no change.

Finally let's add an index notation to f what we get f_n and replace the notation $S_{n,\pm}$ with S_n. Which corresponds the positions of f_n to those of S_n .

This means that it will no longer be necessary to raise the sign $(-)$ in the imaginary exponential, but leave it in front. We just want to adapt the imaginary product to a notation of superimposed hypercomplex numbers that offers associativity.

So the applications associated with the imaginary product will now be:

$$f_n(H) = I \text{ such that: } f_n = \begin{cases} f(0+4p) = 0 \\ f(1+4p) = 1 \\ f(2+4p) = 0 \\ f(3+4p) = 1 \end{cases} \text{; and } S_n(H) = J \text{ such that: } S_n = \begin{cases} S(0+4p) = +1 \\ S(1+4p) = +1 \\ S(2+4p) = -1 \\ S(3+4p) = -1 \end{cases}$$

We will also change the sign $\prod_i = S_1(\alpha_1 + \alpha_1{}') \times S_2(\alpha_2 + \alpha_2{}') \times S_3(\alpha_3 + \alpha_3{}')$ and define as a superposition of signs; in their rotation planes. We will use the following superimposed notation:

$$\prod_i = \langle S_1(\alpha_1 + \alpha_1{}')|S_2(\alpha_2 + \alpha_2{}')|S_3(\alpha_3 + \alpha_3{}')\rangle$$

This allows us to maintain the rules of signs of multiplication in each plane of rotation. This superposition allows us to avoid multiplying signs that are not in the same plane of rotation. I remind you that; the fact that multiplication is linked to rotation in the plane defined by Jean Robert Argan among others. 2

We will define the modified imaginary product as follows:

Let the sets $H = \{0+4p\,;1+4p\,;2+4p\,;3+4p\}$ $(p \in \mathbb{Z})$; $I = \{0\,;1\}$ and $J = \{-1\,;\,+1\}$ and applications f_n ; S_i defined by:

$$f_n(H) = I \text{ such that: } \begin{cases} f(0+4p) = 0 \\ f(1+4p) = 1 \\ f(2+4p) = 0 \\ f(3+4p) = 1 \end{cases} \text{; and } S_n(H) = J \text{ such that: } \begin{cases} S(0+4p) = +1 \\ S(1+4p) = +1 \\ S(2+4p) = -1 \\ S(3+4p) = -1 \end{cases}$$

Then the imaginary product application is defined by:

$$\otimes_e^i(V_i\,;\,V_i') = e^{\frac{\pi}{2}(\alpha_1 v_{1,i}+\alpha_2 v_{2,i}+\alpha_3 v_{3,i})} \times e^{\frac{\pi}{2}(\alpha_1{}' v_{1,i}+\alpha_2{}' v_{2,i}+\alpha_3{}' v_{3,i})}$$

$$\otimes_e^i(V_i\,;\,V_i') = \langle S_1(\alpha_1 + \alpha_1{}')|S_2(\alpha_2 + \alpha_2{}')|S_2(\alpha_3 + \alpha_3{}')\rangle e^{\frac{\pi}{2}(f_1(\alpha_1+\alpha_1{}')\times v_{1,i}+f_2(\alpha_2+\alpha_2{}')\times v_{2,i}+f_3(\alpha_3+\alpha_3{}')\times v_{3,i})}$$

If we note $\prod_i = \langle S_1(\alpha_1 + \alpha_1{}')|S_2(\alpha_2 + \alpha_2{}')|S_3(\alpha_3 + \alpha_3{}')\rangle$ this on the assumption that the signs were positive to begin with.

Taking into account the previous signs we will have:

$$\prod_i = \langle S_1(\alpha_1) \times S_1(\alpha_1{}') \times S_1[f_1(\alpha_1 + \alpha_1{}')]|S_2(\alpha_2) \times S_2(\alpha_2{}') \times S_2[f_2(\alpha_2 + \alpha_2{}')]|S_3(\alpha_3) \times S_3(\alpha_3{}') \times S_3[f_3(\alpha_3$$

We once again obtain a long notation:

$$\text{This is why we will also note: } \begin{cases} \prod_1 = S_1(\alpha_1) \times S_1(\alpha_1{}') \times S_1[f_1(\alpha_1 + \alpha_1{}')] \\ \prod_2 = S_2(\alpha_2) \times S_2(\alpha_2{}') \times S_2[f_2(\alpha_2 + \alpha_2{}')] \\ \prod_3 = S_3(\alpha_3) \times S_3(\alpha_3{}') \times S_3[f_3(\alpha_3 + \alpha_3{}')] \end{cases}$$

So we have $\otimes_e^i(V_i\,;\,V_i') = \prod_i \times e^{\frac{\pi}{2}(f_1(\alpha_1+\alpha_1{}')\times v_{1,i}+f_2(\alpha_2+\alpha_2{}')\times v_{2,i}+f_3(\alpha_3+\alpha_3{}')\times v_{3,i})}$

doi:10.20944/preprints202506.0879.v2

$$\otimes_e^i(V_i\,;\,V_i') = \langle\, \textstyle\prod_1 \mid \prod_2 \mid \prod_3 \rangle \times e^{\frac{\pi}{2}(f_1(\alpha_1+\alpha_1')\times v_{1,i}+f_2(\alpha_2+\alpha_2')\times v_{2,i}+f_3(\alpha_3+\alpha_3')\times v_{3,i})}$$

If we note

$$e^{\frac{\pi}{2}(f_1(\alpha_1+\alpha_1')\times v_{1,i}+f_2(\alpha_2+\alpha_2')\times v_{2,i}+f_3(\alpha_3+\alpha_3')\times v_{3,i})} = \langle f_1(\alpha_1+\alpha_1') | f_2(\alpha_2+\alpha_2') | f_3(\alpha_3+\alpha_3')\rangle$$

We will have $\otimes_e^i(V_i\,;\,V_i') = \langle\, \prod_1 \mid \prod_2 \mid \prod_3 \rangle \times \langle f_1(\alpha_1+\alpha_1') | f_2(\alpha_2+\alpha_2') | f_3(\alpha_3+\alpha_3')\rangle$

Since this notation can be long, we will propose a simpler notation of the imaginary product as follows:

$$\otimes_e^i(V_i\,;\,V_i') = \left\langle \begin{matrix} f_1(\alpha_1+\alpha_1') \\ \prod_1 \end{matrix} \middle| \begin{matrix} f_2(\alpha_2+\alpha_2') \\ \prod_2 \end{matrix} \middle| \begin{matrix} f_3(\alpha_3+\alpha_3') \\ \prod_3 \end{matrix} \right\rangle$$

Each hypercomplex unit is defined by a pair of overlapping values:

(i) A logical component resulting from a binary function f_n

(ii) A phase component associated with $\prod_n$, representing the sign resulting from the superposition and desuperposition phases.

These two values are encoded in overlapping lines. The imaginary product is defined by the rule:

$$\otimes_e^i(V_i\,;\,V_i') = \left\langle \begin{matrix} f_1(\alpha_1+\alpha_1') \\ \prod_1 \end{matrix} \middle| \begin{matrix} f_2(\alpha_2+\alpha_2') \\ \prod_2 \end{matrix} \middle| \begin{matrix} f_3(\alpha_3+\alpha_3') \\ \prod_3 \end{matrix} \right\rangle$$

This rating $\left\langle \begin{matrix} \dots \\ \dots \end{matrix} \middle| \begin{matrix} \dots \\ \dots \end{matrix} \middle| \begin{matrix} \dots \\ \dots \end{matrix} \right\rangle$ indicates an overlapping concatenation.

3.3. Demonstration of Commutativity and Associativity

Let us try to show that imaginary product is commutative and associative.

To begin, let's note that the applications:

- $f_1;\ f_2;\ f_3 \ldots\ldots\ f_n$ are equivalent; the index notation is there to specify that each application f_n is associated with a base $v_{i,n}$.
- $S_1;\ S_2;\ S_3 \ldots\ldots\ S_n$ are equivalent; the index notation is there to specify that each application S_n is associated with the application f_n and therefore with the base $v_{i,n}$.
- We will generalize the notation for a level superposition n with the notation:
- $V_i = \underbrace{\left\langle \begin{matrix} \alpha_1 \\ S_1(\alpha_1) \end{matrix} \middle| \begin{matrix} \cdots\cdots \\ \cdots\cdots \end{matrix} \middle| \begin{matrix} \alpha_n \\ S_n(\alpha_n) \end{matrix} \right\rangle}_{n}$; where $\alpha_n \in \{0; 1\}$ and $n \in \mathbb{N}$

3.3.1. Commutativity

For commutativity we have:

$$\otimes_e^i(V_i\,;\,V_i') = \underbrace{\left\langle \begin{matrix} f_1(\alpha_1+\alpha_1') \\ \prod_1 \end{matrix} \middle| \begin{matrix} \cdots\cdots \\ \cdots\cdots \end{matrix} \middle| \begin{matrix} f_n(\alpha_n+\alpha_n') \\ \prod_n \end{matrix} \right\rangle}_{n}$$

Addition and multiplication being commutative we have:

$$\left\langle \begin{matrix} f_1(\alpha_1+\alpha_1') \\ \prod_1 \end{matrix} \middle| \begin{matrix} \cdots\cdots \\ \cdots\cdots \end{matrix} \middle| \begin{matrix} f_n(\alpha_n+\alpha_n') \\ \prod_n \end{matrix} \right\rangle = \left\langle \begin{matrix} f_1(\alpha_1'+\alpha_1) \\ \prod_1 \end{matrix} \middle| \begin{matrix} \cdots\cdots \\ \cdots\cdots \end{matrix} \middle| \begin{matrix} f_n(\alpha_n'+\alpha_n) \\ \prod_n \end{matrix} \right\rangle$$

We can then conclude that: $\otimes_e^i(V_i\,;\,V_i') = \otimes_e^i(V_i'\,;\,V_i)$

Therefore the imaginary product is commutative for all levels of superposition.

3.3.2. Associativity

For associativity we have:

$$\otimes_e^i[(V_i\,;\,V_i');V_i''] = \left\langle \frac{f_1(\alpha_1+\alpha_1')}{\prod_1}\middle|\begin{matrix}\cdots\cdots\\\cdots\cdots\end{matrix}\middle|\frac{f_n(\alpha_n+\alpha_n')}{\prod_n}\right\rangle\left\langle\frac{\alpha_1''}{S_1(\alpha_1'')}\middle|\begin{matrix}\cdots\cdots\\\cdots\cdots\end{matrix}\middle|\frac{\alpha_n''}{S_n(\alpha_n'')}\right\rangle$$

$$\otimes_e^i[(V_i\,;\,V_i');V_i'']$$
$$= \left\langle \frac{f_1[f_1(\alpha_1+\alpha_1')+\alpha_1'']}{\prod_1\times S_1(\alpha_1'')\times S_1[f_1(\alpha_1+\alpha_1')+\alpha_1'']}\middle|\begin{matrix}\cdots\cdots\\\cdots\cdots\end{matrix}\middle|\frac{f_n[f_n(\alpha_n+\alpha_n')+\alpha_n'']}{\prod_1\times S_n(\alpha_n'')\times S_n(f_n(\alpha_n+\alpha_n')+\alpha_n'')}\right\rangle$$

Then we have:

$$\otimes_e^i[V_i\,;(\,V_i'\,;V_i'')] = \left\langle\frac{\alpha_1}{S_1(\alpha_1)}\middle|\begin{matrix}\cdots\cdots\\\cdots\cdots\end{matrix}\middle|\frac{\alpha_n}{S_n(\alpha_n)}\right\rangle\left\langle\frac{f_1(\alpha_1'+\alpha_1'')}{\prod_1'}\middle|\begin{matrix}\cdots\cdots\\\cdots\cdots\end{matrix}\middle|\frac{f_n(\alpha_n'+\alpha_n'')}{\prod_n'}\right\rangle$$

$$\otimes_e^i[V_i\,;(\,V_i'\,;V_i'')]$$
$$= \left\langle\frac{f_1[\alpha_1+f_1(\alpha_1'+\alpha_1'')]}{S_1(\alpha_1)\times\prod_1'\times S_1[\alpha_1+f_1(\alpha_1'+\alpha_1'')]}\middle|\begin{matrix}\cdots\cdots\\\cdots\cdots\end{matrix}\middle|\frac{f_n[\alpha_n+f_n(\alpha_n'+\alpha_n'')]}{S_n(\alpha_n)\times\prod_n'\times S_n[\alpha_n+f_n(\alpha_n'+\alpha_n'')]}\right\rangle$$

To say that: $\otimes_e^i[V_i\,;(\,V_i^{'}\,;V_i^{''})] = \otimes_e^i[V_i\,;(\,V_i^{'}\,;V_i^{''})]$; is equivalent to having the following two equalities:

$$\begin{cases} f_1[f_1(\alpha_1+\alpha_1')+\alpha_1''] = f_1[\alpha_1+f_1(\alpha_1'+\alpha_1'')] \\ \prod_1\times S_1(\alpha_1'')\times S_1[f_1(\alpha_1+\alpha_1')+\alpha_1''] = S_1(\alpha_1)\times\prod_1'\times S_1[\alpha_1+f_1(\alpha_1'+\alpha_1'')]\end{cases}$$

We will try to test all possible cases. The three elements $\alpha_1\,;\alpha_1^{'};\ \alpha_1^{''}$ belong to the binary set $\{0;1\}$, and the set of arrivals is also binary; so the number of possibilities is $2^3=8$. We will study this equality in the form of a table.

Let us remember that the applications $f_1;\ f_2;\ f_3\ldots\ldots\ f_n$ are equivalent and so are the applications $S_1;\ S_2\,;\ S_3\ldots\ldots\ S_n$. So we will limit the comparison to the application f_1 and the application S_1.

➢ Let's start with the application f_1

Table 1. Possible results of associative equalities for the application f_1.

Possibilities	$f_1[f_1(\alpha_1+\alpha_1')+\alpha_1'']$	$f_1[\alpha_1+f_1(\alpha_1'+\alpha_1'')]$
$\alpha_1=0\ ;\ \alpha_1'=0\ ;\ \alpha_1''=0$	0	0
$\alpha_1=1\ ;\ \alpha_1'=0\ ;\ \alpha_1''=0$	1	1
$\alpha_1=0\ ;\ \alpha_1'=1\ ;\ \alpha_1''=0$	1	1
$\alpha_1=0\ ;\ \alpha_1'=0\ ;\ \alpha_1''=1$	1	1
$\alpha_1=1\ ;\ \alpha_1'=1\ ;\ \alpha_1''=0$	0	0
$\alpha_1=1\ ;\ \alpha_1'=0\ ;\ \alpha_1''=1$	0	0
$\alpha_1=0\ ;\ \alpha_1'=1\ ;\ \alpha_1''=1$	0	0
$\alpha_1=1\ ;\ \alpha_1'=1\ ;\ \alpha_1''=1$	1	1

We can deduce from this table that: $f_1[f_1(\alpha_1+\alpha_1^{'})+\alpha_1^{''}] = f_1[\alpha_1+f_1(\alpha_1^{'}+\alpha_1^{''})]$

➢ Let's finish with the application S_1

Table 2. Possible results of the associative equalities of the application S_1.

Possibilities	$\prod_1\times S_1(\alpha_1'')\times S_1[f_1(\alpha_1+\alpha_1')+\alpha_1'']$	$S_1(\alpha_1)\times\prod_1'\times S_1[\alpha_1+f_1(\alpha_1'+\alpha_1'')]$
$\alpha_1=0\,;\ \alpha_1'=0\,;\ \alpha_1''=0$	+1	+1
$\alpha_1=1\,;\ \alpha_1'=0\,;\ \alpha_1''=0$	+1	+1
$\alpha_1=0\,;\ \alpha_1'=1\,;\ \alpha_1''=0$	+1	+1
$\alpha_1=0\,;\ \alpha_1'=0\,;\ \alpha_1''=1$	+1	+1
$\alpha_1=1\,;\ \alpha_1'=1\,;\ \alpha_1''=0$	-1	-1
$\alpha_1=1\,;\ \alpha_1'=0\,;\ \alpha_1''=1$	-1	-1

 doi:10.20944/preprints202506.0879.v2

$\alpha_1 = 0\,;\ \alpha_1' = 1\,;\ \alpha_1'' = 1$	-1	-1
$\alpha_1 = 1\,;\ \alpha_1' = 1\,;\ \alpha_1'' = 1$	-1	-1

We can deduce from this second table that:

$$\prod_1 \times S_1(\alpha_1'') \times S_1[f_1(\alpha_1 + \alpha_1') + \alpha_1''] = S_1(\alpha_1) \times \prod_1{}' \times S_1[\alpha_1 + f_1(\alpha_1' + \alpha_1'')]$$

From the results of tables 1 and 2: the equality $\otimes_e^i[V_i\,;(V_i'\,;V_i'')] = \otimes_e^i[V_i\,;(V_i'\,;V_i'')]$ is true Therefore the imaginary product is associative for all levels of superposition.

3.3.3. The Actual Product $\otimes_e^r$

For the real product $\otimes_e^r$ to multiply two real parts together; we will no longer need a transformation (rotation and a transformation from imaginary to real writing as proposed in the first article). This is for two reasons: the first because all the real axes are united on a single one and second the new rules (new notation) allow the de-superposition. Thus the basis of the real part for a superimposed hypercomplex number for a level superposition n is written as follows:

$$h_r = \underbrace{\left\langle \begin{matrix} 0 \\ \pm 1 \end{matrix} \left| \begin{matrix} \cdots\cdot\cdots \\ \cdots\cdot\cdots \end{matrix} \right| \begin{matrix} 0 \\ \pm 1 \end{matrix} \right\rangle}_{n}$$

4. Notation and Structure of Algebraic Spaces $\mathbb{S}_{(2\,;\,2)}$ and $\mathbb{S}_{(3\,;\,3)}$

4.1. Algebraic Space $\mathbb{S}_{(2\,;\,2)}$

The set of complex numbers $\mathbb{C}$ and its 3D extension $\mathbb{C}_{(1\,;\,2)}$ as defined in the article "Complex Superposition Algebra" 1 are subsets of the algebraic space $\mathbb{S}_{(2\,;\,2)}$. Here is the commutative table of $\mathbb{C}_{(1\,;\,2)}$:

Table 3. commutative table in $\mathbb{C}_{(1\,;\,2)}$ 1.

$\times$	1	$e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}(i+j)}$	$e^{\frac{\pi}{2}(i-j)}$
1	1	$e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}(i+j)}$	$e^{\frac{\pi}{2}(i-j)}$
$e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}i}$	-1	$e^{\frac{\pi}{2}(i+j)}$	$-e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}j}$
$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}(i+j)}$	-1	$-e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}i}$
$e^{\frac{\pi}{2}(i+j)}$	$e^{\frac{\pi}{2}(i+j)}$	$-e^{\frac{\pi}{2}j}$	$-e^{\frac{\pi}{2}i}$	-1	-1
$e^{\frac{\pi}{2}(i-j)}$	$e^{\frac{\pi}{2}(i-j)}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}i}$	-1	-1

The imaginary numbers $i = e^{\frac{\pi}{2}i}$ and $j = e^{\frac{\pi}{2}j}$ are the primary bases.

Here we note the appearance of new superimposed imaginary numbers $e^{\frac{\pi}{2}(i+j)}$ and $e^{\frac{\pi}{2}(i-j)}$.

To simplify their representation we will note:

- s the first superimposed imaginary number $s = e^{\frac{\pi}{2}(i+j)}$
- t the second imaginary superimposed number $t = e^{\frac{\pi}{2}(i-j)}$.

Table 3 allows to preserve the associativity property during the superposition phase. However, desuperposition phase can lead to the loss of associativity.

This will allow us to give the superimposed notation of the imaginary product and consequently of the hypercomplex numbers in $\mathbb{C}_{(1;\,2)}$

$i = \left\langle \begin{matrix} 1 \\ 1 \end{matrix} \middle| \begin{matrix} 0 \\ 1 \end{matrix} \right\rangle$; $j = \left\langle \begin{matrix} 0 \\ 1 \end{matrix} \middle| \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle$; $s = \left\langle \begin{matrix} 1 \\ 1 \end{matrix} \middle| \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle$ and $t = \left\langle \begin{matrix} 1 \\ 1 \end{matrix} \middle| \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle$

For real numbers we have: $\left\langle \begin{matrix} 0 \\ 1 \end{matrix} \middle| \begin{matrix} 0 \\ 1 \end{matrix} \right\rangle = 1$ and it opposite $\left\langle \begin{matrix} 0 \\ -1 \end{matrix} \middle| \begin{matrix} 0 \\ -1 \end{matrix} \right\rangle = -1$. So we can define a hypercomplex geometry with four signs.

$q_0 = \left\langle \begin{matrix}0\\1\end{matrix} \middle| \begin{matrix}0\\1\end{matrix} \right\rangle = \langle + | + \rangle$ and it opposite $-q_0 = \left\langle \begin{matrix}0\\-1\end{matrix} \middle| \begin{matrix}0\\-1\end{matrix} \right\rangle = \langle - | - \rangle$

$q_1 = \left\langle \begin{matrix}0\\1\end{matrix} \middle| \begin{matrix}0\\-1\end{matrix} \right\rangle = \langle + | - \rangle$ and it opposite $-q_1 = \left\langle \begin{matrix}0\\-1\end{matrix} \middle| \begin{matrix}0\\1\end{matrix} \right\rangle = \langle - | + \rangle$

So some will think that it is multiplying the signs – to obtain the number 1; in the writing of the number $\left\langle \begin{matrix}0\\-1\end{matrix} \middle| \begin{matrix}0\\-1\end{matrix} \right\rangle$.

What we need to understand here is that this number is the square of the imaginary numbers of level 2, so it is logically negative. This being the very definition of an imaginary number.

Let's note here that this looks a lot like binary code.

We can give the following table of signs:

Table 4. commutative table of the four signs in $\mathcal{C}_{(1\,;\,2)}1$.

$\times$	q_0	$-q_0$	q_1	$-q_1$
q_0	q_0	$-q_0$	q_1	$-q_1$
$-q_0$	$-q_0$	q_0	$-q_1$	q_1
q_1	q_1	$-q_1$	q_0	$-q_0$
$-q_1$	$-q_1$	q_1	$-q_0$	q_0

But we will come back to this later to give a logical and geometric explanation but above all its possible applications.

In the meantime, let's give an example of calculation

$$i \times j = \left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}0\\1\end{matrix} \right\rangle \times \left\langle \begin{matrix}0\\1\end{matrix} \middle| \begin{matrix}1\\1\end{matrix} \right\rangle$$

$$i \times j = \left\langle \begin{matrix} f_1(1+0) \\ S_1(1) \times S_1(0) \times S_1(1+0) \end{matrix} \middle| \begin{matrix} f_2(0+1) \\ S_2(0) \times S_2(1) \times S_2(0+1) \end{matrix} \right\rangle$$

$$i \times j = \left\langle \begin{matrix} f_1(1) \\ S_1(1) \times S_1(0) \times S_1(1) \end{matrix} \middle| \begin{matrix} f_2(1) \\ S_2(0) \times S_2(1) \times S_2(1) \end{matrix} \right\rangle = \left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}1\\1\end{matrix} \right\rangle = s$$

Let us try to calculate the applications f and S directly.

$$(i \times j) \times s = \left(\left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}0\\1\end{matrix} \right\rangle \times \left\langle \begin{matrix}0\\1\end{matrix} \middle| \begin{matrix}1\\1\end{matrix} \right\rangle \right) \times \left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}1\\1\end{matrix} \right\rangle = \left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}1\\1\end{matrix} \right\rangle \times \left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}1\\1\end{matrix} \right\rangle = \left\langle \begin{matrix}0\\-1\end{matrix} \middle| \begin{matrix}0\\-1\end{matrix} \right\rangle = -q_0$$

$$i \times (j \times s) = \left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}0\\1\end{matrix} \right\rangle \left(\left\langle \begin{matrix}0\\1\end{matrix} \middle| \begin{matrix}1\\1\end{matrix} \right\rangle \times \left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}1\\1\end{matrix} \right\rangle \right) = \left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}0\\1\end{matrix} \right\rangle \times \left\langle \begin{matrix}1\\1\end{matrix} \middle| \begin{matrix}0\\-1\end{matrix} \right\rangle = \left\langle \begin{matrix}0\\-1\end{matrix} \middle| \begin{matrix}0\\-1\end{matrix} \right\rangle = -q_0$$

The imaginary product is a binary code generator.

So we can table 3 with the following table:

Table 5. commutative table in $\mathcal{C}_{(1\,;\,2)}1$.

$\times$	q_0	i	j	s	t
q_0	q_0	$q_0.i$	$q_0.j$	$q_0.s$	$q_0.t$
i	$q_0.i$	$-q_1$	$q_0.s$	$-q_1.j$	$-q_0.j$
j	$q_0.j$	$q_0.s$	q_1	$q_1.i$	$q_0.i$
s	$q_0.s$	$-q_1.j$	$q_1.i$	$-q_0$	$-q_1$
t	$q_0.t$	$-q_0.j$	$q_0.i$	$-q_1$	$-q_0$

This table allows us to keep information on the superposition of signs; which preserves commutativity and associativity.

4.2. Algebraic Space $\mathbb{S}_{(\mathbf{3}\,;\,\mathbf{3})}$

To simplify their representation we will note:

❖ Real numbers superposition level 3

We have: $\left\langle {0 \atop 1} \middle| {0 \atop 1} \middle| {0 \atop 1} \right\rangle = 1$ and it opposite $\left\langle {0 \atop -1} \middle| {0 \atop -1} \middle| {0 \atop -1} \right\rangle = -1$ ($v_r = 1$ the real number v_r). So we can define a hypercomplex geometry with eight signs.

- $q_0 = \left\langle {0 \atop 1} \middle| {0 \atop 1} \middle| {0 \atop 1} \right\rangle = \langle + | + | + \rangle$ and it opposite $-q_0 = \left\langle {0 \atop -1} \middle| {0 \atop -1} \middle| {0 \atop -1} \right\rangle = \langle - | - | - \rangle$
- $q_1 = \left\langle {0 \atop -1} \middle| {0 \atop 1} \middle| {0 \atop 1} \right\rangle = \langle - | + | + \rangle$ and it opposite $-q_1 = \left\langle {0 \atop 1} \middle| {0 \atop -1} \middle| {0 \atop -1} \right\rangle = \langle + | - | - \rangle$
- $q_2 = \left\langle {0 \atop 1} \middle| {0 \atop -1} \middle| {0 \atop 1} \right\rangle = \langle + | - | + \rangle$ and it opposite $-q_2 = \left\langle {0 \atop -1} \middle| {0 \atop 1} \middle| {0 \atop -1} \right\rangle = \langle - | + | - \rangle$
- $q_3 = \left\langle {0 \atop 1} \middle| {0 \atop 1} \middle| {0 \atop -1} \right\rangle = \langle + | + | - \rangle$ and it opposite $-q_3 = \left\langle {0 \atop -1} \middle| {0 \atop -1} \middle| {0 \atop 1} \right\rangle = \langle - | - | + \rangle$

What we observe here is a superposition of the signs. It will noted that the sign configuration obtained here is very similar to the sign configuration of quarks in particle physics.6

❖ Imaginary numbers

- Superposition of level 1

➢ $i_{1;1} = v_{1,i} = e^{\frac{\pi}{2} i_{1;1}} = \left\langle {1 \atop 1} \middle| {0 \atop 1} \middle| {0 \atop 1} \right\rangle$: the first primary imaginary number

➢ $i_{2;1} = v_{2,i} = e^{\frac{\pi}{2} i_{2;1}} = \left\langle {0 \atop 1} \middle| {1 \atop 1} \middle| {0 \atop 1} \right\rangle$: the second primary imaginary number

➢ $i_{3;1} = v_{3,i} = e^{\frac{\pi}{2} i_{3;1}} = \left\langle {0 \atop 1} \middle| {0 \atop 1} \middle| {1 \atop 1} \right\rangle$: the third primary imaginary number

- Superposition of level 2

➢ $i_{1;2} = s_{1,i} = e^{\frac{\pi}{2}(i_{3;1}+i_{1;1})} = \left\langle {1 \atop 1} \middle| {0 \atop 1} \middle| {1 \atop 1} \right\rangle$: the first superimposed imaginary number of level 2

➢ $i_{2;2} = s_{2,i} = e^{\frac{\pi}{2}(i_{1;1}+i_{2;1})} = \left\langle {1 \atop 1} \middle| {1 \atop 1} \middle| {0 \atop 1} \right\rangle$: the second superimposed imaginary number of level 2

➢ $i_{3;2} = s_{3,i} = e^{\frac{\pi}{2}(i_{2;1}+i_{3;1})} = \left\langle {0 \atop 1} \middle| {1 \atop 1} \middle| {1 \atop 1} \right\rangle$: the third superimposed imaginary number of level 2

➢ $i_{4;2} = s_{4,i} = e^{\frac{\pi}{2}(i_{3;1}-i_{1;1})} = \left\langle {1 \atop -1} \middle| {0 \atop 1} \middle| {1 \atop 1} \right\rangle$: the fourth superimposed imaginary number of level 2

➢ $i_{5;2} = s_{5,i} = e^{\frac{\pi}{2}(i_{1;1}-i_{2;1})} = \left\langle {1 \atop 1} \middle| {1 \atop -1} \middle| {0 \atop 1} \right\rangle$: the fifth superimposed imaginary number of level 2

➢ $i_{6;2} = s_{6,i} = e^{\frac{\pi}{2}(i_{2;1}-i_{3;1})} = \left\langle {0 \atop 1} \middle| {1 \atop 1} \middle| {1 \atop -1} \right\rangle$: the sixth superimposed imaginary number of level 2:

- Superposition of level 3

➢ $i_{1;3} = u_{2,i} = e^{\frac{\pi}{2}(i_{1,1}+i_{2,1}+i_{3,1})} = \left\langle {1 \atop 1} \middle| {1 \atop 1} \middle| {1 \atop 1} \right\rangle$ the first superimposed imaginary number of level 3

➢ $i_{2;3} = u_{1,i} = e^{\frac{\pi}{2}(i_{3,1}+i_{1,1}-i_{2,1})} = \left\langle {1 \atop 1} \middle| {1 \atop -1} \middle| {1 \atop 1} \right\rangle$ the second superimposed imaginary number of level 3

➢ $i_{3;3} = u_{3,i} = e^{\frac{\pi}{2}(i_{1,1}+i_{2,1}-i_{3,1})} = \left\langle {1 \atop 1} \middle| {1 \atop 1} \middle| {1 \atop -1} \right\rangle$ the third superimposed imaginary number of level 3

➢ $i_{4;3} = u_{4,i} = e^{\frac{\pi}{2}(i_{2,1}+i_{3,1}-i_{1,1})} = \left\langle {1 \atop -1} \middle| {1 \atop 1} \middle| {1 \atop 1} \right\rangle$ the fourth superimposed imaginary number of level 3

The imaginary number $i_{p;n}$ will designate the p^th superimposed imaginary number of level n

Calculation example:

$$i_{2;2} \times i_{3;3} = \left\langle {1 \atop 1} \middle| {1 \atop 1} \middle| {0 \atop 1} \right\rangle \times \left\langle {1 \atop 1} \middle| {1 \atop 1} \middle| {1 \atop -1} \right\rangle = \left\langle {0 \atop -1} \middle| {0 \atop -1} \middle| {1 \atop -1} \right\rangle = -q_0 . i_{3;1}$$

Here we must note that the result is level 1 superposition.

$$-q_0 \times i_{3;1} = \left\langle {0 \atop -1} \middle| {0 \atop -1} \middle| {0 \atop -1} \right\rangle \times \left\langle {0 \atop 1} \middle| {0 \atop 1} \middle| {1 \atop 1} \right\rangle = \left\langle {0 \atop -1} \middle| {0 \atop -1} \middle| {1 \atop -1} \right\rangle$$

Which means that in the table established in the article "Complex Superposition Algebra" 14; we must make some modification related to the rules of superposition of signs. However this table is indeed commutative; the negative signs have been multiplied between them so in some related to the de-superposition we will have an inversion of the sign. The new rules established show that the

 doi:10.20944/preprints202506.0879.v2

imaginary product is commutative and associative. So there is no longer the need to establish a table related to commutativity or associativity for real and imaginary products.

5. Logical and Physical Interpretation:

This superimposed binary product models a regular logical operator in a leveled superposition space. The preservation of fundamental algebraic properties opens a path towards a representation of qubits and tbits in a discrete geometric structure, where each level encodes an extended binary logic.6

6. Conclusion

We have presented a superimposed hypercomplex multiplication that satisfies the properties of associativity and commutativity, which is unusual in algebraic extensions beyond complexes 27. This structure opens the way to a logical formalization of quantum operators in a richer framework than that of classical state vectors.6

7. Perspectives

This framework will be explored further in a later article, where it will be shown that this structure allows superposition, spin and quantum mechanics to be naturally linked via a dynamic evolution code.6

Data Availability Statement: All data that support this paper are included within the article.

Conflicts of Interest: The author declares that there is no conflict of interest regarding this paper.

Pre-Print: A preprint has previously been published on Zenodo *[Séqueira, J. A. (2025), link:* https://doi.org/10.5281/zenodo.15464528*].*

Author Contributions: José Antoine Séqueira wrote the main manuscript text, prepared all tables and then reviewed the manuscript.

Funding: No funding was received for this research.

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