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[Mashael A. Alshehri](#) *

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Article

Entropy-Based Evidence Functions for Testing Dilation Order via Cumulative Entropies

Mashaël A. Alshehri

Department of Quantitative Analysis, College of Business Administration, King Saud University, Riyadh, 11362, Saudi Arabia

Abstract

This paper introduces novel entropy-based evidence functions for assessing the dilation order of probability distributions, constructed from cumulative residual entropy (CRE) and cumulative entropy (CE). The proposed test statistics are developed within a rigorous evidential framework, and their asymptotic distributions are established, ensuring a solid foundation for large-sample inference. Beyond their theoretical appeal, these tests act as effective entropy-driven evidence functions, offering a compelling alternative to traditional approaches such as Kullback–Leibler discrepancies. Comprehensive Monte Carlo simulations highlight their robustness and consistently superior power across a wide range of distributional scenarios, including heavy-tailed models where conventional methods often fail. A real data example further illustrates their practical utility, showing how cumulative entropies can provide sharper statistical evidence and clarify stochastic comparisons in applied settings. These results not only advance the theoretical foundation of statistical evidence but also open avenues for applying cumulative entropies to broader classes of stochastic inference problems.

Keywords: dilation order; cumulative residual entropy (CRE); cumulative entropy (CE); evidence functions; statistical inference; HNBUE; L-statistics; test power

1. Introduction

Modern statistical inference increasingly emphasizes the role of statistical evidence, where model comparisons and hypothesis testing are guided by functions that quantify the support provided by data. Traditionally, such evidence functions have been built on information-theoretic discrepancies, most notably the Kullback–Leibler divergence and model selection criteria such as AIC and BIC. While these tools are powerful for assessing model fit, many problems in reliability, finance, and insurance require a broader perspective that addresses not only fit but also the variability and stochastic ordering of probability distributions.

Stochastic orders provide a rigorous framework for comparing probability distributions according to characteristics such as location, variability, and risk. They play an important role in diverse areas including reliability, actuarial science, economics, and finance, where understanding differences in distributional behavior is essential. Among these, the dilation order is particularly significant, as it ranks random variables by their degree of dispersion and thus provides statistical evidence for variability. This makes it especially valuable when analyzing heavy-tailed or complex data, where conventional variance-based methods often lack sensitivity. Recall that a random variable Y is more dispersed in the dilation order than a random variable X (denoted by $X \leq_{dil} Y$) if

$$\mathbb{E}[\phi(X - \mathbb{E}(X))] \leq \mathbb{E}[\phi(Y - \mathbb{E}(Y))], \tag{1}$$

for every convex function ϕ , provided the expectations exist [1]. This ordering formalizes the idea that Y exhibits greater dispersion than X . It is straightforward to see that $X \leq_{dil} Y$ implies $\text{Var}(X) \leq \text{Var}(Y)$, although variance provides only a total ordering, which is less informative than the partial

ordering offered by dilation. For additional details on the dilation order and its connections with other variability and stochastic orders, see Shaked and Shanthikumar [2] and Sordo et al. [3].

Consequently, several statistical tests have been proposed to verify the presence of dilation order [4,5]. At the same time, the measurement of uncertainty and the development of entropy-based functionals have become active areas of research in statistics and information theory. The **Shannon differential entropy**, introduced in [6], is defined for a nonnegative random variable X with pdf $f(x)$ and cdf $F(x)$ as

$$\mathcal{H}(X) = - \int_0^{\infty} f(x) \log f(x) dx, \quad (2)$$

where “ $\log(\cdot)$ ” means natural logarithm. It is well known that $\mathcal{H}(X)$ is location-free, since X and $X + b$ have identical differential entropy for any $b > 0$. However, the differential entropy defined in (2) has limitations as a continuous analogue of discrete entropy, which has motivated the development of alternative information measures. Several extensions have been proposed for continuous distributions, including weighted entropy and its residual and past variants [7,8]. In this direction, Rao et al. [9] introduced the cumulative residual entropy (CRE), defined as:

$$\mathcal{E}(X) = - \int_0^{\infty} S(x) \log S(x) dx = \int_0^{\infty} S(x) \Lambda(x) dx, \quad (3)$$

where $S(x) = P(X > x)$ denotes the survival function, and

$$\Lambda(x) = -\log S(x) = \int_0^x \frac{f(u)}{S(u)} du, x > 0, \quad (4)$$

denotes the cumulative hazard function. The cumulative residual entropy (CRE) has proved to be an effective measure for characterizing information dispersion, particularly in the study of aging problems within reliability theory. Owing to this property, it has found wide applications in reliability analysis and related areas [10,11], including the influential works of Navarro et al. [13–15], among others. From an evidential perspective, CRE serves not only as a measure of uncertainty but also as a functional tool for drawing statistical evidence about variability and aging behavior in random variables. Di Crescenzo and Longobardi [16] introduced the cumulative entropy (CE) as an alternative measure of uncertainty associated with inactivity time. It is obtained by replacing the cumulative distribution function (cdf) in place of the probability density function (pdf) in the Shannon differential entropy, and is defined as:

$$\mathcal{CE}(X) = - \int_0^{\infty} F(x) \log F(x) dx = \int_0^{\infty} F(x) T(x) dx, \quad (5)$$

where

$$T(x) = -\log F(x) = \int_x^{\infty} \frac{f(u)}{F(u)} du, x > 0, \quad (6)$$

denotes the cumulative reversed hazard function. Since the logarithmic argument in its definition is a probability, the cumulative entropy $\mathcal{CE}(X)$, like the CRE, is always nonnegative and takes values in the range $[0, \infty)$. In contrast, the Shannon entropy $\mathcal{H}(X)$ may be negative for continuous variables. Moreover, $\mathcal{CE}(X) = 0$ if and only if X is constant, underscoring its role as a measure of uncertainty. Di Crescenzo and Longobardi [16] further investigated several properties of cumulative entropy and its dynamic version for past lifetimes, establishing results on characterization and stochastic orderings. From the standpoint of statistical evidence, these properties reinforce the usefulness of CE as a functional tool for comparing distributions and quantifying uncertainty in diverse applications.

In this paper, we consider two random variables, X and Y , under the condition that $X \leq_{\text{dil}} Y$. Our objective is to develop and evaluate statistical tests for the null hypothesis $H_0: X \stackrel{\text{dil}}{=} Y$ against the alternative $H_1: X \leq_{\text{dil}} Y$ and $X \neq Y$. The proposed methodology is based on constructing empirical evidence functions from the cumulative residual entropy (CRE) and cumulative entropy (CE), applied to two independent samples of X and Y . Unlike existing approaches for testing the dilation order, such as those of Aly [20], Belzunce et al. [5], Sordo et al. [3], and Zardasht [21], our framework offers clear advantages. In particular, the CRE- and CE-based evidence functions are

straightforward to implement, computationally efficient, and capable of achieving superior performance in detecting differences in variability, especially in heavy-tailed settings. This study makes three contributions. First, it provides a comprehensive review of existing dilation-order tests and their limitations. Second, it develops new entropy-based evidence functions grounded in cumulative entropies, establishing their asymptotic properties and practical implementation. Third, it validates the proposed approach through extensive simulation experiments and a real data application, illustrating both its theoretical soundness and practical utility.

The remainder of the paper is organized as follows. Section 2 presents the theoretical foundations, showing that the dilation order $\leq_{\text{dil}}$ induces a natural relationship between the cumulative residual entropy (CRE) and cumulative entropy (CE) of random variables. In this section, we also formulate the proposed test statistics for the dilation order alternative, establish their asymptotic distributions, and discuss adaptations for practical implementation. Section 3 reports on a Monte Carlo simulation study that evaluates the finite-sample performance of the proposed tests, with particular attention to their power properties under different distributional settings. Section 4 provides an application to a real dataset, illustrating how entropy-based evidence functions can clarify stochastic comparisons in practice. Finally, Section 5 concludes with a summary of the main findings, highlights the advantages of the proposed approach, and suggests directions for future research.

2. Entropy-Based Evidence Functions for the Dilation Order

This section investigates how cumulative residual entropy (CRE) and cumulative entropy (CE) can serve as evidence functions for assessing the dilation order. A key advantage of CRE lies in its connection to the mean residual life (MRL) function, $m(x) = \mathbb{E}(X - x | X > x)$. Asadi and Zohrevand [11] showed that $\mathcal{E}(X) = \mathbb{E}[m(X)]$, a relationship that underscores the relevance of CRE in reliability theory, where the MRL function is widely used to describe system aging. Similarly, CE is closely related to the mean inactivity time (MIT) function, $\tilde{m}(x) = \mathbb{E}(x - X | X \leq x)$ and Di Crescenzo and Longobardi [16] established that $\mathcal{CE}(X) = \mathbb{E}[\tilde{m}(X)]$. These connections highlight the role of CRE and CE not only as measures of uncertainty but also as practical evidence functions for quantifying variability. We now turn to the relationship between the dilation order of two random variables and the ordering of their CRE and CE. Recall that for a random variable X with cumulative distribution function (cdf) F , the quantile function is defined by $F^{-1}(u) = \inf\{x: F(x) \geq u\}$ for $u \in (0,1)$.

Theorem 2.1. Let X and Y be two absolutely continuous nonnegative random variables with respective finite means $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ and with pdfs f and g and cdfs F and G , respectively. Then,

- (i) if $X \leq_{\text{dil}} Y$, then $\mathcal{E}(X) \leq \mathcal{E}(Y)$.
- (ii) if $X \leq_{\text{dil}} Y$, then $\mathcal{CE}(X) \leq \mathcal{CE}(Y)$.

Proof. (i) Given that $\mathbb{E}(X | X > x) = \frac{1}{S(x)} \int_x^\infty uf(u)du$, and $m(x) = \mathbb{E}(X | X > x) - x$, along with the relation $\mathcal{E}(X) = \mathbb{E}[m(X)]$ and Eq. (4), we can rewrite $\mathcal{E}(X)$ as:

$$\begin{aligned} \mathcal{E}(X) &= \int_0^\infty \mathbb{E}(X | X > x) f(x) dx - \mathbb{E}(X) \\ &= \int_0^\infty \frac{1}{S(x)} \int_x^\infty uf(u) du f(x) dx - \mathbb{E}(X) \\ &= \int_0^\infty uf(u) \int_0^u \frac{f(x)}{S(x)} dx du - \mathbb{E}(X) \end{aligned}$$

$$\begin{aligned}
&= - \int_0^\infty u f(u) \log S(u) du - \mathbb{E}(X) \\
&= - \int_0^1 F^{-1}(u) \log(1-u) du - \mathbb{E}(X), (7)
\end{aligned}$$

where the final equality follows from the substitution $u = F(x)$. An analogous identity holds for $\mathcal{E}(Y)$. Recall from Theorem 3.A. 8 of Shaked and Shanthikumar [2] that $X \leq_{dil} Y$ implies

$$\frac{1}{1-p} \int_p^1 [F^{-1}(u) - G^{-1}(u)] du \leq \int_0^1 [F^{-1}(u) - G^{-1}(u)] du, \text{ for all } p \in [0,1].$$

Since $\mathbb{E}(X) = \int_0^1 F^{-1}(u) du$ and $\mathbb{E}(Y) = \int_0^1 G^{-1}(u) du$, integrating both sides of the inequality over $[0,1]$ and applying Fubini's theorem yields

$$\begin{aligned}
\mathbb{E}(X) - \mathbb{E}(Y) &\geq \int_0^1 \frac{1}{1-p} \int_p^1 [F^{-1}(u) - G^{-1}(u)] du dp \\
&= \int_0^1 [F^{-1}(u) - G^{-1}(u)] \int_0^u \frac{1}{1-p} dp du \\
&= - \int_0^1 [F^{-1}(u) - G^{-1}(u)] \log(1-u) du \\
&= \int_0^1 G^{-1}(u) \log(1-u) du - \int_0^1 F^{-1}(u) \log(1-u) du.
\end{aligned}$$

This implies

$$- \int_0^1 F^{-1}(u) \log(1-u) du - \mathbb{E}(X) \leq \int_0^1 G^{-1}(u) \log(1-u) du - \mathbb{E}(Y),$$

or equivalently, $\mathcal{CE}(X) \leq \mathcal{CE}(Y)$, by recalling relation (7).

(ii) Using $\mathbb{E}(X | X \leq x) = \frac{1}{F(x)} \int_0^x u f(u) du$, $\tilde{m}(x) = x - \mathbb{E}(X | X \leq x)$, and Eq. (6), we can rewrite $\mathcal{CE}(X) = \mathbb{E}[\tilde{m}(X)]$, as in Part (i), as follows:

$$\begin{aligned}
\mathcal{CE}(X) &= \int_0^\infty u f(u) \log F(u) du + \mathbb{E}(X) \\
&= \int_0^1 F^{-1}(u) \log(u) du + \mathbb{E}(X), (8)
\end{aligned}$$

where the last equality follows from the substitution $u = F(x)$. The same identity holds for $\mathcal{CE}(Y)$. Recall from Shaked and Shanthikumar [2] that $X \leq_{dil} Y$ yields

$$\int_0^1 [F^{-1}(u) - G^{-1}(u)] du \leq \frac{1}{p} \int_0^p [F^{-1}(u) - G^{-1}(u)] du, \text{ for all } p \in [0,1].$$

Integrating both sides of the preceding relation over $[0,1]$ and applying Fubini's theorem yields

$$\begin{aligned}
\mathbb{E}(X) - \mathbb{E}(Y) &\leq \int_0^1 \frac{1}{p} \int_0^p [F^{-1}(u) - G^{-1}(u)] du dp \\
&= \int_0^1 [F^{-1}(u) - G^{-1}(u)] \int_u^1 \frac{1}{p} dp du \\
&= \int_0^1 G^{-1}(u) \log(u) du - \int_0^1 F^{-1}(u) \log(u) du.
\end{aligned}$$

This implies that

$$\int_0^1 F^{-1}(u) \log(u) du + \mathbb{E}(X) \leq \int_0^1 G^{-1}(u) \log(u) du + \mathbb{E}(Y)$$

or equivalently $\mathcal{CE}(X) \leq \mathcal{CE}(Y)$ by (8). This completes the proof. $\square$

The next theorem highlights a key implication: if two random variables are ordered by dilation and share the same CE, then they must be identical in distribution or differ only by a location shift.

Theorem 2.2. Under the conditions of Theorem 2.1, if $X \leq_{\text{dil}} Y$ and $\mathcal{CE}(X) = \mathcal{CE}(Y)$, then X and Y have the same distribution up to a location parameter.

Proof. Consider the function

$$A_X(p) = \int_0^p [F^{-1}(t) - \mathbb{E}(X)]dt, \text{ for } 0 \leq p \leq 1.$$

Since $A_X(0) = A_X(1) = 0$, and from Eq. (8), we have

$$\mathcal{E}(X) = - \int_0^1 \frac{1}{p} A_X(p) dp. \quad (9)$$

Thus, $\mathcal{CE}(X) = \mathcal{CE}(Y)$ is equivalent to

$$\int_0^1 \frac{1}{p} [A_X(p) - A_Y(p)] dp = 0. \quad (10)$$

Furthermore, by Theorem 2.1 of Ramos and Sordo [17], $X \leq_{\text{dil}} Y$ implies

$$A_X(p) \geq A_Y(p), 0 \leq p \leq 1. \quad (11)$$

From (10) and (11), $A_X(p) = A_Y(p)$ almost everywhere on $[0,1]$. We claim that $A_X(p) = A_Y(p)$ for all $p \in [0,1]$. Otherwise, there exists an interval $(a, b) \subset [0,1]$ such that $A_X(p) > A_Y(p)$ for all $p \in (a, b)$. Then,

$$\int_0^1 \frac{1}{p} [A_X(p) - A_Y(p)] dp \geq \int_a^b \frac{1}{p} [A_X(p) - A_Y(p)] dp > 0.$$

contradicting with (10). Therefore,

$$\int_p^1 [F^{-1}(t) - \mathbb{E}(X)]dt = \int_p^1 [G^{-1}(t) - \mathbb{E}(Y)]dt, \text{ for all } p \in [0,1]. \quad (12)$$

Differentiating (12) with respect to p yields

$$F^{-1}(p) = G^{-1}(p) + c \text{ for all } p \in [0,1],$$

where $c = \mathbb{E}(X) - \mathbb{E}(Y)$, implying that X and Y have the same distribution up to a location parameter. $\square$

For random variables ordered by dilation, equal CRE values imply identical distributions or differences only in location which is proved in the next theorem.

Theorem 2.3. Under the conditions of Theorem 2.1, if $X \leq_{\text{dil}} Y$ and $\mathcal{E}(X) = \mathcal{E}(Y)$, then X and Y have the same distribution up to a location parameter.

Proof. Consider the function

$$\bar{A}_X(p) = \int_p^1 [F^{-1}(t) - \mathbb{E}(X)]dt, \text{ for } 0 \leq p \leq 1.$$

Given that $\bar{A}_X(0) = \bar{A}_X(1) = 0$, and using Eq. (7), we have

$$\mathcal{E}(X) = \int_0^1 \frac{1}{1-p} \bar{A}_X(p) dp \quad (13)$$

Consequently, $\mathcal{E}(X) = \mathcal{E}(Y)$ is equivalent to $\int_0^1 \frac{1}{1-p} [\bar{A}_X(p) - \bar{A}_Y(p)] dp = 0$. Since $\bar{A}_X(p) = -A_X(p)$ for all $0 \leq p \leq 1$, Theorem 2.1 of Ramos and Sordo [17] implies that $X \leq_{\text{dil}} Y$ leads to

$$\bar{A}_X(p) \leq \bar{A}_Y(p), 0 \leq p \leq 1 \quad (14)$$

The remainder of the proof is analogous to that of Theorem 2.2.

It should be noted that Theorems 2.2 and 2.3 imply that if $X \leq_{\text{dil}} Y$ and $\mathcal{CE}(X) = \mathcal{CE}(Y)$ (or $\mathcal{E}(X) = \mathcal{E}(Y)$), then $X \stackrel{\text{dil}}{=} Y$, meaning X and Y are equal in distribution up to a location parameter.

3. Statistical Evidence for the Dilation Order via Cumulative Entropies

Economic and social processes often influence the variability of distributions, such as household spending before and after-tax reforms, or stock returns before and after financial policy changes. A natural question in these contexts is whether such changes significantly alter variability. To address this, we develop tests for the null hypothesis $H_0: X \stackrel{\text{dil}}{=} Y$ (variability remains unchanged) against the alternative $H_1: X \leq_{\text{dil}} Y$ and $X \stackrel{\text{dil}}{\neq} Y$ (variability increases). According to Theorems 2.1–2.3, this comparison can be expressed in terms of entropy-based evidence functions. In particular, the functionals

$$\Delta_1 = \mathcal{E}(Y) - \mathcal{E}(X) \text{ and } \Delta_2 = \mathcal{CE}(Y) - \mathcal{CE}(X),$$

serve as natural measures of departure from H_0 in favor of H_1 . Thus, the null hypothesis should be rejected if Δ_1 or Δ_2 exceed their corresponding critical thresholds. Since the true values of $\mathcal{E}(X)$, $\mathcal{E}(Y)$, $\mathcal{CE}(X)$, and $\mathcal{CE}(Y)$ are generally unknown, we estimate them from independent random samples $\{X_1, X_2, \dots, X_n\}$ and $\{Y_1, Y_2, \dots, Y_n\}$. Replacing the population entropies with their empirical counterparts, we obtain the test statistics

$$\hat{\Delta}_1 = \hat{\mathcal{E}}_n(Y) - \hat{\mathcal{E}}_n(X), \quad (15)$$

and

$$\hat{\Delta}_2 = \hat{\mathcal{CE}}_n(Y) - \hat{\mathcal{CE}}_n(X). \quad (16)$$

where $\hat{\mathcal{E}}_n(\cdot)$ and $\hat{\mathcal{CE}}_n(\cdot)$ denote the empirical estimators of CRE and CE, respectively. The rejection thresholds c_1 and c_2 are determined by the null distributions of $\hat{\Delta}_1$ and $\hat{\Delta}_2$, which are studied in the next subsection.

We reject H_0 when the estimates of $\hat{\Delta}_1$ or $\hat{\Delta}_2$ are sufficiently large. Let $X_1, X_2, \dots, X_n$ be a sequence of independent and identically distributed (i.i.d.) continuous nonnegative random variables, with order statistics $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$. The empirical distribution function corresponding to $F(x)$ is defined as

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n I_{(X_i \leq x)},$$

which can equivalently be expressed as

$$\hat{F}_n(x) = \begin{cases} 0, & x < x_{1:n} \\ \frac{i}{n}, & x_{i:n} \leq x \leq x_{i+1:n}, \quad (i = 1, 2, \dots, n-1) \\ 1, & x > x_{n:n} \end{cases}$$

where I_A denotes the indicator function of event A . An estimator of the CRE, based on a nonparametric approach and derived from the L -functional estimator, is then given by

$$\hat{\mathcal{E}}_n(X) = \int_0^\infty x J(\hat{F}_n(x)) d\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n J\left(\frac{i}{n}\right) X_{i:n},$$

where $J(u) = -\log(1-u) - 1$, for $0 < u < 1$. Similar arguments can be applied to obtain the estimator $\hat{\mathcal{E}}_n(Y)$.

The following theorem establishes the asymptotic normality of the test statistic $\hat{\Delta}_1$, providing the theoretical foundation for its use as an evidence function in testing the dilation order.

Theorem 3.1. Assume that $\mathbb{E}(X^2) < \infty$ and $\mathbb{E}(Y^2) < \infty$ with $\sigma^2(F) > 0$, and $\sigma^2(G) > 0$. Let $S = m + n$ and suppose that for some $0 < \tau < 1$, $\frac{n}{S} \rightarrow \tau$, $\frac{m}{S} \rightarrow 1 - \tau$ as $\min\{n, m\} \rightarrow \infty$. Then, as $\min\{n, m\} \rightarrow \infty$, $\sqrt{S}(\hat{\Delta}_1 - \Delta_1) \rightarrow_d N(0, \sigma^2(F, G))$, where

$$\sigma^2(F, G) = \frac{\sigma^2(F)}{\tau} + \frac{\sigma^2(G)}{1 - \tau},$$

with

$$\sigma^2(F) = \int_0^\infty \int_0^\infty (F(\min(x, y)) - F(x)F(y))J(x)J(y)dx dy, \quad (17)$$

and $\sigma^2(G)$ defined analogously.

Proof. Since the function J is bounded and continuous, Theorems 2 and 3 of Stigler [18] imply that $\sqrt{n}(\hat{\mathcal{E}}_n(X) - \mathcal{E}(X))$ converges in distribution to a normal law with mean zero and finite variance $\sigma^2(F) > 0$ as $n \rightarrow \infty$. A similar convergence holds for $\hat{\mathcal{E}}_m(Y)$, because convergence in distribution is preserved under convolution.

To address the dependence of (19) on the unknown distribution function, we employ a consistent estimator of the variance. Following the representation of Jones and Zitikis [19], we define

$$\hat{\sigma}_n^2(F) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left(\min\left(\frac{i}{n}, \frac{j}{n}\right) - \frac{i}{n} \frac{j}{n} \right) J\left(\frac{i}{n}\right) J\left(\frac{j}{n}\right) (X_{i+1:n} - X_{i:n})(X_{j+1:n} - X_{j:n}),$$

with $\hat{\sigma}_m^2(G)$ defined analogously. The decision rule for rejecting H_0 in favor of H_1 at significance level α is:

$$\left| \frac{\hat{\Delta}_1}{\sqrt{\frac{\hat{\sigma}_n^2(F)}{n} + \frac{\hat{\sigma}_m^2(G)}{m}}} \right| > z_{1-q}, \quad (18)$$

where z_{1-q} represents the $(1 - q)$ -quantile of the standard normal distribution.

We now present the analogous result for the cumulative entropy. To this end, we propose a nonparametric estimator of CE, derived from the L -functional representation, defined as:

$$\widehat{\mathcal{CE}}_n(X) = \int_0^\infty x \bar{J}(\hat{F}_n(x)) d\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n \bar{J}\left(\frac{i}{n}\right) X_{i:n},$$

where $\bar{J}(u) = \log(u) + 1$, for $0 < u < 1$. A similar estimator can be constructed for $\widehat{\mathcal{CE}}_m(Y)$. The asymptotic normality of the CE-based test statistic $\hat{\Delta}_2$ is established in the following theorem. Since its proof closely parallels that of Theorem 3.1, it is omitted here for brevity. ||

Theorem 3.2. Assume that $\mathbb{E}(X^2) < \infty$ and $\mathbb{E}(Y^2) < \infty$ such that $\bar{\sigma}^2(F) > 0$, and $\bar{\sigma}^2(G) > 0$. Let $S = m + n$ and suppose that for some $0 < \tau < 1$, we have

$$\frac{n}{S} \rightarrow \tau, \frac{m}{S} \rightarrow 1 - \tau \text{ as } \min\{n, m\} \rightarrow \infty.$$

Then, as $\min\{n, m\} \rightarrow \infty$, $\sqrt{S}(\hat{\Delta}_2 - \Delta_2)$ is normal with mean zero and the finite variance

$$\bar{\sigma}_{n,m}^2(F, G) = \frac{\bar{\sigma}_n^2(F)}{\tau} + \frac{\bar{\sigma}_m^2(G)}{1 - \tau},$$

where

$$\bar{\sigma}^2(F) = \int_0^\infty \int_0^\infty (F(\min(x, y)) - F(x)F(y))\bar{J}(x)\bar{J}(y)dx dy \quad (19)$$

and $\bar{\sigma}^2(G)$ defined analogously.

The estimator for $\bar{\sigma}^2(F)$ is defined as:

$$\hat{\sigma}_n^2(F) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left(\min\left(\frac{i}{n}, \frac{j}{n}\right) - \frac{i}{n} \frac{j}{n} \right) \bar{J}\left(\frac{i}{n}\right) \bar{J}\left(\frac{j}{n}\right) (X_{i+1:n} - X_{i:n})(X_{j+1:n} - X_{j:n}).$$

Similarly, we estimate $\bar{\sigma}^2(G)$ as $\hat{\sigma}_m^2(G)$. Consequently, the decision rule for rejecting H_0 in favor of H_1 at significance level α is:

$$\left| \frac{\hat{\Delta}_2}{\sqrt{\frac{\hat{\sigma}_n^2(F)}{n} + \frac{\hat{\sigma}_m^2(G)}{m}}} \right| > z_{1-q}, \quad (20)$$

where z_{1-q} is defined previously.

Remark 3.1. Belzunce et al. [5] highlight an important feature of the dilation order: it provides a natural framework for characterizing the harmonic new better than used in expectation (HNBUE) and harmonic new worse than used in expectation (HNWUE) aging classes. Specifically, a random variable X belongs to the HNBUE (or HNWUE) class if and only if it is less than or equal to another random variable YYY in the dilation order, where Y follows an exponential distribution with mean equal to that of X , i.e., $\mathbb{E}(Y) = \mathbb{E}(X)$. Building on this foundational concept, we introduce a test statistic that can be employed to evaluate the null hypothesis:

H'_0 : X follows an exponential distribution *vs* H'_1 : X belongs to the HNBUE or HNWUE class but does not conform to an exponential distribution.

If Y represents a random variable with an exponential distribution with mean of $\mathbb{E}(X)$, we can derive

$$\Delta_1 = \int_0^\infty x \log(1 - F(x)) dF(x),$$

and

$$\Delta_2 = - \int_0^\infty x \log F(x) dF(x) - \mathbb{E}(X) \left(\frac{\pi^2}{16} - 2 \right),$$

where measures quantify deviation from H'_0 to H'_1 . The measures are empirically estimated, respectively, as:

$$\hat{\Delta}_1 = \frac{1}{n} \sum_{i=1}^n J'\left(\frac{i}{n}\right) X_{i:n}, \text{ and } \hat{\Delta}_2 = \frac{1}{n} \sum_{i=1}^n \bar{J}\left(\frac{i}{n}\right) X_{i:n},$$

where $J'(u) = \log(1 - u)$ and $\bar{J}(u) = -\log(u) - \left(\frac{\pi^2}{16} - 2\right)$ for $0 < u < 1$. To obtain scaleinvariant tests, we can use the statistics $\hat{\Delta}_1^{HNBUE} = \hat{\Delta}_1/\bar{X}$ and $\hat{\Delta}_2^{HNBUE} = \hat{\Delta}_2/\bar{X}$, where $\bar{X}$ represents the sample mean. By similar arguments as in the proofs of Theorems 3.1 and 3.2, and applying Slutsky's theorem, we obtain the following asymptotic distributions:

$$\sqrt{n} \left(\hat{\Delta}_1^{HNBUE} - \frac{\Delta_1}{\mathbb{E}(X)} \right) \xrightarrow{d} N \left(0, \frac{\sigma_{J'}^2(F)}{\mathbb{E}^2(X)} \right),$$

and

$$\sqrt{n} \left(\hat{\Delta}_2^{HNBUE} - \frac{\Delta_2}{\mathbb{E}(X)} \right) \xrightarrow{d} N \left(0, \frac{\sigma_{\bar{J}}^2(F)}{\mathbb{E}^2(X)} \right).$$

The null hypothesis H'_0 should be rejected if

$$\left| \sqrt{\frac{n}{\hat{\sigma}_{n,J'}^2(F)}} \hat{\Delta}_1 \right| > z_{1-q}, \text{ and } \left| \sqrt{\frac{n}{\hat{\sigma}_{n,\bar{J}}^2(F)}} \hat{\Delta}_2 \right| > z_{1-q},$$

where $\hat{\sigma}_{n,J'}^2(F)$ and $\hat{\sigma}_{n,\bar{J}}^2(F)$ are the estimators of $\sigma_{J'}^2(F)$ and $\sigma_{\bar{J}}^2(F)$, respectively.

3.1. Simulation Study

To assess the finite-sample performance of the proposed tests in (18) and (20), we carried out a simulation study comparing their power functions across a range of representative probability models. The chosen distributions are widely applied in economics, finance, insurance, and reliability, and together they span scenarios from light-tailed to heavy-tailed behavior. As a natural benchmark, we first considered the exponential distribution, a standard reference model in reliability theory whose tail behavior provides a baseline for detecting departures toward heavier-tailed alternatives. To capture such heavy-tailed phenomena, we included the Pareto distribution, commonly employed in economics, finance, and insurance to model extreme events. Its scale and shape parameters strongly affect dispersion, making it particularly relevant for testing under the dilation order.

We also examined the gamma distribution, a versatile model frequently used in econometrics, Bayesian analysis, and life-testing. Its shape–scale parameterization offers flexibility in modeling waiting times, and in the special case of integer shape parameters it reduces to the Erlang distribution. Finally, we incorporated the Weibull distribution, another classical lifetime model with broad applications in reliability and survival analysis, well known for its ability to describe diverse aging behaviors. Together, these four families, exponential, Pareto, gamma, and Weibull, provide a balanced experimental design that reflects both exponential-tail and long-tail settings. For comparability and to ensure meaningful stochastic orderings, all simulated distributions were standardized to share a common mean, although this constraint is not required for the theoretical validity of the proposed tests

We evaluated our proposed tests by comparing their empirical power against four recent tests for the dilation order. Specifically, we compared our statistics $\hat{\Delta}_1$ and $\hat{\Delta}_2$ to those developed by following test statistics:

- Aly's t_N statistic [20]:

$$t_N = \delta_m(G) - \delta_n(F),$$

where

$$\delta_m(G) = \frac{1}{m^2} \sum_{i=2}^m (i-1)(m-i+1)(X_{i:m} - X_{i-1:m}),$$

and $\delta_n(F)$ is defined similarly.

- The test statistic $\hat{\Delta}_{dil}^\alpha(X, Y)$ proposed by Belzunce et al. [5]:

$$\hat{\Delta}_{dil}^\alpha(X, Y) = \frac{1}{n} \sum_{i=1}^n \left(c_{i,n}^\alpha - \frac{1-\alpha^2}{6} \right) X_{i:n} - \frac{1}{m} \sum_{i=1}^m \left(c_{i,m}^\alpha - \frac{1-\alpha^2}{6} \right) Y_{i:n},$$

where

$$c_{i,r}^\alpha = \begin{cases} \left(\frac{1}{\alpha} - 1 \right) \frac{3r^2\alpha - 3i^2 + 3i - 1}{6r^2} & \text{if } i < k+1 \\ \frac{3\alpha(r-k-1)^2 + (k-r\alpha)^3 + \alpha(3r-3k-2)}{6r^2\alpha} & \text{if } i = k+1 \\ \frac{3(r-i)^2 + 3(r-i) + 1}{6r^2} & \text{if } i > k+1, \end{cases}$$

with $\frac{k}{r} \leq \alpha < \frac{k+1}{r}$.

- The test statistic $\hat{\Delta}(n, m)$ introduced by Sordo et al. [3], which is based on Gini's mean difference:

$$\hat{\Delta}(n, m) = \frac{m+1}{m(m-1)} \sum_{i=1}^m 2 \left(\frac{2i}{m+1} - 1 \right) Y_{i:m} - \frac{n+1}{n(n-1)} \sum_{i=1}^n 2 \left(\frac{2i}{n+1} - 1 \right) X_{i:n}.$$

- Zardasht’s T_{nm}^α statistic [21], defined as:

$$T_{nm}^\alpha = \frac{1}{m} \sum_{i=1}^m J_\alpha\left(\frac{i}{m}\right) Y_{i:m} - \frac{1}{n} \sum_{i=1}^n J_\alpha\left(\frac{i}{n}\right) X_{i:n},$$

where $J_\alpha(u) = (1/(1-\alpha))[1-\alpha(1-u)^{\alpha-1}]$ for all $\alpha > 0$ and $\alpha \neq 1$.

The statistic $\widehat{\Delta}^\alpha \text{dil}(X, Y)$ from Belzunce et al. [5] depends on a parameter $\alpha \in (0, 1)$; since its performance remains largely consistent across different α values, we adopted $\alpha = 0.5$ for our analysis. Similarly, for $T^{\alpha}nm$ from Zardasht [21], we chose $\alpha = 2$. We also simulated the following scenarios which is tabulated in Table 1 and compared the empirical powers of the test statistics.

- (i) Exponential Distribution: For this scenario, $X \sim E(1)$ and $Y \sim E(1/\beta)$ where β is varied from 1 to 2. The null hypothesis is then represented by the case where $\beta = 1$.
- (ii) Pareto Distribution: For this scenario, the random variable $X \sim P(10, 3)$ and $Y \sim P(10/\beta, 3)$ where β varied from 1 to 2. The null hypothesis is then represented by the case where $\beta = 1$.

Gamma Distribution: For this scenario, $X \sim G(2, 1)$ and $Y \sim G(\beta, 1)$ where β varied from 2 to 3. The null hypothesis is then represented by the case where $\beta = 2$.

- (iv) Weibull Distribution: For this scenario, $X \sim W(2, 1)$ and $Y \sim W(\beta, 1)$ where β varied from 1 to 2. The null hypothesis is then represented by the case where $\beta = 2$.

Table 1. Probability distributions with the shape parameter β and the scale parameter γ .

Distribution	Probability Density Function	Support
Exponential	$f(x) = \frac{1}{\gamma} e^{-\frac{x}{\gamma}},$	$x > 0, \gamma > 0$
Pareto	$f(x) = \frac{\beta \gamma^\beta}{(x + \gamma)^{\beta + 1}},$	$x > 0, \gamma, \beta > 0$
Gamma	$f(x) = \frac{1}{\gamma^\beta \Gamma(\beta)} x^{\beta - 1} e^{-\frac{x}{\gamma}}$	$x > 0, \gamma, \beta > 0$
Weibull	$f(x) = \frac{\beta}{\gamma^\beta} x^{\beta - 1} e^{-\left(\frac{x}{\gamma}\right)^\beta},$	$x > 0, \gamma, \beta > 0$

The empirical power of the proposed test statistics was examined under each distributional setting using 5000 independently generated sample pairs with sizes $n = m = 25, 50$, and 100. For each replication, the rejection of the null hypothesis was recorded, and the empirical power was computed as the proportion of rejections across the 5000 replicates. The results, reported in Tables 1 and 5, confirm the expected consistency: the power of all tests increases with larger sample sizes. In particular, the CE-based statistic $\widehat{\Delta}_2$ demonstrates superior performance in most scenarios, especially for the exponential, Pareto, and gamma distributions. This highlights the ability of $\widehat{\Delta}_2$ to effectively distinguish exponential behavior from heavy-tailed or non-exponential alternatives within these families. However, the performance of $\widehat{\Delta}_2$ is less satisfactory for the Weibull distribution, suggesting that it may not be well-suited to data governed by this model. By contrast, the performance of $\widehat{\Delta}_1$ improves notably as the sample size increases. With sufficiently large datasets, $\widehat{\Delta}_1$ becomes more reliable and robust, partially compensating for its relatively weaker performance in small-sample scenarios.

Overall, while $\widehat{\Delta}_2$ emerges as a strong and versatile test statistic, its practical application should be considered in light of both the underlying distribution and the available sample size. In this sense, $\widehat{\Delta}_1$ and $\widehat{\Delta}_2$ may be viewed as complementary tools, each with advantages depending on the context.

Table 2. Power comparisons of the tests in significance level $q = 0.05$.

Exponential							
$n = m$	γ	t_N	$\hat{\Delta}_{\text{dil}}^{0.5}(X, Y)$	$\hat{\Delta}(n, m)$	T_{nm}^2	$\hat{\Delta}_1$	$\hat{\Delta}_2$
25	0.5	0.8150	0.8218	0.8116	0.8092	0.7424	0.8526
	0.6	0.6040	0.6104	0.5986	0.5816	0.5012	0.6428
	0.7	0.3740	0.3796	0.3674	0.3630	0.3402	0.3924
	0.8	0.2144	0.2136	0.2146	0.1972	0.1894	0.2222
	0.9	0.1090	0.1160	0.1074	0.1078	0.0998	0.1074
	1.0	0.0520	0.0450	0.0490	0.0466	0.0554	0.0502
50	0.5	0.9122	0.9124	0.9104	0.9086	0.8732	0.9436
	0.6	0.6628	0.6724	0.6366	0.6530	0.6310	0.7120
	0.7	0.3616	0.3546	0.3342	0.3414	0.3632	0.3898
	0.8	0.1488	0.1452	0.1296	0.1368	0.1754	0.1398
	0.9	0.0422	0.0468	0.0396	0.0452	0.0626	0.0426
	1.0	0.0116	0.0112	0.0108	0.0102	0.0248	0.0114
100	0.5	0.9734	0.9762	0.9670	0.9694	0.9588	0.9850
	0.6	0.7386	0.7404	0.7100	0.7272	0.7532	0.7910
	0.7	0.3210	0.3202	0.2976	0.3136	0.3914	0.3586
	0.8	0.0702	0.0680	0.0630	0.0686	0.1388	0.0804
	0.9	0.0120	0.0116	0.0070	0.0102	0.0278	0.0110
	1.0	0.0016	0.0006	0.0006	0.0008	0.0040	0.0002

Table 3. Power comparisons of the tests in significance level $q = 0.05$.

Pareto							
$n = m$	β	t_N	$\hat{\Delta}_{\text{dil}}^{0.5}(X, Y)$	$\hat{\Delta}(n, m)$	T_{nm}^2	$\hat{\Delta}_1$	$\hat{\Delta}_2$
25	1.0	0.0542	0.0474	0.0456	0.0528	0.0500	0.0436
	1.2	0.1816	0.1724	0.1606	0.1870	0.1626	0.1806
	1.4	0.3642	0.3562	0.3402	0.3610	0.3208	0.3880
	1.6	0.5534	0.5638	0.5500	0.5612	0.4982	0.5796
	1.8	0.7140	0.7344	0.7034	0.7246	0.6464	0.7392
	2.0	0.8260	0.8332	0.8170	0.8348	0.7652	0.8548
50	1.0	0.0154	0.0086	0.0110	0.0136	0.0296	0.0118
	1.2	0.1088	0.1004	0.0966	0.1210	0.1528	0.1114

100	1.4	0.3500	0.3380	0.3030	0.3636	0.3678	0.3500
	1.6	0.6124	0.6198	0.5786	0.6490	0.6114	0.6542
	1.8	0.8164	0.8202	0.7918	0.8348	0.7898	0.8466
	2.0	0.9272	0.9242	0.9124	0.9314	0.9042	0.9428
	1.0	0.0008	0.0014	0.0008	0.0022	0.0090	0.0008
	1.2	0.0558	0.0488	0.0286	0.0586	0.1252	0.0468
	1.4	0.3106	0.3112	0.2590	0.3578	0.4140	0.3330
	1.6	0.6928	0.6784	0.6338	0.7180	0.7436	0.7224
	1.8	0.9150	0.9130	0.8826	0.9228	0.9114	0.9348
	2.0	0.9828	0.9828	0.9748	0.9844	0.9756	0.9890

Table 4. Power comparisons of the tests in significance level $q = 0.05$.

Weibull							
$n = m$	β	t_N	$\widehat{\Delta}_{\text{dil}}^{0.5}(X, Y)$	$\widehat{\Delta}(n, m)$	T_{nm}^2	$\widehat{\Delta}_1$	$\widehat{\Delta}_2$
25	1.0	0.9682	0.9568	0.9602	0.9558	0.9696	0.9106
	1.2	0.8828	0.8768	0.8894	0.8884	0.8980	0.8254
	1.4	0.7472	0.7408	0.7610	0.7266	0.7392	0.6878
	1.6	0.5604	0.5454	0.5622	0.5460	0.5160	0.4992
	1.8	0.3538	0.3504	0.3702	0.3530	0.3050	0.3416
	2.0	0.2160	0.2024	0.2042	0.1942	0.1582	0.2040
50	1.0	0.9944	0.9940	0.9958	0.9946	0.9962	0.9804
	1.2	0.9616	0.9550	0.9590	0.9642	0.9736	0.9192
	1.4	0.8464	0.8276	0.8382	0.8504	0.8542	0.7690
	1.6	0.6004	0.5738	0.5940	0.5962	0.5950	0.5358
	1.8	0.3146	0.2948	0.3128	0.3158	0.2770	0.2822
	2.0	0.1216	0.1184	0.1156	0.1280	0.1066	0.1254
100	1.0	0.9998	0.9998	0.9994	1.0000	0.9998	0.9986
	1.2	0.9948	0.9938	0.9924	0.9964	0.9980	0.9802
	1.4	0.9240	0.9094	0.9236	0.9358	0.9424	0.8698
	1.6	0.6496	0.6268	0.6292	0.6366	0.6658	0.5574
	1.8	0.2586	0.2262	0.2386	0.2512	0.2480	0.2260
	2.0	0.0502	0.0416	0.0464	0.0546	0.0464	0.0480

Table 5. Power comparisons of the tests in significance level $q = 0.05$.

Gamma							
$n = m$	β	t_N	$\hat{\Delta}_{dil}^{0.5}(X, Y)$	$\hat{\Delta}(n, m)$	T_{nm}^2	$\hat{\Delta}_1$	$\hat{\Delta}_2$
25	2.0	0.2054	0.2026	0.2034	0.1812	0.1464	0.2016
	2.2	0.2642	0.2700	0.2676	0.2548	0.1802	0.2874
	2.4	0.3340	0.3330	0.3318	0.3194	0.2266	0.3774
	2.6	0.4072	0.4060	0.4258	0.3852	0.2624	0.4656
	2.8	0.4542	0.4734	0.4818	0.4400	0.2838	0.5676
	3.0	0.5448	0.5404	0.5294	0.5198	0.3324	0.6438
50	2.0	0.1230	0.1180	0.1318	0.1176	0.0950	0.1170
	2.2	0.1908	0.1908	0.1960	0.1878	0.1444	0.2020
	2.4	0.2738	0.2730	0.2880	0.2654	0.1864	0.3340
	2.6	0.3614	0.3850	0.3734	0.3556	0.2444	0.4604
	2.8	0.4738	0.4634	0.4612	0.4614	0.2964	0.5850
	3.0	0.5392	0.5546	0.5588	0.5310	0.3532	0.6926
100	2.0	0.0504	0.0494	0.0528	0.0510	0.0532	0.0524
	2.2	0.1154	0.1138	0.1126	0.1096	0.0782	0.1318
	2.4	0.2016	0.2138	0.2034	0.2024	0.1476	0.2582
	2.6	0.3234	0.3364	0.3234	0.3126	0.2118	0.4396
	2.8	0.4524	0.4588	0.4558	0.4536	0.2684	0.6092
	3.0	0.5780	0.5898	0.5768	0.5680	0.3600	0.7508

3.2. Real Data Example

To illustrate the practical utility of the proposed methodology, we analyze a real dataset on survival times of male RFM strain mice, originally reported by Hoel [22]. The study considered two groups: the first group (X) consisted of mice raised under conventional laboratory conditions, while the second group (Y) was raised in a germ-free environment. In both groups, death was due to thymic lymphoma, allowing a direct comparison of survival variability under different environmental conditions. This dataset is particularly valuable in survival and reliability analysis because it allows us to investigate how external factors, in this case environmental exposure, affect the dispersion of lifetimes. Specifically, our interest lies in testing whether the survival distribution of mice raised in germ-free conditions (Y) is more dispersed than that of mice raised conventionally (X), which corresponds to verifying the dilation order relationship $X \leq_{dil} Y$.

As a preliminary step, we followed the graphical approach recommended by Belzunce et al. [5], which suggested evidence consistent with the dispersive ordering $X \leq_{dil} Y$. Building on this, we applied six test statistics, including the proposed entropy-based measures $\hat{\Delta}_1$ and $\hat{\Delta}_2$, to formally assess the hypothesis. The results, summarized in Table 6, provide strong statistical support for the dilation order. In particular, all six test statistics yielded small p -values, leading to rejection of the null hypothesis of equality and confirming the alternative $X \leq_{dil} Y$. The entropy-based test $\hat{\Delta}_2$ was especially effective, delivering the strongest evidence among the six statistics.

Table 6. Statistical Test Results for Real Data Set.

Test	t_N	$\hat{\Delta}_{dil}^{0.5}(X, Y)$	$\hat{\Delta}(n, m)$	T_{nm}^2	$\hat{\Delta}_1$	$\hat{\Delta}_2$
Statistic	32.63350	5.07340	66.54914	32.01251	41.10225	47.00085
p-value	0.000556	0.000160	0.000200	0.000300	0.003912	0.000038

This real-data application highlights three key insights. First, it demonstrates how cumulative entropies can function as practical evidence measures, capable of validating stochastic orderings in empirical settings. Second, it shows that entropy-based tests can reveal differences in variability between populations that may not be apparent from mean comparisons alone. Third, it illustrates the robustness and versatility of the proposed methodology in survival data analysis, with implications extending to biomedical research, reliability engineering, and actuarial science. This validation on real data underscores that entropy-based evidence functions are not only theoretically sound but also practically reliable, even in complex biological survival settings.

4. Conclusion

This paper introduced new entropy-based test statistics for assessing the dilation order, constructed from cumulative residual entropy (CRE) and cumulative entropy (CE). Their theoretical properties were established by deriving asymptotic distributions and examining behavior under large-sample conditions. Together, these developments provide a rigorous framework for evaluating stochastic variability through evidence functions. The performance of the proposed methods was studied extensively through simulation experiments and compared with several recent alternatives from the literature. The results demonstrated that the CE-based statistic delivers high power and consistency across a wide range of scenarios, while the CRE-based statistic becomes particularly robust and effective with larger sample sizes. These findings underscore the complementary nature of the two approaches and highlight their relevance in both small- and large-sample contexts.

To illustrate their practical utility, the methods were applied to a real dataset involving survival times of RFM strain mice. The analysis confirmed the presence of dilation ordering between groups, reinforcing the applicability of CRE- and CE-based statistics as evidence tools for distinguishing variability in real-world problems. This case study highlights not only the theoretical soundness of the methodology but also its value in survival analysis, reliability, and related applied fields.

In addition, the framework was extended to the important problem of testing exponentiality against HNBUE and HNWUE alternatives, for which asymptotic results were also derived. This broadens the scope of entropy-based approaches, linking stochastic orders with aging properties that are central in reliability and actuarial science.

Several open problems arise from this work and remain of considerable interest for future research:

1. Developing refined small-sample approximations or bootstrap-based procedures to improve the practical accuracy of the proposed tests.
2. Extending the methodology to multivariate distributions, dependent data, and censored survival settings.
3. Examining robustness under model misspecification and embedding the tests within Bayesian evidence frameworks to enhance both theory and practice.
4. Exploring connections between entropy-based orders and other information measures, such as Rényi or Tsallis entropies, to construct new families of test statistics with broader applicability.

Institutional Review Board Statement: This study did not involve human participants or animals.

Data Availability Statement: The data generated or analyzed during this study are included in the article.

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