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The Permanent Rank of a Matrix (Part Three) Note on the Additive Basis Conjecture

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The Permanent Rank of a Matrix (Part Three) Note on the Additive Basis Conjecture

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Abstract

We show that in a vector space over Z_3, the union of any four linear bases is an additive basis, thus proving the Additive Basis Conjecture for $p = 3$, and providing an alternative proof of the weak 3-flow conjecture.

Keywords: Additive Basis Conjecture; weak 3-flow conjecture; permanent rank

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1. Introduction

The Additive Basis Problem is a classical problem in additive combinatorics whose history parallels that of the more famous Arithmetic Progressions Problem. Both have been extensively studied since the 1930's (see e.g. [Erdős]), first for the integers, but later also for other abelian groups; see e.g. [Mesh] (for arithmetic progressions) and [JLPT] (for additive bases) for early work taking this more general viewpoint.

For arithmetic progressions, recent years have seen celebrated progress on the central problem of bounding the sizes of AP-free sets in Z_p^n [CLP, EG]. But there has been no comparable breakthrough on what's perhaps the best-known problem on additive bases in Z_p^n: the following conjecture of Jaeger, Linial, Payan and Tarsi.

Recall that a multiset B is an *additive basis* of a vector space S, if every element of S is a linear combination of elements in B, with each coefficient either 0 or 1.

Conjecture 1 (The Additive Basis Conjecture [JLPT])

For any prime p, there exists a constant $c(p)$, such that in any vector space over Z_p, the multiset union of any $c(p)$ linear bases is an additive basis.

Conjecture 1 was studied in [ALM, Sz, NPT, EVLT, HQ, CKMS]. It is related to a few other problems in discrete mathematics. For example, the case $p = 3$ implies Francois Jaeger's weak 3 Flow Conjecture (proved by Carsten Thomassen in 2012 [Th]).

It is proved in [ALM] that the union of any $c(p) \log n$ linear bases is an additive basis, where n is the dimension of the vector space. Our approach, like that of [ALM], is based on permanents. Define the *perrank* of a matrix $M_{m\times n}$ to be the size of a largest square submatrix with nonzero permanent; if this is equal to m or n, then we say M has *full perrank*.

Conjecture 2 [ALM] If $B_1, B_2, \cdots, B_p$ are nonsingular matrices over a field of characteristic $p \geq 3$, then $\begin{pmatrix} B_1 & B_2 & \cdots & B_p \\ \vdots & \vdots & & \vdots \\ B_1 & B_2 & \cdots & B_p \end{pmatrix}$, where each row repeats $p-1$ times, has full perrank.

Here we give the first constant bounds for both conjectures, and in particular the first proof of *any* case of the Additive Basis Conjecture:

Theorem 3 **(Main Theorem)** If P, R, S, T are nonsingular matrices over a field of characteristic 3, then $\begin{pmatrix} P & R & S & T \\ P & R & S & T \end{pmatrix}$ has full perrank.

Corollary 4 Conjecture 1 holds for $p = 3$, with $c(p) = 4$.

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This corollary follows from Theorem 3 by the Combinatorial Nullstellensatz (see [ALM] section 3 for details), while Conjecture 2 implies Conjecture 1 with $c(p) = p$. Theorem 3 will be proved at the end of the paper, after we have developed the necessary machinery, the main point here being Theorem 7. An early look at the easy derivation of Theorem 3 should help to motivate what precedes it.

This paper is part 3 of the author's series *"The permanent rank of a matrix"*; part 1 was [Yu]; and at this writing part 2 is still in preparation.

2. Definitions and Notation

Given a field, let A^n be the quotient of the polynomial ring in n variables $x_1, x_2, \cdots, x_n$ by the ideal generated by $x_1^2, x_2^2, \cdots, x_n^2$. The k-th degree component of the graded algebra A^n is denoted by A_k^n; we omit n when there is no ambiguity. For an ideal $J \subseteq A$, let $J_k = A_k \cap J$.

For $f \in A$, define $\mathrm{Ker}(f) = \{g : gf = 0\}$, $\mathrm{Im}(f) = \{fg : g \in A\}$.

We introduce two operators. Let ∂_x and E_x be the quotient and remainder of formal division by x; we use ∂_i for ∂_{x_i} and similarly for E_i. For example, if $f = x_1x_2 + x_1x_3 + x_2x_3$, then $\partial_1 f = x_2 + x_3$ and $E_1 f = x_2x_3$.

(We use the letter E because E_i *eliminates* all terms containing x_i).

It is obvious that $E_i(fg) = (E_i f)(E_i g)$, $\partial_i(fg) = (\partial_i f)(E_i g) + (\partial_i g)(E_i f)$, and $E_iE_j = E_jE_i$, $\partial_i\partial_j = \partial_j\partial_i$, $E_i\partial_j = \partial_j E_i$ (but $E_i\partial_j \neq E_j\partial_i$).

For $u \in A_1$, define its *support* to be $\mathrm{supp}(u) := \{x : \partial_x u \neq 0\}$. An element of A_1 is also called a *linear form*.

Let u be a linear form with $\partial_x u = c \neq 0$, set $\partial_x = \partial$, $E_x = E$ for simplicity. For any f, define another division operation: divide f by u w.r.t. x by

$f = Ef + (\partial f)x = Ef + (\partial f)(u - Eu)c^{-1} = c^{-1}(\partial f)u + Ef - c^{-1}(\partial f)(Eu)$ and define $R_{(u,x)}f := Ef - c^{-1}(\partial f)(Eu)$ as the remainder.

Evidently $\partial(Rf) = 0$ and $f - Rf \in \mathrm{Im}(u)$, this is what we need later.

For U and V subspaces of A_i and A_j respectively, define UV to be the subspace of A_{i+j} spanned by $\{uv : u \in U,\ v \in V\}$. Define $\mathrm{Im}(U)$ to be the ideal generated by U, and $\mathrm{Ker}(U) = \{f : fu = 0\ \forall u \in U\}$.

A subspace of A_1 is also called a *linear form space*. For a linear form space U, define its *support* to be $\mathrm{supp}(U) := \bigcup_{u \in U} \mathrm{supp}(u)$; define its *minimum support function* to be

$\mathrm{ms}_i(U) := \min\{|\mathrm{supp}(V)| : V \subseteq U \text{ with } \dim(V) = i\}$ for $i \leq \dim(U)$.

Label the rows and columns of a matrix $M_{m\times n}$ with variables $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_n$ respectively, and view its rows and columns as linear forms in A^n and A^m. Then M has full perrank iff the product of its rows or columns is nonzero in A^n or A^m.

3. Supporting Results and Proofs

From now on, we assume the ground field has characteristic 3 and is infinite, otherwise extend it to infinity. The following theorem is the base step for the main induction in the proof of Theorem 7.

Theorem 5

(1) $\mathrm{Ker}_k(u) = \mathrm{Im}_k(u^2)$ for any linear form u with $|\mathrm{supp}(u)| \geq 2k + 1$.

(2) $\mathrm{Ker}_k(u^2) = \mathrm{Im}_k(u)$ for any linear form u with $|\mathrm{supp}(u)| \geq 2k + 2$.

Proof. Since $u^3 = 0$, one direction is trivial. The other direction is by induction on k, easy to verify when $k = 1$. Pick any $x \in \mathrm{supp}(u)$, and set $\partial_x = \partial$, $E_x = E$ for simplicity. WMA $\partial u = 1$.

(1) Suppose $f \in \mathrm{Ker}_k(u)$, $fu = 0$; take ∂, $Ef + (\partial f)(Eu) = 0$; multiply by Eu, $(\partial f)(Eu)^2 = 0$. By induction hypothesis of (2), $\partial f = g(Eu)$ for some g, then

$f = Ef + (\partial f)x = (\partial f)(x - Eu) = -g(Eu)(Eu - x) = -g(Eu + x)^2 = -gu^2$.

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(2) Suppose $f \in \mathrm{Ker}_k(u^2)$, $fu^2 = 0$; take ∂, $(2Ef + (\partial f)Eu)Eu = 0$. By (1) we have $Ef - (\partial f)Eu = g(Eu)^2$ for some g, then
$f = Ef + (\partial f)x = (\partial f)(Eu + x) + g(Eu)^2 = (\partial f)u + gu(Eu - x) \in \mathrm{Im}(u)$. □

We say a linear form space U *covers* $(a_1, a_2, \cdots, a_k)$ if $\mathrm{ms}_i(U) \geq a_i$ for all $1 \leq i \leq k$.

The following lemma plays a crucial role in the proof of Theorem 7.

Lemma 6 Suppose U is a linear form space with $\dim(U) = n$ covering an increasing sequence $(a_1, a_2, \cdots, a_n)$. Then for each $0 \leq k \leq n$, there exists a subspace $U_k \subseteq U$ with $\dim(U_k) = k$ covering $(a_{n+1-k}, a_{n+2-k}, \cdots, a_n)$.

Proof. Set $U_0 = 0$ and suppose U_k exists.

For each $S \subseteq \mathrm{supp}(U)$ with $|S| = a_{n-k} - 1$, let

$V_S := \{v : v \in U,$ there exists $u \in U_k$ such that $\mathrm{supp}(v + u) \subseteq S\}$;

note $U_k \subseteq V_S$ since $\mathrm{supp}(0) = \varnothing$. Claim V_S is a proper subspace of U, otherwise choose $\{v_i\}$ such that $U = \mathrm{span}(U_k, v_1, v_2, \cdots, v_{n-k})$. For each i, choose $u_i \in U_k$ such that $\mathrm{supp}(v_i + u_i) \subseteq S$. Let $V = \mathrm{span}(\{v_i + u_i\})$, then $\dim(V) = n - k$, $|\mathrm{supp}(V)| < a_{n-k}$, contradiction.

Because the ground field is infinite, U is not a finite union of its proper subspaces. Choose $v \in U$ that is not in any V_S, and let $U_{k+1} := \mathrm{span}(U_k, v)$.

If $u \in U_{k+1} \backslash U_k$, then $|\mathrm{supp}(u)| \geq a_{n-k}$ by our choice of v. If $0 \neq u \in U_k$, then by induction hypothesis, $|\mathrm{supp}(u)| \geq \mathrm{ms}_1(U_k) \geq a_{n+1-k} > a_{n-k}$. So $\mathrm{ms}_1(U_{k+1}) \geq a_{n-k}$.

For any $H \subseteq U_{k+1}$ with $\dim(H) = h \geq 2$, either $\dim(H \cap U_k) = h - 1$ or $H \subseteq U_k$. By induction hypothesis, either

$|\mathrm{supp}(H)| \geq |\mathrm{supp}(H \cap U_k)| \geq \mathrm{ms}_{h-1}(U_k) \geq a_{n+h-1-k}$, or

$|\mathrm{supp}(H)| \geq \mathrm{ms}_h(U_k) \geq a_{n+h-k} > a_{n+h-k-1}$.

So $\mathrm{ms}_h(U_{k+1}) \geq a_{n+h-k-1}$. □

Theorem 7 Suppose U is a linear form space, $\dim(U) = n$ and $k \geq 0$.

(A) If $\mathrm{ms}_i(U) \geq 4i - 2 + 2k$ for $1 \leq i \leq n$, then $\mathrm{Ker}_k(U^{2n}) = \mathrm{Im}_k(U)$.

(B) If $\mathrm{ms}_i(U) \geq 4i - 3 + 2k$ for $1 \leq i \leq n$, then
$\mathrm{Ker}_{2n-2+k}(U) = \mathrm{Im}_{2n-2+k}(U^{2n})$.

The proof is delicate, any mismatch between degree and support or other discrepancy invalidates it. We need to check degree and support (abbrev. CDS) 8 times; 5 times exact match; 3 times there is extra support of exactly one. The induction hypothesis is applied 6 times in the proof.

Proof. One direction is trivial, the other direction is by induction on n. Theorem 5 gives the case $n = 1$. Suppose true for $n - 1$, then induction on k by: $B(n, 0) \longrightarrow A(n, 0) \longrightarrow B(n, 1) \longrightarrow A(n, 1) \longrightarrow B(n, 2) \longrightarrow A(n, 2) \cdots$.

Claim: $A(n, k-1)$ implies $B(n, k)$ for all $k \geq 1$.

Suppose U satisfies condition (B) and $f \in \mathrm{Ker}_{2n-2+k}(U)$. By Lemma 6, there exists $V \subseteq U$ with $\dim(V) = n - 1$, and $\mathrm{ms}_i(V) \geq 4i + 1 + 2k$ for all $1 \leq i \leq n - 1$.

Choose a variable x and a linear basis $\{u_1 + x, u_2, u_3, \cdots, u_{n-1}\}$ of V such that $\partial_x u_i = 0$ for all $1 \leq i \leq n - 1$. Choose any $u \in U \backslash V$ and let $u_n = R_{(u_1+x,x)} u$. Then $\partial_x u_n = 0$ and $\{u_1 + x, u_2, \cdots, u_n\}$ is a linear basis of U. Note $2n - 2 + k = 2(n-1) - 2 + (k+2)$, and $f \in \mathrm{Ker}_{2n-2+k}(V)$ also. Apply $B(n-1, k+2)$ to V, CDS exact match. We get

$$f = g(u_1 + x)^2 u_2^2 \cdots u_{n-1}^2 \tag{1}$$

for some g with $\deg(g) = k$. WMA $\partial_x g = 0$, otherwise replace it with $R_{(u_1+x,x)} g$. Then $fu_n = 0$ gives $gu_n(u_1 + x)^2 u_2^2 \cdots u_{n-1}^2 = 0$.

Apply $A(n-1, k+1)$ to V, CDS extra support of one. We have
$gu_n = a_1(u_1 + x) + a_2 u_2 + \cdots + a_{n-1} u_{n-1}$ for some a_i.

If $k = 0$, then $g = a_i = 0$, $f = 0$. Here we got $B(n, 0)$.

If $k \geq 1$, multiply above by $(u_1 + x)^2$. Let $b_i = R_{(u_1+x,x)} a_i$, we get
$gu_n(u_1 + x)^2 = b_2 u_2 (u_1 + x)^2 + \cdots + b_{n-1} u_{n-1} (u_1 + x)^2$. Then take ∂_x, we get $gu_n u_1 = b_2 u_2 u_1 + \cdot\cdot$

$\cdot + b_{n-1}u_{n-1}u_1$. Apply Theorem 5(1) to u_1, $\deg(gu_n) = k+1$, $u_1 + x \in V$, $|\mathrm{supp}(u_1)| \geq 2k+4$, CDS extra support of one. We have $gu_n = b_2u_2 + \cdots + b_{n-1}u_{n-1} + du_1^2$ (2)
for some d with $\deg(d) = k-1$.

Multiply by $u_2^2 \cdots u_{n-1}^2 u_n^2$, we get $du_1^2u_2^2 \cdots u_n^2 = 0$.

Since $\mathrm{ms}_1(U) \geq 1+2k \geq 3$, $x \notin U$, so $\dim(E_x(U)) = n$. Apply $A(n, k-1)$ to $E_x(U)$, CDS exact match. We get $d = c_1u_1 + c_2u_2 + \cdots + c_nu_n$. Substitue into (2), we get $(g - c_nu_1^2)u_n \in \mathrm{Im}(\mathrm{span}(u_2, \cdots, u_{n-1}))$.

Multiply by $u_2^2 \cdots u_{n-1}^2$, we get $(g - c_nu_1^2)u_2^2 \cdots u_{n-1}^2 u_n = 0$. Introduce a dummy variable y to make $(g - c_nu_1^2)u_2^2 \cdots u_{n-1}^2(u_n + y)^2 = 0$.

Let $Y := \mathrm{span}(u_2, \cdots, u_{n-1}, u_n + y)$. For any $I \subseteq Y$ with $\dim(I) = i$, if $y \notin \mathrm{supp}(I)$, then $I \subseteq \mathrm{span}(u_2, \cdots, u_{n-1}) \subseteq V$ with $|\mathrm{supp}(I)| \geq 4i + 1 + 2k$. If $y \in \mathrm{supp}(I)$, then $|\mathrm{supp}(E_y(I))| \geq 4i - 3 + 2k$ since $E_y(I) \subseteq U$, and so $|\mathrm{supp}(I)| \geq 4i - 2 + 2k$. Apply $A(n-1, k)$ to Y, CDS exact match. We have $g - c_nu_1^2 \in \mathrm{Im}(Y)$. Take E_y, we get $g - c_nu_1^2 \in \mathrm{Im}(U)$; then $g \in \mathrm{Im}(U)$ since $u_1^2 = (u_1 + x)(u_1 - x)$.

Substitute $g \in \mathrm{Im}(U)$ into (1), we get $f = hu_n(u_1 + x)^2u_2^2 \cdots u_{n-1}^2$ (3)
for some h with $\deg(h) = k-1$. Then $fu_n = 0$ gives

$hu_n^2(u_1 + x)^2u_2^2 \cdots u_{n-1}^2 = 0$.

Apply $A(n, k-1)$ to U, CDS extra support of one, we have $h \in \mathrm{Im}(U)$. When $k = 1$, $h = 0$, $f = 0$. When $k \geq 2$, substitute $h \in \mathrm{Im}(U)$ into (3), $f = pu_n^2(u_1 + x)^2u_2^2 \cdots u_{n-1}^2$ for some p. That is, $f \in \mathrm{Im}_{2n-2+k}(U^{2n})$. □

Claim: $B(n,k)$ implies $A(n,k)$ for all $k \geq 0$.

Suppose U satisfies condition (A) and $f \in \mathrm{Ker}_k(U^{2n})$. Choose a variable x and a linear basis $\{u_1 + x, u_2, u_3, \cdots, u_n\}$ of U such that $\partial_x u_i = 0$ for all $1 \leq i \leq n$. Observe $U^{2n} = \mathrm{span}((u_1 + x)^2u_2^2 \cdots u_n^2)$. Let $g = R_{(u_1+x,x)}f$, then $g \in \mathrm{Ker}_k(U^{2n})$ also. Take ∂_x to $g(u_1 + x)^2u_2^2 \cdots u_n^2 = 0$, we get
$gu_1u_2^2 \cdots u_n^2 = 0$. So $gu_2^2 \cdots u_n^2 \in \mathrm{Ker}_{2n-2+k}(E_x(U))$.

Since $\mathrm{ms}_1(U) \geq 2$, $x \notin U$, so $\dim(E_x(U)) = n$. Apply $B(n,k)$ to $E_x(U)$, CDS exact match. We have $gu_2^2 \cdots u_n^2 \in \mathrm{Im}_{2n-2+k}(E_x(U)^{2n})$.

So when $k \leq 1$, $gu_2^2 \cdots u_n^2 = 0$ since $2n - 2 + k < 2n$; and when $k \geq 2$, $gu_2^2 \cdots u_n^2 = hu_1^2u_2^2 \cdots u_n^2$ for some h, then $(g - hu_1^2)u_2^2 \cdots u_n^2 = 0$.

Apply $A(n-1, k)$ to $\mathrm{span}(u_2, \cdots, u_n) \subseteq U$, CDS exact match.

When $k = 0$, $g = 0$, $f = 0$. When $k = 1$, $g \in \mathrm{Im}(U)$, $f \in \mathrm{Im}(U)$. When $k \geq 2$, $g - hu_1^2 \in \mathrm{Im}(U)$. Since $u_1^2 = (u_1 + x)(u_1 - x)$, $g \in \mathrm{Im}(U)$, $f \in \mathrm{Im}(U)$. □

Proof of Theorem 3: Let U be the linear form space spanned by the rows of $(P\ R\ S\ T)_{n \times 4n}$, then $\mathrm{ms}_i(U) \geq 4i$ for all $1 \leq i \leq n$. By applying Theorem 7(A) with $k = 0$, we have $U^{2n} \neq 0$. That is, $\begin{pmatrix} P & R & S & T \\ P & R & S & T \end{pmatrix}$ has full perrank. □

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