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On MultiLine Graphs, MultiLine HyperGraphs, and MultiLine Super-HyperGraphs

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Abstract

A finite hypergraph generalizes an ordinary graph by permitting a hyperedge to connect any nonempty subset of vertices, thereby representing genuine multiway interactions. Extending this idea, a finite SuperHyperGraph is obtained through an iterated powerset construction, so that set-valued objects formed at one level may function as vertices or edge endpoints at the next, providing a natural framework for hierarchical and multilayer relational structures. In contrast, a line graph transforms each edge of a graph into a vertex, with two such new vertices adjacent precisely when the corresponding original edges share an endpoint. In this paper, we introduce the notion of a MultiLine Graph, in which multiple edges can be assigned to a vertex, and then develop its higher-order extensions, namely the MultiLine HyperGraph and the MultiLine Super-HyperGraph. We further investigate their fundamental properties and structural characteristics.

Keywords: HyperGraph; SuperHyperGraph; line graph, MultiLine Graph

MSC: 05C65

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1. Preliminaries

This section establishes the notation used throughout the paper and reviews the basic set-theoretic and combinatorial concepts needed in later sections.

1.1. SuperHyperGraphs

We begin with the set-theoretic operations underlying SuperHyperGraphs and then recall the relevant graph-theoretic definitions.

Definition 1 (Base set)**.** *A* base set *S is the underlying universe of admissible objects in the setting under consideration, i.e.,*

$$S = \{ x \mid x \text{ is an admissible object in the given context} \}.$$

Accordingly, every element of $\mathcal{P}(S)$—and, more generally, of any iterated powerset—is ultimately formed from elements of S.

Definition 2 (Powerset)**.** *(see [1]) For a set S, its* powerset *is the family of all subsets of S:*

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

In particular, $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

Definition 3 (Hypergraph)**.** *[2,3] A* hypergraph *is an ordered pair $H = (V, E)$ where*

- *V is a finite set of* vertices*, and*
- *E is a finite family of nonempty subsets of V, called* hyperedges*.*

Thus a hyperedge may contain more than two vertices, allowing one to model genuinely multiway relations.

Definition 4 (n-fold iterated powerset). *[4,5] Let X be a set. Define $\mathcal{P}^1(X) := \mathcal{P}(X)$ and, for $n \geq 1$, set recursively*

$$\mathcal{P}^{n+1}(X) := \mathcal{P}(\mathcal{P}^n(X)).$$

When excluding the empty set, we write

$$\mathcal{P}^n_*(X) := \mathcal{P}^n(X) \setminus \{\emptyset\}.$$

Definition 5 (n-SuperHyperGraph). *(see [6–8]) Let V_0 be a finite, nonempty base set, and define the iterated powersets by*

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}(\mathcal{P}^k(V_0)) \quad (k \in \mathbb{N} \cup \{0\}).$$

*For $n \geq 0$, an n-*SuperHyperGraph *on V_0 is a pair*

$$\mathrm{SHG}^{(n)} = (V, E)$$

satisfying

$$V \subseteq \mathcal{P}^n(V_0) \qquad \text{and} \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

*Elements of V are called n-*supervertices*, and elements of E are called* superhyperedges*; equivalently, each superhyperedge is a nonempty subset of the supervertex set V.*

1.2. Line SuperHyperGraph

A line hypergraph replaces each hyperedge by a vertex; each original vertex yields a hyperedge of all incident hyperedges [9–14]. A line SuperHyperGraph makes superhyperedges vertices; each supervertex induces a hyperedge consisting of all recursively incident superhyperedges [15,16].

Definition 6 (Line HyperGraph). *(cf. [9,14]) Let $H = (V, E)$ be a (finite) hypergraph, i.e.,*

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

For each vertex $v \in V$, define the star *of v in H by*

$$\mathrm{Star}_H(v) := \{\, e \in E \mid v \in e \,\} \subseteq E.$$

The line hypergraph *of H is the hypergraph*

$$L(H) = (V^L, E^L),$$

where

$$V^L := E, \qquad E^L := \{\, \mathrm{Star}_H(v) \subseteq E \mid v \in V,\ \mathrm{Star}_H(v) \neq \emptyset \,\}.$$

Thus the vertices of $L(H)$ are the hyperedges of H, and each original vertex $v \in V$ induces a hyperedge of $L(H)$ consisting of all hyperedges of H incident to v.

Definition 7 (Line SuperHyperGraph). *[15–17] Let V_0 be a finite nonempty base set and let $n \in \mathbb{N} \cup \{0\}$. Let*

$$H^{(n)} = (V_n, E)$$

be a level-n SuperHyperGraph, i.e.,

$$V_n \subseteq \mathcal{P}^n(V_0) \qquad \text{and} \qquad \emptyset \neq E \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}.$$

For each $v \in V_n$, define the star *of v in $H^{(n)}$ by*

$$\mathrm{Star}_H(v) := \{\, e \in E \mid v \in e \,\} \subseteq E.$$

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The line SuperHyperGraph *of* $H^{(n)}$ *is the pair*

$$L\Big(H^{(n)}\Big) \;=\; (V^L_{n+1}, E^L_{n+1}),$$

defined by

$$V^L_{n+1} := E, \qquad E^L_{n+1} := \big\{\, \mathrm{Star}_H(v) \subseteq E \mid v \in V_n,\ \mathrm{Star}_H(v) \neq \varnothing \,\big\}.$$

Equivalently, $L(H^{(n)})$ *has one* $(n+1)$*-supervertex for each superhyperedge of* $H^{(n)}$*, and for every* n*-supervertex* $v \in V_n$ *it has a superhyperedge consisting of all superhyperedges of* $H^{(n)}$ *that contain* v*.*

Remark 1 (Level shift)**.** *Since* $V_n \subseteq \mathcal{P}^n(V_0)$*, we have*

$$V^L_{n+1} = E \subseteq \mathcal{P}(V_n) \subseteq \mathcal{P}\big(\mathcal{P}^n(V_0)\big) = \mathcal{P}^{n+1}(V_0),$$

so the vertex set of $L(H^{(n)})$ *naturally lives at level* $n+1$*. Moreover, each element of* E^L_{n+1} *is a nonempty subset of* V^L_{n+1} *by construction.*

2. Main Results

In this section, we present our main results.

2.1. MultiLine Graph

In an ordinary graph, adjacency is recorded only at the vertex level. By contrast, in a MultiLine Graph each vertex is equipped with several distinguishable *local lines*, and an edge is formed by pairing two such local lines. This provides a vertex-centered framework in which one can explicitly represent the fact that a vertex may carry several incident lines.

Definition 8 (MultiLine Graph)**.** *Let* V *be a finite nonempty set. Let* $\mathcal{L}$ *be a finite set, and let*

$$\pi : \mathcal{L} \longrightarrow V$$

be a surjective map. For each $v \in V$*, define*

$$\mathcal{L}(v) := \pi^{-1}(v),$$

whose elements are called the local lines attached to v*.*

A MultiLine Graph *is a triple*

$$\mathcal{G} = (V, \mathcal{L}, E)$$

satisfying the following conditions:

(i) *E is a set of* 2*-element subsets of* $\mathcal{L}$*, i.e.,*

$$E \subseteq \big\{\, \{\ell, \ell'\} \subseteq \mathcal{L} \;\big|\; |\{\ell, \ell'\}| = 2 \,\big\};$$

(ii) *for every* $\{\ell, \ell'\} \in E$*, one has*

$$\pi(\ell) \neq \pi(\ell');$$

thus each edge joins local lines belonging to two distinct vertices;

(iii) *if* $e_1, e_2 \in E$ *and* $e_1 \neq e_2$*, then*

$$e_1 \cap e_2 = \varnothing.$$

Hence each local line is used by at most one edge.

The elements of E are called multiline edges*. For each vertex* $v \in V$*, the number*

$$m(v) := |\mathcal{L}(v)|$$

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is called the line multiplicity *of v.*

Remark 2. *A MultiLine Graph is not the same as the classical* line graph*. Here the word "multi-line" means that each vertex carries several distinguishable local lines. In this sense, the structure is closer to a half-edge model with explicit vertex-wise line multiplicities.*

Definition 9 (Incident edge set and degree)**.** *Let $\mathcal{G} = (V, \mathcal{L}, E)$ be a MultiLine Graph. For each $v \in V$, define the incident edge set of v by*

$$\mathrm{Inc}_{\mathcal{G}}(v) := \{\, e \in E \mid e \cap \mathcal{L}(v) \neq \emptyset \,\}.$$

The degree *of v in $\mathcal{G}$ is*

$$\deg_{\mathcal{G}}(v) := |\,\mathrm{Inc}_{\mathcal{G}}(v)|.$$

Proposition 1. *Let $\mathcal{G} = (V, \mathcal{L}, E)$ be a MultiLine Graph. Then, for every $v \in V$,*

$$\deg_{\mathcal{G}}(v) \leq m(v).$$

Proof. Fix $v \in V$. By Definition 8, distinct edges of E are pairwise disjoint. Therefore, if an edge $e \in E$ is incident with v, then it uses a unique local line from $\mathcal{L}(v)$. Since no local line can belong to two different edges, the number of edges incident with v cannot exceed the number of local lines attached to v. Hence

$$\deg_{\mathcal{G}}(v) \leq |\mathcal{L}(v)| = m(v).$$

□

Definition 10 (Saturated and proper MultiLine Graphs)**.** *A MultiLine Graph $\mathcal{G} = (V, \mathcal{L}, E)$ is called* saturated *if*

$$\bigcup_{e \in E} e = \mathcal{L},$$

that is, every local line is used by some multiline edge.

A saturated MultiLine Graph is called proper *if*

$$m(v) \geq 2 \qquad \text{for all } v \in V.$$

Thus, in a proper MultiLine Graph, every vertex carries at least two local lines.

Proposition 2. *If $\mathcal{G} = (V, \mathcal{L}, E)$ is saturated, then for every $v \in V$,*

$$\deg_{\mathcal{G}}(v) = m(v).$$

In particular, if $\mathcal{G}$ is proper, then every vertex is incident with at least two edges:

$$\deg_{\mathcal{G}}(v) \geq 2 \qquad (v \in V).$$

Proof. Assume that $\mathcal{G}$ is saturated. Then every local line in $\mathcal{L}(v)$ belongs to some edge of E. Because distinct edges are disjoint, each local line of $\mathcal{L}(v)$ contributes to exactly one incident edge of v. Hence the number of incident edges at v is precisely the number of local lines attached to v, namely

$$\deg_{\mathcal{G}}(v) = |\mathcal{L}(v)| = m(v).$$

If, moreover, $\mathcal{G}$ is proper, then $m(v) \geq 2$ for all $v \in V$, and therefore

$$\deg_{\mathcal{G}}(v) = m(v) \geq 2.$$

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□

2.2. MultiLine HyperGraph

A MultiLine HyperGraph extends the MultiLine Graph framework by allowing one multiline edge to connect more than two local lines simultaneously. Thus, each vertex may carry several distinguishable local lines, and a multiline hyperedge is formed by selecting a finite set of such local lines attached to pairwise distinct vertices.

Definition 11 (MultiLine HyperGraph). *Let V be a finite nonempty set, let $\mathcal{L}$ be a finite set, and let*

$$\pi : \mathcal{L} \longrightarrow V$$

be a surjective map. For each vertex $v \in V$, define

$$\mathcal{L}(v) := \pi^{-1}(v),$$

whose elements are called the local lines attached to v.

A MultiLine HyperGraph *is a triple*

$$\mathcal{H} = (V, \mathcal{L}, E)$$

such that the following conditions hold:

(i) *E is a finite set of subsets of $\mathcal{L}$ satisfying*

$$E \subseteq \{\, F \subseteq \mathcal{L} \mid |F| \geq 2 \,\};$$

(ii) *for every $F \in E$, the restriction*

$$\pi|_F : F \longrightarrow V$$

is injective; equivalently, if $\ell, \ell' \in F$ and $\ell \neq \ell'$, then

$$\pi(\ell) \neq \pi(\ell');$$

(iii) *distinct multiline hyperedges are pairwise disjoint, i.e., if $F_1, F_2 \in E$ and $F_1 \neq F_2$, then*

$$F_1 \cap F_2 = \varnothing.$$

The elements of E are called multiline hyperedges. *For each $v \in V$, the number*

$$m(v) := |\mathcal{L}(v)|$$

is called the line multiplicity *of v.*

Remark 3. *Condition (ii) means that a single multiline hyperedge cannot use two local lines attached to the same vertex. Hence each multiline hyperedge determines a genuine multiway relation among distinct vertices. Condition (iii) ensures that each local line participates in at most one multiline hyperedge.*

Definition 12 (Incident multiline hyperedges and degree). *Let $\mathcal{H} = (V, \mathcal{L}, E)$ be a MultiLine HyperGraph. For each $v \in V$, define*

$$\mathrm{Inc}_{\mathcal{H}}(v) := \{\, F \in E \mid F \cap \mathcal{L}(v) \neq \varnothing \,\}.$$

The degree *of v in $\mathcal{H}$ is*

$$\deg_{\mathcal{H}}(v) := |\,\mathrm{Inc}_{\mathcal{H}}(v)|.$$

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Proposition 3. *Let $\mathcal{H} = (V, \mathcal{L}, E)$ be a MultiLine HyperGraph. Then, for every $v \in V$,*

$$\deg_{\mathcal{H}}(v) \leq m(v).$$

Proof. Fix $v \in V$. Since distinct multiline hyperedges are pairwise disjoint, each local line in $\mathcal{L}(v)$ can belong to at most one multiline hyperedge. Therefore the number of multiline hyperedges incident with v cannot exceed the number of local lines attached to v. Hence

$$\deg_{\mathcal{H}}(v) \leq |\mathcal{L}(v)| = m(v).$$

□

Definition 13 (Saturated MultiLine HyperGraph)**.** *A MultiLine HyperGraph $\mathcal{H} = (V, \mathcal{L}, E)$ is called* saturated *if*

$$\bigcup_{F \in E} F = \mathcal{L}.$$

Equivalently, every local line is used by some multiline hyperedge.

Proposition 4. *If $\mathcal{H} = (V, \mathcal{L}, E)$ is saturated, then for every $v \in V$,*

$$\deg_{\mathcal{H}}(v) = m(v).$$

Proof. Assume that $\mathcal{H}$ is saturated. Then every local line in $\mathcal{L}(v)$ belongs to some multiline hyperedge. Since distinct multiline hyperedges are disjoint, each local line of $\mathcal{L}(v)$ belongs to exactly one multiline hyperedge, and distinct local lines of $\mathcal{L}(v)$ cannot belong to the same multiline hyperedge by Definition 11(ii). Therefore the number of multiline hyperedges incident with v is exactly $|\mathcal{L}(v)|$. Hence

$$\deg_{\mathcal{H}}(v) = m(v).$$

□

2.3. MultiLine SuperHyperGraph

A MultiLine SuperHyperGraph extends the previous construction from ordinary vertices to higher-level supervertices. In this setting, each n-supervertex carries several distinguishable local lines, and each multiline superhyperedge is a finite set of local lines attached to pairwise distinct n-supervertices.

Definition 14 (MultiLine n-SuperHyperGraph)**.** *Let V_0 be a finite nonempty base set, let $n \in \mathbb{N} \cup \{0\}$, and let*

$$V_n \subseteq \mathcal{P}^n(V_0).$$

Let $\mathcal{L}$ be a finite set, and let

$$\pi : \mathcal{L} \longrightarrow V_n$$

be a surjective map. For each $x \in V_n$, define

$$\mathcal{L}(x) := \pi^{-1}(x),$$

whose elements are called the local lines attached to the n-supervertex x.

A MultiLine n-SuperHyperGraph *is a triple*

$$\mathcal{S}^{(n)} = (V_n, \mathcal{L}, E)$$

satisfying the following conditions:

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(i) *E is a finite set of subsets of $\mathcal{L}$ satisfying*

$$E \subseteq \{\, F \subseteq \mathcal{L} \mid |F| \geq 2 \,\};$$

(ii) *for every $F \in E$, the restriction*

$$\pi|_F : F \longrightarrow V_n$$

is injective; equivalently, if $\ell, \ell' \in F$ and $\ell \neq \ell'$, then

$$\pi(\ell) \neq \pi(\ell');$$

(iii) *distinct multiline superhyperedges are pairwise disjoint, i.e., if $F_1, F_2 \in E$ and $F_1 \neq F_2$, then*

$$F_1 \cap F_2 = \varnothing.$$

The elements of E are called multiline superhyperedges. *For each $x \in V_n$, the number*

$$m(x) := |\mathcal{L}(x)|$$

is called the line multiplicity *of the n-supervertex x.*

Remark 4. *A MultiLine n-SuperHyperGraph is built over the level-n supervertex set $V_n \subseteq \mathcal{P}^n(V_0)$. Thus the local lines do not replace the n-supervertices; rather, they refine their incidence structure by allowing each n-supervertex to possess several distinguishable attachment slots.*

Definition 15 (Incident multiline superhyperedges and degree)**.** *Let $\mathcal{S}^{(n)} = (V_n, \mathcal{L}, E)$ be a MultiLine n-SuperHyperGraph. For each $x \in V_n$, define*

$$\mathrm{Inc}_{\mathcal{S}^{(n)}}(x) := \{\, F \in E \mid F \cap \mathcal{L}(x) \neq \varnothing \,\}.$$

The degree *of x in $\mathcal{S}^{(n)}$ is*

$$\deg_{\mathcal{S}^{(n)}}(x) := |\,\mathrm{Inc}_{\mathcal{S}^{(n)}}(x)|.$$

Proposition 5. *Let $\mathcal{S}^{(n)} = (V_n, \mathcal{L}, E)$ be a MultiLine n-SuperHyperGraph. Then, for every $x \in V_n$,*

$$\deg_{\mathcal{S}^{(n)}}(x) \leq m(x).$$

Proof. The proof is identical to that of Proposition 3. Because distinct multiline superhyperedges are pairwise disjoint, each local line attached to x can belong to at most one multiline superhyperedge. Therefore the number of multiline superhyperedges incident with x cannot exceed the number of local lines attached to x. Hence

$$\deg_{\mathcal{S}^{(n)}}(x) \leq |\mathcal{L}(x)| = m(x).$$

□

Definition 16 (Saturated MultiLine n-SuperHyperGraph)**.** *A MultiLine n-SuperHyperGraph $\mathcal{S}^{(n)} = (V_n, \mathcal{L}, E)$ is called* saturated *if*

$$\bigcup_{F \in E} F = \mathcal{L}.$$

Proposition 6. *If $\mathcal{S}^{(n)} = (V_n, \mathcal{L}, E)$ is saturated, then for every $x \in V_n$,*

$$\deg_{\mathcal{S}^{(n)}}(x) = m(x).$$

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Proof. The proof is the same as in the hypergraph case. Under saturation, every local line attached to x is used by some multiline superhyperedge; by pairwise disjointness of multiline superhyperedges and injectivity of $\pi|_F$ on each $F \in E$, this correspondence is one-to-one. Therefore

$$\deg_{\mathcal{S}^{(n)}}(x) = |\mathcal{L}(x)| = m(x).$$

□

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Use of Generative AI and AI-Assisted Tools: I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

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