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Electromagnetic Gauge-Invariance Is the Local Rotational Symmetry of a Reference Axis

Moab Croft

Physics Department, Illinois State University, 100 N University St., Normal, 61761, IL, USA; mecrofft@ilstu.edu

Abstract

A geometric-algebraic interpretation of electromagnetic gauge-invariance in nonrelativistic quantum electrodynamics is given. Using the *Algebra of Physical Space*, the Pauli spinor is defined as the rotor that carries a fixed reference axis to a massive fermion's spin vector. The spin vector is invariant under *internal* rotations about that reference axis; promoting those internal rotation angles to spacetime-dependent fields yields precisely the usual $\mathrm{Spin}(2) \approx \mathrm{U}(1)$ gauge transformations. From this construction the Schrödinger-Pauli equation (SPE) follows naturally as a differential equation of a rotor field, and a manifestly gauge-covariant SPE is constructed where the gauge group originates from the reference axis' rotational symmetry. So electromagnetic gauge-invariance is interpreted geometrically as the freedom to choose the starting orientation (about the reference axis) rather than as an abstract phase of a complex scalar field. This approach reproduces standard results while giving a concrete geometry that clarifies conceptual foundations. A discussion of these results concludes the paper.

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1. Introduction and Motivation

This paper claims to answer the question, "To what does electromagnetic gauge symmetry correspond, *physically*?" For this, a highly nuanced approach is due; because Group-theoretically and Physics-wise there is nothing novel. So the learned reader might label this work as trivial. It is the author's opinion that the importance of this work is indeed subtle, but not trivial. It *is* subtle, yet nonetheless fills a foundational crack in the understanding of Physics.

The roots of electromagnetic gauge-invariance lie (with Ampère) two-hundred years ago [1], and the modern understanding lies with Fock and Weyl one-hundred years ago [2,3]—it is that a U(1) phase-rotation of a charged particle's field does not affect observables. What does this physically mean? From a review of the history of gauge-invariance [4],

> While for Electroweak Theory and QCD gauge invariance is of paramount importance, its physical meaning in QED per se does not seem to be extremely profound.

This paper proposes a simple-yet-profound answer using a geometric algebra. Within the last half-century, such algebras have given significant physical and interpretational insight to Physics [5–7]. Very recently, a particular geometric algebra (called the Spacetime Algebra) was used to give a physical interpretation to Wigner's little group for photons, which had lacked a true interpretation since its inception [8]. This work will use the *Algebra of Physical Space* (APS), which is the geometric algebra of 3-dimensional Euclidean space. Specifically, the *Pauli spinor* will be defined as the rotation from some *reference axis* to a massive fermion's *spin vector*. This spin vector is invariant under all *internal* rotations about the reference axis. The spin-$\uparrow$ eigenstate will be constructed and the *Pauli-Schrödinger equation* will follow. It will be seen that the internal rotations' phases (about the reference axis), when promoted to fields in time and space, correspond to the usual electromagnetic gauge transformations. To further concretize and verify this correspondence, a construction of the manifestly gauge-invariant Schrödinger-Pauli equation—where the gauge group still comes from the reference axis' rotational

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symmetry—will be offered. The afore-promised nuanced discussion of this proof and its implications will conclude in Section 3.

All algebraic essentials will be covered in Appendix A. To get the most out of this work, the author recommends lightly reading the full paper, then reading the appendix to garner a better mathematical understanding, and finally rereading the full paper.

2. Reference Symmetry and the Schrödinger Equation

Let $\hat{s} \in \mathbb{G}_3^1$ be the *spin vector* of a massive fermion. In a classical application of the geometry, this encodes for the physical orientation of the particle in space; in a quantum application of the geometry, this encodes for the orientation of the particle on the Bloch-sphere.[1] No matter the application—classical or quantum—the algebra herein will be the same, and so will be the geometric interpretations. Because a spin vector's orientation makes no sense in isolation, a *reference axis* $\hat{a} \in \mathbb{G}_3^1$ must be introduced. For any such reference axis, there is a rotor $R_s \in \mathrm{Spin}(3) \approx \mathrm{SU}(2) \subset \mathbb{G}_3$ such that

$$\hat{s} = R_s \hat{a} R_s^\dagger. \tag{1}$$

This rotor is the (operatorial) *Pauli spinor* which rotates the reference axis $\hat{a}$ into the spin vector $\hat{s}$ [6,7,11–14]. Note that for all $\theta \in \mathbb{R}$,

$$\hat{s} = R_s e^{-i\theta\hat{a}} \hat{a} e^{i\theta\hat{a}} R_s^\dagger. \tag{2}$$

This shows that the spin vector is invariant under

$$R_s \mapsto R_s e^{-i\theta\hat{a}}, \tag{3}$$

which is a global $\mathrm{Spin}(2) \approx \mathrm{U}(1)$ symmetry group. Soon $\hat{s}$ will be promoted to a *field* and this global symmetry will become a local *gauge group*. Now, suppose that the spin vector and reference axis are aligned, $\hat{s} \cdot \hat{a} = 1$. Then $R_s = R_\uparrow$ such that $R_\uparrow \hat{a} R_\uparrow^\dagger = \hat{a}$. This is a spin-$\uparrow$ eigenstate *with respect to the reference axis*. Let this eigenspinor depend on the exponential of a linear parameter τ,

$$R_\uparrow(\tau) = e^{-i\omega\tau\hat{a}}, \tag{4}$$

where $\omega \in \mathbb{R}$ is some constant of the eigenstate. The derivative with respect to the parameter gives

$$\partial_\tau R_\uparrow = -i\omega R_\uparrow \hat{a} \quad \implies \quad i\partial_\tau R_\uparrow \hat{a} = \omega R_\uparrow. \tag{5}$$

One may identify the parameter τ with proper time and the eigenvalue ω with a rest-energy value $E_0/\hbar$. Rearranging the above equation:

$$i\hbar\partial_\tau R_\uparrow \hat{a} = E_0 R_\uparrow. \tag{6}$$

This is recognized as the *Schrödinger-Pauli equation* of a free fermion [12]. For convenience, as E_0 has no influence upon dynamics, the convention $E_0 = 0$ is used henceforth. Now, suppose that the spin vector becomes a field over space $\mathbf{x} \in \mathbb{G}_3^1$ and time τ, $\hat{s} \mapsto \hat{s}(\tau, \mathbf{x})$. For a general Hermitian object $\widehat{H} \in \mathbb{G}_3$ (traditionally called an operator), unitary time evolution is assumed:

$$\hat{s} \mapsto \hat{s}(\tau, \mathbf{x}) \quad \leftrightarrow \quad R_\uparrow \mapsto \Psi = e^{-i\widehat{H}\tau}. \tag{7}$$

Then,

$$i\hbar\partial_\tau \Psi\hat{a} = \widehat{H}\Psi\hat{a} \quad \leftrightarrow \quad i\hbar\partial_\tau \Psi = \widehat{H}\Psi. \tag{8}$$

[1] There in fact *is* such a thing as classical spin [9,10].

In the case $\widehat{H} \propto \hat{a}$, the spin remains in its spin-$\uparrow$ eigenstate and this is simply the *Schrödinger equation*. In the general case $\widehat{H} \not\propto \hat{a}$, the spin is not in an eigenstate and this is the *Pauli equation*. This paper will refer to both possibilities using the umbrella-term Schrödinger-Pauli equation (SPE). A quote from [11] is highly relevant:

> Consistency [with Pauli theory] requires that the Schrödinger theory be regarded as describing an electron in an eigenstate of spin.

Hence this realization is not novel, but this method of discussion and derivation is. Recall EQ. 2: It says that $\hat{s}$ is invariant under the global map $R_s \mapsto R_s e^{-i\theta\hat{a}}$. There is an equivalent *local* invariance for $\hat{s}(\tau, \mathbf{x})$ under

$$\Psi \mapsto \Psi' = \Psi e^{-i\theta(\tau,\mathbf{x})\hat{a}}, \tag{9}$$

where $\theta(\tau, \mathbf{x})$ is now a scalar field in time and space. In both the global and local cases, the relevant spin vectors are invariant as a result of the *rotational symmetry* (Spin(2) $\approx$ U(1)) of the reference axis $\hat{a}$.

Due to its simplicity and brevity, the above proof-of-gauge-invariance might leave the reader wanting more, so an explicit construction of a manifestly gauge-covariant SPE—where the gauge group comes from the rotational symmetry of $\hat{a}$—will now be offered. Demonstration of gauge-covariance demands covariant derivatives! A *gauge covariant derivative* $\mathcal{D}$ must satisfy

$$\mathcal{D}'\Psi' = (\mathcal{D}\Psi)e^{-i\theta(\tau,\mathbf{x})\hat{a}}. \tag{10}$$

Here the primes indicate a different gauge. To motivate the form of covariant derivatives satisfying this condition, one takes derivatives of $\Psi' = \Psi e^{-i\theta(\tau,\mathbf{x})\hat{a}}$ and simultaneously considers the known gauge transformations of the electric potential. The known transformation for the electric *scalar* potential $\phi \in \mathbb{R}$ is

$$\phi \mapsto \phi' = \phi - \frac{\hbar}{q}\partial_\tau\theta(\tau, \mathbf{x}). \tag{11}$$

The time derivative of Ψ' is

$$\partial_\tau\Psi' = \partial_\tau\Psi e^{-i\theta(\tau,\mathbf{x})} - i\partial_\tau\theta(\tau, \mathbf{x})\Psi e^{-i\theta(\tau,\mathbf{x})\hat{a}}. \tag{12}$$

Together, these say that the covariant time derivative must be

$$D_\tau(\cdot) = \partial_\tau(\cdot) - i\frac{q}{\hbar}\phi(\cdot)\hat{a}. \tag{13}$$

Plugging in Ψ' and ϕ' gives

$$\begin{aligned} D'_\tau\Psi' &= \left(\partial_\tau\Psi - i\frac{q}{\hbar}(\phi' + \frac{\hbar}{q}\partial_\tau\theta(\tau, \mathbf{x})\Psi\hat{a})e^{-i\theta(\tau,\mathbf{x})\hat{a}}\right) \\ &= \left(\partial_\tau - i\frac{q}{\hbar}\phi\Psi\hat{a}\right)e^{-i\theta(\tau,\mathbf{x})\hat{a}} = (D_\tau\Psi)e^{-i\theta(\tau,\mathbf{x})\hat{a}}, \end{aligned} \tag{14}$$

verifying the construction. The known gauge transformation for the electric *vector* potential $\mathbf{A} \in \mathbb{G}_3^1$ is

$$\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} + \frac{\hbar}{q}\nabla\theta(\tau, \mathbf{x}), \tag{15}$$

and $\nabla \in \mathbb{G}_3^1$ is the gradient (vector derivative with respect to $\mathbf{x}$). By the same reasoning as before, the covariant gradient must be

$$\mathbf{D}(\cdot) = \nabla(\cdot) + i\frac{q}{\hbar}\mathbf{A}(\cdot)\hat{a}. \tag{16}$$

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Verifying with Ψ' and $\mathbf{A}'$:

$$\begin{aligned}\mathbf{D}'\Psi' &= \left(\nabla\Psi + i\frac{q}{\hbar}(\mathbf{A}' - \frac{\hbar}{q}\nabla\theta(\tau,\mathbf{x})\Psi\hat{a})\right)e^{-i\theta(\tau,\mathbf{x})\hat{a}} \\ &= \left(\nabla\Psi + i\frac{q}{\hbar}\mathbf{A}\Psi\hat{a}\right)e^{-i\theta(\tau,\mathbf{x})\hat{a}} = (\mathbf{D}\Psi)e^{-i\theta(\tau,\mathbf{x})\hat{a}}.\end{aligned} \tag{17}$$

So, in the presence of the electromagnetic potential $A = c\phi + \mathbf{A}$,

$$i\hbar D_\tau\Psi = -\frac{\hbar^2}{2m}\mathbf{D}^2\Psi \tag{18}$$

is the manifestly gauge-covariant SPE—for $\widehat{H} = -\hat{a}\hbar^2\mathbf{D}^2/2m$—under

$$\mathrm{Spin}(2)_{\hat{a}} = \{e^{-i\theta(\tau,\mathbf{x})\hat{a}} \mid \theta(\tau,\mathbf{x}) \in \mathbb{R}\}, \tag{19}$$

which is the group of rotational symmetry about the reference axis $\hat{a}$. Further discussion of this fact and its interpretation's relevance are to be written in Section 3.

3. Discussion

To recap, the *Pauli spinor* was defined as the rotation that takes a *reference axis* to a massive fermion's *spin vector* and from this the Schrödinger-Pauli equation (SPE) was derived. The spin vector is invariant under *internal* rotations about the reference axis and it was seen, by noting the covariance of the SPE, that this reference axis' rotational symmetry group (Spin(2) $\approx$ U(1)) was exactly the electromagnetic gauge group.

Electromagnetic gauge-invariance is traditionally interpreted as a simple unchangedness of observables under a phase-rotation of a fermion's field. What this phase represents is not specified and so its meaning is presumed to be relatively superficial. However, the earlier geometric construction shows a clear interpretation: The invariance comes from the rotational symmetry of a reference axis—an effectively *imposed* basis—that is *necessary*[2] for constructing Pauli spinors.

However, one might object and say that this geometric view is overly pedantic. In isolation this might be true, but not when combined with the geometric interpretation of the Pauli spinors: Rotations encoding a fermion's spin-orientation. In the traditional view, the SPE is a differential equation of a complex scalar field and the gauge-invariance is a statement of complex scalar phase-invariance. But in the geometry, the SPE is a differential equation of a rotor field and the gauge-invariance is a statement of the *starting position's* rotational symmetry. The traditional view is an abstraction of this geometric view, and its geometric and physical interpretation is hidden by the abstraction.

Why is this important? Firstly, this work's interpretation of the electromagnetic gauge-invariance now fits within the geometric-relational interpretation of Quantum Mechanics that has slowly been developing [15–18]. Secondly, the more concrete geometry lends the interpretation to deeper intuition and easier pedagogy. It is well-known among the Physics community—among researchers, teachers, students, and even outside purely scientific institutions—that intuition and pedagogy is often abstract and challenging. It is the hope of the author that this simple geometric interpretation will fill the crack in this foundational understanding of Physics and render the interpretation and teaching of Quantum Mechanics slightly easier.

Data Availability Statement: This paper contains no results found through data analysis or any like method.

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[2] Recall that the orientation of the spin vector is nonsensical without a reference; for otherwise the spin vector would have orientation with respect to what?

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Conflicts of Interest: There are neither financial nor non-financial competing interests relevant to the author and their work.

Appendix A. Algebraic Essentials

The *Algebra of Physical Space* (sometimes called the *Pauli Algebra*) is the real geometric algebra $\mathbb{G}_3$. It is isomorphic to the algebra of 2×2 complex matrices, $M_2(\mathbb{C})$, which serves as its irreducible matrix representation. However, matrices are unnecessary for the purposes of this paper and everything may be completed coordinate- and matrix-free. The algebra is generated by three orthonormal basis vectors,

$$\sigma_x, \quad \sigma_y, \quad \sigma_z, \tag{A1}$$

which satisfy the inner product

$$\sigma_j \cdot \sigma_k = \frac{1}{2}(\sigma_j\sigma_k + \sigma_k\sigma_j) = \delta_{jk}, \tag{A2}$$

where δ_{jk} is the Kronecker delta. These basis vectors are written with the same notation as the *Pauli matrices* due to their representation in $M_2(\mathbb{C})$. Any vector $\mathbf{a} \in \mathbb{G}_3^1$ may be written as a real linear sum of the vector basis:

$$\mathbf{a} = a^j\sigma_j = a^x\sigma_x + a^y\sigma_y + a^z\sigma_z, \tag{A3}$$

but in this paper will be only expressed in its coordinate-free form: $\mathbf{a}$. A unit vector is written with a hat, $\hat{a}$. The geometric product of any two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}. \tag{A4}$$

The first term is the traditional *inner product*,

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) = a^j b_j \tag{A5}$$

and the second term is the *outer product*,

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) = \frac{1}{2}\sum_{j \neq k}(a^j b^k - a^k b^j)\sigma_j \wedge \sigma_k. \tag{A6}$$

Between two vectors, the inner product returns a scalar while the outer product returns a *bivector*. The full multivector basis of the algebra is then

$$1, \quad \{\sigma_j\}, \quad \{\sigma_j \wedge \sigma_k\}, \quad \sigma_x \wedge \sigma_y \wedge \sigma_z. \tag{A7}$$

The last term is the *pseudoscalar* and is traditionally written $i = \sigma_{xyz} = \sigma_x \wedge \sigma_y \wedge \sigma_z$ because it commutes with all elements of the algebra such that the center of the algebra is isomorphic to $\mathbb{C}$.

Appendix A.1. Spin(3) *in* $\mathbb{G}_3$

The group $\mathrm{Spin}(3) \approx \mathrm{SU}(2)$ is that of the 3-dimensional Euclidean space's rotations:

$$\mathrm{Spin}(3) = \{R \in \mathbb{G}_3^+ \mid RR^\dagger = R^\dagger R = 1\}. \tag{A8}$$

Here $\mathbb{G}_3^+$ is the *even subalgebra* of $\mathbb{G}_3$, which is the subspace of scalars and bivectors (which is closed under multiplication). Also, $(\cdot)^\dagger$ is the Hermitian conjugate (reversion) which satisfies, for two elements $g, h \in \mathbb{G}_3$,

$$(gh)^\dagger = h^\dagger g^\dagger. \tag{A9}$$

This group exists due to the basis bivectors of EQ. A7, which form the Lie algebra $\mathfrak{spin}(3) \approx \mathfrak{su}(2)$. In general, rotors are exponentials of an arbitrary bivector $\Omega \in \mathbb{G}_3^2$:

$$R = e^{-\frac{1}{2}\Omega}. \tag{A10}$$

Bivectors may also be represented as imaginary vectors using the pseudoscalar i: $\Omega = i\mathbf{w}$. Then, using the *sandwich product*,

$$g \mapsto RgR^{\dagger}, \tag{A11}$$

any multivector can be rotated.

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