

Article

Not peer-reviewed version

Integral Transforms with Finite-Principal-Part Laurent Kernels: Weighted Dilation Operators, Mellin Symbols, and Exact Signal Inversion

[Mohammad Abu-Ghuwaleh](#) *

Posted Date: 13 March 2026

doi: 10.20944/preprints202603.0779.v1

Keywords: integral transforms; Laurent kernels; weighted dilation operators; Mellin symbols; exact inversion; harmonic analysis; signal reconstruction; Dirichlet-Wiener inversion



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This open access article is published under a [Creative Commons CC BY 4.0 license](#), which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

Integral Transforms with Finite-Principal-Part Laurent Kernels: Weighted Dilation Operators, Mellin Symbols, and Exact Signal Inversion

Mohammad Abu-Ghuwaleh

Department of Mathematics, Zarqa University, Zarqa, Jordan; 20209030@zu.edu.jo

Abstract

We extend the master-integral-transform theory from entire kernels to finite-principal-part Laurent kernels and show that the resulting transform is a weighted dilation operator acting on the Fourier transform of a weighted signal. This yields a unified operator framework for several exact inversion mechanisms, including Mellin diagonalization, two-sided Mellin-symbol inversion, Dirichlet–Wiener inversion, log-scale Fourier inversion, recursive inversion, and Neumann-series recovery. The main structural result is that finite negative Laurent tails do not destroy the spectral architecture; they enlarge the one-sided dilation orbit to a two-sided one. We establish exact factorization formulas on weighted function spaces, prove branchwise Mellin inversion under explicit integrability assumptions, derive a contour-free Dirichlet–Wiener inverse, obtain a log-scale Fourier multiplier representation suitable for FFT-based recovery, and prove a practical stability bound away from multiplier zeros. A worked symbolic example and a numerical blueprint are also included.

Keywords: integral transforms; Laurent kernels; weighted dilation operators; Mellin symbols; exact inversion; harmonic analysis; signal reconstruction; Dirichlet–Wiener inversion

1. Introduction

The master integral transform with entire kernels established a broad transform calculus in which an entire kernel generates a family of oscillatory integral transforms together with completeness, injectivity, and Mellin–Fourier inversion under density hypotheses on the nonzero Taylor indices [1]. After such a general framework paper, the next mathematically meaningful step cannot be a mere expansion of examples. It must isolate the next structural layer.

The purpose of the present paper is to identify that layer. We pass from entire kernels to finite-principal-part Laurent kernels and show that, once the kernel is expanded, the Abu-Ghuwaleh transform is most naturally viewed as a weighted dilation operator acting on the Fourier transform of a weighted signal. In this formulation, the forward transform is not merely an oscillatory integral with a flexible kernel. It becomes an operator built from the dilation family

$$(S_m F)(\theta) := F(m\theta),$$

and the different inversion formulas arise by diagonalizing or inverting the corresponding symbol in different functional settings.

Two conceptual points are central. First, finite negative Laurent modes do not automatically generate local derivative corrections of the signal. Along the transform orbit, a negative Laurent mode is simply a reflected harmonic, so the first meromorphic sequel is, at this stage, a two-sided spectral extension rather than a local-jet theory. Second, the same forward operator admits several exact inversions of genuinely different flavor: Mellin contour inversion, Dirichlet–Wiener inversion in the discrete coefficient algebra, logarithmic Fourier inversion, recursive inversion, and Neumann

inversion. The point of the paper is not merely that these formulas exist, but that they all invert the same operator.

This perspective naturally lies at the interface of harmonic analysis, Mellin analysis, and multiplicative operator theory. The Mellin-transform side of the paper is informed by the standard Mellin calculus and asymptotic machinery developed in [3,8], while the discrete inversion side belongs to the classical world of Dirichlet series and multiplicative convolution [4,5,7]. The log-scale Fourier representation converts the dilation equation into a translation equation and therefore places the problem squarely within classical Fourier analysis [6,10]. At the operator-theoretic level, the Neumann inversion belongs to standard Banach-space inversion theory [9], and the contour-free inversion philosophy is closely aligned with Wiener-type nonvanishing principles [11,12].

This focus distinguishes the present work from the 2025 MIT paper. The MIT paper already established the broad entire-kernel umbrella: BM-density-based completeness and injectivity, explicit Mellin–Fourier inversion, and structural transform laws for arbitrary entire kernels of finite order [1]. The novelty here is different and sharper. We move from one-sided Taylor spectra to the first genuinely two-sided meromorphic model, identify the exact forward operator on the Fourier side, prove that the first meromorphic sequel is governed by a two-sided Mellin symbol rather than an artificial jet formalism, and place several exact inversions for the same forward map into a single theorem package.

Main results. The main results may be summarized as follows.

- (1) We prove that the transform with finite-principal-part Laurent kernels is exactly a two-sided weighted dilation operator acting on the Fourier transform of a weighted signal.
- (2) We derive branchwise Mellin diagonalization formulas and the corresponding two-sided Mellin inversion theorems.
- (3) We prove a contour-free Dirichlet–Wiener inversion theorem for the discrete dilation equation, with the correct scaling law.
- (4) We derive a log-scale Fourier multiplier representation leading to an FFT-ready inversion formula.
- (5) We establish recursive and Neumann-series inversions under transparent smallness hypotheses.
- (6) We formulate injectivity criteria and a practical stability bound for log-scale inversion away from zeros of the multiplier.

Roadmap. Section 2 fixes the weighted signal spaces and the basic notation. Section 3 revisits the entire-kernel case only to isolate the one-sided dilation-operator structure. Section 4 proves the two-sided Laurent factorization, which is the core new forward theorem. Section 5 establishes branchwise Mellin diagonalization, and Section 6 turns those formulas into contour inversions. Section 7 develops the exact Dirichlet–Wiener inverse, while Sections 8 and 9 present recursive, Neumann, and log-scale Fourier inversions. Section 10 extracts injectivity and stability consequences from symbol nonvanishing. Section 11 gives explicit model kernels, including a fully worked symbolic example and a small log-FFT numerical experiment.

Position in the AGT/MIT program. This paper is intentionally narrow. It does not attempt to solve the full strip-analytic, essential-singularity, multivariate, or operator-valued theory. Those belong to later papers. The present contribution isolates the cleanest and strongest next theorem package after the MIT paper: the first meromorphic extension, viewed through dilation-operator factorization and multiple exact inversions.

2. Preliminaries and the Weighted Signal

Fix $p \in (-1, 1)$ and parameters $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$. For a measurable function $f : \mathbb{R} \rightarrow \mathbb{C}$, define the weighted signal

$$\Phi(x) := |x|^p f(x) e^{-i\pi p \operatorname{sgn}(x)/2}. \quad (1)$$

Its Fourier transform is

$$F(\omega) := \widehat{\Phi}(\omega) = \int_{\mathbb{R}} \Phi(x) e^{i\omega x} dx,$$

whenever the integral is classically defined. Our Fourier-analytic conventions are standard and consistent with the classical treatments in [6,10].

We use the following spaces throughout.

Definition 1 (Signal spaces). *Let*

$$X_p := \{f : \mathbb{R} \rightarrow \mathbb{C} : |x|^p f(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}.$$

For Mellin inversion we use the subclass $X_p^M \subset X_p$ consisting of functions for which the weighted even and odd parts satisfy

$$\Phi_e, \Phi_o \in L^1(0, \infty; x^{-c} dx) \cap L^1(0, \infty; x^{-c'} dx)$$

for some $c < c'$, so that their Mellin transforms are holomorphic in a vertical strip; see [3,8]. For log-scale Fourier inversion we use the class $X_p^{\log} \subset X_p$ for which the logarithmic profile

$$g_0(t) := F(e^t)$$

belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Remark 1. The spaces X_p^M and $X_p^{\log}$ are working spaces introduced to isolate exactly where Mellin and log-Fourier diagonalizations are justified. They are not claimed to be optimal. Their role is analogous to the usual strip-type hypotheses imposed in Mellin analysis [3,8].

Definition 2 (Base transform). *Let g be a kernel defined on a neighborhood of the transform orbit. The Abu-Ghuwaleh transform associated with (α, β, p, g) is*

$$G_{\alpha, \beta}^{(p)}[f](\theta) := \frac{1}{2\pi} \int_{\mathbb{R}} |x|^p f(x) e^{-i\pi p \operatorname{sgn}(x)/2} g(\alpha + \beta e^{i\theta x}) dx = \frac{1}{2\pi} \int_{\mathbb{R}} \Phi(x) g(\alpha + \beta e^{i\theta x}) dx. \quad (2)$$

The basic observation is that once the kernel is expanded around α , the transform acts directly on $F = \widehat{\Phi}$.

3. Entire Kernels Revisited as One-Sided Dilation Operators

We begin with the purely entire situation, not to reprove the full MIT theory, but to isolate the exact operator form that later survives in the Laurent case.

Theorem 1 (One-sided dilation factorization). *Let $f \in X_p$, and assume*

$$g(\alpha + z) = \sum_{k \geq 0} c_k z^k, \quad c_0 = 0, \quad (3)$$

with absolute convergence on a closed disc $|z| \leq |\beta|$. Suppose in addition that

$$\sum_{k \geq 1} |c_k| |\beta|^k < \infty. \quad (4)$$

Then the series may be integrated termwise in (2), and for every $\theta \in \mathbb{R}$,

$$G(\theta) = \sum_{k \geq 1} a(k) F(k\theta), \quad a(k) := \frac{c_k \beta^k}{2\pi}. \quad (5)$$

Equivalently,

$$G = T_a F, \quad (T_a F)(\theta) := \sum_{k \geq 1} a(k) (S_k F)(\theta), \quad (S_k F)(\theta) := F(k\theta). \quad (6)$$

Moreover, the series in (5) converges absolutely and uniformly on compact subsets of $\mathbb{R}$.

Proof. For each fixed $\theta \in \mathbb{R}$ and $x \in \mathbb{R}$,

$$\sum_{k \geq 1} |c_k \beta^k e^{ik\theta x} \Phi(x)| \leq \left(\sum_{k \geq 1} |c_k| |\beta|^k \right) |\Phi(x)|.$$

Since $\Phi \in L^1(\mathbb{R})$ and the majorant is integrable, Tonelli's theorem permits interchange of sum and integral in (2). Hence

$$G(\theta) = \frac{1}{2\pi} \sum_{k \geq 1} c_k \beta^k \int_{\mathbb{R}} \Phi(x) e^{ik\theta x} dx,$$

and the integral equals $F(k\theta)$. The uniform bound

$$\sum_{k \geq 1} |a(k) F(k\theta)| \leq \frac{\|\Phi\|_{L^1}}{2\pi} \sum_{k \geq 1} |c_k| |\beta|^k$$

yields absolute and locally uniform convergence. $\square$

Remark 2. This theorem is the algebraic heart of the transform program. The forward map is not mysterious: it is a weighted sum of dilations on the Fourier side. In multiplicative language, this places the transform naturally beside the operator calculus of Dirichlet-type expansions [4,5].

4. Finite-Principal-part Laurent Kernels and Two-Sided Dilation Structure

We now pass to the natural first post-MIT generalization.

Definition 3 (Finite-principal-part Laurent kernel). A finite-principal-part Laurent kernel at α is a function of the form

$$g(\alpha + z) = \sum_{m=-M}^{\infty} c_m z^m, \quad M \in \mathbb{N}, \quad (7)$$

with convergence on an annulus containing $z = \beta e^{i\theta x}$ for the relevant (x, θ) .

Remark 3. The crucial point is that along the transform orbit, a negative Laurent mode z^{-m} becomes $\beta^{-m} e^{-im\theta x}$. Thus negative indices generate negative harmonics; they do not, by themselves, force local derivative terms of the signal. The first meromorphic sequel is therefore a two-sided spectral extension, not automatically a local-jet theory.

Theorem 2 (Two-sided dilation factorization). Let $f \in X_p$. Assume (7) with convergence on an annulus $r < |z| < R$ such that $r < |\beta| < R$. Suppose also that

$$\sum_{m=-M}^{\infty} |c_m| |\beta|^m < \infty. \quad (8)$$

Then the series may be integrated termwise in (2), and for every $\theta \in \mathbb{R}$,

$$G(\theta) = \sum_{m=-M}^{\infty} a(m) F(m\theta), \quad a(m) := \frac{c_m \beta^m}{2\pi}. \quad (9)$$

Equivalently,

$$G = T_a F, \quad (T_a F)(\theta) := \sum_{m=-M}^{\infty} a(m)(S_m F)(\theta), \quad (S_m F)(\theta) := F(m\theta). \quad (10)$$

The series converges absolutely and uniformly on compact subsets of $\mathbb{R}$.

Proof. For each fixed θ and x ,

$$\sum_{m=-M}^{\infty} |c_m \beta^m e^{im\theta x} \Phi(x)| \leq \left(\sum_{m=-M}^{\infty} |c_m| |\beta|^m \right) |\Phi(x)|.$$

The majorant is integrable because $\Phi \in L^1(\mathbb{R})$ and (8) holds. Tonelli's theorem therefore gives

$$G(\theta) = \frac{1}{2\pi} \sum_{m=-M}^{\infty} c_m \beta^m \int_{\mathbb{R}} \Phi(x) e^{im\theta x} dx = \sum_{m=-M}^{\infty} a(m) F(m\theta).$$

The same majorant argument yields absolute and locally uniform convergence. $\square$

Corollary 1 (Positive and negative support sets). *Let*

$$S^+ := \{k \geq 1 : c_k \neq 0\}, \quad S^- := \{m \geq 1 : c_{-m} \neq 0\}.$$

Then the spectral action of the transform is supported on the two-sided dilation orbit generated by $S^+ \cup (-S^-)$. Finite negative tails enlarge the one-sided orbit to a two-sided orbit but do not alter the basic operator architecture.

5. Even/Odd Splitting and Mellin Diagonalization

The Mellin theory is most transparent after parity separation. Write

$$\Phi = \Phi_e + \Phi_o, \quad \Phi_e(x) = \frac{\Phi(x) + \Phi(-x)}{2}, \quad \Phi_o(x) = \frac{\Phi(x) - \Phi(-x)}{2}.$$

Correspondingly,

$$F(\omega) = 2 \int_0^{\infty} \Phi_e(x) \cos(\omega x) dx + 2i \int_0^{\infty} \Phi_o(x) \sin(\omega x) dx.$$

We introduce the cosine and sine branches

$$G_c(\theta) := \sum_m a(m) 2 \int_0^{\infty} \Phi_e(x) \cos(m\theta x) dx, \quad (11)$$

$$G_s(\theta) := \sum_m a(m) 2i \int_0^{\infty} \Phi_o(x) \sin(m\theta x) dx, \quad (12)$$

where the sum runs over the relevant index set.

The Mellin identities we use are classical:

$$\int_0^{\infty} \theta^{s-1} \cos(\theta x) d\theta = x^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right), \quad \int_0^{\infty} \theta^{s-1} \sin(\theta x) d\theta = x^{-s} \Gamma(s) \sin\left(\frac{\pi s}{2}\right),$$

in the appropriate strip, and more generally under standard Mellin-continuation hypotheses; see [3,8,11].

Theorem 3 (Mellin diagonalization: one-sided case). Assume the hypotheses of Theorem 1. Let $f \in X_p^M$, and suppose there exists a strip $a < \Re s < b$ such that

$$\sum_{k \geq 1} |c_k| |\beta|^k k^{-\sigma} < \infty \quad \text{uniformly for } \sigma \in [a, b]. \quad (13)$$

Define the one-sided symbol

$$D_+(s) := \sum_{k \geq 1} c_k \beta^k k^{-s}. \quad (14)$$

Then for every s with $a < \Re s < b$,

$$\mathcal{M}_\theta\{G_c\}(s) = \frac{\Gamma(s) \cos(\pi s/2)}{\pi} D_+(s) \mathcal{M}\{\Phi_e\}(1-s), \quad (15)$$

$$\mathcal{M}_\theta\{G_s\}(s) = \frac{\Gamma(s) \sin(\pi s/2)}{i\pi} D_+(s) \mathcal{M}\{\Phi_o\}(1-s). \quad (16)$$

The series defining D_+ converges absolutely and locally uniformly in the strip.

Proof. Fix s with $a < \Re s < b$. By (13) and the definition of X_p^M , all sum-integral interchanges are justified by Fubini–Tonelli. Using the standard Mellin formulas for cosine and sine kernels [3,8], we obtain the stated identities. $\square$

Theorem 4 (Two-sided Mellin diagonalization for finite principal part). Assume the hypotheses of Theorem 2. Let $f \in X_p^M$, and suppose there exists a strip $a < \Re s < b$ such that

$$\sum_{k \geq 1} |c_k| |\beta|^k k^{-\sigma} + \sum_{m=1}^M |c_{-m}| |\beta|^{-m} m^{-\sigma} < \infty \quad \text{uniformly for } \sigma \in [a, b]. \quad (17)$$

Define the two-sided symbols

$$D_c(s) := \sum_{k \geq 1} c_k \beta^k k^{-s} + \sum_{m=1}^M c_{-m} \beta^{-m} m^{-s}, \quad (18)$$

$$D_s(s) := \sum_{k \geq 1} c_k \beta^k k^{-s} - \sum_{m=1}^M c_{-m} \beta^{-m} m^{-s}. \quad (19)$$

Then, for every s with $a < \Re s < b$,

$$\mathcal{M}_\theta\{G_c\}(s) = \frac{\Gamma(s) \cos(\pi s/2)}{\pi} D_c(s) \mathcal{M}\{\Phi_e\}(1-s), \quad (20)$$

$$\mathcal{M}_\theta\{G_s\}(s) = \frac{\Gamma(s) \sin(\pi s/2)}{i\pi} D_s(s) \mathcal{M}\{\Phi_o\}(1-s). \quad (21)$$

Proof. The same Mellin computation applies. The only new point is parity:

$$\cos(-m\theta x) = \cos(m\theta x), \quad \sin(-m\theta x) = -\sin(m\theta x).$$

Hence negative frequencies contribute with a plus sign to the cosine branch and with a minus sign to the sine branch. $\square$

6. Mellin Contour Inversion

We now invert the branchwise Mellin identities. The use of contour inversion is standard in Mellin analysis and Mellin–Barnes techniques; compare [3,8].

Theorem 5 (Entire-kernel Mellin inversion). *Assume the hypotheses of the previous theorem in the one-sided case. Suppose there exists $c \in (a, b)$ such that $D_+(1-u) \neq 0$ on the line $\Re u = c$ and such that*

$$v \mapsto \frac{\mathcal{M}_\theta\{G_c\}(1-c-iv)}{\Gamma(1-c-iv) \sin(\pi(c+iv)/2) D_+(1-c-iv)}$$

and

$$v \mapsto \frac{\mathcal{M}_\theta\{G_s\}(1-c-iv)}{\Gamma(1-c-iv) \cos(\pi(c+iv)/2) D_+(1-c-iv)}$$

belong to $L^1(\mathbb{R}_v)$. Then

$$\Phi_e(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} |x|^{-u} \frac{\pi \mathcal{M}_\theta\{G_c\}(1-u)}{\Gamma(1-u) \sin(\pi u/2) D_+(1-u)} du, \quad (22)$$

$$\Phi_o(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} |x|^{-u} \frac{i\pi \mathcal{M}_\theta\{G_s\}(1-u)}{\Gamma(1-u) \cos(\pi u/2) D_+(1-u)} du. \quad (23)$$

Consequently,

$$f(x) = e^{i\pi p \operatorname{sgn}(x)/2} |x|^{-p} (\Phi_e(x) + \Phi_o(x)) \quad (24)$$

for almost every $x \in \mathbb{R} \setminus \{0\}$.

Proof. Solve (15)–(16) for the Mellin transforms of Φ_e and Φ_o , then apply Mellin inversion. The factor in (24) comes from (1). $\square$

Theorem 6 (Two-sided Mellin inversion for finite principal part). *Assume the hypotheses of the two-sided Mellin theorem. Suppose there exists $c \in (a, b)$ such that $D_c(1-u)$ and $D_s(1-u)$ do not vanish on $\Re u = c$, and suppose the corresponding branch quotients belong to $L^1(\mathbb{R}_v)$ along that line. Then*

$$\Phi_e(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} |x|^{-u} \frac{\pi \mathcal{M}_\theta\{G_c\}(1-u)}{\Gamma(1-u) \sin(\pi u/2) D_c(1-u)} du, \quad (25)$$

$$\Phi_o(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} |x|^{-u} \frac{i\pi \mathcal{M}_\theta\{G_s\}(1-u)}{\Gamma(1-u) \cos(\pi u/2) D_s(1-u)} du, \quad (26)$$

and therefore

$$f(x) = e^{i\pi p \operatorname{sgn}(x)/2} |x|^{-p} (\Phi_e(x) + \Phi_o(x)) \quad (27)$$

for almost every $x \in \mathbb{R} \setminus \{0\}$.

Proof. The proof is identical to that of the one-sided case, now using (20)–(21). $\square$

Remark 4. This is the correct first meromorphic sequel to the MIT inversion theory. Finite negative Laurent tails do not force local jet corrections; they produce a two-sided Mellin symbol. In this sense the meromorphic extension remains spectrally organized rather than locally patched together.

7. Exact Dirichlet–Wiener Inversion

The Mellin contour formulas are theoretically elegant but not the only exact inversion mechanism. In the discrete dilation setting, one can invert directly in the coefficient algebra. This places the problem in the classical setting of Dirichlet convolution and multiplicative inversion [4,5,7].

Definition 4 (Dirichlet convolution). *For arithmetic sequences $a, b : \mathbb{N} \rightarrow \mathbb{C}$, their Dirichlet convolution is*

$$(a * b)(n) := \sum_{d|n} a(d)b(n/d).$$

If $a(1) \neq 0$, there exists a unique Dirichlet inverse $b = a^{-1*}$ satisfying

$$(a * b)(n) = \delta_1(n).$$

Theorem 7 (Dirichlet–Wiener inversion). Assume the one-sided discrete dilation equation

$$G(\theta) = \sum_{k \geq 1} a(k)F(k\theta), \quad (28)$$

where $a(1) \neq 0$, $a \in \ell^1(\mathbb{N})$, and its Dirichlet inverse b belongs to $\ell^1(\mathbb{N})$. Suppose $F \in C_0(\mathbb{R})$ and the series in (28) converges absolutely and uniformly on compact subsets of $\mathbb{R}$. Then

$$F(\theta) = \sum_{n \geq 1} b(n)G(n\theta), \quad (29)$$

with absolute and locally uniform convergence. Consequently,

$$\Phi = \mathcal{F}^{-1}[F], \quad f(x) = e^{i\pi p \operatorname{sgn}(x)/2} |x|^{-p} \Phi(x) \quad (30)$$

for almost every $x \neq 0$.

Proof. Because $a, b \in \ell^1(\mathbb{N})$ and F is bounded, the double series

$$\sum_{n \geq 1} |b(n)| \sum_{k \geq 1} |a(k)| |F(kn\theta)|$$

is absolutely summable, uniformly for θ on compact sets. Tonelli's theorem therefore permits rearrangement:

$$\sum_{n \geq 1} b(n)G(n\theta) = \sum_{m \geq 1} \left(\sum_{n|m} b(n)a(m/n) \right) F(m\theta) = \sum_{m \geq 1} (b * a)(m) F(m\theta) = F(\theta).$$

□

Remark 5. The correct scaling in the exact discrete inverse is $n\theta$, not θ/n . That little scaling gremlin is easy to mishandle, so it is worth making explicit.

The terminology “Dirichlet–Wiener” is justified because the inversion mechanism combines multiplicative convolution inversion with a nonvanishing-symbol philosophy reminiscent of classical Wiener theory [11,12].

8. Recursive Triangular Inversion and Neumann Series

The operator equation also admits direct iterative inversion.

Proposition 1 (Triangular recursion). Assume (28) and $a(1) \neq 0$. Then formally

$$F(\theta) = \frac{1}{a(1)} G(\theta) - \sum_{m \geq 2} \frac{a(m)}{a(1)} F(m\theta). \quad (31)$$

If $F(\omega) \rightarrow 0$ as $|\omega| \rightarrow \infty$ and the tail is suitably summable, then (31) yields a practical large-to-small scale recursion.

Proof. Separate the $m = 1$ term from (28) and solve algebraically for $F(\theta)$. □

Theorem 8 (Neumann-series inversion). *Let X be a Banach function space on $\mathbb{R}$ such that each dilation operator $S_m : F \mapsto F(m \cdot)$ is bounded on X . Assume $a(1) \neq 0$ and*

$$\sum_{m \geq 2} |a(m)| \|S_m\|_{X \rightarrow X} < |a(1)|.$$

Define

$$(AF)(\theta) := \sum_{m \geq 2} a(m)F(m\theta).$$

Then $A \in \mathcal{B}(X)$ with $\|A\| < |a(1)|$, and the equation $(a(1)I + A)F = G$ has the unique solution

$$F = \frac{1}{a(1)} \sum_{j=0}^{\infty} \left(-\frac{A}{a(1)}\right)^j G, \quad (32)$$

with convergence in X .

Proof. Write (28) as $(a(1)I + A)F = G$. Since $\|A/a(1)\| < 1$, the inverse exists by the Neumann lemma:

$$(a(1)I + A)^{-1} = \frac{1}{a(1)} \sum_{j \geq 0} \left(-\frac{A}{a(1)}\right)^j.$$

Applying this to G gives (32). Standard Banach-space operator theory justifies the argument [9]. $\square$

9. Log-Scale Fourier Diagonalization and FFT Inversion

Assume $\theta > 0$ and define

$$\theta = e^t, \quad h(t) := G(e^t), \quad g_0(t) := F(e^t). \quad (33)$$

Then (28) becomes

$$h(t) = \sum_{k \geq 1} a(k)g_0(t + \log k). \quad (34)$$

Thus the dilation equation on $\mathbb{R}_+$ becomes a translation equation on the logarithmic line. This is the key structural move behind the FFT-ready inversion, and it puts the problem into a standard Fourier-multiplier framework [6,10].

Theorem 9 (Log-scale Fourier multiplier). *Assume (28) and let $f \in X_p^{\log}$. Suppose also that $a \in \ell^1(\mathbb{N})$. Then $h, g_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and (34) holds in $L^1 \cap L^2$. Taking Fourier transforms in t yields*

$$\widehat{h}(\xi) = A(\xi)\widehat{g_0}(\xi), \quad A(\xi) := \sum_{k \geq 1} a(k)e^{i\xi \log k}. \quad (35)$$

If $1/A \in L^\infty(\mathbb{R})$, then

$$g_0(t) = \mathcal{F}_t^{-1} \left[\frac{\widehat{h}(\xi)}{A(\xi)} \right](t), \quad F(e^t) = g_0(t). \quad (36)$$

Proof. Since $a \in \ell^1(\mathbb{N})$ and $g_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\sum_{k \geq 1} |a(k)| \|g_0(\cdot + \log k)\|_{L^1 \cap L^2} = \|g_0\|_{L^1 \cap L^2} \sum_{k \geq 1} |a(k)| < \infty.$$

Hence the series in (34) converges absolutely in $L^1 \cap L^2$, and termwise Fourier transformation is justified. Since translation becomes multiplication,

$$\mathcal{F}_t \{g_0(t + \log k)\}(\xi) = e^{i\xi \log k} \widehat{g_0}(\xi),$$

which yields (35). Division by $A(\xi)$ and inverse Fourier transformation give (36). $\square$

Remark 6. *The multiplier*

$$A(\xi) = \sum_{k \geq 1} a(k) e^{i\xi \log k}$$

is a generalized trigonometric series in the logarithmic frequencies $\log k$, and therefore has an almost-periodic flavor in the sense of Bohr [2]. That viewpoint is not needed for the proofs here, but it helps explain why zeros of A govern injectivity and conditioning.

Proposition 2 (Practical stability away from multiplier zeros). *Assume the hypotheses of the preceding theorem. Let $I \subset \mathbb{R}$ be measurable and suppose that*

$$|A(\xi)| \geq \eta > 0 \quad \text{for a.e. } \xi \in I.$$

For exact data $\hat{h}$ and noisy data $\hat{h}_\delta$, define the frequency-windowed reconstructions

$$g_{0,I} := \mathcal{F}_t^{-1} \left[1_I \frac{\hat{h}}{A} \right], \quad g_{0,I}^\delta := \mathcal{F}_t^{-1} \left[1_I \frac{\hat{h}_\delta}{A} \right].$$

Then

$$\|g_{0,I}^\delta - g_{0,I}\|_{L^2(\mathbb{R})} \leq \eta^{-1} \|1_I(\hat{h}_\delta - \hat{h})\|_{L^2(\mathbb{R})}.$$

In particular, the log-scale inversion is L^2 -stable on any frequency window on which the multiplier is bounded away from zero.

Proof. By Plancherel's theorem,

$$\|g_{0,I}^\delta - g_{0,I}\|_{L^2} = \left\| 1_I \frac{\hat{h}_\delta - \hat{h}}{A} \right\|_{L^2} \leq \left\| \frac{1_I}{A} \right\|_{L^\infty} \|1_I(\hat{h}_\delta - \hat{h})\|_{L^2} \leq \eta^{-1} \|1_I(\hat{h}_\delta - \hat{h})\|_{L^2}.$$

$\square$

10. Injectivity and Support Geometry

The operator viewpoint makes injectivity transparent.

Proposition 3 (Injectivity via symbol nonvanishing). *In any of the inversion frameworks above, injectivity of the forward map follows provided the relevant symbol has no zeros on the inversion set:*

- (a) $D_+(1-u) \neq 0$ for Mellin inversion in the one-sided case;
- (b) $D_c(1-u) \neq 0$ and $D_s(1-u) \neq 0$ for two-sided branchwise Mellin inversion;
- (c) $A(\xi) \neq 0$ for log-scale Fourier inversion;
- (d) Dirichlet-invertibility of a for discrete algebraic inversion.

Proof. Each assertion follows directly from inspection of the corresponding inversion formula. $\square$

The density and support geometry familiar from the entire-kernel MIT theory remain relevant here. Finite negative tails do not destroy the positive-index support set; they enlarge the spectral support by finitely many reflected modes. Therefore any completeness theory built on sufficiently rich positive support survives the finite-principal-part extension in a robust form.

Remark 7. *A finite principal part is strong enough to produce a genuine meromorphic extension and weak enough to preserve the algebraic dilation architecture without the technical instability of infinite negative tails or essential singularities. It is therefore the mathematically clean first post-MIT sequel.*

11. Worked Kernel Families

We record three model families illustrating the theory.

Example 1 (Exponential entire kernel). Take $g(\alpha + z) = e^z - 1$, so $c_k = 1/k!$ for $k \geq 1$. Then

$$a(k) = \frac{\beta^k}{2\pi k!}, \quad G(\theta) = \sum_{k \geq 1} \frac{\beta^k}{2\pi k!} F(k\theta).$$

The corresponding Mellin symbol is

$$D_+(s) = \sum_{k \geq 1} \frac{\beta^k}{k!} k^{-s}.$$

The factorial decay makes this model particularly friendly both analytically and numerically.

Example 2 (Simple meromorphic extension). Let

$$g(\alpha + z) = z^{-1} + e^z - 1.$$

Then

$$G(\theta) = \frac{\beta^{-1}}{2\pi} F(-\theta) + \sum_{k \geq 1} \frac{\beta^k}{2\pi k!} F(k\theta).$$

The branch symbols are

$$D_c(s) = \beta^{-1} + \sum_{k \geq 1} \frac{\beta^k}{k!} k^{-s}, \quad D_s(s) = -\beta^{-1} + \sum_{k \geq 1} \frac{\beta^k}{k!} k^{-s}.$$

Example 3 (Fully worked symbolic model). Let

$$g(\alpha + z) = z^{-1} + z, \quad \beta = 1.$$

Then

$$G(\theta) = \frac{1}{2\pi} (F(-\theta) + F(\theta)).$$

If Φ is even, then F is even and therefore

$$G(\theta) = \frac{1}{\pi} F(\theta), \quad F(\theta) = \pi G(\theta).$$

At the level of branch symbols,

$$D_c(s) = 2, \quad D_s(s) = 0.$$

Thus the transform is injective on the even subspace but not on the full space. This tiny example shows, with zero theatrical smoke, how branchwise symbol zeros encode loss of injectivity.

12. A Mini Numerical Experiment for Log-Scale Inversion

Consider the entire kernel

$$g(\alpha + z) = e^z - 1, \quad a(k) = \frac{\beta^k}{2\pi k!},$$

and the weighted signal

$$\Phi(x) = e^{-x^2}, \quad F(\omega) = \hat{\Phi}(\omega) = \sqrt{\pi} e^{-\omega^2/4}.$$

Then

$$G(\theta) = \sum_{k \geq 1} \frac{\beta^k}{2\pi k!} \sqrt{\pi} e^{-k^2 \theta^2 / 4}.$$

Sampling on the logarithmic grid $\theta = e^t$ gives

$$h(t) = G(e^t), \quad g_0(t) = F(e^t) = \sqrt{\pi} e^{-e^{2t} / 4}.$$

A practical numerical inversion proceeds as follows.

- (1) Sample $h(t_j)$ on a uniform grid in t .
- (2) Compute an FFT approximation of $\widehat{h}$.
- (3) Divide by

$$A(\xi) = \sum_{k \geq 1} \frac{\beta^k}{2\pi k!} e^{i\xi \log k},$$

truncated at a value of k for which the factorial tail is negligible.

- (4) Apply the inverse FFT to recover $g_0(t_j)$.
- (5) Reconstruct $F(e^{t_j})$ and then recover f by inverse Fourier inversion.

Because the coefficients decay factorially, the multiplier is smooth and rapidly convergent, and the experiment provides a clean test-bed for the log-scale inversion theory.

13. Discussion

The present paper is deliberately sharp rather than encyclopedic. It proves the exact dilation-operator form of the transform in the entire and finite-principal-part Laurent settings, proves Mellin diagonalization and branchwise inversion formulas, proves exact discrete Dirichlet–Wiener inversion with the correct scaling law, and proves the log-scale Fourier multiplier representation and associated FFT recovery. It also adds a practical stability estimate for windowed log-scale inversion away from zeros of the multiplier.

It does not attempt to settle the full theory of infinitely many negative Laurent modes, essential singularities, branch cuts, strip-analytic kernels, multivariate kernels, or operator-valued symbols. Those belong to later papers. The merit of the present paper is precisely that it stops where the mathematics is cleanest and strongest.

The natural next sequence in the program is also now clearer: Hardy-strip and Wiener–Mellin theories, operator-valued symbols, matrix and Hilbert-space-valued transforms, and then only later the wilder zoo of branch cuts and essential singularities.

14. Conclusion

The correct first post-MIT paper is not a vague broadening of examples. It is the theorem that the transform is a dilation-operator calculus and that its first meromorphic extension is a two-sided Laurent theory with several exact inversions. That is what has been proved here.

The principal message may be compressed into one line:

$$G = T_a(\widehat{\Phi}), \quad \text{and inversion means inverting the symbol of } T_a.$$

This formulation unifies Mellin contour inversion, Dirichlet–Wiener inversion, log-scale FFT inversion, recursive inversion, and Neumann-series inversion. It also shows why finite-principal-part meromorphic kernels are the correct next step after the MIT paper: they preserve the spectral skeleton while genuinely enlarging the transform class. In that sense, the paper sharpens the program rather than merely widening it.

References

1. M. Abu-Ghuwaleh, *The Master Integral Transform with Entire Kernels*, Mathematics **13** (2025), no. 21, Art. 3431.

2. H. Bohr, *Almost Periodic Functions*, Chelsea Publishing Co., New York, 1947.
3. P. Flajolet, X. Gourdon, and P. Dumas, Mellin transforms and asymptotics: harmonic sums, *Theoret. Comput. Sci.* **144** (1995), no. 1–2, 3–58.
4. G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge University Press, Cambridge, 1915.
5. H. Helson, *Dirichlet Series*, Henry Helson, Berkeley, CA, 2005.
6. Y. Katznelson, *An Introduction to Harmonic Analysis*, 3rd ed., Cambridge University Press, Cambridge, 2004.
7. H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2006.
8. R. B. Paris and D. Kaminski, *Asymptotics and Mellin–Barnes Integrals*, Encyclopedia of Mathematics and its Applications, vol. 85, Cambridge University Press, Cambridge, 2001.
9. W. Rudin, *Functional Analysis*, 2nd ed., McGraw–Hill, New York, 1991.
10. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 3rd ed., Chelsea Publishing Co., New York, 1986.
11. D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series, vol. 6, Princeton University Press, Princeton, NJ, 1946.
12. N. Wiener, Tauberian theorems, *Ann. of Math. (2)* **33** (1932), no. 1, 1–100.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.