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Article

Exponential Stability for a Degenerate/Singular Beam-Type Equation in Non-Divergence Form

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Abstract: The paper deals with the stability for a degenerate/singular beam equation in non-divergence form. In particular, we assume that the degeneracy and the singularity are at the same boundary point and we impose clamped conditions where the degeneracy occurs and dissipative conditions at the other endpoint. Using the energy method, we provide some conditions to obtain the stability for the considered problem.

Keywords: degenerate wave equation; singular potentials; stabilization; exponential decay

MSC 2020: 35L80; 35L81; 93D23; 93D15

1. Introduction

This paper is devoted to study the stability of a beam-type degenerate equation with a small singular perturbation through a linear boundary feedback. To be more precise, we consider the following problem:

$$\begin{cases} y_{tt}(t, x) + ay_{xxxx}(t, x) - \frac{\lambda}{d}y(t, x) = 0, & (t, x) \in (0, +\infty) \times (0, 1), \\ y(t, 0) = 0, \quad y_x(t, 0) = 0, & t > 0, \\ \beta y(t, 1) - y_{xxx}(t, 1) + y_t(t, 1) = 0, & t > 0, \\ \gamma y_x(t, 1) + y_{xx}(t, 1) + y_{tx}(t, 1) = 0, & t > 0, \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & x \in (0, 1), \end{cases} \quad (1.1)$$

where β and γ are non negative constants and the function a is such that $a(0) = 0$ and $a(x) > 0$ for all $x \in (0, 1]$. In particular, for the function a we consider two types of degeneracy according to the following definitions.

Definition 1.1. A function $g : [0, 1] \rightarrow \mathbb{R}$ is weakly degenerate at 0, (WD) for short, if $g \in C^0[0, 1] \cap C^1(0, 1]$ is such that $g(0) = 0, g > 0$ on $(0, 1]$ and, if

$$\sup_{x \in (0, 1]} \frac{x|g'(x)|}{g(x)} := K_g, \quad (1.2)$$

then $K_g \in (0, 1)$.

Definition 1.2. A function $g : [0, 1] \rightarrow \mathbb{R}$ is strongly degenerate at 0, (SD) for short, if $g \in C^1[0, 1]$ is such that $g(0) = 0, g > 0$ on $(0, 1]$ and in (1.2) we have $K_g \in [1, 2)$.

Roughly speaking, when $g(x) \sim x^K$, it is (WD) if $K \in (0, 1)$ and (SD) if $K \in [1, 2)$.

Problems similar to (1.1) are considered in several papers (see, for example, [3,4,10–13,23]). In particular, in [10,12] the following Euler-Bernoulli beam equation is considered

$$my_{tt}(t, x) + EIy_{xxxx}(t, x) = 0, \quad x \in (0, 1), \quad t > 0, \quad (1.3)$$

with clamped conditions at the left end

$$y(t, 0) = 0, \quad y_x(t, 0) = 0, \quad (1.4)$$

and with dissipative conditions at the right end

$$\begin{cases} -EIy_{xxx}(t, 1) + \mu_1 y_t(t, 1) = 0, & \mu_1 \geq 0, \\ EIy_{xx}(t, 1) + \mu_2 y_{tx}(t, 1) = 0, & \mu_2 \geq 0. \end{cases} \quad (1.5)$$

Here y is the vertical displacement, y_t is the velocity, y_x is the rotation, y_{tx} is the angular velocity, m is the mass density per unit length, EI is the flexural rigidity coefficient, $-EIy_{xx}$ is the bending moment and $-EIy_{xxx}$ is the shear. In particular, the boundary conditions (1.5) mean that the shear is proportional to the velocity and the bending moment is negatively proportional to the angular moment. Observe that if we consider $\beta = \gamma = 0$ in (1.1), then we have boundary conditions analogous to those in (1.5). Thus, the dissipative conditions at 1 are not surprising. We remark that the conditions $\beta, \gamma \geq 0$ are necessary to study the well posedness of the problem and to prove the equivalence among all the norms introduced in the paper and that are crucial to obtain the stability result.

Observe that in all the references above the equation is non-degenerate; however, there are some papers where the equation is degenerate in the sense that a *degenerate damping* appears in the equation of (1.3) (see, for example, [9,14,22]). The first paper where the equation is degenerate in the sense that the fourth order operator degenerates in a point as in (1.1) is [7]. However, to our knowledge, [5] is the first paper where the *stability* for (1.1) with $\lambda = 0$ is considered. On the other hand, for degenerate wave-equation we refer to [2] (see also the arxiv version of 2015), for problem in divergence form, and to [16], for problem in non-divergence form.

As far as we know, for beam-type equations admitting simultaneously degeneracy and singularity, only controllability problems have been faced (see the recent paper [1]), while nothing has been done for stability. For this reason, in this paper we focus on such a problem, proving that (1.1) permits boundary stabilization, provided that the singular term has a small coefficient (see Theorem 3.3 below). Hence, we may regard this result as a natural continuation of [5] and a perturbation of the related one in [1]. Clearly, the presence of the singular term $\frac{y}{d}$ introduces several difficulties with respect to [5], which let us treat only the case of a function d with weak degeneracy, according to the definition above. For a stability result for a degenerate/singular wave equation we refer to [19].

The paper is organized as follows: in Section 2 we give the functional setting and some preliminary results that we will use in the rest of the paper, together with the existence of solutions. In Section 3 we introduce the energy associated to a solution of the problem and, by a multiplier method, we show that it decays exponentially as time diverges. The last section is devoted to the conclusions and to some open problems.

2. Preliminary Results and Well-Posedness

In this section we introduce the functional setting needed to treat (1.1). However, here our assumptions are more general than those required to obtain the stability result in the next section.

We start assuming a very modest requirement.

Hypothesis 1. *The functions $a, d \in C^0[0, 1]$ are such that*

1. $a(0) = d(0) = 0, a, d > 0$ on $(0, 1]$,

2. there exist $K_a, K_d \in (0, 2)$ such that the functions

$$x \mapsto \frac{x^{K_a}}{a(x)} \quad (2.1)$$

and

$$x \mapsto \frac{x^{K_d}}{d(x)} \quad (2.2)$$

are nondecreasing in a right neighborhood of $x = 0$.

It is clear that, if Hypothesis 1 holds, then

$$\lim_{x \rightarrow 0} \frac{x^\gamma}{a(x)} = 0, \quad (2.3)$$

for all $\gamma > K_a$, and

$$\lim_{x \rightarrow 0} \frac{x^\gamma}{d(x)} = 0, \quad (2.4)$$

for all $\gamma > K_d$.

Let us remark that if a is (WD) or (SD), then (1.2) implies that (2.1) holds on the whole domain $(0, 1]$. Analogously for d .

In order to treat (1.1), let us introduce the following Hilbert spaces with the related inner products and norms given by:

$$L_{\frac{1}{a}}^2(0, 1) := \left\{ u \in L^2(0, 1) \mid \|u\|_{\frac{1}{a}} < \infty \right\},$$

$$\langle u, v \rangle_{\frac{1}{a}} := \int_0^1 uv \frac{1}{a} dx, \quad \|u\|_{\frac{1}{a}}^2 = \int_0^1 \frac{u^2}{a} dx,$$

for all $u, v \in L_{\frac{1}{a}}^2(0, 1)$;

$$H_{\frac{1}{a}}^i(0, 1) := L_{\frac{1}{a}}^2(0, 1) \cap H^i(0, 1), i = 1, 2,$$

$$\langle u, v \rangle_{i, \frac{1}{a}} := \langle u, v \rangle_{\frac{1}{a}} + \sum_{k=1}^i \int_0^1 u^{(k)}(x) v^{(k)}(x) dx$$

and

$$\|u\|_{H_{\frac{1}{a}}^i(0, 1)}^2 := \|u\|_{\frac{1}{a}}^2 + \sum_{k=1}^i \|u^{(k)}\|_{L^2(0, 1)}^2,$$

$\forall u, v \in H_{\frac{1}{a}}^i(0, 1), i = 1, 2$. In addition to the previous ones, we introduce the following important Hilbert spaces:

$$H_{\frac{1}{a}, 0}^1(0, 1) := \left\{ u \in H_{\frac{1}{a}}^1(0, 1) : u(0) = 0 \right\} \quad \text{and}$$

$$H_{\frac{1}{a}, 0}^2(0, 1) := \left\{ u \in H_{\frac{1}{a}, 0}^1(0, 1) \cap H^2(0, 1) : u'(0) = 0 \right\},$$

with the previous inner products $\langle \cdot, \cdot \rangle_{i, \frac{1}{a}}$ and norms $\|\cdot\|_{H_{\frac{1}{a}}^i(0, 1)}, i = 1, 2$. Now, consider the scalar product

$$\langle u, v \rangle_{i, \circ} := \int_0^1 u^{(i)}(x) v^{(i)}(x) dx$$

for all $u, v \in H_{\frac{1}{a}}^i(0, 1)$, which induces the norm

$$\|u\|_{i, \circ} := \|u^{(i)}\|_{L^2(0, 1)}, \quad \forall u \in H_{\frac{1}{a}}^i(0, 1),$$

$i = 1, 2$. Observe that, if a is continuous, $a(0) = 0$ and (2.1) is satisfied, then the norms $\|\cdot\|_{H_{\frac{1}{a}}^i(0,1)}$, $\|\cdot\|_i$ and $\|\cdot\|_{i,\circ}$ are equivalent in $H_{\frac{1}{a}}^i(0,1)$. Here

$$\|u\|_i^2 := \|u\|_{L_{\frac{1}{a}}^2(0,1)}^2 + \|u^{(i)}\|_{L^2(0,1)}^2, \quad \forall u \in H_{\frac{1}{a}}^i(0,1),$$

$i = 1, 2$ (see, e.g., [7]). Clearly, if $i = 1$, the previous equivalence is obviously satisfied. Indeed, $\|\cdot\|_{H_{\frac{1}{a}}^1(0,1)}$ and $\|\cdot\|_1$ coincide and, by [8, Proposition 2.6], one has that there exists $C > 0$ such that

$$\int_0^1 \frac{u^2}{a} dx \leq C \int_0^1 (u')^2 dx, \quad (2.5)$$

for all $u \in H_{\frac{1}{a}}^1(0,1)$. Let

$$C_{HP} \text{ be the best constant of (2.5).} \quad (2.6)$$

Now, assume $i = 2$ and fix $u \in H_{\frac{1}{a}}^2(0,1)$. Proceeding as for $i = 1$ and applying the classical Hardy's inequality to $z := u'$ (observe that $z \in H_{\frac{1}{a}}^1(0,1)$), we have

$$\int_0^1 (u')^2 dx \leq \int_0^1 \frac{z^2}{x^2} dx \leq 4 \int_0^1 (z')^2 dx = 4 \int_0^1 (u'')^2 dx = 4\|u\|_{2,\circ}^2.$$

Hence, $\|\cdot\|_{H_{\frac{1}{a}}^2(0,1)}$ and $\|\cdot\|_2$ are equivalent in $H_{\frac{1}{a}}^2(0,1)$ (actually they are equivalent in $H_{\frac{1}{a}}^2(0,1)$, see, e.g., [7]). Moreover, by the previous inequality,

$$\int_0^1 \frac{u^2}{a} dx \leq C_{HP} \int_0^1 (u')^2 dx \leq 4C_{HP}\|u\|_{2,\circ}^2 \quad (2.7)$$

and the thesis follows. In particular, $\|u\|_1^2 \leq (C_{HP} + 1)\|u\|_{1,\circ}^2$ for all $u \in H_{\frac{1}{a}}^1(0,1)$ and

$$\|u\|_2^2 \leq (4C_{HP} + 1)\|u\|_{2,\circ}^2 \quad (2.8)$$

for all $u \in H_{\frac{1}{a}}^2(0,1)$ (see [5, Proposition 2.1]).

As in [1, Proposition 2.3], one can prove the next result

Proposition 2.1. Assume Hypothesis 1 and take K_a, K_d such that $K_a + K_d \leq 2$. If $u \in H_{\frac{1}{a}}^2(0,1)$, then $\frac{u}{\sqrt{ad}} \in L^2(0,1)$ and there exists a positive constant $C > 0$ such that

$$\int_0^1 \frac{u^2(x)}{a(x)d(x)} dx \leq C \int_0^1 (u''(x))^2 dx. \quad (2.9)$$

Let

$$\tilde{C}_{HP} \text{ be the best constant of (2.9).} \quad (2.10)$$

As in [15,16] or in [17, Chapter V], we assume the next hypothesis:

Hypothesis 2. The constant $\lambda \in \mathbb{R}$ is such that $\lambda \neq 0$ and

$$\lambda < \frac{1}{\tilde{C}_{HP}}. \quad (2.11)$$

Observe that the case $\lambda = 0$ is already considered in [5]. Thus it is not restrictive to assume $\lambda \neq 0$.

Moreover, if $\lambda \in \left(0, \frac{1}{\tilde{C}_{HP}}\right)$, we can take $\epsilon \in (0, 1)$ such that

$$\lambda = \frac{1 - \epsilon}{\tilde{C}_{HP}} > 0. \quad (2.12)$$

Hence, as a consequence of Proposition 2.1, one has the next estimate (see [1, Proposition 2.4]).

Proposition 2.2. Assume Hypothesis 1 and $\lambda \in \left(0, \frac{1}{\tilde{C}_{HP}}\right)$. If $u \in H_{\frac{1}{a},0}^2(0,1)$, then

$$\int_0^1 (u''(x))^2 dx - \lambda \int_0^1 \frac{u^2(x)}{a(x)d(x)} dx \geq \epsilon \int_0^1 (u''(x))^2 dx.$$

Under Hypotheses 1 and 2, one can consider in $H_{\frac{1}{a},0}^2(0,1)$ also the product

$$\langle u, v \rangle_{2,\sim} := \langle u, v \rangle_{2,\circ} - \lambda \int_0^1 \frac{uv}{ad} dx,$$

which induces the norm

$$\|u\|_{2,\sim}^2 = \|u\|_{2,\circ}^2 - \lambda \int_0^1 \frac{u^2}{ad} dx.$$

By Propositions 2.1 and 2.2, one can prove the following equivalence.

Corollary 2.1. Assume Hypotheses 1 and 2 and $K_a + K_d \leq 2$. Then the norms $\|\cdot\|_{H_{\frac{1}{a}}^2(0,1)}$, $\|\cdot\|_{2,\circ}$, $\|\cdot\|_{2,\sim}$ and $\|\cdot\|_{2,\sim}$ are equivalent in $H_{\frac{1}{a}}^2(0,1)$.

In order to study the well-posedness of (1.1), we introduce the operator $A : D(A) \subset L_{\frac{1}{a}}^2(0,1) \rightarrow L_{\frac{1}{a}}^2(0,1)$ by $Au := au''''$, for all $u \in D(A) := \left\{u \in H_{\frac{1}{a},0}^2(0,1) : au'''' \in L_{\frac{1}{a}}^2(0,1)\right\}$, where the next the Gauss-Green formula holds

$$\int_0^1 u'''' v dx = u'''(1)v(1) - u''(1)v'(1) + \int_0^1 u'' v'' dx \quad (2.13)$$

for all $(u, v) \in D(A) \times H_{\frac{1}{a},0}^2(0,1)$ (see [5]). Moreover, consider

$$A_\lambda u := Au - \frac{\lambda}{d} u, \quad \forall u \in D(A_\lambda),$$

where

$$D(A_\lambda) := \left\{u \in H_{\frac{1}{a},0}^2(0,1) \mid A_\lambda u \in L_{\frac{1}{a}}^2(0,1)\right\}. \quad (2.14)$$

Observe that if $u \in H_{\frac{1}{a},0}^2(0,1)$ and $K_a + 2K_d \leq 2$ one has that $\frac{u}{d} \in L_{\frac{1}{a}}^2(0,1)$; hence $u \in D(A_\lambda)$ if and only if $u \in D(A)$, i.e. $D(A_\lambda) = D(A)$ (for more details we refer to [1]). For this reason in the following we assume the next assumption.

Hypothesis 3. Assume Hypothesis 1 and $K_a + 2K_d \leq 2$.

Under this assumption, it is clear that d cannot be (SD). On the other hand, a can be (SD), but in this case K_d has to be very small.

Finally, to prove the well posedness of (1.1), we need to introduce the last Hilbert space $\mathcal{H}_0 := H_{\frac{1}{a},0}^2(0,1) \times L_{\frac{1}{a}}^2(0,1)$, with inner product and norm given by

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle_{\mathcal{H}_0} := \langle u, \tilde{u} \rangle_{2,\circ} + \langle v, \tilde{v} \rangle_{\frac{1}{a}} + \beta u(1)\tilde{u}(1) + \gamma u'(1)\tilde{u}'(1)$$

and

$$\|(u, v)\|_{\mathcal{H}_0}^2 := \|u''\|_{L^2(0,1)}^2 + \|v\|_{\frac{1}{a}}^2 + \beta u^2(1) + \gamma (u'(1))^2$$

for every $(u, v), (\tilde{u}, \tilde{v}) \in \mathcal{H}_0$, where $\beta, \gamma \geq 0$, and the matrix operator $\mathcal{A}_1 : D(\mathcal{A}) \subset \mathcal{H}_0 \rightarrow \mathcal{H}_0$ given by

$$\mathcal{A} := \begin{pmatrix} 0 & Id \\ -A_\lambda & 0 \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \{(u, v) \in D(A) \times H_{\frac{1}{a},0}^2(0,1) : \beta u(1) - u'''(1) + v(1) = 0, \\ \gamma u'(1) + u''(1) + v'(1) = 0\}.$$

Thanks to (2.13) one can prove the next theorem that contains the main properties of the operator $(\mathcal{A}, D(\mathcal{A}))$. Since the proof is similar to the one of [6] or [18], we omit it.

Theorem 2.1. Assume a (WD) or (SD). Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is non positive with dense domain and generates a contraction semigroup $(S(t))_{t \geq 0}$.

Thanks to the previous theorem, one has the next result, that can be proved as in [1, Theorem 2.7].

Theorem 2.2. Hypotheses 2 and 3 hold. If $(y_0, y_1) \in H_{\frac{1}{a},0}^2(0,1) \times L_{\frac{1}{a}}^2(0,1)$, then there exists a unique mild solution

$$y \in C^1([0, +\infty); L_{\frac{1}{a}}^2(0,1)) \cap C([0, +\infty); H_{\frac{1}{a},0}^2(0,1))$$

of (1.1) which depends continuously on the initial data. In addition, if $(y_0, y_1) \in D(\mathcal{A}_1)$, then the solution y is classical, in the sense that

$$y \in C^2([0, +\infty); L_{\frac{1}{a}}^2(0,1)) \cap C^1([0, +\infty); H_{\frac{1}{a},0}^2(0,1)) \cap C([0, +\infty); D(\mathcal{A}))$$

and the equation of (1.1) holds for all $t \geq 0$.

Remark 1. Due to the reversibility in time of the equation, solutions exist with the same regularity for $t < 0$. We will use this fact in the proof of the controllability result, by considering a backward problem whose final time data will be transformed in initial ones: this is the reason for the notation of the initial data in problem (1.1).

The last important result of this section is given by the next proposition. Let us start with

Hypothesis 4. Assume a and d (WD) or (SD) with $K_a + 2K_d \leq 2$, $\lambda \neq 0$ with $\lambda < \frac{1}{C_{HP}}$ and $\beta, \gamma \geq 0$.

Proposition 2.3. Assume Hypothesis 4 and define

$$|||z|||^2 := \int_0^1 (z'')^2 dx - \lambda \int_0^1 \frac{z^2}{ad} dx + \beta z^2(1) + \gamma (z'(1))^2$$

for all $z \in H_{\frac{1}{a},0}^2(0,1)$. Then the norms $||| \cdot |||$ and $\|\cdot\|_{2,\circ}$ are equivalent in $H_{\frac{1}{a},0}^2(0,1)$. Moreover, for every $\rho, \mu \in \mathbb{R}$, the variational problem

$$\int_0^1 z'' \varphi'' dx - \lambda \int_0^1 \frac{z \varphi}{ad} dx + \beta z(1) \varphi(1) + \gamma z'(1) \varphi'(1) = \rho \varphi(1) + \mu \varphi'(1), \quad \forall \varphi \in H_{\frac{1}{a},0}^2(0,1),$$

admits a unique solution $z \in H_{\frac{1}{a},0}^2(0,1)$ which satisfies the estimates

$$\|z\|_{L_{\frac{1}{a}}^2(0,1)}^2 \leq \frac{4C_{HP}}{C_e} (|\rho| + |\mu|)^2, \quad \text{and} \quad |||z|||^2 \leq (|\rho| + |\mu|)^2, \quad (2.15)$$

where

$$C_e := \begin{cases} 1, & \lambda < 0, \\ \epsilon, & \lambda > 0. \end{cases} \quad (2.16)$$

In addition $z \in D(A_{1,\lambda})$ and solves

$$\begin{cases} A_{1,\lambda}z = 0, \\ \beta z(1) - z'''(1) = \rho, \\ \gamma z'(1) + z''(1) = \mu. \end{cases} \quad (2.17)$$

Proof. As a first step observe that for all $z \in H_{\frac{1}{a},0}^2(0,1)$ one has

$$|z'(x)| = \left| \int_0^x z''(t) dt \right| \leq \|z''\|_{L^2(0,1)} = \|z\|_{2,\circ}, \quad (2.18)$$

and

$$|z(x)| = \left| \int_0^x z'(t) dt \right| = \left| \int_0^x \int_0^t z''(\tau) d\tau dt \right| \leq \|z''\|_{L^2(0,1)} = \|z\|_{2,\circ} \quad (2.19)$$

for all $x \in [0,1]$. Thus, $||| \cdot |||$ and $\|\cdot\|_{2,\circ}$ are equivalent. Indeed for all $z \in H_{\frac{1}{a},0}^2(0,1)$, if $\lambda < 0$, one has immediately that

$$\|z\|_{2,\circ}^2 = \|z''\|_{L^2(0,1)}^2 \leq |||z|||^2. \quad (2.20)$$

If $\lambda \in \left(0, \frac{1}{\tilde{C}_{HP}}\right)$, by Proposition 2.2, one has

$$\int_0^1 (z''(x))^2 dx - \lambda \int_0^1 \frac{z^2(x)}{a(x)d(x)} dx \geq \epsilon \int_0^1 (z''(x))^2 dx. \quad (2.21)$$

for all $z \in H_{\frac{1}{\sigma},0}^2(0,1)$, and so

$$\|z\|_{2,\circ}^2 \leq \frac{1}{\epsilon} |||z|||^2, \quad (2.22)$$

for all $z \in H_{\frac{1}{\sigma},0}^2(0,1)$. In conclusion, we have

$$\|z\|_{2,\circ}^2 \leq \frac{1}{C_\epsilon} |||z|||^2. \quad (2.23)$$

Now, we prove that there exists $C > 0$ such that

$$|||z|||^2 \leq C \|z\|_{2,\circ}^2,$$

for all $z \in H_{\frac{1}{\sigma},0}^2(0,1)$. Clearly, (2.19) and (2.18) imply $\beta z^2(1) \leq \beta \|z\|_{2,\circ}^2$ and $\gamma(z'(1))^2 \leq \gamma \|z\|_{2,\circ}^2$, respectively; hence, if $\lambda > 0$, one has immediately $|||z|||^2 \leq (1 + \beta + \gamma) \|z\|_{2,\circ}^2$; if $\lambda < 0$, by (2.9), then

$$|||z|||^2 \leq (1 - \lambda \tilde{C}_{HP} + \beta + \gamma) \|z\|_{2,\circ}^2.$$

In any case, the claim holds.

Now, consider the bilinear and symmetric form $\Lambda : H_{\frac{1}{a},0}^2(0,1) \times H_{\frac{1}{a},0}^2(0,1) \rightarrow \mathbb{R}$ such that

$$\Lambda(z, \phi) := \int_0^1 z'' \phi'' dx - \lambda \int_0^1 \frac{z\phi}{ad} dx + \beta z(1)\phi(1) + \gamma z'(1)\phi'(1).$$

As in [18] or in [19], one can easily prove that Λ is coercive and continuous. Now, consider the linear functional

$$\mathcal{L}(\phi) := \rho\phi(1) + \mu\phi'(1),$$

with $\varphi \in H_{\frac{1}{a},0}^2(0,1)$ and $\rho, \mu \in \mathbb{R}$. Clearly, $\mathcal{L}$ is continuous and linear. Thus, by the Lax-Milgram Theorem, there exists a unique solution $z \in H_{\frac{1}{a},0}^2(0,1)$ of

$$\Lambda(z, \varphi) = \mathcal{L}(\varphi) \quad (2.24)$$

for all $\varphi \in H_{\frac{1}{a},0}^2(0,1)$. In particular,

$$|||z|||^2 = \Lambda(z, z) = \int_0^1 (z'')^2 dx - \lambda \int_0^1 \frac{z^2}{ad} dx + \beta z^2(1) + \gamma (z'(1))^2 = \rho z(1) + \mu z'(1). \quad (2.25)$$

Concerning the other estimates, by (2.18)-(2.20) and (2.25), we have $|||z|||^2 = \rho z(1) + \mu z'(1) \leq (|\rho| + |\mu|) |||z|||$; thus

$$|||z||| \leq |\rho| + |\mu|. \quad (2.26)$$

Moreover, by the equivalence of the norms in $H_{\frac{1}{a},0}^2(0,1)$, Proposition 2.2 and by (2.7), one has

$$|||z|||^2 = \|z\|_{2,\circ}^2 - \lambda \int_0^1 \frac{z^2}{ad} dx + \beta z^2(1) + \gamma (z'(1))^2 \geq C_\epsilon \|z\|_{2,\circ}^2 \geq \frac{C_\epsilon}{4C_{HP}} \|z\|_{L_{\frac{1}{a}}^2(0,1)}^2,$$

where C_ϵ is as in (2.16). Thus, by (2.26), $\|z\|_{L_{\frac{1}{a}}^2(0,1)}^2 \leq \frac{4C_{HP}}{C_\epsilon} |||z|||^2 \leq \frac{4C_{HP}}{C_\epsilon} (|\rho| + |\mu|)^2$.

Now, we will prove that z belongs to $D(A)$ and solves (2.17). To this aim, consider again (2.24); clearly, it holds for every $\varphi \in C_c^\infty(0,1)$, so that $\int_0^1 z'' \varphi'' dx - \lambda \int_0^1 \frac{z\varphi}{ad} dx = 0$ for all $\varphi \in C_c^\infty(0,1)$. Thus $z'''' = \lambda \frac{z}{ad}$ a.e. in $(0,1)$ (see, e.g., [20, Lemma 1.2.1]) and so $az'''' - \lambda \frac{z}{d} = 0$ a.e. in $(0,1)$, in particular $A_\lambda z = 0 \in L_{\frac{1}{a}}^2(0,1)$; this implies that $z \in D(A)$.

Now, coming back to (2.24) and using (2.13) and the fact that $A_\lambda z = 0$, we have

$$\begin{aligned} \int_0^1 z'' \varphi'' dx - \lambda \int_0^1 \frac{z\varphi}{ad} dx + \beta z(1)\varphi(1) + \gamma z'(1)\varphi'(1) &= \rho \varphi(1) + \mu \varphi'(1) \\ \iff -z'''(1)\varphi(1) + (z''\varphi')(1) + \beta z(1)\varphi(1) + \gamma (z'\varphi')(1) &= \rho \varphi(1) + \mu \varphi'(1) \end{aligned}$$

for all $\varphi \in H_{\frac{1}{a},0}^2(0,1)$. Thus, $-z'''(1) + \beta z(1) = \rho$ and $\gamma z'(1) + z''(1) = \mu$, that is z solves (2.17). $\square$

3. Energy Estimates and Exponential Stability

In this section we prove the main result of the paper. In particular, proving some estimates of the energy associated (1.1), we obtain the exponential stability.

To begin with, we give the next definition.

Definition 3.1. For a mild solution y of (1.1) we define its energy as the continuous function

$$E_y(t) := \frac{1}{2} \int_0^1 \left(\frac{y_t^2(t,x)}{a(x)} + y_{xx}^2(t,x) - \frac{\lambda}{ad} y^2(t,x) \right) dx + \frac{\beta}{2} y^2(t,1) + \frac{\gamma}{2} y_x^2(t,1), \quad \forall t \geq 0. \quad (3.1)$$

Recalling that $\beta, \gamma \geq 0$, one has that if y is a mild solution and if $\beta, \gamma \neq 0$, then

$$y^2(t,1) \leq \frac{2}{\beta} E_y(t) \quad \text{and} \quad y_x^2(t,1) \leq \frac{2}{\gamma} E_y(t);$$

on the other hand, thanks to (2.19), (2.18) and (2.20), for all $\beta \geq 0$ and $\gamma \geq 0$,

$$y^2(t,1) \leq \frac{2}{C_\epsilon} E_y(t) \quad \text{and} \quad y_x^2(t,1) \leq \frac{2}{C_\epsilon} E_y(t),$$

where C_ϵ is as in (2.16). Thus, we have

$$y^2(t, 1) \leq C_\beta E_y(t) \quad \text{and} \quad y_x^2(t, 1) \leq C_\gamma E_y(t), \quad (3.2)$$

$$\text{where } C_\beta := \begin{cases} 2 \min\left\{\frac{1}{C_\epsilon}, \frac{1}{\beta}\right\}, & \beta \neq 0, \\ \frac{2}{C_\epsilon}, & \beta = 0 \end{cases} \quad \text{and } C_\gamma := \begin{cases} 2 \min\left\{\frac{1}{C_\epsilon}, \frac{1}{\gamma}\right\}, & \gamma \neq 0, \\ \frac{2}{C_\epsilon}, & \gamma = 0. \end{cases}$$

Observe that if $\beta > 1$, being $\frac{1}{\beta} < 1$ and $\frac{1}{C_\epsilon} \geq 1$ (recall that $\epsilon \in (0, 1)$), one has that $\min\left\{\frac{1}{C_\epsilon}, \frac{1}{\beta}\right\} = \frac{1}{\beta}$. Analogously for γ .

As in [6, Theorem 3.1], it is possible to prove that the energy is a non-increasing function.

Theorem 3.1. Assume Hypothesis 4 and let y be a classical solution of (1.1). Then the energy is non-increasing. In particular,

$$\frac{dE_y(t)}{dt} = -y_t^2(t, 1) - y_{tx}^2(t, 1), \quad \forall t \geq 0.$$

Actually, one can prove that the previous monotonicity result holds also under weaker assumptions on the functions a and d .

Theorem 3.2. Assume Hypothesis 4. Fixed $T > 0$, if y is a classical solution of (1.1), then

$$\begin{aligned} 0 &= 2 \int_0^1 \left[y_t \frac{x}{a} y_x \right]_{t=s}^{t=T} dx - \frac{1}{a(1)} \int_s^T y_t^2(t, 1) dt + \int_{Q_s} \frac{y_t^2}{a} \left(1 - \frac{xa'}{a} \right) dx dt - \frac{\lambda}{a(1)d(1)} \int_0^T y^2(t, 1) dt \\ &+ 3 \int_{Q_s} y_{xx}^2 dx dt + 2\beta \int_s^T y_x(t, 1) y(t, 1) dt + 2 \int_s^T y_x(t, 1) y_t(t, 1) dt \\ &+ 2\gamma \int_s^T y_x^2(t, 1) dt + 2 \int_s^T y_x(t, 1) y_{tx}(t, 1) dt - \int_s^T y_{xx}^2(t, 1) dt \\ &+ \lambda \int_{Q_s} \left(1 - \frac{xa'}{a} - \frac{xd'}{d} \right) \frac{y^2}{ad} dx dt, \end{aligned} \quad (3.3)$$

for every $0 < s < T$. Here $Q_s := (s, T) \times (0, 1)$.

Proof. Since some computations are similar to the ones of [6, Proposition 4.7], we sketch them. Fix $s \in (0, T)$. Multiplying the equation in (1.1) by $\frac{xy_x}{a}$ and integrating over Q_s , we have

$$0 = \int_{Q_s} y_{tt} \frac{xy_x}{a} dx dt + \int_{Q_s} ay_{xxxx} \frac{xy_x}{a} dx dt - \lambda \int_{Q_s} \frac{xy y_x}{ad} dx dt. \quad (3.4)$$

As in [6], one has that

$$\begin{aligned} &\int_{Q_s} y_{tt} \frac{xy_x}{a} dx dt + \int_{Q_s} ay_{xxxx} \frac{xy_x}{a} dx dt \\ &= \int_0^1 \left[y_t \frac{xy_x}{a} \right]_{t=s}^{t=T} dx - \frac{1}{2a(1)} \int_s^T y_t^2(t, 1) dt + \frac{1}{2} \int_{Q_s} \frac{y_t^2}{a} \left(1 - \frac{xa'}{a} \right) dx dt \\ &+ \frac{3}{2} \int_{Q_s} y_{xx}^2 dx dt + \beta \int_s^T y_x(t, 1) y(t, 1) dt + \int_s^T y_x(t, 1) y_t(t, 1) dt \\ &+ \gamma \int_s^T y_x^2(t, 1) dt + \int_s^T y_x(t, 1) y_{tx}(t, 1) dt - \frac{1}{2} \int_s^T y_{xx}^2(t, 1) dt. \end{aligned}$$

Hence it remains to compute $-\lambda \int_{Q_s} \frac{xyy_x}{ad} dx dt$. As in [19], one has

$$\begin{aligned} -\lambda \int_{Q_s} \frac{xyy_x}{ad} dx dt &= -\frac{\lambda}{2} \int_s^T \left[\frac{xy^2}{ad} \right]_{x=0}^{x=1} dt + \frac{\lambda}{2} \int_{Q_s} \left(\frac{ad - x(a'd + ad')}{(ad)^2} \right) y^2 dx dt \\ &= -\frac{\lambda}{2} \int_s^T \left[\frac{xy^2}{ad} \right]_{x=0}^{x=1} dt + \frac{\lambda}{2} \int_{Q_s} \left(1 - \frac{xa'}{a} - \frac{xd'}{d} \right) \frac{y^2}{ad} dx dt. \end{aligned}$$

By [19, Lemma 1]

$$\lambda \int_0^T \left[y^2 \frac{x}{ad} \right]_{x=0}^{x=1} = \frac{\lambda}{a(1)d(1)} \int_0^T y^2(t, 1) dt$$

and the thesis follows. $\square$

As a consequence of the previous equality, we have the next relation.

Proposition 3.1. Assume Hypothesis 4. If y is a classical solution of (1.1), then for every $0 < s < T$ we have

$$\int_{Q_s} \frac{y_t^2}{a} \left(\frac{K_a}{2} + 1 - \frac{xa'}{a} \right) dx dt + \int_{Q_s} y_{xx}^2 \left(3 - \frac{K_a}{2} \right) dx dt + \lambda \int_{Q_s} \left(1 - \frac{xa'}{a} - \frac{xd'}{d} + \frac{K_a}{2} \right) \frac{y^2}{ad} dx dt = (B.T.), \quad (3.5)$$

where

$$\begin{aligned} (B.T.) &= \frac{K_a}{2} \int_0^1 \left[\frac{yy_t}{a} \right]_{t=s}^{t=T} dx - 2 \int_0^1 \left[y_t \frac{x}{a} y_x \right]_{t=s}^{t=T} dx + \frac{K_a \beta}{2} \int_s^T y^2(t, 1) dt \\ &\quad + \frac{K_a}{2} \int_s^T y(t, 1) y_t(t, 1) dt + \gamma \left(\frac{K_a}{2} - 2 \right) \int_s^T y_x^2(t, 1) dt \\ &\quad + \left(\frac{K_a}{2} - 2 \right) \int_s^T y_x(t, 1) y_{tx}(t, 1) dt + \int_s^T \frac{y_t^2(t, 1)}{a(1)} dt - 2\beta \int_s^T y_x(t, 1) y(t, 1) dt \\ &\quad - 2 \int_s^T y_x(t, 1) y_t(t, 1) dt + \int_s^T y_{xx}^2(t, 1) dt + \frac{\lambda}{a(1)d(1)} \int_s^T y^2(t, 1) dt. \end{aligned}$$

Proof. Let y be a classical solution of (1.1) and fix $s \in (0, T)$. Multiplying the equation in (1.1) by $\frac{y}{a}$, integrating by parts over Q_s and using (2.13), we obtain

$$\begin{aligned} 0 &= \int_0^1 \left[y_t \frac{y}{a} \right]_{t=s}^{t=T} dx - \int_{Q_s} \frac{y_t^2}{a} dx dt + \int_s^T (yy_{xxx})(t, 1) dt \\ &\quad - \int_s^T (y_x y_{xx})(t, 1) dt + \int_{Q_s} y_{xx}^2 dx dt - \lambda \int_{Q_s} \frac{y^2}{ad} dx dt. \end{aligned} \quad (3.6)$$

Obviously, all the previous integrals make sense and multiplying (3.6) by $\frac{K_a}{2}$, one has

$$\begin{aligned} 0 &= \frac{K_a}{2} \int_{Q_s} \left(-\frac{y_t^2}{a} + y_{xx}^2 - \lambda \frac{y^2}{ad} \right) dx dt + \frac{K_a}{2} \int_0^1 \left[y_t \frac{y}{a} \right]_{t=s}^{t=T} dx \\ &\quad + \frac{K_a}{2} \int_s^T y(t, 1) y_{xx}(t, 1) dt - \frac{K_a}{2} \int_s^T y_x(t, 1) y_{xx}(t, 1) dt. \end{aligned} \quad (3.7)$$

By summing (3.3) and (3.7) and using the boundary conditions at 1, we get the thesis. $\square$

An immediate consequence of (3.5) is the next result. However, to prove it we assume an additional hypothesis on functions a and d .

Hypothesis 5. Assume a and d (WD) or (SD) with $K_a + 2K_d < 2$, $\lambda \neq 0$ with $\lambda < \frac{1}{\bar{C}_{HP}}$ and $\beta, \gamma \geq 0$.

Observe that this hypothesis is more restrictive than Hypothesis 4, indeed in Hypothesis 5 we exclude the case $K_a + 2K_d = 2$. In fact, as we can see already from the next result, the condition $K_a + 2K_d < 2$ is important for the technique used in the following.

Proposition 3.2. Assume Hypothesis 5 and let y be a classical solution of (1.1). Then, for every $0 < s < T$ and for all $\varepsilon_0 \in (0, 2 - K_a - 2K_d)$, one has

$$\begin{aligned} \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda \frac{y^2}{ad} \right) dx dt &\leq \left(4\vartheta + \varrho + \frac{C_\gamma}{2} \left(2 - \frac{K_a}{2} \right) \right) E_y(s) \\ &+ \left(\frac{K_a}{4} + \beta + \frac{K_a\beta}{2} + \frac{\lambda}{a(1)d(1)} \right) \int_s^T y^2(t, 1) dt + (\beta + 1 + 2\gamma^2) \int_s^T y_x^2(t, 1) dt, \end{aligned}$$

if $\lambda > 0$, and

$$\begin{aligned} \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda \frac{y^2}{ad} \right) dx dt &\leq \left(4\vartheta + \varrho + \frac{C_\gamma}{2} \left(2 - \frac{K_a}{2} \right) \right) E_y(s) \\ &+ \left(\frac{K_a}{4} + \beta + \frac{K_a\beta}{2} \right) \int_s^T y^2(t, 1) dt + (\beta + 1 + 2\gamma^2) \int_s^T y_x^2(t, 1) dt \\ &- 4\lambda \tilde{C}_{HP} \left(1 + \frac{3}{2} K_a + K_d \right) \int_s^T E_y(t) dt, \end{aligned}$$

if $\lambda < 0$.

$$\text{Here } \vartheta := \max \left\{ \frac{1}{\varepsilon} \left(\frac{4}{a(1)} + K_a C_{HP} \right), 1 + \frac{K_a}{4} \right\} \text{ and } \varrho := \max \left\{ 2, \frac{K_a}{4} + 1 + \frac{1}{a(1)} \right\}.$$

Proof. By assumption we can take $\varepsilon_0 \in (0, 2 - K_a - 2K_d)$; thus

$$\begin{aligned} 1 - \frac{xa'}{a} + \frac{K_a}{2} &> \frac{\varepsilon_0}{2}, \\ 3 - \frac{K_a}{2} &> \frac{\varepsilon_0}{2} \\ 1 - \frac{xa'}{a} - \frac{xd'}{d} + \frac{K_a}{2} &\geq \frac{\varepsilon_0}{2}. \end{aligned} \tag{3.8}$$

Now, we distinguish between the case $\lambda > 0$ and $\lambda < 0$.

Case $\lambda > 0$.

In this case, the distributed terms in (3.5) can be estimated from below in the following way:

$$\begin{aligned} \int_{Q_s} \frac{y_t^2}{a} \left(\frac{K_a}{2} + 1 - \frac{xa'}{a} \right) dx dt + \int_{Q_s} y_{xx}^2 \left(3 - \frac{K_a}{2} \right) dx dt + \lambda \int_{Q_s} \left(1 - \frac{xa'}{a} - \frac{xd'}{d} + \frac{K_a}{2} \right) \frac{y^2}{ad} dx dt \\ \geq \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{a} + y_{xx}^2 + \lambda \frac{y^2}{ad} \right) dx dt \geq \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda \frac{y^2}{ad} \right) dx dt. \end{aligned} \tag{3.9}$$

Now, we estimate the boundary terms in (3.5) from above. First of all consider the integral $\int_0^1 \left(-2y_t \frac{x}{a} y_x + \frac{K_a}{2} \frac{yy_t}{a} \right) (\tau, x) dx$ for all $\tau \in [s, T]$. Using the fact that $\frac{x^2}{a(x)} \leq \frac{1}{a(1)}$, together with the classical Hardy's inequality, (2.6) and proceeding as in [5, Proposition 3.3], one has

$$\begin{aligned} \int_0^1 \left(-2y_t \frac{x}{a} y_x + \frac{K_a}{2} \frac{yy_t}{a} \right) (\tau, x) dx &\leq \\ &\leq \frac{4}{a(1)} \int_0^1 y_{xx}^2(\tau, x) dx + \left(1 + \frac{K_a}{4} \right) \int_0^1 \frac{y_t^2}{a}(\tau, x) dx + K_a C_{HP} \int_0^1 y_{xx}^2(\tau, x) dx. \end{aligned}$$

Hence, by Proposition 2.2,

$$\begin{aligned}
 & \int_0^1 \left(-2y_t \frac{x}{a} y_x + \frac{K_a}{2} \frac{y y_t}{a} \right) (\tau, x) dx \\
 & \leq \left(\frac{4}{a(1)} + K_a C_{HP} \right) \int_0^1 y_{xx}^2(\tau, x) dx + \left(1 + \frac{K_a}{4} \right) \int_0^1 \frac{y_t^2}{a}(\tau, x) dx \\
 & \leq \frac{1}{\epsilon} \left(\frac{4}{a(1)} + K_a C_{HP} \right) \left(\int_0^1 y_{xx}^2(\tau, x) dx - \lambda \int_0^1 \frac{y^2}{ad}(\tau, x) dx \right) + \left(1 + \frac{K_a}{4} \right) \int_0^1 \frac{y_t^2}{a}(\tau, x) dx \\
 & \leq 2 \max \left\{ \frac{1}{\epsilon} \left(\frac{4}{a(1)} + K_a C_{HP} \right), 1 + \frac{K_a}{4} \right\} E_y(\tau).
 \end{aligned}$$

for all $\tau \in [s, T]$. Hence, since the energy is non-increasing,

$$\int_0^1 \left[-2y_t \frac{x}{a} y_x + \frac{K_a}{2} \frac{y y_t}{a} \right]_{t=s}^{t=T} dx \leq 4 \max \left\{ \frac{1}{\epsilon} \left(\frac{4}{a(1)} + K_a C_{HP} \right), 1 + \frac{K_a}{4} \right\} E_y(s). \quad (3.10)$$

Now, by (3.2) and the fact that $K_a < 2$, we have

$$\begin{aligned}
 & \gamma \left(\frac{K_a}{2} - 2 \right) \int_s^T y_x^2(t, 1) dt + \left(\frac{K_a}{2} - 2 \right) \int_s^T y_x(t, 1) y_{tx}(t, 1) dt \\
 & \leq \left(\frac{K_a}{2} - 2 \right) \frac{1}{2} \int_s^T (y_x(t, 1)^2)_t dt = \left(\frac{K_a}{2} - 2 \right) \frac{1}{2} (y_x^2(T, 1) - y_x^2(s, 1)) \\
 & \leq \left(2 - \frac{K_a}{2} \right) \frac{1}{2} y_x^2(s, 1) \leq \left(2 - \frac{K_a}{2} \right) \frac{C_\gamma}{2} E_y(s).
 \end{aligned} \quad (3.11)$$

Obviously

$$\frac{K_a}{2} \int_s^T y(t, 1) y_t(t, 1) dt \leq \frac{K_a}{4} \int_s^T y^2(t, 1) dt + \frac{K_a}{4} \int_s^T y_t^2(t, 1) dt, \quad (3.12)$$

$$-\beta \int_s^T 2y_x(t, 1) y(t, 1) dt \leq \beta \int_s^T 2|y_x(t, 1) y(t, 1)| dt \leq \beta \int_s^T y_x^2(t, 1) dt + \beta \int_s^T y^2(t, 1) dt \quad (3.13)$$

and

$$-\int_s^T 2y_x(t, 1) y_t(t, 1) dt \leq \int_s^T 2|y_x(t, 1) y_t(t, 1)| dt \leq \int_s^T y_x^2(t, 1) dt + \int_s^T y_t^2(t, 1) dt. \quad (3.14)$$

Furthermore, recalling that $\gamma y_x(t, 1) + y_{xx}(t, 1) + y_{tx}(t, 1) = 0$,

$$\int_s^T y_{xx}^2(t, 1) dt \leq 2\gamma^2 \int_s^T y_x^2(t, 1) dt + 2 \int_s^T y_{tx}^2(t, 1) dt. \quad (3.15)$$

Hence, by (3.5), (3.9)-(3.15) and Theorem 3.1, we have

$$\begin{aligned}
& \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda \frac{y^2}{ad} \right) dx dt \leq 4 \max \left\{ \frac{1}{\epsilon} \left(\frac{4}{a(1)} + K_a C_{HP} \right), 1 + \frac{K_a}{4} \right\} E_y(s) \\
& + \left(2 - \frac{K_a}{2} \right) \frac{C_\gamma}{2} E_y(s) + \left(\frac{K_a}{4} + 1 + \frac{1}{a(1)} \right) \int_s^T y_t^2(t, 1) dt + 2 \int_s^T y_{tx}^2(t, 1) dt \\
& + \left(\frac{K_a}{4} + \beta + \frac{K_a \beta}{2} + \frac{\lambda}{a(1)d(1)} \right) \int_s^T y^2(t, 1) dt + (\beta + 1 + 2\gamma^2) \int_s^T y_x^2(t, 1) dt \\
& \leq 4 \max \left\{ \frac{1}{\epsilon} \left(\frac{4}{a(1)} + K_a C_{HP} \right), 1 + \frac{K_a}{4} \right\} E_y(s) + \left(2 - \frac{K_a}{2} \right) \frac{C_\gamma}{2} E_y(s) \\
& + \max \left\{ 2, \frac{K_a}{4} + 1 + \frac{1}{a(1)} \right\} \int_s^T -\frac{d}{dt} E_y(t) dt \\
& + \left(\frac{K_a}{4} + \beta + \frac{K_a \beta}{2} + \frac{\lambda}{a(1)d(1)} \right) \int_s^T y^2(t, 1) dt + (\beta + 1 + 2\gamma^2) \int_s^T y_x^2(t, 1) dt \\
& \leq 4 \max \left\{ \frac{1}{\epsilon} \left(\frac{4}{a(1)} + K_a C_{HP} \right), 1 + \frac{K_a}{4} \right\} E_y(s) + \left(2 - \frac{K_a}{2} \right) \frac{C_\gamma}{2} E_y(s) \\
& + \max \left\{ 2, \frac{K_a}{4} + 1 + \frac{1}{a(1)} \right\} E_y(s) \\
& + \left(\frac{K_a}{4} + \beta + \frac{K_a \beta}{2} + \frac{\lambda}{a(1)d(1)} \right) \int_s^T y^2(t, 1) dt + (\beta + 1 + 2\gamma^2) \int_s^T y_x^2(t, 1) dt.
\end{aligned}$$

Hence

$$\begin{aligned}
& \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda \frac{y^2}{ad} \right) dx dt \leq \left(4\vartheta + \varrho + \left(2 - \frac{K_a}{2} \right) \frac{C_\gamma}{2} \right) E_y(s) \\
& + \left(\frac{K_a}{4} + \beta + \frac{K_a \beta}{2} + \frac{\lambda}{a(1)d(1)} \right) \int_s^T y^2(t, 1) dt + (\beta + 1 + 2\gamma^2) \int_s^T y_x^2(t, 1) dt
\end{aligned}$$

and the thesis follows.

Case $\lambda < 0$. In this case, by definition of energy and (2.9), one has

$$\int_{Q_s} \frac{y^2}{ad} dx dt \leq \tilde{C}_{HP} \int_{Q_s} y_{xx}^2(t, x) dx dt \leq 2\tilde{C}_{HP} \int_s^T E_y(t) dt,$$

hence

$$-\lambda \int_{Q_s} \frac{y^2}{ad} dx dt \leq -2\lambda \tilde{C}_{HP} \int_s^T E_y(t) dt. \quad (3.16)$$

Moreover, by (3.5) and (3.8), one has

$$\begin{aligned}
& \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda \frac{y^2}{ad} \right) dx dt \leq \int_{Q_s} \frac{y_t^2}{a} \left(\frac{K_a}{2} + 1 - \frac{xa'}{a} \right) dx dt + \int_{Q_s} y_{xx}^2 \left(3 - \frac{K_a}{2} \right) dx dt \\
& - \lambda \int_{Q_s} \left(1 - \frac{xa'}{a} - \frac{xd'}{d} + \frac{K_a}{2} \right) \frac{y^2}{ad} dx dt \\
& = (B.T.) - 2\lambda \int_{Q_s} \left(1 - \frac{xa'}{a} - \frac{xd'}{d} + \frac{K_a}{2} \right) \frac{y^2}{ad} dx dt,
\end{aligned} \quad (3.17)$$

where (B.T.) are the boundary terms in (3.5). Now, by (3.16),

$$-2\lambda \int_{Q_s} \left(1 - \frac{xa'}{a} - \frac{xd'}{d} + \frac{K_a}{2}\right) \frac{y^2}{ad} dx dt \leq -4\lambda \tilde{C}_{HP} \left(1 + \frac{3}{2}K_a + K_d + M\right) \int_s^T E_y(t) dt.$$

Proceeding as for the case $\lambda > 0$, and using the fact that $\frac{\lambda}{a(1)d(1)} < 0$, one can estimate the boundary terms in the following way

$$\begin{aligned} (B.T.) &\leq 4 \max \left\{ \frac{1}{\epsilon} \left(\frac{4}{a(1)} + K_a C_{HP} \right), 1 + \frac{K_a}{4} \right\} E_y(s) \\ &\quad + \left\{ \max \left\{ 2, \frac{K_a}{4} + 1 + \frac{1}{a(1)} \right\} + \left(2 - \frac{K_a}{2} \right) \frac{C_\gamma}{2} \right\} E_y(s) \\ &\quad + \left(\frac{K_a}{4} + \beta + \frac{K_a \beta}{2} \right) \int_s^T y^2(t, 1) dt + (\beta + 1 + 2\gamma^2) \int_s^T y_x^2(t, 1) dt. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\epsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda \frac{y^2}{ad} \right) dx dt &\leq \left(4\theta + \varrho \left(2 - \frac{K_a}{2} \right) \frac{C_\gamma}{2} \right) E_y(s) \\ &\quad + \left(\frac{K_a}{4} + \beta + \frac{K_a \beta}{2} \right) \int_s^T y^2(t, 1) dt + (\beta + 1 + 2\gamma^2) \int_s^T y_x^2(t, 1) dt \\ &\quad - 4\lambda \tilde{C}_{HP} \left(1 + \frac{3}{2}K_a + K_d \right) \int_s^T E_y(t) dt \end{aligned}$$

and the thesis follows. $\square$

In the next proposition, we will find an estimate from above for $\int_s^T y^2(t, 1) dt + \int_s^T y_x^2(t, 1) dt$. To this aim, set

$$\begin{aligned} \tilde{C}_\beta &:= \begin{cases} \frac{1}{\beta}, & \beta \neq 0, \text{ for all considered } \lambda, \\ \frac{1}{C_\epsilon}, & \beta = 0, \text{ for all considered } \lambda, \end{cases} \\ \tilde{C}_\gamma &:= \begin{cases} \frac{1}{\gamma}, & \gamma \neq 0, \text{ for all considered } \lambda, \\ \frac{1}{C_\epsilon}, & \gamma = 0, \text{ for all considered } \lambda, \end{cases} \end{aligned}$$

and

$$\nu := \frac{1}{\tilde{C}_\beta + \tilde{C}_\gamma}. \quad (3.18)$$

Proposition 3.3. Assume Hypothesis 5. If y is a classical solution of (1.1), then for every $0 < s < T$ and for every $\delta \in (0, \nu)$ we have

$$\begin{aligned} \int_s^T y^2(t, 1) dt + \int_s^T y_x^2(t, 1) dt &\leq \frac{\delta}{C_\delta} \int_s^T E_y(t) dt \\ &\quad + \frac{1}{C_\delta} \left[\frac{2}{C_\epsilon} (1 + 2C_{HP}(C_\beta + C_\gamma)) + \frac{1}{\delta} \left(\frac{4C_{HP}}{C_\epsilon} + \frac{1}{2} \right) \right] E_y(s), \end{aligned}$$

where

$$C_\delta := 1 - \delta(\tilde{C}_\beta + \tilde{C}_\gamma)$$

Proof. Set $\rho = y(t, 1)$, $\mu = y_x(t, 1)$, where $t \in [s, T]$, and let $z = z(t, \cdot) \in H_{\frac{1}{a}, 0}^2(0, 1)$ be the unique solution of

$$\int_0^1 z_{xx} \varphi'' dx - \lambda \int_0^1 \frac{z\varphi}{\sigma d} dx + \beta z(t, 1) \varphi(1) + \gamma z'(t, 1) \varphi'(1) = \rho \varphi(1) + \mu \varphi'(1), \quad \forall \varphi \in H_{\frac{1}{a}, 0}^2(0, 1).$$

By Proposition 2.3, $z(t, \cdot) \in D(A_{1,\lambda})$ for all t and solves

$$\begin{cases} A_{1,\lambda} z = 0, \\ \beta z(t, 1) - z_{xx}(t, 1) = \rho, \\ \gamma z_x(t, 1) + z_{xx}(t, 1) = \mu. \end{cases} \quad (3.19)$$

By (2.15), we also have

$$\|z(t)\|_{L^2_{\frac{1}{a}}(0,1)}^2 \leq \frac{8C_{HP}}{C_\epsilon} (y^2(t, 1) + y_x^2(t, 1)) \quad \text{and} \quad |||z(t)|||^2 \leq 2(y^2(t, 1) + y_x^2(t, 1)), \quad (3.20)$$

where C_ϵ is defined in (2.16). Moreover, if $\beta \neq 0$ and $\gamma \neq 0$, and by Proposition 2.2 if $\lambda > 0$, one has

$$z^2(t, 1) \leq \frac{1}{\beta} |||z|||^2 \leq \frac{2}{\beta} (y^2(t, 1) + y_x^2(t, 1))$$

and

$$z_x^2(t, 1) \leq \frac{1}{\gamma} |||z|||^2 \leq \frac{2}{\gamma} (y^2(t, 1) + y_x^2(t, 1)).$$

On the other hand, if $\beta = 0$ and $\gamma = 0$, then by (2.18), (2.19) and (2.23), it results $z^2(t, 1), z_x^2(t, 1) \leq \|z\|_{2,\circ}^2 \leq \frac{1}{C_\epsilon} |||z|||^2 \leq \frac{1}{C_\epsilon} (|\rho| + |\mu|)^2 \leq \frac{2}{C_\epsilon} (y^2(t, 1) + y_x^2(t, 1))$. In every case, for all considered λ , we have

$$z^2(t, 1) \leq 2\tilde{C}_\beta (y^2(t, 1) + y_x^2(t, 1)), \quad (3.21)$$

and

$$z_x^2(t, 1) \leq 2\tilde{C}_\gamma (y^2(t, 1) + y_x^2(t, 1)). \quad (3.22)$$

Finally, observe that for all considered λ and all $t > 0$, we have

$$\frac{1}{2} \int_0^1 \left(\frac{1}{a} y_t^2(t, x) + y_{xx}^2(t, x) \right) dx \leq \frac{1}{C_\epsilon} E_y(t), \quad (3.23)$$

Indeed, consider, first of all $\lambda < 0$, then

$$\frac{1}{2} \int_0^1 \left(\frac{1}{a} y_t^2(t, x) + y_{xx}^2(t, x) \right) dx \leq E_y(t).$$

If $\lambda \in \left(0, \frac{1}{\tilde{C}_{HP}}\right)$ and $\epsilon \in (0, 1)$ is as in (2.12), we obtain (2.21), which implies that

$$\begin{aligned} 2E_y(t) &\geq \epsilon \int_0^1 y_{xx}^2(t, x) dx + \int_0^1 \frac{y_t^2(t, x)}{a(x)} dx + \beta y^2(t, 1) + \gamma y_x^2(t, 1) \\ &\geq \epsilon \left(\int_0^1 y_{xx}^2(t, x) dx + \int_0^1 \frac{y_t^2(t, x)}{a(x)} dx \right) + \beta y^2(t, 1) + \gamma y_x^2(t, 1); \end{aligned}$$

in particular,

$$\frac{1}{2} \int_0^1 \left(\frac{1}{a} y_t^2(t, x) + y_{xx}^2(t, x) \right) dx \leq \frac{1}{\epsilon} E_y(t)$$

for all $t \geq 0$.

Now, multiplying the equation in (1.1) by $\frac{z}{a}$ and integrating over Q_s , we have

$$\begin{aligned} 0 &= \int_{Q_s} y_{tt} \frac{z}{a} dx dt + \int_{Q_s} z y_{xxxx} dx dt - \lambda \int_{Q_s} \frac{y}{ad} z dx dt \\ &= \int_0^1 \left[y_t \frac{z}{a} \right]_{t=s}^{t=T} dx - \int_{Q_s} \frac{y_t z_t}{a} dx dt + \int_s^T z(t, 1) y_{xxx}(t, 1) dt - \int_s^T z_x(t, 1) y_{xx}(t, 1) dt \\ &\quad + \int_{Q_s} z_{xx} y_{xx} dx dt - \lambda \int_{Q_s} \frac{y}{ad} z dx dt. \end{aligned} \quad (3.24)$$

Hence, (3.24) reads

$$\begin{aligned} &\int_0^1 \left[y_t \frac{z}{a} \right]_{t=s}^{t=T} dx - \int_{Q_s} \frac{y_t z_t}{a} dx dt - \lambda \int_{Q_s} \frac{y}{ad} z dx dt \\ &= - \int_s^T z(t, 1) y_{xxx}(t, 1) dt + \int_s^T z_x(t, 1) y_{xx}(t, 1) dt - \int_{Q_s} z_{xx} y_{xx} dx dt. \end{aligned} \quad (3.25)$$

On the other hand, multiplying the equation in (3.19) by $\frac{y}{a}$ and integrating over Q_s , we have $\int_{Q_s} z_{xxxx} y dx dt = 0$. By (2.13), we get $\int_{Q_s} z_{xx} y_{xx} dx dt = - \int_s^T z_{xxx}(t, 1) y(t, 1) dt + \int_s^T y_x(t, 1) z_{xx}(t, 1) dt$. Substituting in (3.25), using the fact that $z_{xxx}(t, 1) = \beta z(t, 1) - \rho$, $z_{xx}(t, 1) = -\gamma z_x(t, 1) + \mu$, $\rho = y(t, 1)$ and $\mu = y_x(t, 1)$, and proceeding as in [5], we have

$$\begin{aligned} &\int_0^1 \left[y_t \frac{z}{a} \right]_{t=s}^{t=T} dx - \int_{Q_s} \frac{y_t z_t}{a} dx dt - \lambda \int_{Q_s} \frac{y}{ad} z dx dt \\ &= - \int_s^T z(t, 1) y_{xxx}(t, 1) dt + \int_s^T z_x(t, 1) y_{xx}(t, 1) dt + \int_s^T y(t, 1) [\beta z(t, 1) - \rho] dt \\ &\quad - \int_s^T y_x(t, 1) [-\gamma z_x(t, 1) + \mu] dt \\ &= \int_s^T z(t, 1) [\beta y(t, 1) - y_{xxx}(t, 1)] dt \\ &\quad + \int_s^T z_x(t, 1) [y_{xx}(t, 1) + \gamma y_x(t, 1)] dt - \int_s^T y^2(t, 1) dt - \int_s^T y_x^2(t, 1) dt. \end{aligned}$$

Then

$$\begin{aligned} \int_s^T y^2(t, 1) dt + \int_s^T y_x^2(t, 1) dt &= - \int_s^T (y_t z)(t, 1) dt - \int_s^T (z_x y_{tx})(t, 1) dt \\ &\quad - \int_0^1 \left[y_t \frac{z}{a} \right]_{t=s}^{t=T} dx + \int_{Q_s} \frac{y_t z_t}{a} dx dt + \lambda \int_{Q_s} \frac{y}{ad} z dx dt. \end{aligned} \quad (3.26)$$

Thus, in order to estimate $\int_s^T y^2(t, 1) dt + \int_s^T y_x^2(t, 1) dt$, we have to consider the four terms in the previous equality.

So, by (2.15), (3.2) and Theorem 3.1 we have, for all $\tau \in [s, T]$,

$$\begin{aligned} \int_0^1 \left| \frac{y_t z}{a}(\tau, x) \right| dx &\leq \frac{1}{2} \int_0^1 \frac{y_t^2(\tau, x)}{a(x)} dx + \frac{1}{2} \int_0^1 \frac{z^2(\tau, x)}{a(x)} dx \\ &\leq \frac{1}{2} \int_0^1 \frac{y_t^2(\tau, x)}{a(x)} dx + \frac{2C_{HP}}{C_\epsilon} (y^2(\tau, 1) + y_x^2(\tau, 1)) \\ &\leq \frac{1}{C_\epsilon} E_y(\tau) + \frac{2C_{HP}}{C_\epsilon} (C_\beta + C_\gamma) E_y(\tau) \\ &\leq (1 + 2C_{HP}(C_\beta + C_\gamma)) \frac{1}{C_\epsilon} E_y(s). \end{aligned}$$

By Theorem 3.1,

$$\left| \int_0^1 \left[\frac{y_t z}{a} \right]_{t=s}^{t=T} dx \right| \leq 2(1 + 2C_{HP}(C_\beta + C_\gamma)) \frac{1}{C_\epsilon} E_y(s). \quad (3.27)$$

Moreover, for any $\delta > 0$ we have

$$\begin{aligned} \int_s^T |(y_t z)(t, 1)| dt &\leq \frac{1}{2\delta} \int_s^T y_t^2(t, 1) dt + \frac{\delta}{2} \int_s^T z^2(t, 1) dt \\ &\leq \frac{1}{2\delta} \int_s^T y_t^2(t, 1) dt + \delta \tilde{C}_\beta \int_s^T (y^2 + y_x^2)(t, 1) dt, \end{aligned} \quad (3.28)$$

by (3.21). In a similar way, using (3.22), it is possible to find the next estimate

$$\int_s^T |(z_x y_{tx})(t, 1)| dt \leq \frac{1}{2\delta} \int_s^T y_{tx}^2(t, 1) dt + \delta \tilde{C}_\gamma \int_s^T (y^2 + y_x^2)(t, 1) dt. \quad (3.29)$$

Therefore, summing (3.28) and (3.29) and applying Theorem 3.1 we obtain

$$\begin{aligned} &\int_s^T |(y_t z)(t, 1)| dt + \int_s^T |(z_x y_{tx})(t, 1)| dt \\ &\leq \frac{1}{2\delta} \int_s^T -\frac{d}{dt} E_y(t) dt + \delta \left(\tilde{C}_\beta + \tilde{C}_\gamma \right) \int_s^T (y^2 + y_x^2)(t, 1) dt \\ &\leq \frac{E_y(s)}{2\delta} + \delta \left(\tilde{C}_\beta + \tilde{C}_\gamma \right) \int_s^T (y^2 + y_x^2)(t, 1) dt. \end{aligned} \quad (3.30)$$

Finally, we estimate the integral $\int_{Q_s} \left| \frac{y_t z_t}{a} \right| dx dt$. To this aim, consider again problem (2.17) and differentiate with respect to t . Thus

$$\begin{cases} a(z_t)_{xxxx} - \lambda \frac{z_t}{d} = 0, \\ \beta z_t(t, 1) - (z_t)_{xxx}(t, 1) = y_t(t, 1), \\ \gamma(z_t)_x(t, 1) + (z_t)_{xx}(t, 1) = (y_x)_t(t, 1). \end{cases}$$

Clearly, z_t satisfies (3.20), in particular

$$\|z_t(t)\|_{L^2_{\frac{1}{d}}(0,1)}^2 \leq \frac{8C_{HP}}{C_\epsilon} (y_t^2(t, 1) + y_{tx}^2(t, 1))$$

and

$$|||z_t(t)|||^2 \leq 2(y_t^2(t, 1) + y_{tx}^2(t, 1)).$$

Thus, by (3.23) and the previous estimate, for $\delta > 0$ we find

$$\begin{aligned} \int_{Q_s} \left| \frac{y_t z_t}{a} \right| dx dt &\leq \frac{\delta}{2} \int_{Q_s} \frac{y_t^2}{a} dx dt + \frac{1}{2\delta} \int_{Q_s} \frac{z_t^2}{a} dx dt \\ &\leq \delta \int_s^T E_y(t) dt + \frac{4C_{HP}}{\delta C_\epsilon} \int_s^T (y_t^2(t, 1) + y_{tx}^2(t, 1)) dt \\ &= \delta \int_s^T E_y(t) dt + \frac{4C_{HP}}{\delta C_\epsilon} \int_s^T -\frac{d}{dt} E_y(t) dt \\ &\leq \delta \int_s^T E_y(t) dt + \frac{4C_{HP}}{\delta C_\epsilon} E_y(s). \end{aligned} \quad (3.31)$$

Coming back to (3.26) and using (3.27), (3.30) and (3.31),

$$\begin{aligned} \int_s^T (y^2 + y_x^2)(t, 1) dt &\leq 2 \left(\frac{1}{C_\epsilon} + \frac{2C_{HP}}{C_\epsilon} (C_\beta + C_\gamma) \right) E_y(s) + \frac{E_y(s)}{2\delta} + \delta \left(\tilde{C}_\beta + \tilde{C}_\gamma \right) \int_s^T (y^2 + y_x^2)(t, 1) dt \\ &\quad + \delta \int_s^T E_y(t) dt + \frac{4C_{HP}}{\delta C_\epsilon} E_y(s). \end{aligned}$$

Hence, for every $\delta \in (0, \nu)$

$$C_\delta \int_s^T (y^2(t, 1) + y_x^2(t, 1)) dt \leq 2 \left(\frac{1}{C_\epsilon} + \frac{2\tilde{C}_{HP}}{C_\epsilon} (C_\beta + C_\gamma) \right) E_y(s) \\ + \delta \int_s^T E_y(t) dt + \frac{1}{\delta} \left(\frac{4C_{HP}}{C_\epsilon} + \frac{1}{2} \right) E_y(s),$$

and the thesis follows. $\square$

As a consequence of Propositions 3.2 and 3.3, we can formulate the main result of the paper, whose proof is based on [21, Theorem 8.1].

Set

$$C_1 := \frac{1}{C_\delta} \left[\frac{2}{C_\epsilon} (1 + 2C_{HP}(C_\beta + C_\gamma)) + \frac{1}{\delta} \left(\frac{4C_{HP}}{C_\epsilon} + \frac{1}{2} \right) \right], \\ C_2 := \left(4\vartheta + \varrho + \frac{C_\gamma}{2} \left(2 - \frac{K_a}{2} \right) \right) \\ C_3 := \begin{cases} \left(\frac{K_a\beta}{2} + \frac{K_a}{4} + \beta + \varepsilon_0 \frac{\beta}{2} + \frac{\lambda}{a(1)d(1)} \right), & \lambda > 0, \\ \left(\frac{K_a}{4} + \beta + \frac{K_a\beta}{2} + \varepsilon_0 \frac{\beta}{2} \right), & \lambda < 0, \end{cases}$$

and

$$C_4 := \left(\beta + 1 + 2\gamma^2 + \varepsilon_0 \frac{\gamma}{2} \right).$$

Theorem 3.3. Assume Hypothesis 5 and if $\lambda < 0$, then $\lambda \in \left(\frac{-\varepsilon_0}{4\tilde{C}_{HP}(1+\frac{3}{2}K_a+K_d)}, 0 \right)$. Let y be a mild solution of (1.1). Then, for all $t > 0$ and for all $\delta \in (0, \min\{\nu, \mu\})$,

$$E_y(t) \leq E_y(0)e^{1-\frac{t}{M}}, \quad (3.32)$$

where

$$\mu := \begin{cases} \frac{\varepsilon_0 C_\delta}{\max\{C_3, C_4\}}, & \lambda > 0, \\ \frac{\varepsilon_0 + 4\lambda\tilde{C}_{HP}(1+\frac{3}{2}K_a+K_d)}{\max\{C_3, C_4\} + (\varepsilon_0 + 4\lambda\tilde{C}_{HP}(1+\frac{3}{2}K_a+K_d))(\tilde{C}_\beta + \tilde{C}_\gamma)}, & \lambda < 0 \end{cases}$$

and

$$M := \begin{cases} \frac{C_\delta(C_2 + \max\{C_3, C_4\}C_1)}{\varepsilon_0 C_\delta - \delta \max\{C_3, C_4\}}, & \lambda > 0, \\ \frac{C_\delta(C_2 + \max\{C_3, C_4\}C_1)}{C_\delta(\varepsilon_0 + 4\lambda C_{HP}(1+\frac{3}{2}K_a+K_d)) - \delta \max\{C_3, C_4\}}, & \lambda < 0. \end{cases}$$

Here ν is defined as in (3.18).

Proof. As a first step, consider y a classical solution of (1.1) and $\lambda > 0$. Take $\delta \in \left(0, \min\left\{\nu, \frac{\varepsilon_0 C_\delta}{\max\{C_3, C_4\}}\right\}\right)$. Then, by definition of E_y and Propositions 3.2, 3.3, we have

$$\begin{aligned}\varepsilon_0 \int_s^T E_y(t) dt &= \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2(t, x)}{a(x)} + y_{xx}^2(t, x) - \frac{\lambda}{ad} y^2(t, x) \right) dx dt \\ &+ \varepsilon_0 \frac{\beta}{2} \int_s^T y^2(t, 1) dt + \varepsilon_0 \frac{\gamma}{2} \int_s^T y_x^2(t, 1) dt \\ &\leq C_2 E_y(s) + C_3 \int_s^T y^2(t, 1) dt + C_4 \int_s^T y_x^2(t, 1) dt \\ &\leq C_2 E_y(s) + \max\{C_3, C_4\} \left(\int_s^T y^2(t, 1) + y_x^2(t, 1) \right) dt \\ &\leq C_2 E_y(s) + \max\{C_3, C_4\} \frac{\delta}{C_\delta} \int_s^T E_y(t) dt + \max\{C_3, C_4\} C_1 E_y(s).\end{aligned}$$

This implies $\left[\varepsilon_0 - \max\{C_3, C_4\} \frac{\delta}{C_\delta} \right] \int_s^T E_y(t) dt \leq (C_2 + \max\{C_3, C_4\} C_1) E_y(s)$. Hence, we can apply [21, Theorem 8.1] with $M := \frac{C_\delta(C_2 + \max\{C_3, C_4\} C_1)}{\varepsilon_0 C_\delta - \delta \max\{C_3, C_4\}}$ and (3.32) holds.

Now, consider $\lambda < 0$. By Propositions 3.2, 3.3, we have

$$\begin{aligned}\varepsilon_0 \int_s^T E_y(t) dt &= \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2(t, x)}{a(x)} + y_{xx}^2(t, x) - \frac{\lambda}{ad} y^2(t, x) \right) dx dt \\ &+ \varepsilon_0 \frac{\beta}{2} \int_s^T y^2(t, 1) dt + \varepsilon_0 \frac{\gamma}{2} \int_s^T y_x^2(t, 1) dt \\ &\leq C_2 E_y(s) + \max\{C_3, C_4\} \left(\int_s^T y^2(t, 1) dt + \int_s^T y_x^2(t, 1) dt \right) \\ &- 4\lambda \tilde{C}_{HP} \left(1 + \frac{3}{2} K_a + K_d \right) \int_s^T E_y(t) dt \\ &\leq (C_2 + \max\{C_3, C_4\} C_1) E_y(s) + \max\{C_3, C_4\} \frac{\delta}{C_\delta} \int_s^T E_y(t) dt \\ &- 4\lambda \tilde{C}_{HP} \left(1 + \frac{3}{2} K_a + K_d \right) \int_s^T E_y(t) dt.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\varepsilon_0 + 4\lambda \tilde{C}_{HP} \left(1 + \frac{3}{2} K_a + K_d \right) - \max\{C_3, C_4\} \frac{\delta}{C_\delta} \right) \int_s^T E_y(t) dt \\ &\leq (C_2 + \max\{C_3, C_4\} C_1) E_y(s).\end{aligned}$$

Recalling that $C_\delta := 1 - \delta(\tilde{C}_\beta + \tilde{C}_\gamma)$, where $\delta < \mu$, one has that

$$\varepsilon_0 + 4\lambda \tilde{C}_{HP} \left(1 + \frac{3}{2} K_a + K_d \right) - \max\{C_3, C_4\} \frac{\delta}{C_\delta} > 0.$$

Hence, again by [21, Theorem 8.1] with $M := \frac{C_\delta(C_2 + \max\{C_3, C_4\} C_1)}{C_\delta(\varepsilon_0 + 4\lambda \tilde{C}_{HP}(1 + \frac{3}{2} K_a + K_d)) - \delta \max\{C_3, C_4\}}$, (3.32) holds.

If y is the mild solution of the problem, we can proceed as in [18], obtaining the thesis. $\square$

4. Conclusions and a Open Problems

In this paper we study the exponential stability of the energy related to (1.1). In particular, in Theorem 3.3 we show that

*If λ is small and a, b are not too degenerate, then
the energy of the solution to (1.1) converges exponentially to 0 as time diverges.*

This result leads to some open problems. The first one is to prove the stability if $K_a + 2K_d \geq 2$ is not satisfied. As we have seen the condition $K_a + 2K_d < 2$ is crucial in Proposition 3.2 and the condition $K_a + 2K_d \leq 2$ is important to obtain that the two domains $D(A)$ and $D(A_\lambda)$ coincide. On the other hand, $K_a + K_d \leq 2$ is crucial to obtain the estimate given in Proposition 2.1.

Another important open problem is to prove the stability of (1.1), when $\lambda \leq \frac{-\varepsilon_0}{4\tilde{C}_{HP}(1+\frac{3}{2}K_a+K_d)}$.

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References

1. M. Akil, G. Fragnelli, A. Sbair, *Boundary controllability for degenerate/singular beam equations*, submitted for publication.
2. F. Alabau-Boussouira, P. Cannarsa, Günter Leugering, *Control and stabilization of degenerate wave equations*, SIAM J. Control Optim., **55** (2017), 2052–2087.
3. C. Behn, J. Steigenberger, C. Will, *Effects of Boundary Damping on Natural Frequencies in Bending Vibrations of Intelligent Vibrissa Tactile Systems*, International Journal on Advances in Intelligent Systems **8** (2015), 245–254.
4. A. Biselli, M.P. Coleman, *The Exact Frequency Equations for the Euler-Bernoulli Beam Subject to Boundary Damping*, International Journal of Acoustics and Vibration **25** (2020), 183–189.
5. A. Camasta, G. Fragnelli, *A stability result for a degenerate beam equation*, SIAM J. Control Optim. **62** (2024), 630–649.
6. A. Camasta, G. Fragnelli, *Boundary controllability for a degenerate beam equation*, Math. Meth. Appl. Sci. **47** (2024), 907–927.
7. A. Camasta, G. Fragnelli, *Fourth-order differential operators with interior degeneracy and generalized Wentzell boundary conditions*, Electron. J. Differential Equations, **2022** (2022), paper No. 87, 1–22.
8. P. Cannarsa, G. Fragnelli, D. Rocchetti, *Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form*, J. Evol. Equ., **8** (2008), 583–616.
9. M.M. Cavalcanti, V.N. Domingos Cavalcanti, M.A. Jorge Silva, V. Narciso, *Stability for extensible beams with a single degenerate nonlocal damping of Balakrishnan-Taylor type*, J. Differential Equations **290** (2021) 197–222.
10. G. Chen, S.G. Krantz, D.W. Ma, C.E. Wayne, H.H. West, *The Euler-Bernoulli beam equation with boundary energy dissipation*, Operator methods for optimal control problems (New Orleans, La., 1986), 67–96, Lecture Notes in Pure and Appl. Math., 108, Dekker, New York, 1987.
11. G. Chen, M.C. Delfour, A.M. Krall, G. Payre, *Modeling, stabilization and control of serially connected beams*, SIAM J. Control Optim. **25** (1987), 526–546.
12. G. Chen, J. Zhou, *The wave propagation method for the analysis of boundary stabilization in vibrating structures*, SIAM J. Appl. Math. **50** (1990), 1254–1283.
13. M.P. Coleman, L.A. McSweeney, *The Exact Frequency Equations for the Rayleigh and Shear Beams with Boundary Damping*, International Journal of Acoustics and Vibration **25** (2020), 3–8.
14. Z. Cong, S. Chunyou, *Stability for a Class of Extensible Beams with Degenerate Nonlocal Damping*, J. Geom. Anal. **33** (2023), <https://doi.org/10.1007/s12220-023-01353-3>.
15. G. Fragnelli, *Interior degenerate/singular parabolic equations in nondivergence form: well-posedness and Carleman estimates*, J. Differential Equations, **260** (2016), 1314–1371.
16. G. Fragnelli, D. Mugnai, *Carleman estimates for singular parabolic equations with interior degeneracy and non smooth coefficients*, Adv. Nonlinear Anal., **6** (2017), 61–84.
17. G. Fragnelli, F. Mugnai, *Control of Degenerate and Singular Parabolic Equations. Carleman Estimates and Observability*, SpringerBriefs in Mathematics: Springer International Publishing, 2021.
18. G. Fragnelli, D. Mugnai, *Linear stabilization for a degenerate wave equation in non divergence form with drift*, Bulletin of Mathematical Sciences. DOI: 10.1142/S1664360725500018
19. G. Fragnelli, D. Mugnai, A. Sbair, *Stabilization for degenerate equations with drift and small singular term* Math. Meth. Appl. Sci. DOI: <https://doi.org/10.1002/mma.10593>.

20. J. Jost, X. Li-Jost, "Calculus of variations", Cambridge University (1998).
21. V. Komornik, "Exact Controllability and Stabilization: The Multiplier Method", RAM Res. Appl. Math. 36, Masson, Paris, John Wiley, Chichester, UK, 1994.
22. V. Narciso, D. Ekinici, E. Piskin, *On a beam model with degenerate nonlocal nonlinear damping*, Evolution Equations and Control Theory **12** (2023), 732-751.
23. S.H. Sandilo, W.T.V. Horssen, *On Boundary Damping for an Axially Moving Tensioned Beam*, Journal of Vibration and Acoustics **134** (2012), 8 pp.

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