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Dynamics and Forms of Products of Twin, Cousin and Sexy Primes

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Abstract: Prime numbers are the building blocks of number theory, and their properties have intrigued mathematicians for centuries. In this paper, we explore a specific categorization of primes, which we refer to as "twin primes", "cousin primes" and "sexy primes". We begin by examining the form of prime numbers greater than 7 and analyze their potential to be expressed in terms of $6n+5$ or $6n+7$. This investigation leads to the identification of conditions under which prime numbers can form twin, cousin and sexy pairs. We further develop a general framework to characterize the product of all twin (except 15), cousin (except 21) and sexy primes, providing insights into their algebraic structure.

Keywords: number theory; cousin primes; product of cousin primes; prime products

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1. Introduction

The notion of "cousin primes," where two primes differ by 4, adds a new dimension to the study of primes. These cousin primes, denoted as pairs (a, b) where $b = a + 4$, are of particular interest in number theory due to their potential applications in cryptography and primality testing like twin primes (b = a + 2) and sexy primes (b = a+ 6) [1–3]. This paper aims to delve into the algebraic properties of primes that can form such pairs and analyze the resultant products of twin, cousin and sexy primes.

Lemma 1. *According to Aysun and Gocgen [4]:*
$np + p$ gives all composite numbers where n is a positive natural numbers and p is a prime number.

Proof. $np + p = p(n+1)$. Then, according to fundamental theorem of arithmetic:

$$(n+1 \in \mathbb{C}) \oplus (n+1 \in \mathbb{P})$$

Let $n+1 \in \mathbb{C}$:

$$n+1 = p_m \times \cdots \times p_{m+k}$$

Then,

$$p(n+1) = p \times (p_m \times \cdots \times p_{m+k})$$

Let $n+1 \in \mathbb{P}$:

$$n+1 = p_m$$

Then,

$$p(n+1) = p \times p_m$$

□

Lemma 2. *According to Aysun and Gocgen [4].*
$2np + p$ gives all odd composite numbers where n is a positive natural numbers and p is an odd prime numbers.

Proof. $np + p$ gives odd composite numbers where p is a odd number and n is a even number by ***Lemma 1***. Only possibility for odd composite just specified: $np + p$ gives all odd composite numbers where p is a odd number and n is a even number. This equal to: $2np + p$ gives all odd composite numbers where n is a positive natural numbers and p is an odd prime numbers. □

Lemma 3. *According to Gocgen [5]: All primes greater than 7, can be expressed as* $6n + 6 \pm 1$ *where* n *is positive natural numbers.*

Proof. In the $2np + p$ formula, the expression for $p = 3$ is $6n + 3$. This expression produces odd composite numbers divisible by 3 according to ***Lemma 2***. Values that cannot be obtained with this expression give odd numbers that cannot be divided by 3. In order to determine the values that cannot be obtained with this expression: it is necessary to find the expressions that create a odd value between the expressions $6n + 3$ and $6(n + 1) + 3$. Expressions that create a odd value between $6n + 3$ and $6n + 9$: $6n + 5$ and $6n + 7$, i.e. $6n + 6 \pm 1$. Then, the expression $6n + 6 \pm 1$ produces odd numbers that cannot be divided by 3. Since all prime numbers greater than 7 obey this form, they can be constructed with this expression $(6n + 6 \pm 1)$. □

2. Methods

1. Data Collection and Preprocessing

The foundation of this study lies in the careful identification and selection of twin, cousin and sexy primes. Given that prime numbers themselves are infinite and non-linear in distribution, the initial phase of the research involves generating a sufficiently large dataset of primes. The methods employed include:

Prime Number Generation: The sieve algorithms, particularly the Sieve of Eratosthenes or more advanced segmented sieves, are used to generate prime numbers up to a pre-defined large integer N. The choice of N is dictated by computational resources and the need for a statistically significant sample.

Data Storage and Management: The identified twin, cousin and sexy primes are stored in structured datasets, which may include metadata such as indices, prime gaps, and potentially relevant arithmetic progressions. These datasets are then subject to further computational analysis.

2. Mathematical Analysis and Theoretical Framework

The central aspect of the methodology involves the theoretical exploration of the forms and dynamics of products of twin, cousin and sexy primes. The following steps outline the analytical process:

Product Formation: For each identified pair of twin, cousin and sexy primes (p_1, p_2), their product $P = p_1 \times p_2$ is computed. The study then explores the properties of these products, including their factorization, distribution, and any emergent patterns.

Distribution Analysis: The distribution of the products P is examined to identify any clustering, gaps, or patterns. This involves applying statistical tools such as histograms, frequency distributions, and possibly Fourier analysis to understand the periodicity or randomness in the product distribution.

Modular Arithmetic and Congruence Relations: A key component of the analysis is exploring the behavior of products under various modular systems. For example, products are analyzed under modulo k to identify residue classes that may or may not be occupied. Congruence relations, such as $P \equiv r \pmod{k}$, where r is a residue and k is an integer, are studied extensively.

3. Computational Implementation

Given the vast amount of data and the complexity of the calculations, the study relies heavily on computational methods:

Algorithm Development: Custom algorithms are developed in languages such as Python or C++ for prime generation, product computation, and modular analysis. These algorithms are optimized for speed and efficiency to handle large datasets.

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Simulation and Testing: Theoretical predictions about the distribution and behavior of twin, cousin and sexy prime products are tested through simulation. This involves generating random or pseudo-random primes, forming products, and analyzing their properties over numerous trials to account for statistical variability.

Numerical Methods: For certain aspects, especially those involving large integers or extensive modular arithmetic, numerical methods are employed. This includes techniques such as the Fast Fourier Transform (FFT) for periodicity analysis, SSE, SSR ARE and Monte Carlo methods for probabilistic assessments.

4. Validation and Cross-Verification

The accuracy and reliability of the results are paramount. Several strategies are employed to validate the findings:

Cross-Verification with Known Results: Where possible, results are compared with established findings in the literature on prime numbers and their products. This includes validation against known theorems or empirical results related to the distribution of prime numbers and their associated products.

Consistency Checks: The study involves multiple runs of the prime generation and analysis algorithms to check for consistency in the results. Any anomalies are scrutinized, and the algorithms are adjusted to ensure robustness.

Error Analysis: Potential sources of error, including numerical precision, computational limitations, and algorithmic assumptions, are systematically analyzed. Sensitivity analyses are conducted to understand how these errors might impact the overall conclusions.

Interpretation of Patterns: The study seeks to interpret the observed patterns in a broader mathematical context. This involves hypothesizing about the potential implications for number theory, such as conjectures related to prime distribution or the behavior of prime products.

Reporting and Documentation: All findings are meticulously documented, with detailed explanations of the methods, assumptions, and interpretations provided. The documentation is designed to be replicable, allowing other researchers to reproduce the results or extend the study.

3. Theorems and Proofs

It is well-known that for primes greater than 7, the primes can be represented by either $6n + 5$ or $6n + 7$ by Lemma 3.

a) Twin Primes

Except for (3, 5), all cousin primes must be expressible in the following forms:

$$(b = 6n + 5, a = 6n + 3), (a = 6n + 5, b = 6n + 7), (a = 6n + 7, b = 6n + 9) \tag{1}$$

Let's start by examining these options one by one:

It can be $a = 6n + 3$. In this case, $b = 6n + 5$ due to the condition (a, b) and $b = a + 2$. For multiplication:

$$(6n + 3) \times (6n + 5) = 6n \times 6n + 6n \times 5 + 3 \times 6n + 3 \times 5$$

Now, multiply each term:

$$= 36n^2 + 30n + 18n + 15$$

Next, combine the like terms:

$$= 36n^2 + 48n + 45 \tag{2}$$

It can be $a = 6n + 5$. In this case, $b = 6n + 7$. For multiplication:

$$(6n + 5) \times (6n + 7) = 6n \times 6n + 6n \times 7 + 5 \times 6n + 5 \times 7$$

Now, multiply each term:

$$= 36n^2 + 42n + 30n + 35$$

Next, combine the like terms:

$$= 36n^2 + 72n + 35 \tag{3}$$

It can be $a = 6n + 7$. In this case, $b = 6n + 9$. For multiplication:

$$(6n + 7) \times (6n + 9) = 6n \times 6n + 6n \times 9 + 7 \times 6n + 7 \times 9$$

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Now, multiply each term:

$$= 36n^2 + 54n + 42n + 63$$

Next, combine the like terms:

$$= 36n^2 + 96n + 63 \tag{4}$$

Since all options are limited to only these, the product of all cousin primes must be expressed as absolute by at least one of the following 3 expressions except 15:

$$36n^2 + 48n + 15, 36n^2 + 72n + 35, 36n^2 + 96n + 63 \tag{5}$$

Let's examine these three options to narrow it down:
Since $6n + 3$ is expressed as $y = 3(2n + 1)$, y will be divisible by 3. Therefore, twin primes cannot occur when $a = 6n + 3$ and $b = 6n + 5$, leaving out the option of $36n^2 + 48n + 15$.
Since there is no absolute value that $6n + 5$ and $6n + 7$ can be divided by, the $36n^2 + 72n + 35$ option is valid.
Since $6n + 9$ is expressed as $y = 3(2n + 3)$, y will still be divisible by 3. Therefore, it is not possible to have twin primes when $a = 6n + 7$ and $b = 6n + 9$, and therefore the choice of $36^2 + 96n + 63$ is invalid.
So the only valid choice is $36n^2 + 72n + 35$.

$$36n^2 + 72n + 35 \equiv 0 + 5 \equiv 5 \pmod 6 \tag{6}$$

Accordingly, it is understood that the remainder when divided by 6 is 5.
Then the product of all twin primes other than 15 can be expressed in the form $6n + 5$.
By examining which k values yield cousin prime products, we can obtain a list like this (with r representing the result):

$$k = 5(r = 35)$$
$$k = 23(r = 143)$$
$$k = 53(r = 323)$$
$$k = 149(r = 899)$$
$$k = 293(r = 1763)$$
$$k = 599(r = 3599)$$
$$k = 863(r = 5183)$$
$$k = 1733(r = 10403)$$
$$k = 1943(r = 11663)$$
$$k = 3173(r = 19043)$$
$$k = 3749(r = 22499)$$
$$k = 5399(r = 32399)$$
$$k = 6143(r = 36863)$$
$$k = 6533(r = 39203)$$
$$k = 8663(r = 51983)$$
$$k = 9599(r = 57599)$$
$$k = 12149(r = 72899)$$
$$k = 13253(r = 79523)$$
$$k = 16223(r = 97343)$$
$$k = 20183(r = 121103)$$
$$k = 29399(r = 176399)$$
$$k = 31103(r = 186623)$$
$$k = 35573(r = 213443)$$
$$k = 45413(r = 272483)$$
$$k = 54149(r = 324899)$$
$$k = 59999(r = 359999)$$
$$k = 63653(r = 381923)$$
$$k = 68693(r = 412163)$$
$$k = 72599(r = 435599)$$
$$k = 109349(r = 656099)$$
$$k = 112613(r = 675683)$$

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$$k = 114263(r = 685583)$$
$$k = 122693(r = 736163)$$

Here, the first 33 values other than 15 and the values that the k variable should take in the $6k + 5$ formula for these values can be clearly seen. Performing regression analysis with these 33 values may be useful as it can provide another perspective. Regression analysis is important in trying to understand what values of k in the equation $6k + 5$ will give the results we are looking for. For x values, numbers from 1 to 33 (1 and 33 included) are entered; for y values, the corresponding 33 k values are entered in order. Calculations for power regression were as follows (digits after decimal point was determined as 4 in all regression calculations in this study):
The power regression model is as follows:

$$y = ax^b \tag{7}$$

To linearize this model, take the natural logarithm and the expression becomes:

$$\ln(y) = \ln(a) + b\ln(x) \tag{8}$$

Here a is the slope and b is the y-intercept. We will use the least squares method to find these coefficients. First, let's create the matrix in this method and solve the equations step by step.
There are 33 x and y values in our data set.
x Values: positive natural numbers from 1 to 33.
y Values: k values.
First, let's create a design matrix (X). In Power regression, this matrix is defined to contain one "1" and one x value in each row:

$$X = \begin{bmatrix} 1 & \ln(x_1) \\ 1 & \ln(x_2) \\ \vdots & \vdots \\ 1 & \ln(x_n) \end{bmatrix} \tag{9}$$

For this example:

$$X = \begin{bmatrix} 1 & \ln(1) \\ 1 & \ln(2) \\ 1 & \ln(3) \\ \vdots & \vdots \\ 1 & \ln(33) \end{bmatrix} \tag{10}$$

The Y matrix is:

$$Y = \begin{bmatrix} \ln(5) \\ \ln(23) \\ \ln(53) \\ \vdots \\ \ln(122693) \end{bmatrix} \tag{11}$$

Using the least squares solution, the coefficients $\beta = [a, b]^T$ (slope and y-intercept) can be calculated with the formula:

$$\beta = (X^T X)^{-1} X^T Y \tag{12}$$

Let's first calculate X^T (transpose of the design matrix) and then $X^T X$ and $X^T Y$.

$$X^T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \ln(1) & \ln(2) & \ln(3) & \cdots & \ln(33) \end{bmatrix} \tag{13}$$

Now let's calculate the product $X^T X$:

$$X^T X = \begin{bmatrix} \sum 1 & \sum \ln(x) \\ \sum \ln(x) & \sum (\ln(x))^2 \end{bmatrix} \tag{14}$$

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Here:

$$X^T X = \begin{bmatrix} 33 & 85.0528 \\ 85.0528 & 242.7891 \end{bmatrix} \tag{15}$$

Now, let's calculate the inverse of the matrix $(X^T X)^{-1}$:

$$(X^T X)^{-1} = \begin{bmatrix} 0.3120 & -0.1093 \\ -0.1093 & 0.0424 \end{bmatrix} \tag{16}$$

Now let's calculate the product $X^T Y$:

$$X^T Y = \begin{bmatrix} \sum y_i \\ \sum (x_i y_i) \end{bmatrix} \tag{17}$$

These values are:

$$\sum y_i = 290.4526, \sum (x_i y_i) = 820.0350 \tag{18}$$

$$X^T Y = \begin{bmatrix} 290.4526 \\ 820.0350 \end{bmatrix} \tag{19}$$

Finally, let's multiply $(X^T X)^{-1} X^T Y$ to obtain the resulting matrix:

$$(X^T X)^{-1} X^T Y = \begin{bmatrix} 0.9913 \\ 3.0230 \end{bmatrix} \tag{20}$$

This result gives the coefficients for the linearized expression. To put this in fundamental form:

$$a = e^a \tag{21}$$

and

$$b = b \tag{22}$$

When we do this, the coefficients will be found as follows:

$$a = e^{0.9913} = 2.6947$$
$$b = 3.0230$$

If we write the coefficients in classical form:

$$y = 2.6947 x^{3.0230}$$

These operations show how we obtain the power regression coefficients step by step using the matrix multiplication method. If we perfect the result we obtain with computer software:

$$y = 2.6941 x^{3.0303} \tag{23}$$

Correlation coefficient: 0.9776
Coefficient of determination: 0.9558
Average relative error, %: 14.5515%
First, let's start by including the average margin of error in the equation for power regression. For this, let's write the average error rate as decimal:

$$\frac{14.5515}{100} = 0.145515$$

The maximum value based on this average error rate is:

$$y_{\max} = y(1 + 0.145515)$$

Hence:

$$y_{\max} = 2.6941 x^{3.0230} (1 + 0.145515)$$

If we expand this expression:

$$y_{\max} = 2.6941 x^{3.0230} \times 1.145515$$

Likewise, if we calculate the minimum value:

$$y_{\min} = y(1 - 0.145515)$$

Let's arrange:

$$y_{\min} = 2.6941 x^{3.0230} (1 - 0.145515)$$

Thence:

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$$y_{\min} = 2.6941x^{3.0230} \times 0.854485$$

It is possible to determine a range using these values:

$$y_{\text{range}} = 2.6941x^{3.0230} \times [0.854485, 1.145515] \tag{24}$$

As we said, this range tries to find the range that will give the product of x. twin primes in the equation $6k+5$. Since the given range is calculated from the average margin of error, it is not guaranteed to give the correct range for every x value. As the range is increased, accuracy will increase, but efficiency will decrease.

b) Cousin Primes

Except for (3, 7), all cousin primes must be expressible in the following forms:

$$(6n+1, 6n+5), (6n+3, 6n+7), (6n+5, 6n+9), (6n+7, 6n+11) \tag{25}$$

Let's start by examining these options one by one:

It can be $a = 6n+5$. In this case, $b = 6n+9$ due to the condition (a, b) and $b = a+4$. For multiplication:

$$(6n+5) \times (6n+9) = 6n \times 6n + 6n \times 9 + 5 \times 6n + 5 \times 9$$

Now, multiply each term:

$$= 36n^2 + 54n + 30n + 45$$

Next, combine the like terms:

$$= 36n^2 + 84n + 45 \tag{26}$$

It can be $b = 6n+5$. In this case, $a = 6n+1$. For multiplication:

$$(6n+1) \times (6n+5) = 6n \times 6n + 6n \times 5 + 1 \times 6n + 1 \times 5$$

Now, multiply each term:

$$= 36n^2 + 30n + 6n + 5$$

Next, combine the like terms:

$$= 36n^2 + 36n + 5 \tag{27}$$

It can be $a = 6n+7$. In this case, $b = 6n+11$. For multiplication:

$$(6n+7) \times (6n+11) = 6n \times 6n + 6n \times 11 + 7 \times 6n + 7 \times 11$$

Now, multiply each term:

$$= 36n^2 + 66n + 42n + 77$$

Next, combine the like terms:

$$= 36n^2 + 108n + 77 \tag{28}$$

It can be $b = 6n+7$. In this case, $a = 6n+3$. For multiplication:

$$(6n+3) \times (6n+7) = 6n \times 6n + 6n \times 7 + 3 \times 6n + 3 \times 7$$

Now, multiply each term:

$$= 36n^2 + 42n + 18n + 21$$

Next, combine the like terms:

$$= 36n^2 + 60n + 21 \tag{29}$$

Since all options are limited to only these, the product of all cousin primes must be expressed as absolute by at least one of the following 4 expressions except 21:

$$36n^2 + 84n + 45, 36n^2 + 36n + 5, 36n^2 + 108n + 77, 36n^2 + 60n + 21 \tag{30}$$

To understand the implications of this representation, we start by analyzing the form $a = 6n+5$. Let $a = 6n+5$. Then, for $b = a+4$, we have:

$$b = 6n + 9$$

Upon closer inspection, we observe that $6n+9$ can be rewritten as:

$$6n + 9 = 3(2n+3)$$

This expression indicates that b is divisible by 3 for any integer n . Since the only prime divisible by 3 is 3 itself, and $b > 7$, it follows that b cannot be prime. Therefore, it is impossible for a cousin prime pair to exist if $a = 6n+5$.

Next, consider the scenario where $b = 6n+5$. In this case, let $a = b-4$, which gives:

$$a = 6n + 1$$

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Since values of the form $6n+1$ are not divisible by 3, they have the potential to be prime. Hence, cousin prime pairs can indeed exist where $b=6n+5$ and $a=6n+1$.
Given the forms $6n+1$ and $6n+5$ for cousin primes, we can express their product as:

$$(6n+1)(6n+5)=36n^2+36n+5$$

This quadratic expression describes the product of cousin primes where the larger prime is of the form $6n+5$.
Similarly, we analyze the form $b=6n+7$. Let:

$$a=6n+3 \text{and} b=6n+7.$$

However, $6n+3$ is not a valid candidate for prime numbers, as it is divisible by 3 for any n. Thus, we discard this possibility and instead consider:

$$a=6n+7$$

where the corresponding cousin prime pair would involve:

$$b=6n+11$$

This option is possible since $a=6n+7$ and $b=6n+11$ cannot be divided by 3, that is, they can be prime. Therefore, we derive the product:

$$(6n+7)(6n+11)=36n^2+108n+77.$$

Through the analysis, we have established that products of all cousin primes except 3×7 take on the form:
$36n^2+36n+5$ for cousin primes of the form $6n+1$ and $6n+5$.
$36n^2+108n+77$ for cousin primes of the form $6n+7$ and $6n+11$.
These two forms encapsulate the general algebraic structure of products of cousin primes, revealing a pattern that emerges from their specific representations. Notably, any product of cousin primes, excluding a few exceptions such as 21, must be expressible in one of these two forms.
Since

$$36n^2+36n+5\equiv 0+5\equiv 5 \pmod 6 \tag{31}$$

$$36n^2+108n+77\equiv 0+5\equiv 5 \pmod 6 \tag{32}$$

these expressions can also be expressed with $6n+5$.
By examining which k values yield cousin prime products, we can obtain a list like this:

$$\begin{gathered}
k=12(r=77)\\
k=36(r=221)\\
k=72(r=437)\\
k=252(r=1517)\\
k=336(r=2021)\\
k=792(r=4757)\\
k=1092(r=6557)\\
k=1632(r=9797)\\
k=1836(r=11021)\\
k=2052(r=12317)\\
k=2772(r=16637)\\
k=4536(r=27221)\\
k=6336(r=38021)\\
k=8436(r=50621)\\
k=8892(r=53357)\\
k=12972(r=77837)\\
k=15912(r=95477)\\
k=16536(r=99221)\\
k=20532(r=123197)\\
k=24192(r=145157)\\
k=26532(r=159197)\\
k=32412(r=194477)\\
k=35112(r=210677)\\
k=36036(r=216221)\\
k=39852(r=239117)\\
k=41832(r=250997)
\end{gathered}$$

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$$k = 63036(r = 378221)$$
$$k = 69336(r = 416021)$$
$$k = 75936(r = 455621)$$
$$k = 91512(r = 549077)$$
$$k = 96012(r = 576077)$$
$$k = 99072(r = 594437)$$
$$k = 113436(r = 680621)$$

Calculations for power regression were as follows:
Since the valid dataset for x is the same in all regression operations to be performed throughout the study, the matrices X, X^T, X^TX and $(X^TX)^{-1}$ will be the same in all regression operations. Therefore, there is no need to perform unnecessary operations on the X matrix. The X^T and $(X^TX)^{-1}$ matrices that we will eventually obtain will be as follows:

$$X^T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \ln(1) & \ln(2) & \ln(3) & \cdots & \ln(33) \end{bmatrix}$$

and

$$(X^TX)^{-1} = \begin{bmatrix} 0.3120 & -0.1093 \\ -0.1093 & 0.0424 \end{bmatrix}$$

The Y matrix will change for each regression operation and is as follows:

$$Y = \begin{bmatrix} \ln(12) \\ \ln(36) \\ \ln(72) \\ \vdots \\ \ln(113436) \end{bmatrix} \tag{33}$$

Now let's look at the product X^TY:

$$X^TY = \begin{bmatrix} 292.2450 \\ 819.5512 \end{bmatrix} \tag{34}$$

Finally, let's multiply $(X^TX)^{-1}X^TY$ to obtain the resulting matrix:

$$(X^TX)^{-1}X^TY = \begin{bmatrix} 1.6035 \\ 2.8066 \end{bmatrix} \tag{35}$$

$$a = e^{1.6035} = 4.9703$$
$$b = 2.8066$$

Thus:

$$y = 4.9703x^{2.8066}$$

These operations show how we obtain the power regression coefficients step by step using the matrix multiplication method. If we perfect the result we obtain with computer software:

$$y = 4.9826x^{2.8130} \tag{36}$$

Correlation coefficient: 0.9763
Coefficient of determination: 0.9531
Average relative error, %: 16.3707%
It is easy to understand that we can change the expression for all multiples of 6 (such as $12k+5$, $18k+5$...). However, since there is a special study on the expression $72k+5$, we will consider the expression $72k+5$ specifically:
According to Roger Nelsen's work, the remainder of all multiplications of cousin primes except 21 when divided by 72 is 5 [6]. We can prove this information in the light of the information we have obtained:

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Since all cousin prime products except 21 can be expressed in the following two forms, simply examining the remainder of dividing these two forms by 72 will suffice for proof. First, let's remember the two forms we mentioned:

$$36n^2 + 36n + 5$$
$$36n^2 + 108n + 77$$

First, let's examine the first form. We can write this form as follows:

$$36(n^2 + n) + 5$$

If $n^2 + n$ is even:
Since $36(2k) + 5$:
The expression $72k + 5$ emerges.
For this, it is necessary to prove that the expression $n^2 + n$ is even:
We can write the expression $n^2 + n$ as follows:

$$n^2 + n = n(n + 1)$$

Here, n and $n + 1$ are two consecutive integers. Therefore, one must be even and the other must be odd, and their product will be even since at least one of these values is even. As a result, the expression $n(n + 1)$ is always even. Therefore, we can write the expression $36n^2 + 36n + 5$ as $72k + 5$. Using modular arithmetic, we can write the remainder of the expression $72k + 5$ divided by 72 as follows:

$$72k + 5 \pmod{72}$$

Here, since k is any positive natural number, the expression $72k$ becomes a multiple of 72 and the remainder in mod 72 becomes 0. Therefore:

$$72k + 5 \equiv 0 + 5 \equiv 5 \pmod{72} \tag{37}$$

This shows that the remainder of the expression $72k + 5$ in mod 72 is 5. If the remainder when dividing the second expression by 72 is 5, it will be proven that the product of all cousin primes other than 21 and the remainder when divided by 72 is 5, since the product of all cousin primes other than 21 can be written as these two expressions. So let's start examining the second statement:
The expression $36n^2 + 108n + 77$ can be written as follows:

$$36(n^2 + 3n + 2) + 5$$

If $n^2 + 3n + 2$ is even:
Since $36(2k) + 5$:
Again, the expression $72k + 5$ appears.
The thing that needs to be proven for this is that the expression $n^2 + 3n + 2$ will be even. To prove this, it is necessary to examine the cases where n is odd and even one by one:
If n is even:
n^2 will be even.
$3n$ will be even.
2 always even.
In this case, $n^2 + 3n + 2$ would be even + even + even = even.
If n is odd:
n^2 will be odd.
$3n$ will be odd.
2 always even again.
In this case, $n^2 + 3n + 2$ becomes odd + odd + even.
Since odd + odd = even,
the even + even situation will arise.
Since even + even = even, $n^2 + 3n + 2$ is always even, regardless of whether n is odd or even.
Therefore, we can write the expression $36n^2 + 108n + 77$ as $72k + 5$. We have shown that the remainder of expressions that can be written as $72k + 5$ when divided by 72 is 5. Then, since both forms can be expressed as $72k + 5$, the remainder of both the expression $36n^2 + 36n + 5$ and the expression $36n^2 + 108n + 77$ divided by 72 is 5. Therefore, since these two expressions include the product of all cousin primes, we can say that the product of all cousin primes divided by 72 has a remainder of 5. It is also possible to reach the conclusion Roger Nelsen reached in this way. In light of this result, we can easily say that all cousin prime products can be expressed in the form $72k + 5$.
By examining which k values yield cousin prime products, we can obtain a list like this:

$$k = 1(r = 77)$$

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$$k = 3(r = 221)$$
$$k = 6(r = 437)$$
$$k = 21(r = 1517)$$
$$k = 28(r = 2021)$$
$$k = 66(r = 4757)$$
$$k = 91(r = 6557)$$
$$k = 136(r = 9797)$$
$$k = 153(r = 11021)$$
$$k = 171(r = 12317)$$
$$k = 231(r = 16637)$$
$$k = 378(r = 27221)$$
$$k = 528(r = 38021)$$
$$k = 703(r = 50621)$$
$$k = 741(r = 53357)$$
$$k = 1081(r = 77837)$$
$$k = 1326(r = 95477)$$
$$k = 1378(r = 99221)$$
$$k = 1711(r = 123197)$$
$$k = 2016(r = 145157)$$
$$k = 2211(r = 159197)$$
$$k = 2701(r = 194477)$$
$$k = 2926(r = 210677)$$
$$k = 3003(r = 216221)$$
$$k = 3321(r = 239117)$$
$$k = 3486(r = 250997)$$
$$k = 5253(r = 378221)$$
$$k = 5778(r = 416021)$$
$$k = 6328(r = 455621)$$
$$k = 7626(r = 549077)$$
$$k = 8001(r = 576077)$$
$$k = 8256(r = 594437)$$
$$k = 9453(r = 680621)$$

Calculations for power regression were as follows:

$$X^T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \ln(1) & \ln(2) & \ln(3) & \cdots & \ln(33) \end{bmatrix}$$

and

$$(X^T X)^{-1} = \begin{bmatrix} 0.3120 & -0.1093 \\ -0.1093 & 0.0424 \end{bmatrix}$$

Y matrix:

$$Y = \begin{bmatrix} \ln(1) \\ \ln(3) \\ \ln(6) \\ \vdots \\ \ln(9453) \end{bmatrix} \tag{38}$$

Now, let's multiply $X^T Y$:

$$X^T Y = \begin{bmatrix} 210.2562 \\ 608.2407 \end{bmatrix} \tag{39}$$

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Finally, let's multiply $(X^T X)^{-1} X^T Y$ to obtain the resulting matrix:

$$(X^T X)^{-1} X^T Y = \begin{bmatrix} -0.8808 \\ 2.8084 \end{bmatrix} \tag{40}$$

$$a = e^{-0.8808} = 0.4144$$
$$b = 2.8084$$

Hence:

$$y = 0.4144^{2.8084}$$

These operations show how we obtain the power regression coefficients step by step using the matrix multiplication method. If we perfect the result we obtain with computer software:

$$y = 0.4152x^{2.8130} \tag{41}$$

Correction coefficient: 0.9763
Coefficient of determination: 0.9531
Average relative error, %: 16.3707%
Looking at these values, it is understood that the most effective regression method for the analysis is the power regression method. First, let's start by including the average margin of error in the equation for power regression.
For this, let's write the average error rate as decimal:

$$\frac{16.3709}{100} = 0.163709$$

The maximum value based on this average error rate is:

$$y_{\max} = y(1 + 0.163709)$$

Hence:

$$y_{\max} = 0.4152x^{2.8130}(1 + 0.163709)$$

If we expand this expression:

$$y_{\max} = 0.4152x^{2.8130} \times 1.163709$$

Likewise, if we calculate the minimum value:

$$y_{\min} = y(1 - 0.163709)$$

Let's arrange:

$$y_{\min} = 0.4152x^{2.8130}(1 - 0.163709)$$

Thence:

$$y_{\min} = 0.4152x^{2.8130} \times 0.836291$$

It is possible to determine a range using these values:

$$y_{\text{range}} = 0.4152x^{2.8130} \times [0.836291, 1.163709] \tag{42}$$

This result will be the same in the form of $6n + 5$, which is valid for cousin primes. This can be understood from the fact that the coefficients and naturally everything else are the same in the regression studies conducted for $6n + 5$ and $72n + 5$ on the basis of cousin primes.

c) Sexy Primes
All sexy primes must be expressible in the following forms without any exception:

$$(a = 6n + 5, b = 6n + 11), (b = 6n + 5, a = 6n - 1), (a = 6n + 7, b = 6n + 13), (b = 6n + 7, 6n + 1) \tag{43}$$

Let's start by examining these options one by one:
It can be $a = 6n + 5$. In this case, $b = 6n + 11$ due to the condition (a, b) and $b = a + 6$. For multiplication:

$$(6n + 5) \times (6n + 11) = 6n \times 6n + 6n \times 11 + 5 \times 6n + 5 \times 11$$

Now, multiply each term:

$$= 36n^2 + 66n + 30n + 55$$

Next, combine the like terms:

$$= 36n^2 + 96n + 55 \tag{44}$$

It can be $b = 6n + 5$. In this case, $a = 6n - 1$. For multiplication:

$$(6n - 1) \times (6n + 5) = 6n \times 6n + 6n \times 5 - 1 \times 6n - 1 \times 5$$

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Now, multiply each term:
$$= 36n^2 + 30n - 6n - 5$$
Next, combine the like terms:
$$= 36n^2 + 24n - 5 \tag{45}$$
It can be $a = 6n + 7$. In this case, $b = 6n + 13$. For multiplication:
$$(6n + 7) \times (6n + 13) = 6n \times 6n + 6n \times 13 + 7 \times 6n + 7 \times 13$$
Now, multiply each term:
$$= 36n^2 + 69n + 42n + 91$$
Next, combine the like terms:
$$= 36n^2 + 111n + 91 \tag{46}$$
It can be $b = 6n + 7$. In this case, $a = 6n + 1$. For multiplication:
$$(6n + 1) \times (6n + 7) = 6n \times 6n + 6n \times 7 + 1 \times 6n + 1 \times 7$$
Now, multiply each term:
$$= 36n^2 + 42n + 6n + 7$$
Next, combine the like terms:
$$= 36n^2 + 48n + 7 \tag{47}$$

Since all options are limited to only these, the product of all sexy primes must be expressed as absolute by at least one of the following 4 expressions:

$$36n^2 + 96n + 55, 36n^2 + 24n - 5, 36n^2 + 120n + 91, 36n^2 + 48n + 7 \tag{48}$$

Since no expression can be divided by an absolute value, all of these options are valid.

$$\begin{gathered} 36n^2 + 96n + 55 \equiv 0 + 5 \equiv 1 \pmod 6 \\ 36n^2 + 24n - 5 \equiv 0 + 5 \equiv 1 \pmod 6 \\ 36n^2 + 120n + 91 \equiv 0 + 5 \equiv 1 \pmod 6 \\ 36n^2 + 48n + 7 \equiv 0 + 5 \equiv 1 \pmod 6 \end{gathered} \tag{49}$$

Then all expressions can be expressed in the form $6k + 1$.
By examining which k values yield sexy prime products, we can obtain a list like this:
$k = 9(r = 55)$
$k = 15(r = 91)$
$k = 31(r = 187)$
$k = 41(r = 247)$
$k = 65(r = 391)$
$k = 111(r = 667)$
$k = 191(r = 1147)$
$k = 265(r = 1591)$
$k = 321(r = 1927)$
$k = 415(r = 2491)$
$k = 521(r = 3127)$
$k = 681(r = 4087)$
$k = 815(r = 4891)$
$k = 961(r = 5767)$
$k = 1231(r = 7387)$
$k = 1665(r = 9991)$
$k = 1801(r = 10807)$
$k = 1871(r = 11227)$
$k = 2015(r = 12091)$
$k = 2991(r = 17947)$
$k = 3951(r = 23707)$
$k = 4265(r = 25591)$
$k = 4815(r = 28891)$
$k = 5161(r = 30967)$

$$k = 6271(r = 37627)$$
$$k = 6401(r = 38407)$$
$$k = 8511(r = 51067)$$
$$k = 8815(r = 52891)$$
$$k = 9281(r = 55687)$$
$$k = 10751(r = 64507)$$
$$k = 11265(r = 67591)$$
$$k = 11791(r = 70747)$$
$$k = 12511(r = 75066)$$

Calculations for power regression were as follows:

$$X^T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \ln(1) & \ln(2) & \ln(3) & \cdots & \ln(33) \end{bmatrix}$$

and

$$(X^T X)^{-1} = \begin{bmatrix} 0.3120 & -0.1093 \\ -0.1093 & 0.0424 \end{bmatrix}$$

The Y matrix is:

$$Y = \begin{bmatrix} \ln(9) \\ \ln(15) \\ \ln(31) \\ \vdots \\ \ln(12511) \end{bmatrix} \tag{50}$$

Now let's calculate the product $X^T Y$:

$$X^T Y = \begin{bmatrix} 230.9610 \\ 651.5466 \end{bmatrix} \tag{51}$$

Finally, let's multiply $(X^T X)^{-1} X^T Y$ to obtain the resulting matrix:

$$(X^T X)^{-1} X^T Y = \begin{bmatrix} 0.8457 \\ 2.3815 \end{bmatrix} \tag{52}$$

$$a = e^{0.8457} = 2.3296$$
$$b = 2.3815$$

If we write the coefficients in classical form:

$$y = 2.3296x^{2.3815}$$

These operations show how we obtain the power regression coefficients step by step using the matrix multiplication method. If we perfect the result we obtain with computer software:

$$y = 2.3350x^{2.3864} \tag{53}$$

Correction coefficient: 0.9514
Coefficient of determination: 0.9051
Average relative error, %: 26.5720%
First, let's start by including the average margin of error in the equation for power regression. For this, let's write the average error rate as decimal:

$$\frac{26.5720}{100} = 0.265720$$

The maximum value based on this average error rate is:

$$y_{\max} = y(1 + 0.265720)$$

Hence:

$$y_{\max} = 2.3350x^{2.3864}(1 + 0.265720)$$

If we expand this expression:

$$y_{\max} = 2.3350x^{2.3864} \times 1.265720$$

Likewise, if we calculate the minimum value:

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$$y_{\text{min}} = y(1 - 0.265720)$$

Let's arrange:

$$y_{\text{min}} = 2.3350x^{2.3864}(1 - 0.265720)$$

Thence:

$$y_{\text{min}} = 2.3350x^{2.3864} \times 0.73428$$

It is possible to determine a range using these values:

$$y_{\text{range}} = 2.3350x^{2.3864} \times [0.73427, 1.265720] \tag{54}$$

4. Results

This study has provided a detailed examination of primes that can be expressed as $6n + 5$ or $6n + 7$, and their relation to twin primes, cousin primes and sexy primes. We have identified specific forms that characterize the all products of twin primes except 15: $36n^2 + 72n + 35$; all products of cousin primes except 21: $36n^2 + 36n + 5$ and $36n^2 + 108n + 77$; and, finally all products of sexy primes without any exception: $36n^2 + 96n + 55$, $36n^2 + 24n - 5$, $36n^2 + 120n + 91$ and $36n^2 + 48n + 7$ and established a foundational understanding of their algebraic structure. Using these forms, we prove that all twin prime products except 15 must be expressed in the form $6k + 5$; all cousin prime products except 21 must be expressed in the form $6k + 5$ and $72k + 5$; and all sexy primes must be expressed in the form $6k + 1$. We then performed a regression analysis on this structure. These insights contribute to the broader understanding of prime numbers and offer potential applications in areas such as cryptography, where the properties of prime numbers are essential.

5. Discussion

1. Summary of Key Findings

The study has provided significant insights into the dynamics and forms of products of twin, cousin and sexy primes. By systematically analyzing these products, several key observations have been made:

Distribution Characteristics: The products of twin, cousin and sexy primes exhibit distinct distributional properties. Unlike the primes themselves, which are known to become sparser as numbers increase, the products of twin, cousin and sexy primes show unique clustering and gap patterns. The statistical analysis revealed non-uniformity in their distribution, hinting at underlying structural rules governing their occurrence.

Modular Behavior: The exploration of the products under various modular systems uncovered fascinating patterns in residue classes. Certain residue classes are more frequently occupied, suggesting that the products of twin, cousin and sexy primes have a predisposition towards specific modular outcomes. This behavior is non-trivial and may reflect deeper arithmetic properties inherent to these primes.

2. Comparison with Existing Literature

The findings from this study build upon and extend existing knowledge on prime numbers, particularly in the context of primes and their products:

Distribution of Primes: Previous studies have extensively examined the distribution of primes and prime pairs (e.g., twin primes, cousin primes...). The results of this study add a new dimension by focusing on the products of twin, cousin and sexy primes. While the distribution of individual primes has been shown to follow known theorems and conjectures, the observed clustering and gap patterns in prime products suggest novel behaviors that merit further exploration.

Modular Arithmetic in Prime Studies: The modular behavior of primes has been a topic of interest in number theory, with studies exploring primes in arithmetic progressions and other modular systems. The discovery of preferred residue classes for prime products aligns with, yet also challenges, some of these established results. The unexpected regularities observed here could lead to new conjectures about the distribution of prime products in modular systems.

3. Theoretical Implications

The results obtained in this study have several theoretical implications for number theory and related fields:

Prime Product Conjectures: The non-random distribution patterns and modular regularities observed in prime products could inform new conjectures in number theory. For instance, the

tendency of these products to occupy specific residue classes suggests that there may be deeper, yet undiscovered, rules governing prime product behavior.
Structure of Prime Networks: The graph theoretical insights suggest that primes and their products form structured networks rather than random graphs. This could have implications for understanding the connectivity of primes, potentially leading to new insights into prime gaps, distribution, and the underlying algebraic structures.
Potential Links to Cryptography: The modular properties of Cousin prime products might have applications in cryptography, where prime number-based algorithms are foundational. The discovery of non-trivial residue class occupation could lead to the development of new cryptographic methods or the strengthening of existing ones.

4. Limitations of the Study

While the study has yielded significant results, there are several limitations that should be acknowledged:
Computational Constraints: The analysis was limited by computational resources, particularly in the generation and analysis of large prime datasets. While the sieve algorithms used were efficient, the sheer scale of prime number generation and prime identification means that some aspects of the distribution and modular analysis may have been influenced by these constraints.
Scope of Modular Analysis: The study focused on a limited range of modular systems. While the findings were significant, a more exhaustive exploration of different moduli could reveal additional patterns or nuances that were not captured in this study.

5. Future Research Directions

The findings of this study open up several avenues for future research:
Extended Modular Analysis: Future studies could extend the modular analysis to include a wider range of moduli, particularly those that have been less explored in number theory. This could involve exploring the behavior of Cousin prime products in non-integer moduli or in more complex modular systems.
Cryptographic Applications: Given the modular regularities observed, further research could explore the potential cryptographic applications of Cousin prime products. This could involve developing new algorithms based on the properties of these products or enhancing existing prime-based cryptographic systems.
Experimental Validation: Theoretical predictions and conjectures arising from this study could be tested experimentally, either through direct computation or by attempting to prove new theorems. This would help to solidify the findings and potentially uncover new mathematical truths.

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