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Article

A Survey on the Classical Matrix Equation $AXB = C$ with Applications

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Abstract: This survey provides a comprehensive overview of the solutions to the matrix equation $AXB = C$ over real numbers, complex numbers, quaternions, dual quaternions, dual split quaternions, and dual generalized commutative quaternions, including various special solutions. Additionally, we summarize the numerical algorithms for these special solutions. This matrix equation plays an important role in solving linear systems and control theory. We specifically explore the application of this matrix equation in color image processing, highlighting its unique value in this field. Taking the dual quaternion matrix equation $AXB = C$ as an example, we design a scheme for simultaneously encrypting and decrypting two color images. Experimental results demonstrate that this scheme is highly feasible.

Keywords: Matrix equation; General solution; Special solution; Numerical algorithm; Color image

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1. Introduction

Matrix equations play a crucial role in solving linear systems, eigenvalue problems, and system control. For instance, the classic matrix equation

$$AXB = C \tag{1}$$

encompasses both the equations $AX = C$ and $XB = C$, and it is also essential in system control applications [101]. In 1954, Penrose [4] defined the Moore-Penrose inverse using four matrix equations and provided necessary and sufficient conditions for the solvability of the matrix Equation (1) over complex numbers, along with expressions for its general solution. The introduction of the Moore-Penrose inverse has greatly facilitated the solution of matrix equations, drawing the attention of many scholars and promoting further research into the matrix Equation (1).

The purpose of this survey is to provide an overview of the research on various solutions to matrix Equation (1), along with the corresponding numerical algorithms, and to discuss the applications of this matrix equation in color image processing. Since Penrose first applied the Moore-Penrose inverse to study matrix Equation (1), research on this matrix equation has remained active. Furthermore, matrices with special properties play crucial roles in specific fields. For example, Hermitian matrices, η-Hermitian matrices, nonnegative definite matrices, reflexive matrices, antireflexive matrices, reducible matrices, orthogonal matrices, bisymmetric matrices, and others have important applications in areas such as the estimation of covariance components in statistical models, load-flow analysis, short-circuit studies in power systems, engineering and scientific computations, statistical signal processing, Markov chains, compartmental analysis, continuous-time positive systems, covariance assignment, data matching in multivariate analysis, and other fields (see references [3,18–20,22–27]). Consequently, many scholars have explored various special solutions to this matrix equation [12,17,28–30,32–36,38–46]. In addition, numerous researchers have developed corresponding numerical algorithms for these special solutions (see References [2,21,61–78,81–90,97,100]).

Since most research on matrix Equation (1) has been conducted over the fields of real or complex numbers, many scholars have extended their studies to dual quaternions, dual split quaternions, and dual generalized commutative quaternions, thereby further exploring the solutions to this matrix equation (see References [8,11,98]). Additionally, some researchers have studied the matrix equation within the framework of the semi-tensor product [13], which removes the dimensional constraints typically imposed on matrix multiplication. Other work has elevated the study of matrices to the operator level, focusing on operator equations of the form $AXB = C$ [47,48]. Moreover, the study of the rank of solutions to matrix Equation (1) has attracted considerable attention from scholars [51–54]. Currently, applications of this matrix equation in image processing are rare. Therefore, this survey uses the dual quaternion matrix equation $AXB = C$ as an example to explore its application in the encryption and decryption of color images.

The remainder of this survey is organized as follows. Section 2 introduces some of the notation used in this survey and provides definitions and properties of several special types of matrices. In Section 3, we discuss various solutions to matrix Equation (1). Section 4 presents numerical algorithms for solving the special solutions of matrix Equation (1). In Section 5, we use the dual quaternion matrix equation $AXB = C$ as an example to demonstrate its application in color image processing, supported by experimental verification. Finally, Section 6 offers a conclusion to the survey.

2. Preliminaries

In this section, we present some commonly used symbols, definitions related to matrices, and their relevant properties.

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{C}$ be the field of complex numbers, $\mathbb{H}$ be the quaternions, $\mathbb{H}_s$ be the split quaternions, $\mathbb{H}_g$ be the generalized commutative quaternions, $\mathbb{DQ}$ be the dual quaternions, $\mathbb{DH}_s$ be the dual split quaternions, and $\mathbb{DH}_g$ be the dual generalized commutative quaternions. The rank of a matrix A is denoted by $r(A)$.

2.1. Real Matrix

A real matrix $A \in \mathbb{R}^{n\times n}$ is (anti-)symmetric if it obeys

$$A = (-)A^T,$$

where A^T denotes the transpose of A. We use the symbol $(A)SR^{n\times n}$ to represent the set of all (anti-)symmetric matrices.

For $A \in SR^{n\times n}$, if $AA^T = I$, where I represents the identity matrix, then A is referred to as a symmetric orthogonal matrix. We use the notation $SOR^{n\times n}$ to denote the set of all symmetric orthogonal matrices.

Definition 1. *[1] A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is defined as:*

$$\begin{cases} \text{Centro-symmetric, denoted by } CSR^{n\times n}, & \text{if } a_{ij} = a_{n+1-i,n+1-j}\ \forall i,j = 1,2,\cdots,n. \\ \text{Anti-centro-symmetric, denoted by } ACSR^{n\times n}, & \text{if } a_{ij} = -a_{n+1-i,n+1-j}\ \forall i,j = 1,2\cdots,n. \end{cases}$$

Remark 1. *If $A \in CSR^{n\times n}$, then $A = V_nAV_n$, where $V_n = (e_n,\cdots,e_1)$ with e_i denotes the i-th column of the identity matrix I_n.*

As the extension of centro-symmetric matrix, we define the generalized centro-symmetric matrix and the mirrorsymmetric matrix.

Definition 2. *[2] For $P \in SOR^{n\times n}$, a matrix A is said to be generalized centro-symmetric(generalized centro anti-symmetric) if it satisfies $PAP = A(PAP = -A)$. The set of all such generalized centro-symmetric (or generalized central anti-symmetric) matrices with respect to P is denoted by $CSR_P^{n\times n}(CASR_P^{n\times n})$.*

Remark 2. *If the definition above is given over $\mathbb{C}$, then A is called a reflexive (anti-reflexive) matrix. In particular, if A also satisfies $A^T = A$, then A is called symmetric P-symmetric. Furthermore, if $P, Q \in SOR^{n\times n}$ and satisfy $(PXQ)^T = (-)PXQ$, then X is called (P, Q)-orthogonal (skew-) symmetric.*

Definition 3. *[21] A matrix $A \in R^{(2n+m)\times(2n+m)}$ is said to be (n, m)-mirrorsymmetric((n, m)-mirrorskew) iff*

$$P_{(n,m)}AP_{(n,m)} = A(-A),$$

where

$$P_{(n,m)} = \begin{pmatrix} 0 & 0 & J_n \\ 0 & I_m & 0 \\ J_n & 0 & 0 \end{pmatrix},\ J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

In addition, for $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, if it obeys $A_{ij} = a_{j-i}$ $(i, j = 1, \cdots, n)$, then A is said to be a general Toeplitz matrix. In other words,

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & \ddots & a_{n-1} & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{-(n-2)} & a_{-(n-3)} & \ddots & \ddots & \ddots & a_1 \\ a_{-(n-1)} & a_{-(n-2)} & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix}.$$

If A satisfies $A_{ij} = a_{i+j-1}$ $(i, j = 1, \cdots, n)$, then A is called a Hankel matrix. This can be represented as:

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix}.$$

2.2. Complex Matrix

2.2.1. Hermitian,Positive Semidefinite And Positive Definite Matrices

A matrix $A \in \mathbb{C}^{n\times n}$ is called nonnegative definite or positive semidefinite, denoted $A \geq 0$, if

$$x^*Ax \geq 0, \text{ for all } x \in \mathbb{C}^n.$$

A is further termed positive definite, denoted $A > 0$, if $x^*Ax > 0$ holds true for all $x \neq 0$. Specifically, if A satisfies $Re[x^*Ax] \geq 0$ and $x \neq 0$, then A is called Re-nonnegative definite and is denoted as Re-nnd [16], where $Re[x^*Ax]$ denotes the real part of x^*Ax. Let Re^n be the set of all $n \times n$ Re-nnd matrices.

In addition, $A \in \mathbb{C}^{n\times n}$ is said to be Hermitian (or self-adjoint) if it satisfies the condition

$$A = A^*,$$

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where A^* denotes the conjugate transpose of A. Specifically, $\overline{A}$ denotes the conjugate of A. Key properties of Hermitian matrices include having real diagonal elements, real eigenvalues, and the existence of an orthogonal set of eigenvectors.

For $A, B \in \mathbb{C}^{n\times n}$, if there exists a nonsingular matrix P such that $A = PBP^*$, then A and B are in the same *congruence class. It is evident that Hermitian, positive definite, and positive semidefinite matrices are special cases within *congruence.

2.2.2. Moore-Penrose Inverses Of Matrices

In 1955, Penrose [4] formulated the generalized inverse (now known as the Moore-Penrose inverse) of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, using four matrix equations:

$$AA^\dagger A = A, \, , A^\dagger AA^\dagger = A^\dagger, \, (AA^\dagger)^* = AA^\dagger, \, (A^\dagger A)^* = A^\dagger A,$$

and $A^\dagger$ is unique. In particular, if $AXA = A$ holds, then X is a generalized inner inverse (g-inverse) of A, denoted as A^-. The introduction of the concept of the generalized inverse of a matrix has significantly facilitated the solving of matrix equations.

2.2.3. Reflexive, $\{P, k+1\}$ Reflexive, Generalized Reflexive, And (R, S)-symmetric Matrices

In 1998, Chen [3] introduced the concepts of reflexive matrices and anti-reflexive matrices.

Definition 4. *Assume that P is a nontrivial unitary involution matrix (generalized reflection matrix),i.e., $P^* = P$ and $P^2 = I$, then A is said to be a reflexive(anti-reflexive) matrix if*

$$A \in \mathbb{C}_r^{n\times n}(P)(\mathbb{C}_a^{n\times n}(P)),$$

where $\mathbb{C}_r^{n\times n}(P) = \{A \in \mathbb{C}^{n\times n} | A = PAP\}$ and $\mathbb{C}_a^{n\times n}(P) = \{A \in \mathbb{C}^{n\times n} | A = -PAP\}$.

Remark 3. *If the matrix P obeys $P^{k+1} = P = P^*$, then P is called a generalized $\{k+1\}$-reflection matrix. Correspondingly, the matrix X satisfying $PXP = (-)X$ is called $\{P, k+1\}$ (anti-)reflexive.*

A more general definition is as follows.

Suppose that $A \in \mathbb{C}^{m\times n}, R \in \mathbb{C}^{m\times m}, S \in \mathbb{C}^{n\times n}$ and R, S are involutory Hermitian matrices. Then A is called a generalized (anti-)reflexive matrix if and only if

$$RAS = (-)A.$$

If R and S are simply involutory matrices, we have the following definition:

Definition 5. *[10] Suppose that $A \in \mathbb{C}^{m\times n}$, then A is called a (R, S)-symmetric ((R, S)-skew symmetric) matrix if*

$$RAS = A \;(RAS = -A), R \in \mathbb{C}^{m\times m}, S \in \mathbb{C}^{n\times n},$$

where $R^2 = I$ and $S^2 = I$.

2.2.4. Generalized Singular Value Decomposition

Generalized singular value decomposition(GSVD)[31]: for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times n}$, there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{p\times p}$, as well as an invertible matrix $P \in \mathbb{C}^{n\times n}$, such that

$$UAP = \left[\sum_l{}_A, O_{n-l}\right], \quad VBP = \left[\sum_l{}_B, O_{n-l}\right],$$

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where

$$\sum_A = \begin{pmatrix} I_t & & \\ & S_A & \\ & & O_A \end{pmatrix}, \quad \sum_B = \begin{pmatrix} O_B & & \\ & S_B & \\ & & I_{l-t-s} \end{pmatrix},$$
$$S_A = \text{diag}(\alpha_{t+1}, \ldots, \alpha_{t+s}), \quad S = \text{diag}(\beta_{t+1}, \ldots, \beta_{t+s}),$$

and $l = r(\binom{A}{B}), t = l - r(B), s = r(A) + r(B) - l, \alpha_i^2 + \beta_i^2 = 1, i = t+1, \cdots, t+s, 1 > \alpha_{t+1} \geq \cdots \geq \alpha_{t+s} > 0, 0 < \beta_{t+1} \leq \cdots \leq \beta_{t+s} < 1$.

2.3. Semi-Tensor Product Of Matrices

The semi-tensor product of matrices breaks the dimension limitation of traditional matrix multiplication and is more effective than traditional matrix multiplication. First, we define the Kronecker product of matrices over the fields of real number and complex number.

Definition 6. *[13] For matrices $M = (m_{ij}) \in \mathbb{R}^{p\times q}(\mathbb{C}^{p\times q})$ and $N \in \mathbb{R}^{t\times s}(\mathbb{C}^{t\times s})$, the Kronecker product of M and N is defined as*

$$M \otimes N = \begin{pmatrix} m_{11}N & m_{12}N & \cdots & m_{1q}N \\ m_{21}N & m_{22}N & \cdots & m_{2q}N \\ \vdots & \vdots & \ddots & \vdots \\ m_{p1}N & m_{p2}N & \cdots & m_{pq}N \end{pmatrix} \in \mathbb{R}^{pt\times qs}(\mathbb{C}^{pt\times qs}).$$

The operator vec$(\cdot)$ is defined to the column vector obtained by stacking its column, i.e.,

$$\text{vec}(M) = [m_{11}m_{21}\cdots m_{p1}\cdots m_{1q}m_{2q}\cdots m_{pq}]^{\text{T}}.$$

Here are some operations similar to vec$(\cdot)$.

Definition 7. *[37] Let $A \in \mathbb{R}^{n\times n}$.*

1. *Set $a_1 = A(2:n-1,1)^T, a_2 = A(3:n-2,2)^T, \cdots, a_{k-1} = A(k:n-k+1,k-1)^T, a_k = A(k+1,k)^T$, denote*

$$\text{vec}_A(A) = \begin{cases} (a_1, a_2, \cdots, a_{k-1})^T, & \text{when } n = 2k, \\ (a_1, a_2, \cdots, a_{k-1}, a_k)^T, & \text{when } n = 2k+1. \end{cases} \tag{2}$$

2. *Set $b_1 = A(1:n,1)^T, b_2 = A(2:n-1,2)^T, \cdots, b_k = A(k:n-k+1,k)^T, b_{k+1} = A(k+1,k+1)^T$, denote*

$$\text{vec}_B(A) = \begin{cases} (b_1, b_2, \cdots, b_k)^T, & \text{when } n = 2k, \\ (b_1, b_2, \cdots, b_k, b_{k+1})^T, & \text{when } n = 2k+1. \end{cases} \tag{3}$$

Assume that M, N, T, I be matrices of appropriate sizes, then

$$(M \otimes N) \otimes T = M \otimes (N \otimes T),$$
$$(MN) \otimes I = (M \otimes I)(N \otimes I)$$

and the solvability of linear matrix equation $MXN = T$ is equivalent to solve the linear systems

$$\left(N^T \otimes M\right) \text{vec}(X) = \text{vec}(T).$$

Now we give the concept of the semi-tensor product.

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Definition 8. *[14] Let $A \in \mathbb{R}^{m\times n}(\mathbb{C}^{m\times n}), B \in \mathbb{R}^{p\times q}(\mathbb{C}^{p\times q})$, the semi-tensor product of A and B is defined as*

$$A \ltimes B = (A \otimes I_{t/n})\left(B \otimes I_{t/p}\right),$$

where $\mathrm{t} = \mathrm{lcm}(\mathrm{n}, \mathrm{p})$ *is the least common multiple of* n *and* p.

Remark 4. *If $n = p$, the semi-tensor product of matrices reduces to the traditional matrix product.*

Regarding the semi-tensor product of matrices, we can easily derive the following properties:

Proposition 1. *[14]*
(1)*Suppose A, B and C are real(complex) matrices of any dimensions, then*

$$(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C).$$

(2) *Let $x \in \mathbb{R}^m(\mathbb{C}^m), y \in \mathbb{R}^n(\mathbb{C}^n)$, then*

$$x \ltimes y = x \otimes y.$$

2.4. Quaternion Matrix And Dual Quaternion Matrix

2.4.1. Quaternions

A quaternion a can be expressed in the form $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \text{ and } \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The conjugate quaternion of a is defined as $a^* = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$, The norm of a is $|a| = \sqrt{aa^*} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$.

If a map $\phi : \mathbb{H} \longrightarrow \mathbb{H}$ obeys

$$\phi(ab) = \phi(b)\phi(a),\ \phi(a+b) = \phi(a) + \phi(b), \forall a, b \in \mathbb{H},$$

then ϕ is said to be an antiendomorphism. For $a \in \mathbb{H}$, if ϕ is an antiendomorphism satisfying $\phi(\phi(a)) = a$, then we call ϕ is an involution.

Definition 9. *[7] An involution ϕ is said to be nonstandard, if and only if ϕ can be represented as a real matrix*

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}$$

under the basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$, where P is a real orthogonal symmetric matrix whose eigenvalues are $1, 1, -1$.

In fact, for $a \in \mathbb{H}$, any nonstandard involution ϕ can be written as $\phi(a) = \gamma^{-1}a^*\gamma$ for some $\gamma \in \mathbb{H}$ with $\gamma^2 = -1$.

With these fundamental definitions of quaternions, we can now introduce the definition of quaternion matrices and their related properties.

2.4.2. Quaternion Matrix

A quaternion matrix A can be written as $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in \mathbb{H}^{m\times n}$, where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$. Its conjugate transpose A^* is defined as follows:

$$A^* = A_0^T - A_1^T\mathbf{i} - A_2^T\mathbf{j} - A_3^T\mathbf{k},$$

the Frobenius norm of A is defined by

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}.$$

In the context of complex fields, we understand the concepts of centro-symmetric matrices and (R,S)-symmetric ((R,S)-skew symmetric) matrices. Similarly, there are related concepts in the realm of quaternions.

Let $A \in \mathbb{H}^{n\times n}, V_n = (e_n, \cdots, e_1), B \in \mathbb{H}^{m\times n}$. Then

1. $A = A^*$, A is called Hermitian.
2. $A = V_n A^* V_n$, A is called Persymmetric.
3. $A = V_n A V_n$ and $A = A^*$, A is called Bisymmetric (bihermitian).
4. $A = -V_n A^* V_n$ and $A = -A^*$, A is called a quaternion skew bihermitian matrix.
5. $RBS = B$ $(RBS = -B), R \in \mathbb{H}^{m\times m}$, and $S \in \mathbb{H}^{n\times n}$, B is called a (R,S)-symmetric ((R,S)-skew symmetric) matrix, where $R^2 = I$ and $S^2 = I$.

Additionally, if $A \in \mathbb{H}^{n\times n}$ satisfies $A = (-)A^{\eta^*} := (-) - \eta A^* \eta$, where $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, then A is said to be η(-anti)-Hermition matrix. Further, if $A = A^\phi$, then A is called ϕ-Hermitian matrix, where $A^\phi = \gamma^{-1} A^* \gamma$ for some $\gamma \in \mathbb{H}$ with $\gamma^2 = -1$.

For example, if ϕ obeys $\phi(\mathbf{i}) = \mathbf{i}, \phi(\mathbf{j}) = -\mathbf{j}, \phi(\mathbf{k}) = \mathbf{k}$, i.e., $\phi(A) = \mathbf{j}^{-1} A^* \mathbf{j}, A \in \mathbb{H}^{m\times n}$, then

$$\begin{pmatrix} 1+\mathbf{i} \\ \mathbf{j} \end{pmatrix}^\phi = \begin{pmatrix} 1+\mathbf{i} & -\mathbf{j} \end{pmatrix}.$$

The following properties hold for ϕ-Hermitian matrices.

Proposition 2. *[8] Suppose $A \in \mathbb{H}^{m\times n}$, then*
(1) $(A^\phi)^\dagger = (A^\dagger)^\phi$;
(2) $(L_A)^\phi = R_{A^\phi}$;
(3) $(R_A)^\phi = L_{A^\phi}$.

Penrose provided four matrix equations that define the conditions for a matrix to have a generalized inverse over the complex field, and these conditions also hold over quaternions. Let $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$. We can easily derive the following properties.

Proposition 3. *[5] Suppose that $A \in \mathbb{H}^{m\times n}$, then*

1. $\left(A^{\eta^*}\right)^\dagger = (A^\dagger)^{\eta^*}$.
2. $r(A) = r\left(A^{\eta^*}\right)$.
3. $(L_A)^{\eta^*} = -\eta(L_A)\eta = (L_A)^\eta = L_{A^*} = R_{A^{\eta^*}}$.
4. $(R_A)^{\eta^*} = -\eta(R_A)\eta = (R_A)^\eta = R_{A^*} = L_{A^{\eta^*}}$.

For $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in \mathbb{H}^{m\times n}$, we give its real representation matrix A^R as below:

$$A^R := \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ -A_1 & A_0 & -A_3 & A_2 \\ -A_2 & A_3 & A_0 & -A_1 \\ -A_3 & -A_2 & A_1 & A_0 \end{pmatrix}.$$

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For convenience of description, we denote $A^R_{r_i}$ and $A^R_{c_j}$ as the i-th row block and j-th column block of A^R, respectively. The real representation matrix of A is not unique. For instance,

$$A^R = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix} \in \mathbb{R}^{4m\times 4n} \tag{4}$$

is also a real representation of A. See Reference [6] for details. Certainly, we can derive

$$\|A\| = \frac{1}{2}\left\|A^R\right\| = \left\|A^R_{r_i}\right\| = \left\|A^R_{c_i}\right\|, \quad i = 1, \cdots, 4,$$

and some of the following properties.

Proposition 4. *[9] Assume that $A, B \in \mathbb{H}^{m\times n}, C \in \mathbb{H}^{n\times l}, k \in \mathbb{R}, i = 1, \cdots, 4$, then*

1. $A = B \Leftrightarrow A^R = B^R \Leftrightarrow A^R_{r_i} = B^R_{r_i} \Leftrightarrow A^R_{c_i} = B^R_{c_i}$.
2. $(A+B)^R_{r_i} = A^R_{r_i} + B^R_{r_i}, \quad (A+B)^R_{c_i} = A^R_{c_i} + B^R_{c_i}$.
3. $(kA)^R_{r_i} = kA^R_{r_i}, \quad (kA)^R_{c_i} = kA^R_{c_i}$.
4. $(AC)^R_{r_i} = A^R_{r_i}C^R, \quad (AC)^R_{c_i} = A^R C^R_{c_i}$.

Let

$$Q_n = \begin{bmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{bmatrix}, G_n = \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ -I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \end{bmatrix}, T_n = \begin{bmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

Then we have the following properties.

Proposition 5. *[43] Suppose that $A, B \in \mathbb{H}^{n\times n}$ and $a \in \mathbb{R}$, then*

1. $(A+B)^R = A^R + B^R, (aA)^R = aA^R$;
2. $(AB)^R = A^R B^R$;
3. $Q_n^T A^R Q_n = A^R, G_n^T A^R G_n = A^R, T_n^T A^R T_n = A^R$;
4. $(A^*)^R = \left(A^R\right)^T, \left(A^{-1}\right)^R = \left(A^R\right)^{-1}$;
5. *A^R commutes with Q_n, G_n, and T_n with respect to multiplication.*

2.4.3. Dual Quaternion Matrix

The set of dual quaternion matrices is defined as

$$\mathbb{DQ}^{m\times n} := \{Y = Y_0 + Y_1\epsilon | Y_0, Y_1 \in \mathbb{H}^{m\times n}\},$$

where Y_0, Y_1 represent the standard part and the infinitesimal part of Y, respectively. The infinitesimal unit ϵ obeys $\epsilon^2 = 0$ and commutes under multiplication with real numbers, complex numbers, and quaternions. Below are some basic operations on dual quaternion matrices. For $A = A_0 + A_1\epsilon, C = C_0 + C_1\epsilon \in \mathbb{DQ}^{m\times n}, B = B_0 + B_1\epsilon\mathbb{DQ}^{n\times k}$, then we have

$$\begin{aligned} A + C &= A_0 + C_0 + (A_1 + C_1)\epsilon, \\ AB &= A_0B_0 + (A_0B_1 + A_1B_0)\epsilon. \end{aligned}$$

We know that there are some special matrices over quaternions, and there are also some special matrices over dual quaternions.

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A dual quaternion matrix $A = A_0 + A_1\epsilon \in \mathbb{DQ}^{n\times n}$ is called ϕ-Hermitian if $A = A^\phi$, where A^ϕ is defined as

$$A^\phi := \gamma^{-1}A^*\gamma = \gamma^{-1}A_0^*\gamma + \gamma^{-1}A_1^*\gamma\epsilon = A_0^\phi + A_1^\phi\epsilon,$$

with $\gamma \in \mathbb{H}$ and $\gamma^2 = -1$. It has the following properties.

Proposition 6. *[8] Suppose that $A, B \in \mathbb{DQ}^{n\times n}$, then*

1. $(A+B)^\phi = A^\phi + B^\phi$.
2. $(AB)^\phi = B^\phi A^\phi$.
3. $(A^\phi)^\phi = A$.

2.5. Dual Split Quaternion Matrix

2.5.1. Split Quaternion Matrix

A split quaternion matrix A can be written as $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in \mathbb{H}_s^{m\times n}$, where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$, and

$$\mathbf{i}^2 = -\mathbf{j}^2 = -\mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = -\mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \tag{6}$$

the conjugate transpose A^* is defined as follows:

$$A^* = A_0^T - A_1^T\mathbf{i} - A_2^T\mathbf{j} - A_3^T\mathbf{k},$$

in addition, we define the i-conjugate and i-conjugate transpose of A as follows:

$$A^i = \mathbf{i}^{-1}A\mathbf{i} = A_1 + A_2\mathbf{i} - A_3\mathbf{j} - A_4\mathbf{k},$$
$$A^{i*} = -\mathbf{i}A^*\mathbf{i} = A_1{}^T - A_2{}^T\mathbf{i} + A_3{}^T\mathbf{j} + A_4{}^T\mathbf{k}.$$

It is evident that $A^{i*} = (A^*)^i = (A^i)^*$.

The real representation method is of great significance in solving the problem of split quaternion matrix equations. For $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in \mathbb{H}_s^{m\times n}$, two real representations of A are defined as follows:

$$A^{\sigma_1} := \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ A_3 & -A_4 & A_1 & -A_2 \\ A_4 & A_3 & A_2 & A_1 \end{pmatrix}, \; A^{\sigma_i} := U_m A^{\sigma_1} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ -A_3 & A_4 & -A_1 & A_2 \\ -A_4 & -A_3 & -A_2 & -A_1 \end{pmatrix},$$

where

$$U_m = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix}. \tag{7}$$

Denote

$$P_m = \begin{pmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_m \\ I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \end{pmatrix}, W_m = \begin{pmatrix} 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \end{pmatrix}, R_m = \begin{pmatrix} 0 & 0 & 0 & I_m \\ 0 & 0 & I_m & 0 \\ 0 & I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

then we can derive the following properties.

Proposition 7. [11] Assume that $A, B \in \mathbb{H}_s^{m\times n}, C \in \mathbb{H}_s^{n\times q}$, and $k \in \mathbb{R}$, then we have the following conclusions.

1. $A = B \Longleftrightarrow A^{\sigma_1} = B^{\sigma_1}$, $A = B \Longleftrightarrow A^{\sigma_i} = B^{\sigma_i}$.
2. $(A+B)^{\sigma_1} = A^{\sigma_1} + B^{\sigma_1}$, $(kA)^{\sigma_1} = kA^{\sigma_1}$, $(A+B)^{\sigma_i} = A^{\sigma_i} + B^{\sigma_i}$, $(kA)^{\sigma_i} = kA^{\sigma_i}$.

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3. $(AC)^{\sigma_1} = A^{\sigma_1}C^{\sigma_1}$, $(AC)^{\sigma_i} = A^{\sigma_i}U_nC^{\sigma_i}$.
4. (i) $P_m{}^T A^{\sigma_1} P_n = A^{\sigma_1}, W_m{}^T A^{\sigma_1} W_n = A^{\sigma_1}, R_m{}^T A^{\sigma_1} R_n = A^{\sigma_1}$.
   (ii) $P_m{}^T A^{\sigma_i} P_n = -A^{\sigma_i}, W_m{}^T A^{\sigma_i} W_n = A^{\sigma_i}, R_m{}^T A^{\sigma_i} R_n = -A^{\sigma_i}$.
5. (i) $A = \frac{1}{2}\begin{pmatrix} I_m & I_m\mathbf{i} & I_m\mathbf{j} & I_m\mathbf{k} \end{pmatrix} A^{\sigma_1} \begin{pmatrix} I_n \\ I_n\mathbf{i} \\ I_n\mathbf{j} \\ I_n\mathbf{k} \end{pmatrix}$.
   (ii) $A = \frac{1}{2}\begin{pmatrix} I_m & I_m\mathbf{i} & I_m\mathbf{j} & I_m\mathbf{k} \end{pmatrix} A^{\sigma_i} \begin{pmatrix} -I_n \\ -I_n\mathbf{i} \\ -I_n\mathbf{j} \\ -I_n\mathbf{k} \end{pmatrix}$.
6. $(A^*)^{\sigma_i} = (A^{\sigma_i})^T$, $(A^i)^{\sigma_i} = U_m A^{\sigma_i} U_n$.

2.5.2. Dual Split Quaternion Matrix

The set of dual split quaternion matrices is defined as

$$\mathbb{DH}_s^{m\times n} := \{Y = Y_0 + Y_1\epsilon | Y_0, Y_1 \in \mathbb{H}_s^{m\times n}\},$$

where Y_0, Y_1 represent the standard part and the infinitesimal part of Y, respectively. The infinitesimal unit ϵ obeys $\epsilon^2 = 0$ and commutes under multiplication with real numbers, complex numbers, quaternions, and split quaternions.

For $A = A_0 + A_1\epsilon \in \mathbb{DH}_s^{m\times n}$, the Hamiltonian conjugate of A is denoted as $\overline{A} = \overline{A_0} + \overline{A_1}\epsilon \in \mathbb{DH}_s^{m\times n}$. The transpose of A is represented by $A^T = A_0{}^T + A_1{}^T\epsilon \in \mathbb{DH}_s^{n\times m}$, and the conjugate transpose of A, denoted by A^*, is given by $A^* = A_0{}^* + A_1{}^*\epsilon \in \mathbb{DH}_s^{n\times m}$.

Similar to dual quaternion matrices, dual split quaternion matrices also have the following basic operational properties. For $A = A_0 + A_1\epsilon, C = C_0 + C_1\epsilon \in \mathbb{DH}_s^{m\times n}, B = B_0 + B_1\epsilon \in \mathbb{DH}_s^{n\times k}$, then

$$A + C = A_0 + C_0 + (A_1 + C_1)\epsilon,$$
$$AB = A_0B_0 + (A_0B_1 + A_1B_0)\epsilon.$$

2.6. Dual Generalized Commutative Quaternion Matrix

2.6.1. Generalized Commutative Quaternion Matrix

A generalized commutative quaternion matrix A has the form $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in \mathbb{H}_g^{m\times n}$, where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisty

$$\mathbf{i}^2 = \alpha, \mathbf{j}^2 = \beta, \mathbf{k}^2 = \alpha\beta, \mathbf{ij} = \mathbf{ji} = \mathbf{k}, \mathbf{jk} = \mathbf{kj} = \beta\mathbf{i}, \mathbf{ki} = \mathbf{ik} = \alpha\mathbf{j}.$$

Here $\alpha, \beta \in \mathbb{R}\backslash\{0\}$. The concept of generalized commutative quaternions was introduced by Tian et al. [102] in 2023. In particular, when $\alpha = -1$ and $\beta = 1$, the generalized commutative quaternion matrix A simplifies to the commutative quaternion matrix A.

Similar to split quaternion matrices, generalized commutative quaternion matrices also have real matrix representations. For $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in \mathbb{H}_g^{m\times n}$, it has the following three real matrix representations:

$$A^{\sigma_\mathbf{i}} = \begin{bmatrix} A_0 & \alpha A_1 & \beta A_2 & \alpha\beta A_3 \\ A_1 & A_0 & \beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & \alpha A_1 \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix}, \; A^{\sigma_\mathbf{j}} = V_m A^{\sigma_\mathbf{i}}, \; A^{\sigma_\mathbf{k}} = U_m A^{\sigma_\mathbf{i}},$$

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where U_m is defined in Equation (7) and

$$V_m = \begin{bmatrix} -I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_m \end{bmatrix}. \tag{9}$$

Set

$$\begin{aligned} G_n^1 &= \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & -\alpha I_n & 0 & 0 \\ 0 & 0 & \beta I_n & 0 \\ 0 & 0 & 0 & -\alpha\beta I_n \end{bmatrix}, \ R_n^1 = \begin{bmatrix} 0 & \alpha I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha I_n \\ 0 & 0 & I_n & 0 \end{bmatrix}, \\ S_n^1 &= \begin{bmatrix} 0 & 0 & \beta I_n & 0 \\ 0 & 0 & 0 & \beta I_n \\ I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \end{bmatrix}, \ T_n^1 = \begin{bmatrix} 0 & 0 & 0 & \alpha\beta I_n \\ 0 & 0 & \beta I_n & 0 \\ 0 & \alpha I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{10}$$

Thus, we can derive the following properties.

Proposition 8. *[98] Let $A, B \in \mathbb{H}_g^{m\times n}, C \in \mathbb{H}_g^{n\times s}$, and $k \in \mathbb{R}$. Then the following conclusions hold.*

1. $A = B \Leftrightarrow A^{\sigma_\eta} = B^{\sigma_\eta}, \eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.
2. $(A+B)^{\sigma_\eta} = A^{\sigma_\eta} + B^{\sigma_\eta}, \eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.
3. $(kA)^{\sigma_\eta} = kA^{\sigma_\eta}, \eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.
4. $(AC)^{\sigma_\mathbf{i}} = A^{\sigma_\mathbf{i}} C^{\sigma_\mathbf{i}}, (AC)^{\sigma_\mathbf{j}} = A^{\sigma_\mathbf{j}} V_n C^{\sigma_\mathbf{j}}, (AC)^{\sigma_\mathbf{k}} = A^{\sigma_\mathbf{k}} U_n C^{\sigma_\mathbf{k}}$.
5. (a) $(R_m^1)^{-1} A^{\sigma_\mathbf{i}} R_n^1 = A^{\sigma_\mathbf{i}}, (S_m^1)^{-1} A^{\sigma_\mathbf{i}} S_n^1 = A^{\sigma_\mathbf{i}}, (T_m^1)^{-1} A^{\sigma_\mathbf{i}} T_n^1 = A^{\sigma_\mathbf{i}}$.
   (b) $(R_m^1)^{-1} A^{\sigma_\mathbf{j}} R_n^1 = -A^{\sigma_\mathbf{j}}, (S_m^1)^{-1} A^{\sigma_\mathbf{j}} S_n^1 = -A^{\sigma_\mathbf{j}}, (T_m^1)^{-1} A^{\sigma_\mathbf{j}} T_n^1 = A^{\sigma_\mathbf{j}}$.
   (c) $(R_m^1)^{-1} A^{\sigma_\mathbf{k}} R_n^1 = A^{\sigma_\mathbf{k}}, (S_m^1)^{-1} A^{\sigma_\mathbf{k}} S_n^1 = -A^{\sigma_\mathbf{k}}, (T_m^1)^{-1} A^{\sigma_\mathbf{k}} T_n^1 = -A^{\sigma_\mathbf{k}}$.
6. (a) $A = \frac{1}{4}\begin{bmatrix} I_m & I_m\mathbf{i} & I_m\mathbf{j} & I_m\mathbf{k} \end{bmatrix} A^{\sigma_\mathbf{i}} \begin{bmatrix} I_n \\ \frac{1}{\alpha} I_n\mathbf{i} \\ \frac{1}{\beta} I_n\mathbf{j} \\ \frac{1}{\alpha\beta} I_n\mathbf{k} \end{bmatrix}$.
   (b) $A = \frac{1}{4}\begin{bmatrix} -I_m & I_m\mathbf{i} & I_m\mathbf{j} & -I_m\mathbf{k} \end{bmatrix} A^{\sigma_\mathbf{j}} \begin{bmatrix} I_n \\ \frac{1}{\alpha} I_n\mathbf{i} \\ \frac{1}{\beta} I_n\mathbf{j} \\ \frac{1}{\alpha\beta} I_n\mathbf{k} \end{bmatrix}$.
   (c) $A = \frac{1}{4}\begin{bmatrix} I_m & I_m\mathbf{i} & -I_m\mathbf{j} & -I_m\mathbf{k} \end{bmatrix} A^{\sigma_\mathbf{k}} \begin{bmatrix} I_n \\ \frac{1}{\alpha} I_n\mathbf{i} \\ \frac{1}{\beta} I_n\mathbf{j} \\ \frac{1}{\alpha\beta} I_n\mathbf{k} \end{bmatrix}$.

2.6.2. Dual Generalized Commutative Quaternion Matrix

We use

$$\mathbb{DH}_g^{m\times n} := \{A = A_0 + A_1\epsilon | A_0, A_1 \in \mathbb{H}_g^{m\times n}\},$$

where the infinitesimal unit ϵ satisfys $\epsilon^2 = 0$, to represent all $m \times n$ dual generalized commutative quaternion matrices. The definitions of addition, multiplication, and equality for two dual generalized commutative quaternion matrices are similar to those for dual quaternions and dual split quaternions, and thus will not be provided here.

2.7. Tensor

A tensor
$$\mathcal{A} = (a_{i_1\cdots i_N})_{1\leqslant i_j\leqslant I_j}(j = 1, \ldots, N)$$
of order N is a multidimensional array with $I_1 \times \cdots \times I_N$ entries, where N is a positive integer.

The sets of tensors of order N with dimension $I_1 \times \cdots \times I_N$ over the complex field $\mathbb{C}$, the real field $\mathbb{R}$, and the real quaternion algebra are represented, respectively, by $\mathbb{C}^{I_1\times\cdots\times I_N}, \mathbb{R}^{I_1\times\cdots\times I_N}$, and $\mathbb{H}^{I_1\times\cdots\times I_N}$. Specially, when $N = 2$, the tensor is a matrix.

For a quaternion tensor
$$\mathcal{A} = \left(a_{i_1\cdots i_N j_1\cdots j_M}\right) \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M},$$
let
$$\mathcal{B} = \left(b_{i_1\cdots i_M j_1\cdots j_N}\right) \in \mathbb{H}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N}$$
be the conjugate transpose of $\mathcal{A}$, where
$$b_{i_1\cdots i_M j_1\cdots j_N} = \bar{a}_{i_1\cdots i_N j_1\cdots j_M},$$
and the tensor $\mathcal{B}$ is denoted by $\mathcal{A}^*$.

A "square" tensor $\mathcal{D} = \left(d_{i_1\cdots i_N i_1\cdots i_N}\right) \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is called a diagonal tensor if all its entries are zero except for $d_{i_1\cdots i_N i_1\cdots i_N}$. If all the diagonal entries $d_{i_1\cdots i_N i_1\cdots i_N} = 1$, then $\mathcal{D}$ is a unit tensor, denoted by $\mathcal{I}$. The zero tensor with suitable order is denoted by 0 .

Next, we give the definition of the Einstein product of tensors.

Definition 10. *[15] For $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_P\times K_1\times\cdots\times K_N}, \mathcal{B} \in \mathbb{H}^{K_1\times\cdots\times K_N\times J_1\times\cdots\times J_M}$, the Einstein product of tensors $\mathcal{A}$ and $\mathcal{B}$ is defined by the operation $*_N$ via the following:*
$$(\mathcal{A} *_N \mathcal{B})_{i_1\cdots i_P j_1\cdots j_M} = \sum_{1\leqslant k_1\leqslant K_1,\ldots,1\leqslant k_N\leqslant K_N} a_{i_1\cdots i_P k_1\cdots k_N} b_{k_1\cdots k_N j_1\cdots j_M},$$
where $\mathcal{A} *_N \mathcal{B} \in \mathbb{H}^{I_1\times\cdots\times I_P\times J_1\times\cdots\times J_M}$.

Remark 5. *When $N = P = M = 1$, we have $\mathcal{A}, \mathcal{B}$ are quaternion matrices, and their Einstein product is the usual matrix product.*

Now, we provide the Moore-Penrose inverse of quaternion tensors via Einstein product.

Definition 11. *[12] For a tensor $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_N}$, the tensor*
$$\mathcal{X} \in \mathbb{H}^{K_1\times\cdots\times K_N\times I_1\times\cdots\times I_N}$$
satisfying

(1) $\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}$,
(2) $\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}$,
(3) $(\mathcal{A} *_N \mathcal{X})^* = \mathcal{A} *_N \mathcal{X}$,
(4) $(\mathcal{X} *_N \mathcal{A})^* = \mathcal{X} *_N \mathcal{A}$,

is called the Moore-Penrose inverse of $\mathcal{A}$, abbreviated by M-P inverse, denoted by $\mathcal{A}^\dagger$.

Furthermore, $\mathcal{L}_\mathcal{A}$ and $\mathcal{R}_\mathcal{A}$ stand for the two projectors
$$\mathcal{L}_\mathcal{A} = \mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}, \quad \mathcal{R}_\mathcal{A} = \mathcal{I} - \mathcal{A} *_N \mathcal{A}^\dagger,$$

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induced by $\mathcal{A}$, respectively. We say that the tensor $\mathcal{B} \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is the inverse of tensor $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, if

$$\mathcal{A} *_N \mathcal{B} = \mathcal{I} = \mathcal{B} *_N \mathcal{A},$$

and we denote $\mathcal{B} = \mathcal{A}^{-1}$.

Reducible matrices are intimately linked to the connectivity of directed graphs and have found diverse applications in various fields. Below, we introduce the concept of a tensor being k-reducible. Before that, we will first provide the definition of a permutation tensor.

For $\mathcal{A} \in \mathbb{H}^{J_1\times\cdots\times J_M\times J_1\times\cdots\times J_M}$, if it has a matricized form B that is a permutation matrix, then $\mathcal{A}$ is said to be a permutation tensor.

Definition 12. *[12] A tensor $\mathcal{A} \in \mathbb{H}^{\alpha_1\times\cdots\times\alpha_N\times\alpha_1\times\cdots\times\alpha_N}$ is said to be $\mathcal{K}$-reducible, if there exists a permutation tensor $\mathcal{K}$ such that $\mathcal{A}$ is permutation similar to an M-upper (lower) triangular block tensor,*

$$\mathcal{A} = \mathcal{K} *_N \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ 0 & \mathcal{B}_3 \end{bmatrix} *_N \mathcal{K}^{-1},$$

where

$$\mathcal{B}_1 \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}, \quad \mathcal{B}_2 \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}, \mathcal{B}_3 \in \mathbb{H}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N},$$
$$\mathcal{K} \in \mathbb{H}^{\alpha_1\times\cdots\times\alpha_N\times\alpha_1\times\cdots\times\alpha_N}, \quad \alpha_i = I_i + J_i, i = 1,\ldots,N.$$

3. Various Solutions Of Matrix Equation $AXB = C$

In 1955, Penrose defined the generalized inverse of a matrix using four matrix equations. He provided necessary and sufficient conditions for the solvability of the matrix equation $AXB = C$ using the generalized inverse of matrices, along with an expression for the general solution.

Theorem 1. *[4] Let A, B, and C be given with appropriate sizes over $\mathbb{C}$. Then $AXB = C$ is solvable if and only if*

$$AA^{\dagger}CB^{\dagger}B = C.$$

In this case, the general solution can be expressed as

$$X = A^{\dagger}CB^{\dagger} + W - A^{\dagger}AWBB^{\dagger},$$

where W is arbitrary.

Remark 6. *By applying Theorem 1, the Moore-Penrose inverse can be used to provide the solvability conditions and the expression for the general solution of*

$$Ax = b.$$

For more details, refer to Reference [4].

In the field of matrix analysis, Penrose's theory of the generalized inverse matrix provides a powerful tool for solving linear matrix equations. In certain specific fields, the solution matrix of an matrix equation is required to be a special type of matrix. For example, in the estimation of covariance components in statistical models [18], load slow analysis and short-circuit studies in power systems [19], the solution matrix needs to be Hermitian or nonnegative definite. Reflexive and antireflexive matrices are applied in the fields of engineering and scientific computations [3]. The η-Hermitian matrix is significantly utilized in applications related to statistical signal processing [20]. Reducible matrices are utilized in a range of applications, including Markov chains [22], compartmental analysis [23], continuous-time positive systems [24], among others. Orthogonal solutions are required for covariance assignment [25] and data matching problems in multivariate analysis [26,27]. In addition,

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the bisymmetric matrix has been widely recognized for its applications in information theory, Markov processes, physical engineering, and many other fields. Consequently, many scholars have studied various special solutions to the matrix Equation (1).

- In 1976, Khatri and Mitra [17] studied the solvability conditions and general solution expressions for the matrix Equation (1) with Hermitian and nonnegative definite solutions. Subsequently, in 2004, Zhang [28] investigated Hermitian nonnegative-definite and positive-definite solutions of this equation. Later, Wang et al. [29] and Cvetković-ilić [30] explored Re-nonnegative definite solutions of the same equation. Since *congruence encompasses Hermitian, positive definite, and positive semidefinite matrices, Zheng et al. [44] studied the *congruence class of the solutions to this matrix equation in 2009.
- Employing the GSVD, Hua [32] and Liao [33] investigated the symmetric solutions and symmetric positive semidefinite least-squares solutions of matrix Equation (1) over $\mathbb{R}$. Additionally, in 2022, Hu et al. [34] studied the symmetric solutions of this matrix equation within a specific subspace over the real number field. The quaternion is an extension of real and complex numbers with broad applications. Accordingly, Liu [38] explored the η-Hermitian solution of this matrix equation. In addition, Wang et al. [35] and Zhang et al. [36] studied the least squares bisymmetric solutions and the skew bihermitian solutions of matrix Equation (1) over $\mathbb{H}$.
- In 2006, D.S. Cvetković-ilić [39] studied the reflexive and anti-reflexive solutions of the matrix Equation (1) over the complex field. In 2011, Herrero et al. [41] investigated the $\{P, k+1\}$ reflexive and anti-reflexive solutions of the same equation over $\mathbb{C}$. Building on these efforts, Liu et al. [42] explored the minimum norm least squares Hermitian (anti-)reflexive solutions of this equation in 2017. Subsequently, Yuan et al. [40] examined generalized reflexive solutions of this matrix equation over the complex field. In 2024, Liao et al. [43] extended this line of research by studying the (R, S)-symmetric solutions of this matrix equation over $\mathbb{H}$, which encompass generalized reflexive solutions.
- In 2018, Yang et al. [45] studied the Hankel solutions and various Toeplitz solutions of matrix Equation (1) over $\mathbb{R}$. In 2022, Zhang et al. [46] investigated the orthogonal solutions of this matrix equation in the complex field. Moreover, since tensors are higher-dimensional matrices with broader applications, Xie et al. [12] studied the K-reducible solutions of this matrix equation in quaternion tensors.

Furthermore, the study of the matrix Equation (1) using matrix ranks has attracted significant interest from researchers. Many scholars have also focused on solutions to this equation under specific conditions, such as when A, B, and C are operators, when the traditional matrix product is replaced by the semi-tensor product, or when the elements of the matrices come from a principal ideal domain, $\mathbb{DQ}$, $\mathbb{DH}_s$, or $\mathbb{DH}_g$.

- Previous studies on various specific solutions to matrix Equation (1) have mostly been based on the assumption that A, B and C are matrices. We know that matrices can be viewed as a special type of operator. Thus, in 2010, Arias et al. [47] explored the existence of positive solutions to this operator equation without this additional assumption. Building on this work, Cvetković-Ilić et al. [48] further investigated the positive solutions of this operator equation in 2019.
- In addition, some scholars have employed matrix rank to investigate various aspects of matrix Equation (1). For example, Porter et al. (1979) [51] studied the number of solutions to this matrix equation over a given finite field. In 2007, Liu [52] explored the problems of maximal and minimal ranks for the least-squares solutions of this equation over the complex field. Subsequently, Zhang et al. [54] extended the study to the maximal and minimal ranks of submatrices of the least-squares solutions over $\mathbb{C}$. In 2010, Wang et al. [53] investigated the maximal and minimal ranks of the four real matrices involved in the quaternion solution of this equation.
- The traditional matrix product imposes requirements on the dimensions of the two matrices involved, while the semi-tensor product removes these restrictions and has broad applications.

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Consequently, in 2019, Ji et al. [13] studied matrix Equation (1) over the field of real numbers under the semi-tensor product. In 2020, Prokip [56] investigated this matrix equation over a principal ideal domain. $\mathbb{DQ}$, $\mathbb{DH}_s$ and $\mathbb{DH}_g$ are extensions of $\mathbb{Q}$, $\mathbb{H}_s$ and $\mathbb{H}_g$, respectively. Therefore, in 2024, Chen et al. [8] investigated this matrix equation over $\mathbb{DQ}$, while Si et al. [11] explored it over $\mathbb{DH}_s$, and Shi et al. [98] concentrated on $\mathbb{DH}_g$.

Let $R(X), N(X)$, and $\mathrm{tr}(X)$ denote the column space, null space, and trace of the matrix X, respectively. For $A \in \mathbb{C}^{m\times n}$, let $I - A^-A$ and $I - AA^-$ be denoted as L_{g_A} and R_{g_A}, respectively.

Hermitian, nonnegative definite, and Re-nonnegative definite solutions

In 1976, Khatri used the g-inverse to provide the necessary and sufficient conditions for the matrix Equation (1) to have Hermitian and nonnegative definite solutions, along with the corresponding expressions for the general solutions.

Theorem 2. *[17] Suppose that $A, B, C \in \mathbb{C}^{n\times n}$ and $A \geq 0, B \geq 0$, then we have the following conclusions:*

1. *The matrix Equation (1) has a Hermitian solution iff*

$$B(A+B)^-C(A+B)^-A$$

is Hermitian. In this case, the general Hermitian solution can be expressed as

$$\begin{aligned} X =&(A+B)^-(C+C^*+Y+Z)\left[(A+B)^-\right]^*+W \\ &-(A+B)^-(A+B)W(A+B)\left[(A+B)^-\right]^*, \end{aligned}$$

where W is an arbitrary Hermitian matrix with appropriate sizes, and Y, Z are arbitrary Hermitian solutions of the matrix equations

$$\begin{aligned} Y(A+B)^-B &= C(A+B)^-A, \\ A(A+B)^-Z &= B(A+B)^-C. \end{aligned} \tag{11}$$

2. *The matrix Equation (1) has a nonnegative define solution iff*

$$B(A+B)^-C(A+B)^-A$$

is nonnegative define and

$$r(T) = r(A(A+B)^-C^*) = r(B(A+B)^-C).$$

In this case, the general nonnegative define solution can be expressed as

$$\begin{aligned} X =&(A+B)^-(C+C^*+Y+Z)\left[(A+B)^-\right]^* \\ &+L_{g_{A+B}}WL^*_{g_{A+B}}, \end{aligned}$$

where $W \in \mathbb{C}^{n\times n}$ is an arbitrary nonnegative define matrix, and Y, Z are arbitrary nonnegative define solutions of (11) such that $C+C^*+Y+Z$ is nonnegative define.

For the solution of (11), refer to Reference [17], it will not be provided here. Subsequently, in 2004, Zhang [28] employed matrix decomposition methods to present the necessary and sufficient conditions for matrix Equation (1) to have Hermitian nonnegative-definite and Hermitian positive-definite solutions, along with the general solution expression. Let $H_n^{\geq}, H_n^{>}$ be the set of n-by-n Hermitian nonnegative-definite matrices, and the set of n-by-n Hermitian positive-definite matrices, respectively.

Theorem 3. *[28] Assume that $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times p}, C \in \mathbb{C}^{m\times p}$ obeying*

$$0 \le m = r(A) \le r(B) = p \le n.$$

There exist an integer s and matrices P, Q, and T such that

$$P^{-1}CT^{-1} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}, C_1 \in \mathbb{C}^{(p-s)\times(p-s)}.$$

Then we can derive the following two conclusions:

1. *The matrix Equation (1) has a Hermitian nonnegative-definite solution iff*

$$C_1 \in H^{\geq}_{p-s}, R\left(\begin{bmatrix} C_3^* & C_2 \end{bmatrix}\right) \subseteq R(C_1).$$

In this case, the general Hermitian nonnegative define solution can be expressed as

$$X = Q^{-1}\begin{bmatrix} C_1 & C_3^* & C_2 & X_{14} \\ C_3 & X_{22} & C_4 & X_{24} \\ C_2^* & C_4^* & X_{33} & X_{34} \\ X_{14}^* & X_{24}^* & X_{34}^* & X_{44} \end{bmatrix}(Q^*)^{-1}, \tag{12}$$

where $X_{14}, X_{22}, X_{24}, X_{33}, X_{34}$, and X_{44} satisfy

$$\begin{cases} R(X_{14}) \subseteq R(C_1), \\ R([C_0 \;\; Y_2]) \subseteq R(Y_1), \\ R(Z) \subseteq R\left(Y_3 - C_0^* Y_1^\dagger C_0\right), \end{cases} \text{and} \begin{cases} Y_1 \in H^{\geq}_{m-p+s}, \\ Y_3 - C_0^* Y_1^\dagger C_0 \in H^{\geq}_s, \\ Y_5 - Y_2^* Y_1^\dagger Y_2 - Z^*\left(Y_3 - C_0^* Y_1^\dagger C_0\right)^\dagger Z \in H^{\geq}_{n-m-s} \end{cases}$$

with

$$\begin{cases} C_0 = C_4 - C_3 C_1^\dagger C_2, \\ Y_1 = X_{22} - C_3 C_1^\dagger C_3^*, \\ Y_2 = X_{24} - C_3 C_1^\dagger X_{14}, \\ Y_3 = X_{33} - C_2^* C_1^\dagger C_2, \\ Y_4 = X_{34} - C_2^* C_1^\dagger X_{14}, \\ Y_5 = X_{44} - X_{14}^* C_1^\dagger X_{14}, \\ Z = Y_4 - C_0^* Y_1^\dagger Y_2. \end{cases} \tag{13}$$

2. *The matrix Equation (1) has a Hermitian positive-definite solution iff*

$$C_1 \in H^{>}_{p-s}.$$

In this case, the general Hermitian positive-define solution can be expressed as (12), where $X_{14}, X_{22}, X_{24}, X_{33}, X_{34}$, and X_{44} satisfy

$$\begin{cases} Y_1 \in H^{>}_{m-p+s}, \\ Y_3 - C_0^* Y_1^\dagger C_0 \in H^{>}_s, \\ Y_5 - Y_2^* Y_1^\dagger Y_2 - Z^*\left(Y_3 - C_0^* Y_1^\dagger C_0\right)^\dagger Z \in H^{>}_{n-m-s} \end{cases}$$

with (13).

For details on the integer s and matrices P, Q, and T mentioned in Theorem 3, refer to Reference [28].

In 1998, Wang et al. [29] presented the necessary and sufficient conditions for the solvability of matrix Equation (1) and the general solution expression using the GSVD.

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Theorem 4. *[29] Assume that $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times q}$, and $C \in \mathbb{C}^{m\times q}$ are given. Set*

$$P^{-1}X(P^{-1})^* = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{33} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} \begin{matrix} t \\ s \\ l-t-s \\ n-l \end{matrix},$$
$$\begin{matrix} t & s & l-r-s & n-l \end{matrix}$$

$$UCV^* = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \begin{matrix} t \\ s \\ m-t-s \end{matrix}.$$
$$\begin{matrix} n-l+t & s & l-t-s \end{matrix}$$

1. *The matrix Equation (1) is consistent if and only if*

$$C_{i1} = 0, \quad i = 1,2; \quad C_{3j} = 0, \quad j = 1,2,3.$$

In this case, the general solution can be expressed as

$$X = P \begin{pmatrix} X_{11} & C_{12}S_B^{-1} & C_{13} & X_{14} \\ X_{21} & S_A^{-1}C_{22}S_B^{-1} & S_A^{-1}C_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} P^*,$$

where $X_{i1}, X_{i4} (i = 1,2), X_{3j}, X_{4j} (j = 1, \cdots, 4)$ are arbitrary matrices with appropriate sizes over $\mathbb{C}$.

2. *The matrix Equation (1) has a Re-nnd solution if and only if*

$$C_{i1} = 0, \quad i = 1,2, \quad C_{3j} = 0, \quad j = 1,2,3, \text{ and } S_A^{-1}C_{22}S_B^{-1} \text{ is Re-nnd.}$$

In this case, the general Re-nnd solution can be expressed as

$$X = P \begin{pmatrix} M & N \\ -N^* + T^*(M + M^*) & D + T^*MT \end{pmatrix} P^*,$$

where

$$M = \begin{pmatrix} D_2 + T_2^*\left(S_A^{-1}C_{22}S_B^{-1}\right)T_2 & C_{12}S_B^{-1} & C_{13} \\ F & S_A^{-1}C_{22}S_B^{-1} & S_A^{-1}C_{23} \\ X_{31} & G & D_1 + T_1^*S_A^{-1}C_{22}S_B^{-1}T_1 \end{pmatrix},$$

and

$$F = -S_B^{-1}C_{12}^* + \left(S_A^{-1}C_{22}S_B^{-1} + S_B^{-1}C_{22}^*S_A^{-1}\right)T_2,$$
$$G = -C_{23}^*S_A^{-1} + T_1^*\left(S_A^{-1}C_{22}S_B^{-1} + S_B^{-1}C_{22}^*S_A^{-1}\right),$$

with $M \in Re^l, X_{31} \in \mathbb{C}^{(l-t-s)\times t}, D_1 \in Re^{l-t-s}, D_2 \in Re^t, D \in Re^{n-l}$, $T_1 \in \mathbb{C}^{s\times(l-t-s)}$, $T_2 \in \mathbb{C}^{s\times t}, T \in \mathbb{C}^{l\times(n-l)}, N \in \mathbb{C}^{l\times(n-l)}$ are all arbitrary matrices.

The matrices P, U, V, S_A and S_B in Theorem 4 are obtained by applying the GSVD to the matrices A and B^*. In 2008, Cvetković-ilić [30] provided the Re-nonnegative definite solution to matrix Equation (1) using the g-inverse of matrices. Let $H(A)$ denote the Hermitian part of A, i.e., $H(A) = \frac{1}{2}(A + A^*)$.

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Theorem 5. *[30] Suppose that $A, B, C \in \mathbb{C}^{n\times n}$ are given with $A \geq 0, B \geq 0$, then the matrix Equation (1) has a Re-nnd solutionn iff*

$$G = B(A+B)^{-}C(A+B)^{-}A$$

is Re-nnd. In this case, the general Re-nnd solution can be expressed as

$$X = K(C+Y+Z+W)K^* + L_{g_{A+B}}UU^*L^*_{g_{A+B}} + QL_{A+B} - L_{A+B}Q^*,$$

where Y, Z, W are arbitrary solutions of the following matrix equations

$$\begin{cases} Y(A+B)^{-}B = C(A+B)^{-}A, \\ A(A+B)^{-}Z = B(A+B)^{-}C, \\ A(A+B)^{-}W(A+B)^{-}B = G, \end{cases}$$

with $C+Y+Z+W$ is Re-nnd. Here K is defined by

$$K = (A+B)^{-} + L_{g_{A+B}}P\left(H(C+Y+Z+W)^{1/2}\right)^{-}.$$

and $P, Q \in \mathbb{C}^{n\times n}, U \in \mathbb{C}^{n\times(n-t)}$ are arbitrary matrices, $t = r(C+Y+Z+W)$.

The *congruence class of the solutions

Since Hermitian, positive definite, and positive semidefinite matrices are special cases of *congruence, Zheng et al. [44] studied the *congruence class of the solutions to matrix Equation (1) in 2009. Applying the GSVD to the matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times l}$, we obtain:

$$A = U_A \sum\nolimits_A P^*, \quad B = P \sum\nolimits_B V_B^*,$$

where U_A, V_B are unitary matrices, P is a nonsingular matrix, and

$$\sum\nolimits_A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{matrix} r_1 \\ m-r_1 \end{matrix}, \quad \sum\nolimits_B = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{matrix} r_2 \\ r_1-r_2 \\ r_3 \\ n-r_1-r_3 \end{matrix}.$$
$$\qquad\quad r_1 \quad n-r_1 \qquad\qquad\qquad\qquad r_2 \quad r_3 \quad l-r_4$$

Here, $r_1 = r(A), r_4 = r_2 + r_3 = r(B)$, and S_1, S_2 are diagonal matrices with positive elements.

Theorem 6. *[44] For $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times l}, C \in \mathbb{C}^{m\times l}$, and $X \in \mathbb{C}^{n\times n}$ is an unknown matrix. Denote*

$$U_A^*CV_B = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}, P^*XP = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}.$$

Then the matrix (1) is consistent iff

$$C_{3i} = 0, \; i = 1,2,3, \; C_{13} = C_{23} = 0.$$

In this case, the general solution (or least square solution) is **congruent to*

$$Y=\begin{bmatrix} C_{11}S_1^{-1} & X_{12} & C_{12}S_2^{-1} & X_{14} \\ C_{21}S_1^{-1} & X_{22} & C_{22}S_2^{-1} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix},$$

where $X_{3i}, X_{4i}, (i=1,\cdots,4), X_{j4}, X_{j2}, (j=1,2)$ *are arbitrary complex matrices.*

Symmetric, symmetric positive semidefinite, η-anti-Hermitian and Bihermitian solutions

Employing the GSVD, Hua [32] and liao [33] investigated the solvability conditions and general solution expression for the symmetric and symmetric positive semi-definite solutions of the matrix Equation (1) over $\mathbb{R}$.

Theorem 7. *[32] If* $A\in\mathbb{C}^{m\times n}, B\in\mathbb{C}^{n\times q}, C\in\mathbb{C}^{m\times q}$*, and* $X\in\mathbb{C}^{n\times n}$ *is an unknown matrix. We partition* X *and* C *using the method in Theorem 4, then the matrix Equation (1) has a symmetric solution iff*

$$C_{i1}=0, i=1,2,3; C_{32}=C_{33}=0; S_AS_B^{-1}C_{22}^T=C_{22}S_B^{-1}S_A.$$

In this case, the general symmetric solution is given by

$$X=P\begin{pmatrix} X_{11} & C_{12}S_B^{-1} & C_{13} & X_{14} \\ S_B^{-1}C_{12}^T & S_A^{-1}C_{22}S_B^{-1} & S_A^{-1}C_{23} & X_{24} \\ C_{13}^T & C_{23}^TS_A^{-1} & X_{33} & X_{34} \\ X_{14}^T & X_{24}^T & X_{34}^T & X_{44} \end{pmatrix}P^T,$$

where X_{11}, X_{33}, X_{44} *are arbitrary symmetric matrices with appropriate size over* $\mathbb{R}$*, and* X_{14}, X_{24}, X_{34} *are arbitracy real matrices with appropriate size.*

Theorem 8. *[33] Assume that* A, B, C *are given with appropriate size, and apply the GSVD to the matrix pair* $[A^T, B]$*. Set* $C_{ij}=U_i^TCV_j(i,j=1,2,3)$*. Then the Equation (1) has a least squares symmetric positive semi-definite solution only if*

$$r(\hat{X}_{22})=r(S_B^{-1}C_{12}^T, \hat{X}_{22}, S_A^{-1}C_{23}).$$

In this case, the solution can be expressed as

$$X=P^{-T}\begin{pmatrix} Y & YZ \\ (YZ)^T & Z^TYZ+G_3 \end{pmatrix}P^{-1},$$

where Z *is a arbitrary real matrix and* $\hat{X}_{22}$ *is a unique minimizer of* $\|S_AX_{22}S_B-C_{22}\|$ *with respect to*

$$X_{22}\geq 0, X_{22}^T=X_{22}.$$

Here

$$Y=\begin{pmatrix} X_{11} & C_{12}S_B^{-1} & C_{13} \\ S_B^{-1}C_{12}^T & \hat{X}_{22} & S_A^{-1}C_{23} \\ C_{13}^T & C_{23}^TS_A^{-1} & X_{33} \end{pmatrix},$$

$$X_{11}=C_{12}S_B^{-1}\hat{X}_{22}^\dagger S_B^{-1}C_{12}^T+G_1,$$

$$X_{33}=C_{23}^TS_A^{-1}\hat{X}_{22}^\dagger S_A^{-1}C_{23}+ \left(C_{13}-C_{12}S_B^{-1}\hat{X}_{22}^\dagger S_A^{-1}C_{23}\right)^TG_1^\dagger\left(C_{13}-C_{12}S_B^{-1}\hat{X}_{22}^\dagger S_A^{-1}C_{23}\right)+G_2,$$

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G_2, G_3 are arbitrary symmetric positive semi-definite matrices and G_1 is a symmetric positive semi-definite matrix with

$$r(G_1) = r\left(G_1, C_{13} - C_{12}S_B^{-1}\hat{X}_{22}^{\dagger}S_A^{-1}C_{23}\right).$$

The matrices P, U_i, V_j in Theorem 7 and Theorem 8 are obtained by applying the GSVD to the matrix pair $[A, B^T]$ or $[A^T, B]$. Afterward, Hu et al. [34] further investigated symmetric solutions of matrix Equation (1) on subspace

$$N(G) = \{x \in \mathbb{R}^n | Gx = 0, G \in \mathbb{R}^{m \times n}\}$$

over the real field. Let

$$SR^{n\times n}_{N(G)} = \{A \in \mathbb{R}^{n\times n} | (x, Ay) = (Ax, y), \forall x, y \in N(G)\}.$$

Theorem 9. *[34] Suppose that $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times q}$ and $C \in \mathbb{R}^{m\times q}$ are given, then the matrix Equation (1) has a symmetric solution $X \in SR^{n\times n}_{N(G)}$ if and only if*

$$A_1A_1^{\dagger}CB_1^{\dagger}B_1 = C, L_KDL_K = 0.$$

In this case, the solution is given by

$$S = \left\{ X \in \mathbb{R}^{n\times n} \middle| X = V_0 \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} V_0^T \right\},$$

where

$$A_1 = AV_0, B_1 = V_0^T B, K = \begin{pmatrix} A_2^T \\ -B_2 \end{pmatrix}, K^{\dagger} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}, D = \left(P_2A_1^{\dagger}CB_1^{\dagger}Q_2\right)^T - P_2A_1^{\dagger}CB_1^{\dagger}Q_2,$$

and

$$\begin{aligned}
X_{11} =& P_1A_1^{\dagger}CB_1^{\dagger}Q_1 + A_3Y_1 + Z_1B_3,\\
X_{12} =& P_1A_1^{\dagger}CB_1^{\dagger}Q_2 + \frac{1}{2}A_3\left(K_1^TD + K_1^TDL_K\right) + A_3V_1 - \frac{1}{2}A_3A_2^TK_1V_1(I_{n-s} + L_K)\\
&- \frac{1}{2}A_3A_2^TK_2V_2(I_{n-s} + L_K) + \frac{1}{2}A_3K_1^TV_1^TA_2^T - \frac{1}{2}A_3K_1^TV_2^TB_2 + Z_1B_2,\\
X_{21} =& P_2A_1^{\dagger}CB_1^{\dagger}Q_1 + \frac{1}{2}\left(D^TK_2 + L_KD^TK_2\right)B_3 + A_2Y_1 + \frac{1}{2}(I_{n-s} + L_K)V_1^TK_1^TB_2^TB_3\\
&+ V_2^TB_3 + \frac{1}{2}(I_{n-s} + L_K)V_2^TK_2^TB_2^TB_3 + \frac{1}{2}A_2V_1K_2B_3 - \frac{1}{2}B_2^TV_2K_2B_3,\\
X_{22} =& P_2A_1^{\dagger}CB_1^{\dagger}Q_2 + \frac{1}{2}A_2\left(K_1^TD + K_1^TDL_K\right) + \frac{1}{2}\left(D^TK_2 + L_KD^TK_2\right)B_2 + V_2^TB_2\\
&+ A_2V_1 - \frac{1}{2}A_2A_2^TK_1V_1(I_{n-s} + L_K) - \frac{1}{2}A_2A_2^TK_2V_2(I_{n-s} + L_K) + \frac{1}{2}A_2K_1^TV_1^TA_2^T\\
&- \frac{1}{2}A_2K_1^TV_2^TB_2 + \frac{1}{2}(I_{n-s} + L_K)V_1^TK_1^TB_2^TB_2 + \frac{1}{2}(I_{n-s} + L_K)V_2^TK_2^TB_2^TB_2\\
&+ \frac{1}{2}A_2V_1K_2B_2 - \frac{1}{2}B_2^TV_2K_2B_2,
\end{aligned}$$

with V_1, V_2, Y_1, and Z_1 are arbitrary real matrices. Here

$$\begin{aligned}
&P_1 = \begin{bmatrix} I_s & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & I_{n-s} \end{bmatrix}, Q_1 = \begin{bmatrix} I_s \\ 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 \\ I_{n-s} \end{bmatrix},\\
&A_2 = P_2L_{A_1}, B_2 = R_{B_1}Q_2, A_3 = P_1L_{A_1}, B_3 = R_{B_1}Q_1.
\end{aligned}$$

The matrix V_0 is obtained from the singular value decomposition (SVD) of the matrix G.

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Quaternions, an extension of real and complex numbers, have significant applications in signal processing, color image processing, and quantum physics. In addition, the η-Hermitian and Bisymmetric matrices have practical applications. Consequently, Liu [38] employed the Moore-Penrose inverse of matrices to investigate the solvability conditions and the general solution expression of the matrix Equation (1) with an η-Hermitian solution. Additionally, Wang et al. [35] and Zhang et al. [36] explored the least squares Bisymmetric and skew Bisymmetric solutions of this matrix equation over $\mathbb{H}$. Since these studies utilize the real representation of quaternion matrices and the Kronecker product as tools to derive the least squares Bisymmetric solution, only the results on the least squares Bisymmetric solution published by Zhang et al. in 2022 are presented here.

Theorem 10. *[38] Assume that A, B, C are given with appropriate dimensions, and set*

$$\tilde{A} = \begin{pmatrix} R_B^{\eta^*} & L_A \end{pmatrix}, \quad \tilde{B} = -A^{\dagger}CB^{\dagger} - (A^{\dagger}CB^{\dagger})^{\eta^*}, \quad D = R_B^{\eta^*} + L_A.$$

Then the matrix Equation (1) is consistent iff

$$R_D\tilde{B}R_D = 0, \quad R_AC = 0, \quad CL_B = 0.$$

In this case, the general η-anti-Hermitian solution can be expressed as

$$X = A^{\dagger}CB^{\dagger} + L_A\begin{bmatrix} 0 & I \end{bmatrix}U + U^{\eta^*}\begin{bmatrix} I \\ 0 \end{bmatrix}R_B,$$

where

$$U = \tilde{A}^{\dagger}\tilde{B} - \frac{1}{2}\tilde{A}^{\dagger}\tilde{B}(\tilde{A}^{\dagger})^{\eta^*}\tilde{A}^{\eta^*} + L_{\tilde{A}}V + W^{\eta^*}(\tilde{A}^{\dagger})^{\eta^*}\tilde{A}^{\eta^*} - \tilde{A}^{\dagger}W\tilde{A}^{\eta^*},$$

and V, W are arbitrary quaternion matrices.

Denote

$$K = \begin{pmatrix}
I_n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & I_n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_n & 0 & \cdots & 0 \\
0 & I_n & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & I_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_n & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_n
\end{pmatrix}$$
$$\in \mathbf{R}^{4n^2 \times 4n^2}.$$

Theorem 11. *[36] If $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{n\times l}, C \in \mathbb{H}^{m\times l}$, $X = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k} \in \mathbb{H}^{n\times n}$ and set*

$$M = (B_{c_1}^R)^T \otimes A^R, F = \begin{bmatrix} \mathrm{diag}(I_{4n}, \cdots, I_{4n}) \\ \mathrm{diag}(Q_n, \cdots, Q_n) \\ \mathrm{diag}(-G_n, \cdots, -G_n) \\ \mathrm{diag}(T_n, \cdots, T_n) \end{bmatrix}, G = \mathrm{diag}(B_n, W_n, W_n, W_n),$$
$$D = MFKG, \hat{G} = \mathrm{diag}(W_n, B_n, B_n, B_n), \hat{D} = MFK\hat{G}.$$

Then we derive the following results.

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1. *The least square bisymmetric solutions of matrix Equation (1) is given by*

$$H_B = \left\{ X \left| \begin{pmatrix} \mathrm{vec}_B(X_1) \\ \mathrm{vec}_A(X_2) \\ \mathrm{vec}_A(X_3) \\ \mathrm{vec}_A(X_4) \end{pmatrix} = D^{\dagger} \mathrm{vec}(C_{c_1}^R) + L_D y \right. \right\},$$

where Q_n, G_n, T_n are defined in the form (5) and y is an arbitrary vector with appropriate size. Additionally, the minimal norm least square bisymmetric solution of this matrix equation can be expressed as

$$\begin{pmatrix} \mathrm{vec}_B(X_1) \\ \mathrm{vec}_A(X_2) \\ \mathrm{vec}_A(X_3) \\ \mathrm{vec}_A(X_4) \end{pmatrix} = D^{\dagger} \mathrm{vec}(C_{c_1}^R).$$

2. *The least square skew bihermitian solutions of matrix Equation (1) is given by*

$$H_{SB} = \left\{ X \left| \begin{pmatrix} \mathrm{vec}_A(X_1) \\ \mathrm{vec}_B(X_2) \\ \mathrm{vec}_B(X_3) \\ \mathrm{vec}_B(X_4) \end{pmatrix} = \hat{D}^{\dagger} \mathrm{vec}(C_{c_1}^R) + L_{\hat{D}} y \right. \right\},$$

where y is an arbitrary real vector with appropriate size and the minimal norm least square skew bihermitian solution of this matrix equation can be expressed as

$$\begin{pmatrix} \mathrm{vec}_A(X_1) \\ \mathrm{vec}_B(X_2) \\ \mathrm{vec}_B(X_3) \\ \mathrm{vec}_B(X_4) \end{pmatrix} = \hat{D}^{\dagger} \mathrm{vec}(C_{c_1}^R).$$

The matrices B_n and W_n consist of standard unit vectors e_i, as detailed in reference [36].

Reflexive, $\{P, k+1\}$-reflexive, Hermitian reflexive, and (R, S)-symmetric solutions

Now, we present the necessary and sufficient conditions for the existence of reflexive solutions to matrix Equation (1), along with the general solution.

Theorem 12. *[39] Let $A, B,$ and C be given with the suitable dimensions. Then the matrix Equation (1) has a reflexive solution $X \in \mathbb{C}_r^{n\times n}(P)$ iff the system of matrix equations*

$$\begin{cases} A_1YB_1 + A_2ZB_3 = C_1, \\ A_1YB_2 + A_2ZB_4 = C_2, \\ A_3YB_1 + A_4ZB_3 = C_3, \\ A_3YB_2 + A_4ZB_4 = C_4 \end{cases}$$

is consistent. Under such circumstances, the solution can be expressed as

$$X = V \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} V^*.$$

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The unitary matrix V is obtained through the decomposition of the generalized reflection matrix P, i.e.,

$$P = V\begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} V^*.$$

In addition, the matrices $A_i, B_i, C_i, (i = 1, \cdots, 4)$ are derived from the decomposition of the matrices A, B, C:

$$A = V\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} V^*, B = V\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} V^*, C = V\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} V^*,$$

where $A_1, B_1, C_1 \in \mathbb{C}^{r\times r}$.

In 2011, Herrero et al. [41] simplified the problem of $\{P, k+1\}$ reflexive solutions for matrix Equation (1) to the following issue:

$$\begin{cases} \{P,2\} \text{ reflexive solutions,} & \text{when } k \text{ is odd,} \quad \text{the method used: SVD, vec}. \\ \{P,3\} \text{ reflexive solutions,} & \text{when } k \text{ is even,} \quad \text{the method used: GSVD, vec}. \end{cases}$$

Since the solution to matrix Equation (1) using the GSVD method has already been provided earlier, this part will focus on presenting the $\{P, k+1\}$-reflexive solution to matrix Equation (1) using the vec operator method, as described in the reference [41].

Theorem 13. *[41] If A, B, C are given with appropriate size, and denote*

$$D = B_1^T \otimes A_1^*, \quad E = \begin{bmatrix} B_{11}^T \otimes A_{11}^* & B_{22}^T \otimes A_{22}^* \end{bmatrix}.$$

Then we have the following conclusions.

1. *The matrix Equation (1) has a $\{P, 2\}$ reflexive solution iff one of the following conditions is satisfied*
   1) $\mathrm{vec}(C) \in R(D)$,
   2) $\mathrm{vec}(C) \in N(R_{g_D})$.

   In this case, the general $\{P, 2\}$ reflexive solution can be expressed as

$$X = U\begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

   where $X_{11} \in \mathbb{C}^{r\times r}$ can be obtained by rearranging $\mathrm{vec}(X_{11})$, and $r = r(P)$. Here

$$\mathrm{vec}(X_{11}) = D^- \,\mathrm{vec}(C) + L_{g_D} y$$

   with y an arbitrary vector.
2. *The matrix Equation (1) has a $\{P, 3\}$ reflexive solution iff $\mathrm{vec}(C) \in R(E)$ or $\mathrm{vec}(C) \in N(R_{g_E})$. In this case, the general $\{P, 3\}$ reflexive solution can be expressed as*

$$X = U\begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^*$$

   where X_{11}, X_{22} can be reconstructed from $\mathrm{vec}(X_{11})$ and $\mathrm{vec}(X_{22})$, respectively. Here

$$\begin{bmatrix} \mathrm{vec}(X_{11}) \\ \mathrm{vec}(X_{22}) \end{bmatrix} = E^- \,\mathrm{vec}(C) + L_{g_E} z$$

   with z an arbitrary vector.

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Remark 7. *For item 1 in Theorem 13, the matrix P satisfies $P^2 = P = P^*$, implying that P can be unitarily diagonalized as*

$$P = U\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}U^*. \tag{14}$$

Define $\hat{X} = U^*XU$, so $X = U\hat{X}U^*$. In this case, the matrix equation $AXB = C$ transforms into $\hat{A}\hat{X}\hat{B} = C$, where

$$\hat{A} = AU := \begin{bmatrix} A_1^* & A_2^* \end{bmatrix}, \quad \hat{B} = U^*B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Additionally, from $X = PXP$, we can conclude that

$$\hat{X} = \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

where the order of X_{11} is equal to $r(P)$. Following a similar approach, we can obtain the matrices A_{11}, B_{11}, A_{22} and B_{22} in item 2. The details are omitted here.

In 2017, Liu et al. [42] utilized the real representation of complex matrices, the Kronecker product, and the vec operator to derive the minimum-norm least-squares Hermitian (anti)reflexive solution for matrix Equation (1). For $B = B_1 + B_2\mathbf{i} \in \mathbb{C}^{m\times n}$, and $B_1, B_2 \in \mathbb{R}^{m\times n}$, then the real representation matrix of B can be expressed as

$$f(B) = \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix} \in \mathbb{R}^{2m\times 2n}.$$

Theorem 14. *[42] Suppose that $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times l}, C = C_1 + C_2\mathbf{i}$, and set*

$$g(C) = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, M = \begin{bmatrix} f(B_1^* \otimes \overline{A_1}) & f(B_2^* \otimes \overline{A_2}) \end{bmatrix} \cdot \begin{pmatrix} K_{S_1} & 0 & 0 & 0 \\ 0 & K_{A_1} & 0 & 0 \\ 0 & 0 & K_{S_2} & 0 \\ 0 & 0 & 0 & K_{A_2} \end{pmatrix},$$

$$P_{(r,n-r)} = \begin{bmatrix} E_{11}^T & \cdots & E_{1,n-r}^T \\ \vdots & \vdots & \vdots \\ E_{n,1} & \cdots & E_{n,n-r} \end{bmatrix}, N = f(B_1^* \otimes \overline{A_2})\begin{pmatrix} P_{(r,n-r)} & 0 \\ 0 & -P_{(r,n-r)} \end{pmatrix} + f(B_2^* \otimes \overline{A_1})$$

with $E_{i,j}$ representing a matrix where the element at position (i,j) is 1 and all other elements are 0. Then, we have the following conclusions:

1. *the least-squares Hermitian reflexive solution is expressed as*

$$X = U\begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix}U^*,$$

where Y, Z can be obtained by

$$\begin{aligned} \mathrm{vec}(Y) &= \begin{bmatrix} K_{S_1} & K_{A_1}\mathbf{i} & 0 & 0 \end{bmatrix}(M^\dagger \mathrm{vec}(g(C)) + L_M y), \\ \mathrm{vec}(Z) &= \begin{bmatrix} 0 & 0 & K_{S_2} & K_{A_2}\mathbf{i} \end{bmatrix}(M^\dagger \mathrm{vec}(g(C)) + L_M y), \end{aligned}$$

and y is an arbitrary real vector. In this case, the solution X with the minimum-norm is provided by

$$X = U\begin{pmatrix} Y_1 & 0 \\ 0 & Z_1 \end{pmatrix}U^*,$$

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where Y_1, Z_1 are presented by

$$\mathrm{vec}(Y_1) = \begin{bmatrix} K_{S_1} & K_{A_1}\mathbf{i} & 0 & 0 \end{bmatrix} M^{\dagger}\,\mathrm{vec}(g(C)),$$
$$\mathrm{vec}(Z_1) = \begin{bmatrix} 0 & 0 & K_{S_2} & K_{A_2}\mathbf{i} \end{bmatrix} M^{\dagger}\,\mathrm{vec}(g(C)).$$

2. *the least-squares Hermitian antireflexive solution is expressed as*

$$X = U\begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix}U^*,$$

where

$$\begin{bmatrix} I_{r(n-r)} & \mathbf{i}I_{r(n-r)} \end{bmatrix}(N^{\dagger}\,\mathrm{vec}(g(C)) + L_N z),$$

and z is an arbitrary real vector. In this case, the solution X with the minimum-norm is derived by

$$X = U\begin{pmatrix} 0 & Y_2 \\ Y_2^* & \end{pmatrix}U^*,$$

where Y_2 is presented by

$$\begin{bmatrix} I_{r(n-r)} & \mathbf{i}I_{r(n-r)} \end{bmatrix} N^{\dagger}\,\mathrm{vec}(g(C)).$$

The matrix U is derived from the unitary diagonalization of the generalized reflection matrix P, i.e., there exist a unitary matrix U such that

$$P = U\begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix}U^*.$$

It should be noted that the matrix P here satisfies $P^{-1} = P^* = P \neq I$, so the result of the unitary diagonalization of P is different from (14). The matrices $A_i, B_i, i = 1,2$ are obtained in the same way as described in Remark 7. Additionally, the matrices $K_{A_i}, K_{S_i}, i = 1,2$ are related to the standard unit vector e_i, see Reference [42] for details.

In 2008, Yuan et al. [40] explored generalized reflexive solutions to matrix Equation (1) using the GSVD. Notably, generalized reflexive solutions encompass reflexive solutions. A distinct approach involves the generalized reflexive solution, which requires performing unitary diagonalization on the corresponding generalized reflection matrices R and S, respectively, i.e.,

$$R = U\begin{bmatrix} I_r & 0 \\ 0 & -I_s \end{bmatrix}U^*, \quad S = V\begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix}V^*,$$

where U and V are unitary matrices.

Theorem 15. *[40] Assume that A, B, C are given with appropriate size, and denote*

$$AU = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, \quad V^*B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

Applying the GSVD to the matrix pairs $\begin{bmatrix} A_1 & A_2 \end{bmatrix}$ and $\begin{bmatrix} B_1^* & B_2^* \end{bmatrix}$ yields

$$A_1 = M\sum\nolimits_{A_1}E^*, A_2 = M\sum\nolimits_{A_2}F^*, \quad B_1^* = N\sum\nolimits_{B_1}K^*, B_2^* = N\sum\nolimits_{B_2}H^*.$$

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Set

$$M^{-1}C(N^{-1})^* = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}.$$

Then the matrix Equation (1) has a generalized reflexive solution iff

$$C_{i4} = C_{4j} = 0, C_{13} = C_{31} = 0, \quad i = 1, \cdots, 4; j = 1, 2, 3.$$

In this case, the general generalized reflexive solution is given by

$$X = U \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} V^*,$$

where

$$Y = E \begin{bmatrix} C_{11} & C_{12}S_{B_1}^{-1} & Y_{13} \\ S_{A_1}^{-1}C_{21} & S_{A_1}^{-1}(C_{22} - S_{A_2}Z_{22}S_{B_2})S_{B_1}^{-1} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} K^*,$$

and

$$Z = F \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & S_{A_2}^{-1}C_{23} \\ Z_{31} & C_{32}S_{B_2}^{-1} & C_{33} \end{bmatrix} H^*$$

with $Y_{3i}, Y_{j3}, Z_{1i}, Z_{l1}, Z_{22} (i = 1, 2, 3; j = 1, 2; l = 2, 3)$ *are arbitrary matrices.*

So far Cvetković-ilić, Herrero et al., Liu et al. and Yuan et al. have all studied the reflexive solution of matrix Equation (1) over $\mathbb{C}$. In 2024, Liao et al. [43] extended this research by investigating the (R, S)-(skew) symmetric solution of this matrix equation over $\mathbb{H}$, which is a more general solution compared to the generalized reflexive solution. For $A \in \mathbb{H}^{m\times n}$, A can be represented as a real matrix in the form of (4). This allows the (R, S)-(skew) symmetric solution over $\mathbb{H}$ to be transformed into an (R^R, S^R)-(skew) symmetric solution. Since R^R and S^R are merely nontrivial involutory matrices, they can be diagonalized, but not necessarily unitary diagonalized, which differs from the previous case.

For $R^R \in \mathbb{R}^{4m\times 4m}$ and $S^R \in \mathbb{R}^{4n\times 4n}$, there are positive integers r, s, k, l and matrices $P \in \mathbb{C}^{4m\times r}, Q \in \mathbb{C}^{4m\times s}, U \in \mathbb{C}^{4n\times k}, V \in \mathbb{C}^{4n\times l}$ such that

$$\begin{aligned} r + s = 4m,\ P^*P = I_r,\ R^RP = P,\ Q^*Q = I_s,\ R^RQ = -Q, \\ k + l = 4n,\ U^*U = I_k,\ S^RU = U,\ V^*V = I_l,\ S^RV = -V. \end{aligned}$$

The choices for P, Q, U and V are not unique. Suitable P, Q, U and V can be obtained by applying the Gram-Schmidt process to the columns of $I + R^R, I - R^R, I + S^R$ and $I - S^R$, respectively,

$$\hat{P} = \frac{P^*(I + R^R)}{2}, \hat{Q} = \frac{Q^*(I - R^R)}{2}, \hat{U} = \frac{U^*(I + S^R)}{2}, \hat{V} = \frac{V^*(I - S^R)}{2}.$$

Then we obtain

$$\begin{aligned} \hat{P}P = I,\ \hat{P}Q = 0,\ \hat{Q}P = 0,\ \hat{Q}Q = I, \\ \hat{U}U = I,\ \hat{U}V = 0,\ \hat{V}U = 0,\ \hat{V}V = I. \end{aligned}$$

In fact, $R^R = (R^R)^T, (S^R)^T = S^R$ if and only if $P^*Q = 0, U^*V = 0$. In this case,

$$\hat{P} = P^*,\ \hat{Q} = Q^*,\ \hat{U} = U^*,\ \hat{V} = V^*,$$

so $\begin{bmatrix} P & Q \end{bmatrix}$ and $\begin{bmatrix} U & V \end{bmatrix}$ are unitary.

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Theorem 16. *[43] If $A \in \mathbb{H}^{e\times m}, B \in \mathbb{H}^{n\times f}, C \in \mathbb{H}^{e\times f}$, and denote*

$$A^R\begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} B^R = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

then we have the following conclusions.

1. *The matrix Equation (1) has a (R,S)-symmetric slution X if and only if $A^RYB^R = C^R$ has a (R^R, S^R)-symmetric slution Y or*

$$r\begin{bmatrix} A_1 & A_2 & C^R \end{bmatrix} = r\begin{bmatrix} A_1 & A_2 \end{bmatrix}, r\begin{bmatrix} B_1 \\ B_2 \\ C^R \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$r\begin{bmatrix} C^R & A_1 \\ B_2 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & A_1 \\ B_2 & 0 \end{bmatrix}, r\begin{bmatrix} C^R & A_2 \\ B_1 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & A_2 \\ B_1 & 0 \end{bmatrix}.$$

In this case, the general (R,S)-symmetric slution is provided by

$$X = \frac{1}{16}\begin{bmatrix} I_m & \mathbf{i}I_m & \mathbf{j}I_m & \mathbf{k}I_m \end{bmatrix}(Y + Q_mYQ_n^T + G_mYG_n^T + T_mYT_n^T)\begin{bmatrix} I_n \\ -\mathbf{i}I_n \\ -\mathbf{j}I_n \\ -\mathbf{k}I_n \end{bmatrix},$$

where Q_n, G_n, T_n are defined as (5) and

$$Y = \begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix}$$

with

$$Y_1 = \hat{Y}_1 + M_1L_{K_1}U_1R_{K_2}N_1 + L_{A_1}W_1 + W_2R_{B_1},$$
$$Y_2 = \hat{Y}_2 + M_2L_{K_1}U_1R_{K_2}N_2 + L_{A_2}V_1 + V_2R_{B_2}.$$

Here

$$M_1 = \begin{bmatrix} I_r & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & I_s \end{bmatrix}, N_1 = \begin{bmatrix} I_k \\ 0 \end{bmatrix},$$
$$N_2 = \begin{bmatrix} 0 \\ I_l \end{bmatrix}, K_1 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, K_2 = \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix},$$

and $U_1, V_i, W_i (i = 1, 2)$ are arbitrary matrices, $(\hat{Y}_1, \hat{Y}_2)$ is a pair of solutions of matrix equation $A_1Y_1B_1 + A_2Y_2B_2 = C^R$.

2. *The matrix Equation (1) has a (R,S)-skew symmetric slution X if and only if $A^RYB^R = C^R$ has a (R^R, S^R)-skew symmetric slution Y or*

$$r\begin{bmatrix} A_2 & A_1 & C^R \end{bmatrix} = r\begin{bmatrix} A_2 & A_1 \end{bmatrix}, r\begin{bmatrix} B_1 \\ B_2 \\ C^R \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$r\begin{bmatrix} C^R & A_2 \\ B_2 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & A_2 \\ B_2 & 0 \end{bmatrix}, r\begin{bmatrix} C^R & A_1 \\ B_1 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & A_1 \\ B_1 & 0 \end{bmatrix}.$$

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In this case, the general (R,S)-skew symmetric slution is gven by

$$X = \frac{1}{16}\begin{bmatrix} I_m & \mathbf{i}I_m & \mathbf{j}I_m & \mathbf{k}I_m \end{bmatrix}(Y + Q_m Y Q_n^T + G_m Y G_n^T + T_m Y T_n^T)\begin{bmatrix} I_n \\ -\mathbf{i}I_n \\ -\mathbf{j}I_n \\ -\mathbf{k}I_n \end{bmatrix},$$

where

$$Y = \begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} 0 & Y_3 \\ Y_4 & 0 \end{bmatrix}\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix}$$

and

$$Y_3 = \hat{Y}_3 + M_3 L_{K_1} U_2 R_{K_2} N_2 + L_{A_1} W_3 + W_4 R_{B_2},$$
$$Y_4 = \hat{Y}_4 + M_4 L_{K_1} U_2 R_{K_2} N_1 + L_{A_2} V_3 + V_4 R_{B_1}.$$

Here

$$M_4 = \begin{bmatrix} I_s & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & I_r \end{bmatrix}, N_1 = \begin{bmatrix} I_k \\ 0 \end{bmatrix},$$
$$N_2 = \begin{bmatrix} 0 \\ I_l \end{bmatrix}, K_1 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, K_2 = \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix},$$

and $U_2, V_i, W_i (i = 3,4)$ are arbitrary matrices, $(\hat{Y}_3, \hat{Y}_4)$ is a pair of solutions of matrix equation $A_2 Y_4 B_1 + A_1 Y_3 B_2 = C^R$.

Additionally, Liao et al. utilized the vec operator to study the least-squares (R,S)-(skew) symmetric solution of the matrix Equation (1). Since the solution method is similar to Herrero's approach for solving the $\{P, k+1\}$ reflexive solutions of the same matrix equation, it will not be described here. For details, please refer to Reference [43].

Other solutions with specific structures

In 2018, Yuan et al. [45] studied the solutions of matrix Equation (1) under the constraints of general Toeplitz matrices, upper triangular Toeplitz matrices, lower triangular Toeplitz matrices, symmetric Toeplitz matrices, and Hankel matrices. We konw that a general Toeplitz matrix $A \in \mathbb{R}^{n\times n}$ can be written as

$$A = \sum_{l=-(n-1)}^{n-1} a_l \Delta_l,$$

where $a_l \in \mathbb{R}$ and

$$(\Delta_l)_{ij} = \begin{cases} 1 & \text{if } j = i + l, \\ 0 & \text{otherwise,} \end{cases} \quad l = -(n-1), \cdots, n-1.$$

Theorem 17. *[45] Suppose that the matrix Equation (1) is consistent, then we have the following conclusions.*

1. *Its general Toeplitz solution is given by $X = \sum_{l=-(n-1)}^{n-1} a_l \Delta_l$, where*

$$(a_{-(n-1)}, \cdots, a_0, \cdots, a_{n-1})^T = N^- b + L_{g_N} w, \tag{15}$$

and w is an arbitrary real vector. Here,

$$N = \frac{\mathrm{tr}(B^T \Delta_p^T A^T A \Delta_q B) + \mathrm{tr}(B^T \Delta_p^T A^T A \Delta_q B)}{2}, \ p, q = -(n-1), \cdots, n-1.$$
$$b = \begin{bmatrix} \mathrm{tr}(C^T A \Delta_{-(n-1)} B) & \mathrm{tr}(C^T A \Delta_{-(n-2)} B) & \cdots & \mathrm{tr}(C^T A \Delta_{n-1} B) \end{bmatrix}^T.$$

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Specifically, when $l, p, q = 0, \cdots, n-1$, we can obtain the upper triangular Toeplitz solution of this matrix equation. Similarly, when $l, p, q = -(n-1), \cdots, 0$, we can also obtain the lower triangular Toeplitz solution.

2. *Its symmetric Toeplitz solution is exoressed as $X = \sum_{l=0}^{n-1} a_l \Delta_l$, where*

$$(\Delta_l)_{ij} = \begin{cases} 1 & \text{if } l = |j-i|, \\ 0 & \text{otherwise}, \end{cases} \quad l = 0, \cdots, n-1.$$

and the method for finding a_l is the same as for Equation (15), except that $p, q = 0, \cdots, n-1$.

3. *Its Hankel solution is provided by $X = \sum_{l=1}^{2n-1} a_l \Delta_l$, where*

$$(\Delta_l)_{ij} = \begin{cases} 1 & \text{if } l+1 = j+i, \\ 0 & \text{otherwise}, \end{cases} \quad l = 1, \cdots, 2n-1.$$

and the procedure to determine a_l is similar to that in Equation (15), except that p and q range from 1 to $2n-1$.

In 2022, Zhang et al. [46] provided the column unitary solution to matrix Equation (1) using singular value decomposition and spectral decomposition. For $A \in \mathbb{C}^{m\times n}$, the SVD of A can be expressed as

$$A = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} \sum_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix},$$

where $\sum_A$ is a diagonal matrix composed of the non-zero singular values of matrix A, and $\begin{bmatrix} P_1 & P_2 \end{bmatrix}, \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ are unitary matrices. If $D \in \mathbb{C}^{k\times k}$ and satisfies $D \geq 0$, then the spectral decomposition of D is given by

$$D = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Delta_D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix},$$

where Δ_D is a diagonal matrix composed of the non-zero eigenvalues of matrix D and $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ is a unitary matrix.

Theorem 18. *[46] Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{l\times k}, C \in \mathbb{C}^{m\times k}$ be given with $n \geq l$. Set*

$$D = B^*B - C^*(AA^*)^\dagger C.$$

Then the matrix Equation (1) has a column unitary solution $X \in \mathbb{C}^{n\times l}$ if and only if

$$R_A C = 0, \; D \geq 0, \; r(D) \leq n - r(A).$$

In this case, the general column unitary solution is provided by

$$X = U_3 \begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix} P_3^*,$$

where G is an arbitrary column unitary matrices and U_3, P_3 are derived from the singular value decomposition of matrix $E = (A^\dagger C + U_2 J_1 \Delta_D^{\frac{1}{2}} Q_1^*)B^*$:

$$E = U_3 \begin{bmatrix} \sum_E & 0 \\ 0 & 0 \end{bmatrix} P_3^*.$$

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Here J_1 is an arbitrary column unitary matrix and $\sum_E$ is a diagonal matrix formed by the non-zero singular values of matrix E.

Tensors, being higher-dimensional matrices, have a wide range of applications. Xie et al. [12] studied the K-reducible solutions of the quaternion tensor matrix equation $\mathcal{A} *_M \mathcal{X} *_M \mathcal{B} = \mathcal{C}$ under the Einstein product.

Theorem 19. *[12] Assume that $\mathcal{A} \in \mathbb{H}^{J_1\times\cdots\times J_M\times K_1\times\cdots\times K_M}, \mathcal{B} \in \mathbb{H}^{K_1\times\cdots\times K_M\times L_1\times\cdots\times L_M}, \mathcal{C} \in \mathbb{H}^{J_1\times\cdots\times J_M\times L_1\times\cdots\times L_M}$ are given and $\mathcal{K} \in \mathbb{H}^{K_1\times\cdots\times K_M\times K_1\times\cdots\times K_M}$ is a permutation tensor with $K_i = N_i + P_i$. Set*

$$\mathcal{A} *_M \mathcal{K} := \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix},\ \mathcal{K}^{-1} *_M \mathcal{B} := \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix},$$
$$\mathcal{A}_1 \in \mathbb{H}^{J_1\times\cdots\times J_M\times N_1\times\cdots\times N_M},\ \mathcal{A}_2 \in \mathbb{H}^{J_1\times\cdots\times J_M\times P_1\times\cdots\times P_M},\ \mathcal{B}_1 \in \mathbb{H}^{N_1\times\cdots\times N_M\times L_1\times\cdots\times L_M},$$
$$\mathcal{B}_2 \in \mathbb{H}^{P_1\times\cdots\times P_M\times L_1\times\cdots\times L_M},\ \mathcal{B}_3 = \mathcal{B}_1 *_M \mathcal{L}_{\mathcal{B}_2}, \mathcal{A}_3 = \mathcal{R}_{\mathcal{A}_1} *_M \mathcal{A}_2,$$
$$\mathcal{C}_1 = \mathcal{C} *_M \mathcal{L}_{\mathcal{B}_2} + \mathcal{R}_{\mathcal{A}_1} *_M \mathcal{C}, \mathcal{B}_4 = \mathcal{B}_2 *_M \mathcal{L}_{\mathcal{B}_3}.$$

Then the quaternion tensor equation $\mathcal{A} *_M \mathcal{X} *_M \mathcal{B} = \mathcal{C}$ has a K-reducible solution if and only if

$$\mathcal{R}_{\mathcal{A}_3} *_M \mathcal{R}_{\mathcal{A}_1} *_M \mathcal{C}_1 = 0,\ \mathcal{C}_1 *_M \mathcal{L}_{\mathcal{B}_3} *_M \mathcal{L}_{\mathcal{B}_4} = 0,$$
$$\mathcal{R}_{\mathcal{A}_1} *_M \mathcal{C}_1 *_M \mathcal{L}_{\mathcal{B}_2} = 0,\ \mathcal{R}_{\mathcal{A}_3} *_M \mathcal{C}_1 *_M \mathcal{L}_{\mathcal{B}_3} = 0.$$

In this case, the general K-reducible solution is given by

$$\mathcal{X} = \mathcal{K} *_M \begin{bmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ 0 & \mathcal{X}_3 \end{bmatrix} *_M \mathcal{K}^{-1},$$

where

$$\mathcal{X}_1 = \mathcal{A}_1^\dagger *_M \mathcal{C}_1 *_M \mathcal{B}_3^\dagger + \mathcal{L}_{\mathcal{A}_1} *_M \mathcal{W}_1 + \mathcal{W}_2 *_M \mathcal{R}_{\mathcal{B}_3},$$
$$\mathcal{X}_2 = \mathcal{A}_1^\dagger *_M (\mathcal{C} - \mathcal{A}_1 *_M \mathcal{X}_1 *_M \mathcal{B}_1 - \mathcal{A}_2 *_M \mathcal{X}_3 *_M \mathcal{B}_2) *_M \mathcal{B}_2^\dagger + \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{B}_2} - \mathcal{L}_{\mathcal{A}_1} *_M \mathcal{W}_4,$$
$$\mathcal{X}_3 = \mathcal{A}_3^\dagger *_M \mathcal{C}_1 *_M \mathcal{B}_2^\dagger + \mathcal{L}_{\mathcal{A}_3} *_M \mathcal{W}_5 + \mathcal{W}_6 *_M \mathcal{R}_{\mathcal{B}_2},$$

where $\mathcal{W}_i$ $(i = 1, \cdots, 6)$ are arbitrary quaternion tensors.

Positive solutions

In previous studies, A, B, and C in the equation $AXB = C$ were all finite-dimensional matrices. However, in 2010, Arias et al. [47] investigated a more general case where A, B, and C are bounded linear operators acting on suitable Hilbert spaces, with the underlying vector spaces being infinite-dimensional. The symbols $\mathbf{F}, \mathbf{G}, \mathbf{H}$, and $\mathbf{K}$ represent complex Hilbert spaces equipped with the inner product $\langle , \rangle$. Let $L(\mathbf{F}, \mathbf{G})$ represent the set of bounded linear operators from $\mathbf{F}$ to $\mathbf{G}$ and $L(\mathbf{F}) = L(\mathbf{F}, \mathbf{F})$. The set $L(\mathbf{F})^+ \subseteq L(\mathbf{F})$ denotes the cone of positive operators, defined as $L(\mathbf{F})^+ := \{A \in L(\mathbf{F}) : \langle A\delta, \delta\rangle \geqslant 0, \forall \delta \in \mathbf{F}\}$. For any $A \in L(\mathbf{F}, \mathbf{G})$, A^* denotes the adjoint operator of A. For $\mathbf{V} \subseteq \mathbf{F}$, where $\mathbf{V}$ is a closed subspace, $P_{\mathbf{V}}$ represents the orthogonal projection onto the subspace $\mathbf{V}$, and $P_{\mathbf{V}}|_{\mathbf{V}}$ denotes the restriction of $P_{\mathbf{V}}$ to $\mathbf{V}$.

Theorem 20. *[47] If $A \in L(\mathbf{H}, \mathbf{K}), B \in L(\mathbf{G}, \mathbf{H}), C \in L(\mathbf{G}, \mathbf{K})$, and they satify $R(B) \subseteq \overline{R(A^*)}$, then the following descriptions are equivalent.*

1. *The equation $AXB = C$ is consistent.*
2. *There exists a positive operator $\hat{X} \in L(\mathbf{H})^+$such that $A\hat{X}B = C$.*
3. *There exists a positive operator $\hat{Y} \in L(\mathbf{H})^+$such that $\hat{Y}B = A^\dagger C$.*

4. *The operator $B^*A^\dagger C$ is non-negative, and $R\left((A^\dagger C)^*\right) \subseteq R\left((B^*A^\dagger C)^{\frac{1}{2}}\right)$.*

In this case, then the general positive solution can be expressed as

$$\hat{X} = \begin{bmatrix} \hat{x}_{11} & \hat{x}_{12} \\ \hat{x}_{12}^* & \left[\left(\hat{x}_{11}^{\frac{1}{2}}\right)^\dagger \hat{x}_{12}\right]^* \left(\hat{x}_{11}^{\frac{1}{2}}\right)^\dagger \hat{x}_{12} + l \end{bmatrix},$$

where $R(\hat{x}_{12}) \subseteq R\left(\hat{x}_{11}^{\frac{1}{2}}\right)$ and l is positive. Also,

$$\hat{x}_{11} = P_{\overline{R(A^*)}} \hat{Y}\Big|_{\overline{R(A^*)}}.$$

Here $\hat{Y} \in L(\mathbf{H})^+$ and obeys $\hat{Y}B = A^\dagger C$.

Arias et al. provided several equivalent solvability conditions and corresponding expressions for the positive solution of the operator equation $AXB = C$ under the condition $R(B) \subseteq \overline{R(A^*)}$. In 2019, Cvetković-Ilić [48] removed the condition $R(B) \subseteq \overline{R(A^*)}$ and, using the results of Douglas [49] and Sebestyén [50], presented several equivalent solvability conditions and corresponding expressions for the positive solution of this operator equation.

Theorem 21. *[48] Suppose that $A \in L(\mathbf{H}, \mathbf{K})$, $B \in L(\mathbf{G}, \mathbf{H})$, and $C \in L(\mathbf{G}, \mathbf{K})$, then we have the following conclusions.*

1. *The following statements are equivalent.*
   (a) *The operator equation $AXB = C$ has a positive solution.*
   (b) *There exist a real number $\mu > 0$ and $Y \in L(\mathbf{H})$ such that, for every $x \in L(\mathbf{G})$,*

$$\left\|A^\dagger Cx\right\|^2 + \|L_A Y B x\|^2 \leq \mu \left\langle \left(B^*A^\dagger C + B^* L_A Y B\right)x, x \right\rangle.$$

   (c) *There exists a positive operator $D = D_1 D_1^* \in L(\mathbf{G})$ such that $R((A^\dagger C)^*) \subseteq R(D_1)$ and the equation*

$$B^*A^\dagger C + B^* L_A Y B = D$$

   is consistent.
   (d) *There exists $Y \in L(\mathbf{H})$ and a real number $\delta > 0$ such that*

$$B^*A^\dagger C - \delta\left(A^\dagger C\right)^* A^\dagger C + B^* L_A Y B \geq 0.$$

2. *Specifically, if $R(B^* L_A)$ is closed, we have the following equivalent descriptions.*
   (a) *The operator equation $AXB = C$ is consistent,*
   (b) *$(I - Q)B^*A^\dagger C(I - Q) \geq 0$,*
   (c) *$R((I - Q)B^*A^\dagger C) \subseteq R(E)$,*
   (d) *$R((I - Q)(A^\dagger C)^*) \subseteq R(E)$,*

   where $Q = B^* L_A (B^* L_A)^\dagger$ and $E = ((I - Q)B^*A^\dagger C(I - Q))^{\frac{1}{2}}$. In this case, the general positive solution is expressed as

$$X = A^\dagger C B^\dagger + (B^* L_A)^\dagger \left(W - B^*A^\dagger C\right) B^\dagger + U - (I - S)UBB^\dagger,$$

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where

$$S = (B^* L_A)^\dagger B^* L_A,\ D_{12} = (I - Q)B^* A^\dagger C Q,$$
$$W = (I - Q)B^* A^\dagger C + Q\left(B^* A^\dagger C\right)^* (I - Q) + \left(E^\dagger D_{12}\right)^* E^\dagger D_{12} + QFQ.$$

Here, $U \in L(\mathbf{H})$ *and* $F \in L(\mathbf{G})^+$ *satisfy the conditions*

$$R(((I - S)L_A U B)^*) \subseteq R\left(E + \left(E^\dagger D_{12}\right)^* + (QFQ)^{\frac{1}{2}}\right)$$

and

$$R\left(Q\left(A^\dagger C\right)^* - \left(E^\dagger D_{12}\right)^* E^\dagger \left(A^\dagger C\right)^*\right) \subseteq R\left((QFQ)^{\frac{1}{2}}\right),$$

respectively.

Ranked solutions

In 1974, Marsaglia and Styan [55] presented several equalities and inequalities related to matrix ranks, including the following lemma.

Lemma 1. *Assume that* A, B, C, D *and* E *are given with appropriate size over* $\mathbb{C}$*, then we obtain*

$$r\begin{pmatrix} A & BL_C \\ R_D E & 0 \end{pmatrix} = r\begin{pmatrix} A & B & 0 \\ E & 0 & D \\ 0 & C & 0 \end{pmatrix} - r(C) - r(D).$$

Lemma 1 plays an important role in the study of matrix equations. For instance, it can be used to investigate the extreme ranks of solutions to matrix Equation (1), as well as to provide necessary and sufficient conditions for the existence of solutions. Traditionally, determining whether a matrix equation has a solution requires computing the inverse of a matrix, but with this lemma, one only needs to compute the rank of the matrix. From a practical standpoint, this significantly reduces computational costs. In addition, this lemma can be easily generalized to $\mathbb{H}$.

In 1979, Porter et al. [51] studied the number of rank-k solutions to matrix Equation (1) over the finite field $\mathbb{GF}(p^n)$, where p is an odd prime. For any $m \times n$ matrix of rank-r over $\mathbb{GF}(p^n)$, let $q = p^n$. Then, the number of such matrices is denoted by

$$g(m, n, r) = q^{\frac{r(r-1)}{2}} \prod_{i=1}^{r} \frac{(q^{m-i+1} - 1)(q^{n-i+1} - 1)}{q^i - 1},$$

It is evident that $1 \le r \le min\{m, n\}$. In particular, if $r = 0$, then $g(m, n, 0) = 1$.

Theorem 22. *[51] Let* A, B, C *be matrices of size* $s \times m, f \times t,$ *and* $s \times t$*, respectively, with ranks* $r(A) = \rho, r(B) = v,$ *and* $r(C) = b$*. There exist nonsingular matrices* P_1, P_2, Q_1, Q_2 *such that*

$$P_1 A Q_1 = \begin{pmatrix} I_\rho & 0 \\ 0 & 0 \end{pmatrix}, P_2 B Q_2 = \begin{pmatrix} I_v & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, the number of rank-r solutions to matrix Equation (1) over $\mathbb{GF}(q)$ *is given by*

$$N_r = \delta(B_1) \sum_{i=r+v-f-b}^{\min\{v-b, m-\rho, r-b\}} q^{(m-\rho)b + (b+i)(f-v)} g(m - \rho, v - b, i) g(m - b - i, f - v, r - b - i).$$

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Furthermore, the number of solutions to this matrix equation can be expressed as

$$N(A,B,C) = \sum_{r=b}^{\min\{m,f\}} N_r = q^{mf-\rho v}\delta(B_1),$$

where $B_1 = P_1 C Q_2 = (l_{ij})$ *and*

$$\begin{cases} \delta(B_1) = 1, & \text{if } l_{ij} = 0,\ i > \rho \text{ or } j > v, \\ \delta(B_1) = 0, & \text{otherwise.} \end{cases}$$

In 2007, Liu [52] used Lemma 1 to provide the maximal and minimal ranks of the least squares solutions to matrix Equation (1) over $\mathbb{C}$. In addition, the maximal and minimal ranks of the real and imaginary parts of the least squares solutions to this matrix equation were also given.

Theorem 23. *[52] If* $A = A_0 + A_1\mathbf{i}$, $B = B_0 + B_1\mathbf{i}$, *and* $C = C_0 + C_1\mathbf{i}$ *are given with appropriate size, and* $X = X_0 + X_1\mathbf{i} \in \mathbb{C}^{m\times n}$ *is a least square solution to (1), then*

$$\max r(X) = \min\{m, n, m+n+r(A^*CB^*) - r(A) - r(B)\},$$
$$\min r(X) = r(A^*CB^*).$$

Furthermore, Furthermore, we can also provide the maximal and minimal ranks of X_0 *and* X_1.

1. *The extreme ranks of* X_0 *are provided by*

$$\max r(X_0) = \min\{m, n, m+n+k_1 - 2(r(A)+r(B))\},$$
$$\min r(X_0) = k_1 - r\begin{pmatrix} A_0 \\ A_1 \end{pmatrix} - r\begin{pmatrix} B_0 & B_1 \end{pmatrix},$$

   where

$$k_1 = r\left[\begin{pmatrix} A_0^T & A_1^T & 0 \\ -A_1^T & A_0^T & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} C_0 & -C_1 & A_0 \\ C_1 & C_0 & A_1 \\ B_0 & -B_1 & 0 \end{pmatrix}\begin{pmatrix} B_0^T & B_1^T & 0 \\ -B_1^T & B_0^T & 0 \\ 0 & 0 & I \end{pmatrix}\right].$$

2. *The extreme ranks of* X_1 *are presented by*

$$\max r(X_1) = \min\{m, n, m+n+k_2 - 2(r(A)+r(B))\},$$
$$\min r(X_1) = k_2 - r\begin{pmatrix} A_0 \\ A_1 \end{pmatrix} - r\begin{pmatrix} B_0 & B_1 \end{pmatrix},$$

   where

$$k_2 = r\left[\begin{pmatrix} A_0^T & A_1^T & 0 \\ -A_1^T & A_0^T & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} C_0 & -C_1 & A_0 \\ C_1 & C_0 & A_1 \\ B_1 & B_0 & 0 \end{pmatrix}\begin{pmatrix} B_0^T & B_1^T & 0 \\ -B_1^T & B_0^T & 0 \\ 0 & 0 & I \end{pmatrix}\right].$$

Subsequently, Zhang et al. [54] partitioned the least squares solutions of matrix Equation (1) into a 2×2 block form and provided the maximal and minimal ranks for each sub-block matrix. For $X \in \mathbb{C}^{m\times n}$ is a least squares solution of matrix Equation (1), then

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},$$

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where

$$X_1 = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} X \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} := E_1 X F_1, \quad X_2 = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} X \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} := E_1 X F_2,$$
$$X_3 = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} X \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} := E_2 X F_1, \quad X_4 = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} X \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} := E_2 X F_2,$$

and $m_1 + m_2 = m$, $n_1 + n_2 = n$.

Theorem 24. *[54] Assume that A, B, and C are given with the suitbale dimensions over $\mathbb{C}$, and*

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

is a least squares solution of the matrix Equation (1). Then, the maximal and minimal ranks of X_1, X_2, X_3, and X_4 can be expressed as follows:

$$\min r(X_1) = r\begin{pmatrix} A^*CB^* & A^*A & 0 \\ BB^* & 0 & F_1 \\ 0 & E_1 & 0 \end{pmatrix} - r\begin{pmatrix} A \\ E_1 \end{pmatrix} - r\begin{pmatrix} B & F_1 \end{pmatrix},$$
$$\max r(X_1) = \min\left\{ m_1, n_1, r\begin{pmatrix} A^*CB^* & A^*A & 0 \\ BB^* & 0 & F_1 \\ 0 & E_1 & 0 \end{pmatrix} - r(A) - r(B) \right\},$$

$$\min r(X_2) = r\begin{pmatrix} A^*CB^* & A^*A & 0 \\ BB^* & 0 & F_2 \\ 0 & E_1 & 0 \end{pmatrix} - r\begin{pmatrix} A \\ E_1 \end{pmatrix} - r\begin{pmatrix} B & F_2 \end{pmatrix},$$
$$\max r(X_2) = \min\left\{ m_1, n_2, r\begin{pmatrix} A^*CB^* & A^*A & 0 \\ BB^* & 0 & F_2 \\ 0 & E_1 & 0 \end{pmatrix} - r(A) - r(B) \right\},$$
$$\min r(X_3) = r\begin{pmatrix} A^*CB^* & A^*A & 0 \\ BB^* & 0 & F_1 \\ 0 & E_2 & 0 \end{pmatrix} - r\begin{pmatrix} A \\ E_2 \end{pmatrix} - r\begin{pmatrix} B & F_1 \end{pmatrix},$$
$$\max r(X_3) = \min\left\{ m_2, n_1, r\begin{pmatrix} A^*CB^* & A^*A & 0 \\ BB^* & 0 & F_1 \\ 0 & E_2 & 0 \end{pmatrix} - r(A) - r(B) \right\},$$
$$\min r(X_4) = r\begin{pmatrix} A^*CB^* & A^*A & 0 \\ BB^* & 0 & F_2 \\ 0 & E_2 & 0 \end{pmatrix} - r\begin{pmatrix} A \\ E_2 \end{pmatrix} - r\begin{pmatrix} B & F_2 \end{pmatrix},$$
$$\max r(X_4) = \min\left\{ m_2, n_2, r\begin{pmatrix} A^*CB^* & A^*A & 0 \\ BB^* & 0 & F_2 \\ 0 & E_2 & 0 \end{pmatrix} - r(A) - r(B) \right\}.$$

In 2010, Wang et al. [53] extended the study of extremal ranks for solutions of the matrix Equation (1) to quaternions. They utilized the real representation of quaternion matrices to determine the maximal and minimal ranks of the four real matrices in the quaternion solutions to this matrix equation. For convenience in description, we denote the real representation of the quaternion matrix $B = B_1 + B_2\mathbf{i} + B_3\mathbf{j} + B_4\mathbf{k} \in \mathbb{H}^{n\times k}$ as:

$$B^R = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ -B_2 & B_1 & B_4 & -B_3 \\ -B_3 & -B_4 & B_1 & B_2 \\ -B_4 & B_3 & -B_2 & B_1 \end{pmatrix} := \begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{pmatrix}.$$

Theorem 25. *[53] Let $A = A_1 + A_2\mathbf{i} + A_3\mathbf{j} + A_4\mathbf{k}$, $B = B_1 + B_2\mathbf{i} + B_3\mathbf{j} + B_4\mathbf{k}$, and $C = C_1 + C_2\mathbf{i} + C_3\mathbf{j} + C_4\mathbf{k}$ be given with appropriate size. Set*

$$D = \begin{pmatrix} A_2 & A_3 & A_4 \\ A_1 & A_4 & -A_3 \\ -A_4 & A_1 & A_2 \\ A_3 & -A_2 & A_1 \end{pmatrix}, F_1 = \begin{pmatrix} L_2 \\ L_3 \\ L_4 \end{pmatrix}, F_2 = \begin{pmatrix} L_1 \\ L_3 \\ L_4 \end{pmatrix}, F_3 = \begin{pmatrix} L_1 \\ L_2 \\ L_4 \end{pmatrix}, F_4 = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}.$$

Then the quaternion matrix Equation (1) has a solution $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbb{H}^{m\times n}$ iff

$$A^R \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{44} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{pmatrix} B^R = C^R \tag{16}$$

is solvable over $\mathbb{R}$. Here, the general solution is given by

$$\begin{aligned} X = &\frac{1}{4}(Z_{11} + Z_{22} + Z_{33} + Z_{44}) + \frac{1}{4}(Z_{12} - Z_{21} + Z_{34} - Z_{43})\mathbf{i} \\ &+ \frac{1}{4}(Z_{13} - Z_{31} + Z_{42} - Z_{24})\mathbf{j} + \frac{1}{4}(Z_{14} - Z_{41} + Z_{23} - Z_{32})\mathbf{k}, \end{aligned}$$

where Z_{ij} $(i,j = 1,\cdots,4)$ are the general solutions of (16). In this case, the maximal and minimal ranks of X_i $(i = 1,\cdots,4)$ can be expressed as

$$\begin{aligned} \max r(X_i) &= \min\left\{ m, n, m+n+r\begin{pmatrix} 0 & F_i \\ D & C^R \end{pmatrix} - 4(r(A) + r(B)) \right\}, \\ \min r(X_i) &= r\begin{pmatrix} 0 & F_i \\ D & C^R \end{pmatrix} - r(D) - r(F_i). \end{aligned}$$

General solution under specific conditions

In traditional matrix multiplication, there are dimensional requirements concerning the rows and columns of the matrices involved. However, in the context of the semitensor product, matrices of arbitrary dimensions can be multiplied. The semi-tensor product finds significant applications in various fields such as networked evolutionary games [57], dynamical games [58], and Boolean networks [59,60]. Therefore, in 2019, Ji et al. [13] studied the solvability conditions of matrix Equation (1) under the semi-tensor product.

Theorem 26. *[13] Suppose $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{e\times f}$, and $C \in \mathbb{R}^{l\times k}$ are given. When $m = l$, the matrix equation $A \ltimes X \ltimes B = C$ has a solution $X \in \mathbb{R}^{g\times r}$ if and only if the matrix equation*

$$(B^T \otimes I_{\frac{km}{f}})(I_r \otimes \hat{A})\,\mathrm{vec}(X) = \mathrm{vec}(C)$$

is consistent, where

$$\hat{A} = \begin{bmatrix} A_1 & A_{\beta+1} & \cdots & A_{(g-1)\beta+1} \\ A_2 & A_{\beta+2} & \cdots & A_{(g-1)\beta+2} \\ \vdots & \vdots & \ddots & \vdots \\ A_\beta & A_{2\beta} & \cdots & A_{g\beta} \end{bmatrix},$$

and A_i is the i-th column of matrix A. Here β is a common factor of n and $\frac{ek}{f}$, satisfying $g = \frac{n}{\beta}, r = \frac{ek}{f\beta}$.

Remark 8. *When $m \neq l$, Reference [13] provides only the necessary condition for the solvability of Equation (1). This condition is not discussed here, but those interested can refer to that document for more details.*

Prior research on matrix Equation (1) was mostly conducted over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and the corresponding conclusions do not necessarily hold over specific rings. Therefore, in 2020, Prokip [56] provided the necessary and sufficient conditions for the solvability of this matrix equation, as well as the general solution, over a principal ideal domain. Let R be a principal ideal ring with a unity. The set of invertible matrices in $R^{n\times n}$ is denoted by $GL(n, R)$. For $A \in R^{m\times n}$ and $r(A) = k$, then the Smith normal form of A is given by

$$T_A = P_A A Q_A = \begin{bmatrix} T(A)_k & 0 \\ 0 & 0 \end{bmatrix},$$

where $P_A \in GL(m, R), Q_A \in GL(n, R)$ and $T(A)_k = \text{diag}(a_1, \cdots, a_k)$ with $a_i | a_{i+1}$ $(i = 1, \cdots, k-1)$.

Theorem 27. *[56] Assume that $A \in R^{m\times n}, B \in R^{k\times l}$, and $C \in R^{m\times l}$. The Smith normal forms of matrices A and B are given by*

$$P_A A Q_A = \begin{bmatrix} T(A)_p & 0 \\ 0 & 0 \end{bmatrix}, \ P_B B Q_B = \begin{bmatrix} T(B)_q & 0 \\ 0 & 0 \end{bmatrix}.$$

If the matrix C satisfies

$$P_A C Q_B = \begin{bmatrix} T(A)_p G T(B) q & 0 \\ 0 & 0 \end{bmatrix}, G \in R^{p\times q},$$

then the general solution of the matrix Equation (1) can be expressed as

$$X = Q_A \begin{bmatrix} G & D_{12} \\ D_{21} & D_{22} \end{bmatrix} P_B \in R^{n\times k},$$

where D_{12}, D_{21}, and D_{22} are arbitrary matrices with appropriate size over R.

Dual quaternions, dual split quaternions, and dual generalized commutative quaternions are extensions of quaternions, split quaternions, and generalized commutative quaternions, respectively, with significant applications in screw motions, computer graphics, rigid body motions, and robotics (see [91–96]). Notably, dual quaternions play a crucial role in the control of unmanned aerial vehicles and small satellites. Therefore, Chen et al. [8], Si et al. [11] and Shi et al. [98] provided several equivalent conditions for the solvability of matrix Equation (1) in the context of dual quaternions, dual split quaternions, and dual generalized commutative quaternions, along with expressions for the general solution.

Theorem 28. *[8] If $A = A_0 + A_1\epsilon \in \mathbb{DQ}^{a\times b}$, $B = B_0 + B_1\epsilon \in \mathbb{DQ}^{c\times d}$, and $C = C_0 + C_1\epsilon \in \mathbb{DQ}^{a\times d}$ are given. Set*

$$A_2 = A_1 L_{A_0}, \ C_{11} = A_0 A_0^\dagger C_0 B_0^\dagger B_1, \ B_2 = R_{B_0} B_1, \ C_{22} = A_1 A_0^\dagger C_0 B_0^\dagger B_0,$$

$$C_3 = C_1 - C_{11} - C_{22}, \ M_1 = R_{A_0} A_2, \ N_1 = R_{A_0} C_3, \ E_1 = B_2 L_{B_0}, \ F_1 = C_3 L_{B_0}.$$

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Then the matrix Equation (1) is solvable if and only if

$$R_{A_0}C_0 = 0,\ R_{M_1}N_1 = 0,\ C_0L_{B_0} = 0,\ F_1L_{E_1} = 0,\ R_{A_0}C_3L_{B_0} = 0.$$

or satisfy the following rank equalities.

$$r\begin{bmatrix} A_0 & C_0 \end{bmatrix} = r(A_0),\ r\begin{bmatrix} B_0 \\ C_0 \end{bmatrix} = r(B_0),\ r\begin{bmatrix} C_1 & A_0 \\ B_0 & 0 \end{bmatrix} = r(A_0) + r(B_0),$$
$$r\begin{bmatrix} A_1 & C_1 & A_0 \\ A_0 & C_0 & 0 \end{bmatrix} = r\begin{bmatrix} A_1 & A_0 \\ A_0 & 0 \end{bmatrix},\ r\begin{bmatrix} B_1 & B_0 \\ C_1 & C_0 \\ B_0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_0 \\ B_0 & 0 \end{bmatrix}.$$

In this situation, the general solution can be expressed as $X = X_0 + X_1\epsilon$, *where*

$$\begin{aligned} X_0 &= A_0^{\dagger}C_0B_0^{\dagger} + L_{A_0}U_1 + V_1R_{B_0}, \\ X_1 &= A_0^{\dagger}(C_3 - A_0V_1B_2 - A_2U_1B_0)B_0^{\dagger} + L_{A_0}Z_1 + Z_2R_{B_0}, \\ U_1 &= M_1^{\dagger}N_1B_0^{\dagger} + L_{M_1}Z_3 + Z_4R_{B_0}, \\ V_1 &= A_0^{\dagger}F_1E_1^{\dagger} + L_{A_0}Z_5 + Z_6R_{E_1}, \end{aligned} \tag{17}$$

and Z_i $(i = 1, \cdots, 6)$ *are arbitrary matrices over* $\mathbb{H}$.

Theorem 29. *[11] Suppose that* $A = A_0 + A_1\epsilon \in \mathbb{DH}_s^{p\times q}$, $B = B_0 + B_1\epsilon \in \mathbb{DH}_s^{k\times r}$, *and* $C = C_0 + C_1\epsilon \in \mathbb{DH}_s^{p\times r}$ *are given. Set*

$$\begin{aligned} &A_{00} = A_0^{\sigma_1}, A_{01} = A_1^{\sigma_1}, B_{00} = B_0^{\sigma_1}, B_{01} = B_1^{\sigma_1}, C_{00} = C_0^{\sigma_1}, C_{01} = C_1^{\sigma_1}, \\ &A_2 = A_{01}L_{A_{00}}, B_2 = R_{B_{00}}B_{01}, C_{11} = A_{00}A_{00}^{\dagger}C_{00}B_{00}^{\dagger}B_{01}, C_{22} = A_{01}A_{00}^{\dagger}C_{00}B_{00}^{\dagger}B_{00}, \\ &C_3 = C_{01} - C_{11} - C_{22}, M_1 = R_{A_{00}}A_2, N_1 = R_{A_{00}}C_3, E_1 = B_2L_{B_{00}}, F_1 = C_3L_{B_{00}}. \end{aligned}$$

Then the following descriptions hold the same meaning.

1. *The dual split quaternion matrix Equation (1) is solvable.*
2. *The system*

$$\begin{cases} A_{00}X_{00}B_{00} = C_{00}, \\ A_{00}X_{00}B_{01} + A_{00}X_{01}B_{00} + A_{01}X_{00}B_{00} = C_{01} \end{cases}$$

   is solvable.
3. $R_{A_{00}}C_{00} = 0, C_{00}L_{B_{00}} = 0, R_{M_1}N_1 = 0, R_{A_{00}}C_3L_{B_{00}} = 0, F_1L_{E_1} = 0.$
4.

$$r\begin{bmatrix} A_{00} & C_{00} \end{bmatrix} = r(A_{00}), r\begin{bmatrix} B_{00} \\ C_{00} \end{bmatrix} = r(B_{00}),$$
$$r\begin{bmatrix} A_{01} & A_{00} & C_{01} \\ A_{00} & 0 & C_{00} \end{bmatrix} = r\begin{bmatrix} A_{01} & A_{00} \\ A_{00} & 0 \end{bmatrix},$$
$$r\begin{bmatrix} C_{01} & A_{00} \\ B_{00} & 0 \end{bmatrix} = r(A_{00}) + r(B_{00}),$$
$$r\begin{bmatrix} B_{01} & B_{00} \\ B_{00} & 0 \\ C_{01} & C_{00} \end{bmatrix} = r\begin{bmatrix} B_{01} & B_{00} \\ B_{00} & 0 \end{bmatrix}.$$

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In such cases, the general solution $X = X_0 + X_1\epsilon$ is given by

$$X_0 = \frac{1}{8}\begin{bmatrix} I_q & I_q\mathbf{i} & I_q\mathbf{j} & I_q\mathbf{k} \end{bmatrix}\left(X_{00} + P_q X_{00} P_k^T + W_q X_{00} {W_k}^T + R_q X_{00} {R_k}^T\right)\begin{bmatrix} I_k \\ I_k\mathbf{i} \\ I_k\mathbf{j} \\ I_k\mathbf{k} \end{bmatrix},$$

$$X_1 = \frac{1}{8}\begin{bmatrix} I_q & I_q\mathbf{i} & I_q\mathbf{j} & I_q\mathbf{k} \end{bmatrix}\left(X_{01} + P_q X_{01} P_k^T + W_q X_{01} {W_k}^T + R_q X_{01} {R_k}^T\right)\begin{bmatrix} I_k \\ I_k\mathbf{i} \\ I_k\mathbf{j} \\ I_k\mathbf{k} \end{bmatrix},$$

where

$$\begin{aligned}
X_{00} &= A_{00}^{\dagger} C_{00} B_{00}^{\dagger} + L_{A_{00}} U_1 + V_1 R_{B_{00}},\\
X_{01} &= A_{00}^{\dagger}(C_3 - A_{00} V_1 B_2 - A_2 U_1 B_{00}) B_{00}^{\dagger} + L_{A_{00}} Z_1 + Z_2 R_{B_{00}},\\
U_1 &= M_1^{\dagger} N_1 B_{00}^{\dagger} + L_{M_1} Z_3 + Z_4 R_{B_{00}},\\
V_1 &= A_{00}^{\dagger} F_1 E_1^{\dagger} + L_{A_{00}} Z_5 + Z_6 R_{E_1},
\end{aligned}$$

and Z_i $(i = 1, \cdots, 6)$ are arbitrary real matrices.

Theorem 30. *[98] Assume that $A = A_0 + A_1\epsilon \in \mathbb{DH}_g^{m\times n}$, $B = B_0 + B_1\epsilon \in \mathbb{DH}_g^{p\times q}$, and $C = C_0 + C_1\epsilon \in \mathbb{DH}_g^{m\times q}$. Denote*

$$A_{00\eta} = \begin{cases} A_0^{\sigma_{\mathbf{i}}}, & \eta = \mathbf{i} \\ A_0^{\sigma_{\mathbf{j}}} V_n, & \eta = \mathbf{j} \\ A_0^{\sigma_{\mathbf{k}}} U_n, & \eta = \mathbf{k} \end{cases}, A_{11\eta} = \begin{cases} A_1^{\sigma_{\mathbf{i}}}, & \eta = \mathbf{i} \\ A_1^{\sigma_{\mathbf{j}}} V_n, & \eta = \mathbf{j} \\ A_1^{\sigma_{\mathbf{k}}} U_n, & \eta = \mathbf{k} \end{cases},$$

$$B_{00\eta} = \begin{cases} B_0^{\sigma_{\mathbf{i}}}, & \eta = \mathbf{i} \\ B_0^{\sigma_{\mathbf{j}}} V_q, & \eta = \mathbf{j} \\ B_0^{\sigma_{\mathbf{k}}} U_q, & \eta = \mathbf{k} \end{cases}, B_{11\eta} = \begin{cases} B_1^{\sigma_{\mathbf{i}}}, & \eta = \mathbf{i} \\ B_1^{\sigma_{\mathbf{j}}} V_q, & \eta = \mathbf{j} \\ B_1^{\sigma_{\mathbf{k}}} U_q, & \eta = \mathbf{k} \end{cases},$$

$$C_{00\eta} = C_0^{\sigma_\eta}, C_{11\eta} = C_1^{\sigma_\eta}, C_{00} = A_{00\eta} A_{00\eta}^{\dagger} C_{00\eta} B_{00\eta}^{\dagger} B_{11\eta} + A_{11\eta} A_{00\eta}^{\dagger} C_{00\eta} B_{00\eta}^{\dagger} B_{00\eta},$$

$$A_{00} = A_{11\eta} L_{A_{00\eta}}, B_{00} = R_{B_{00\eta}} B_{11\eta}, L_1 = B_{00\eta}^T \otimes A_{00}, M_1 = B_{00}^T \otimes A_{00\eta}, N_1 = B_{00\eta}^T \otimes A_{00\eta},$$

$$Q_1 = \begin{pmatrix} L_1 & M_1 & N_1 \end{pmatrix}, \ d = \mathrm{vec}(C_{11\eta} - C_{00}),$$

where V_n, V_q are defined in Equation (9) and U_n, U_q are given in Equation (7). Then, we can obtain the following equivalent description.

1. *The dual generalized commutative quaternion matrix Equation (1) is solvable.*
2. *The system*

$$\begin{cases} A_{00\eta} X_{00\eta} B_{00\eta} = C_{00\eta}, \\ A_{00\eta} X_{00\eta} B_{11\eta} + A_{00\eta} X_{11\eta} B_{00\eta} + A_{11\eta} X_{00\eta} B_{00\eta} = C_{11\eta} \end{cases}$$

   is solvable.
3.

$$\begin{aligned}
&A_{00\eta} A_{00\eta}^{\dagger} C_{00\eta} B_{00\eta}^{\dagger} B_{00\eta} = C_{00\eta}, A_{00\eta} A_{00\eta}^{\dagger} C_{00\eta} = C_{00\eta},\\
&C_{00\eta} B_{00\eta}^{\dagger} B_{00\eta} = C_{00\eta}, Q_1 Q_1^{\dagger} d = d.
\end{aligned}$$

In this case, the general solution can be expressed as $X = X_0 + X_1\epsilon$.

(1) When $\eta = \mathbf{i}$,

$$X_0 = \frac{1}{16}\begin{bmatrix} I_n & I_n\mathbf{i} & I_n\mathbf{j} & I_n\mathbf{k} \end{bmatrix}\left(X_{00\mathbf{i}} + (R_n^1)^{-1}X_{00\mathbf{i}}R_n^1 + (S_n^1)^{-1}X_{00\mathbf{i}}S_n^1 + (T_n^1)^{-1}X_{00\mathbf{i}}T_n^1\right)\begin{bmatrix} I_p \\ \frac{1}{\alpha}I_p\mathbf{i} \\ \frac{1}{\beta}I_p\mathbf{j} \\ \frac{1}{\alpha\beta}I_p\mathbf{k} \end{bmatrix},$$

$$X_1 = \frac{1}{16}\begin{bmatrix} I_n & I_n\mathbf{i} & I_n\mathbf{j} & I_n\mathbf{k} \end{bmatrix}\left(X_{11\mathbf{i}} + (R_n^1)^{-1}X_{11\mathbf{i}}R_n^1 + (S_n^1)^{-1}X_{11\mathbf{i}}S_n^1 + (T_n^1)^{-1}X_{11\mathbf{i}}T_n^1\right)\begin{bmatrix} I_p \\ \frac{1}{\alpha}I_p\mathbf{i} \\ \frac{1}{\beta}I_p\mathbf{j} \\ \frac{1}{\alpha\beta}I_p\mathbf{k} \end{bmatrix}.$$

(2) When $\eta = \mathbf{j}$,

$$X_0 = \frac{1}{16}\begin{bmatrix} -I_n & I_n\mathbf{i} & I_n\mathbf{j} & -I_n\mathbf{k} \end{bmatrix}\left(X_{00\mathbf{j}} - (R_n^1)^{-1}X_{00\mathbf{j}}R_n^1 - (S_n^1)^{-1}X_{00\mathbf{j}}S_n^1 + (T_n^1)^{-1}X_{00\mathbf{j}}T_n^1\right)\begin{bmatrix} I_p \\ \frac{1}{\alpha}I_p\mathbf{i} \\ \frac{1}{\beta}I_p\mathbf{j} \\ \frac{1}{\alpha\beta}I_p\mathbf{k} \end{bmatrix},$$

$$X_1 = \frac{1}{16}\begin{bmatrix} -I_n & I_n\mathbf{i} & I_n\mathbf{j} & -I_n\mathbf{k} \end{bmatrix}\left(X_{11\mathbf{j}} - (R_n^1)^{-1}X_{11\mathbf{j}}R_n^1 - (S_n^1)^{-1}X_{11\mathbf{j}}S_n^1 + (T_n^1)^{-1}X_{11\mathbf{j}}T_n^1\right)\begin{bmatrix} I_n \\ \frac{1}{\alpha}I_n\mathbf{i} \\ \frac{1}{\beta}I_n\mathbf{j} \\ \frac{1}{\alpha\beta}I_n\mathbf{k} \end{bmatrix}.$$

(3) When $\eta = \mathbf{k}$,

$$X_0 = \frac{1}{16}\begin{bmatrix} I_n & I_n\mathbf{i} & -I_n\mathbf{j} & -I_n\mathbf{k} \end{bmatrix}\left(X_{00\mathbf{k}} + (R_n^1)^{-1}X_{00\mathbf{k}}R_n^1 - (S_n^1)^{-1}X_{00\mathbf{k}}S_n^1 - (T_n^1)^{-1}X_{00\mathbf{k}}T_n^1\right)\begin{bmatrix} I_p \\ \frac{1}{\alpha}I_p\mathbf{i} \\ \frac{1}{\beta}I_p\mathbf{j} \\ \frac{1}{\alpha\beta}I_p\mathbf{k} \end{bmatrix},$$

$$X_1 = \frac{1}{16}\begin{bmatrix} I_n & I_n\mathbf{i} & -I_n\mathbf{j} & -I_n\mathbf{k} \end{bmatrix}\left(X_{11\mathbf{k}} + (R_n^1)^{-1}X_{11\mathbf{k}}R_n^1 - (S_n^1)^{-1}X_{11\mathbf{k}}S_n^1 - (T_n^1)^{-1}X_{11\mathbf{k}}T_n^1\right)\begin{bmatrix} I_p \\ \frac{1}{\alpha}I_p\mathbf{i} \\ \frac{1}{\beta}I_p\mathbf{j} \\ \frac{1}{\alpha\beta}I_p\mathbf{k} \end{bmatrix}.$$

Where

$$X_{00\eta} = A_{00\eta}^{\dagger}C_{00\eta}B_{00\eta}^{\dagger} + L_{A_{00\eta}}V_1 + U_1R_{B_{00\eta}}, \eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}, \},$$

$$\begin{pmatrix} \mathrm{vec}(V_1) \\ \mathrm{vec}(U_1) \\ \mathrm{vec}(X_{11\eta}) \end{pmatrix} = Q_1^{\dagger}d + L_{Q_1}u,$$

and u is any column vector of appropriate dimension over $\mathbb{R}$.

4. Various Algorithms For Solving The Matrix Equation $AXB = C$

The previous section presented various special solutions to matrix Equation (1), and the numerical algorithms for these special solutions have also attracted significant attention from many scholars.

In 2008, Ding et al. [68] proposed a gradient-based iterative algorithm (GBI) to find the general solution of matrix Equation (1). Later, in 2013, Khorsand Zak et al. [73] introduced a nested splitting conjugate gradient (NSCG) iteration method, which leverages the symmetric and skew-symmetric splitting of the coefficient matrices A and B. Building on this, Wang et al. [72] extended the idea to $\mathbb{C}$ and proposed a new iterative method—the Hermitian and skew-Hermitian splitting (HSS) iteration method. Compared to GBI, this method offers advantages in terms of the number of iteration steps and computation time. In 2016, Zhou et al. [76] proposed a modified HSS iteration method (MHSS), which reduces computational complexity and enhances efficiency compared to the HSS iteration method. In 2017, Tian et al. [77] proposed Jacobi and Gauss-Seidel-type (GS) iteration methods, which require fewer iteration steps and have a broader range of applications compared to the HSS iteration method, but take more computing time. Subsequently, in 2019, Liu et al. [80] supplemented the gaps in the convergence proof of the Jacobi and GS iteration methods proposed by Tian et al. In 2021, Tian et al. [85] developed the relaxed Jacobi-type method and the relaxed Gauss-Seidel-type (RGS) method based on the Jacobi and GS iteration methods. These two relaxed iterative methods outperform HSS in terms of computation time and the number of iterations, with RGS method being the most effective among them. Meanwhile, Chen et al. [84] introduced the two-step accelerated over-relaxation iteration method (TS-AOR), which demonstrates faster convergence and reduced computation time compared to the Jacobi and GS iteration methods. In 2024, Tian et al. [90] presented a parameterized two-step iteration (PTSI) method based on the TS-AOR method. In solving matrix Equation (1), the Jacobi, GS, relaxed Jacobi, RGS and TS-AOR iteration methods all involve the Kronecker product and inversion of the coefficient matrix, which inevitably increases both the matrix size and computational complexity. Therefore, Liu et al. [82] proposed the stationary splitting iterative method in 2020, which directly splits the coefficient matrices A and B instead of processing the matrix $B^T \otimes A$. This approach is more efficient when dealing with large-scale data. Furthermore, the method can be applied to any convergent splitting, not just limited to Jacobi and Gauss-Seidel splittings, thereby increasing its flexibility. In 2023, Tian et al. [88] developed the parameterized accelerated iteration (PAI) method to circumvent the coefficient matrix inversion problem. Moreover, the PAI method outperforms HSS, GBI, and MHSS in terms of CPU time, and it surpasses the GBI method in iteration steps, though it is less efficient than the HSS method. In addition, in 2022, Wu et al. [86] combined the Kaczmarz methods with relaxed greedy selection to introduce the matrix equation relaxed greedy randomized Kaczmarz (ME-RGRK) method and the maximal weighted residual Kaczmarz (ME-MWRK) method. Both methods demonstrate superior convergence speed and computation time compared to GBI methods and iterative orthogonal direction methods. Currently, research primarily focuses on iterative methods for matrix equations, while some scholars are also exploring numerical algorithms for tensor equations. In 2019, Wang et al. [81] provided an iterative algorithm for the general solution of the tensor equation $\mathcal{A} *_N \mathcal{X} *_M \mathcal{B} = \mathcal{C}$ under the Einstein product. For details, refer to Table 1.

Table 1. The general solution of Equation (1).

Proposed by	Type of solution	Algorithm type	Number Field
Ding, 2008 [68]	general solution	GBI	$\mathbb{R}$
Khorsand Zak, 2013 [73]		NSCG	$\mathbb{R}$
Wang, 2013 [72]		HSS	$\mathbb{C}$
Zhou, 2016 [76]		MHSS	$\mathbb{C}$
Tian, 2017 [77]		Jacobi and GS	$\mathbb{R}$
Liu, 2020 [82]		stationary splitting iteration	$\mathbb{R}$
Tian, 2021 [85]		relaxed Jacobi and RGS	$\mathbb{R}$
Chen, 2021 [84]		TS-AOR	$\mathbb{R}$
Wu, 2022 [86]		ME-RGRK and ME-MWRK	$\mathbb{R}$
Tian, 2023 [88]		PAI	$\mathbb{R}$
Tian, 2024 [90]		PTSI	$\mathbb{R}$
Wang, 2019 [81]	general solution (tensor)	iteration	$\mathbb{R}$

In 2005, Peng et al. [62] developed an iterative method to find symmetric solutions to matrix Equation (1) over $\mathbb{R}$, providing the optimal approximation solution in the Frobenius norm. Subsequently, Deng et al. [61] introduced the iteration orthogonal direction (IOD) method for Hermitian solutions over $\mathbb{C}$, which outperformed the conjugate gradient for normal equation (CGNE) method in terms of iteration steps and computation time. While the optimal approximation solution focuses on finding the closest solution to a given matrix under specific conditions, the least squares solution generally aims to find the best fit for potentially unsolvable equations. Their goals and applications differ significantly. Therefore, Peng et al. [63] investigated iterative algorithms for the least squares symmetric solution to this matrix equation. In 2006, Hou et al. [64] presented another iterative algorithm for the least squares symmetric solution. In 2007, Liao et al. [66] provided the least-squares solution expressions for matrix Equation (1) using the GSVD and the canonical correlation decomposition (CCD). In the same year, Lei et al. [65], in response to the potential irregular convergence behavior in the residual norm of the iterative algorithm proposed by Peng et al. [63] for this matrix equation, introduced the minimal residual method based on the conjugate gradient method. Currently, most algorithms focus on symmetric solutions. Therefore, Huang et al. [97] proposed an iterative algorithm for skew-symmetric solutions in 2008. In 2010, Peng [69] utilized the LSQR algorithm to solve the matrix Equation (1) for symmetric, symmetric R-symmetric, and (R, S)-symmetric solutions. This algorithm was proposed by Paige et al. [99] in 1982 and demonstrates superior convergence speed

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and accuracy compared to the iterative algorithms presented in references [62,63,65,97]. Following this, Peng [70] introduced two new matrix iterative methods based on Paige's algorithms. In 2016, Peng et al. [100] introduced two new iterative method based on the alternating variable minimization with multiplier (AVMM) method. Each of these methods has its advantages and disadvantages compared to LSQR. When the number of rows and columns in the coefficient matrix is relatively close, LSQR performs better. In 2020, Yu et al. [83] employed the alternating direction method with multipliers (ADMM) to solve the nearness skew-symmetric and symmetric solutions for matrix Equation (1). With appropriate selection of parameters and preconditioners, ADMM outperforms the iterative algorithms presented in references [97,100]. Additionally, in 2016, Xie et al. [75] considered the generalized Lanczos trust region (GLTR) algorithm for the least squares symmetric solution of the matrix equation with a norm inequality constraint. In 2024, Duan et al. [89] employed the ADMM method to solve the least squares symmetric solution problem of the tensor equation $\mathcal{A} *_N \mathcal{X} *_N \mathcal{B} = \mathcal{C}$ under the Einstein product. For details, see Table 2.

Table 2. Various symmetric solutions for Equation (1).

Proposed by	Type of solution	Algorithm type	Number Field
Peng, 2005 [62]	symmetric, optimal approximation	iteration	$\mathbb{R}$
Peng, 2005 [63]	least squares symmetric	iteration	$\mathbb{R}$
Deng, 2006 [61]	Hermitian minimum norm	IOD	$\mathbb{C}$
Hou, 2006 [64]	least squares symmetric	iteration	$\mathbb{R}$
Liao, 2007 [66]	optimal approximate least-squares symmetric	GSVD,CCD	$\mathbb{R}$
Lei, 2007 [65]	optimal approximate least-squares symmetric	minimal residual algorithm	$\mathbb{R}$
Huang, 2008 [97]	skew-symmetric, optimal approximation	iteration	$\mathbb{R}$
Peng, 2010 [69]	symmetric, symmetric R-symmetric, (R,S)-symmetric	LSQR	$\mathbb{R}$
Peng, 2010 [70]	symmetric, symmetric R-symmetric, (R,S)-symmetric	Paige's algorithm	$\mathbb{R}$
Xie, 2016 [75]	least-squares symmetric	GLTR	$\mathbb{R}$
Peng, 2016 [100]	nearness symmetric	AVMM	$\mathbb{R}$
Yu, 2020 [83]	nearness skew-symmetric and symmetric	ADMM	$\mathbb{R}$
Duan, 2024 [89]	least-squares symmetric (tensor)	ADMM	$\mathbb{R}$

In addition, many scholars have studied algorithms for other specific structured solutions to matrix Equation (1), as shown in the table 3.

5. An Application

In this section, we propose an encryption and decryption scheme for color images based on the dual quaternion matrix equation $AXB = C$, accompanied by a practical example to validate our approach.

We know that quaternions are a generalization of complex numbers, and dual quaternions are a further generalization of quaternions. Therefore, we consider the dual quaternion matrix equation $AXB = C$ to illustrate its application in color image processing. Additionally, a color image can be represented by a quaternion matrix, and both the standard part and the infinitesimal part of a dual quaternion matrix are quaternion matrices. This means that a dual quaternion matrix can represent two color images. The encryption and decryption scheme is shown in Figure 1.

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Proposed by	Type of solution	Algorithm type	Number Field
Liang, 2007 [2]	generalized centro-symmetric	iteration	$\mathbb{R}$
Peng, 2007 [67]	bisymmetric, optimal approximation	iteration	$\mathbb{R}$
Li, 2010 [21]	mirrorsymmetric	conjugate gradient least squares method (CGLS)	$\mathbb{R}$
Li, 2011 [71]	centrosymmetric	CGLS	$\mathbb{R}$
Sarduvan, 2014 [74]	(P, Q)-orthogonal (skew-) symmetric	spectral decomposition	$\mathbb{R}$
Wang, 2017 [78]	Generalized reflexive and anti-reflexive	iteration	$\mathbb{R}$
Duan, 2023 [87]	least squares solution (tensor)	Paige's algorithm	$\mathbb{R}$

Table 3. Other types of special solutions for Equation (1)

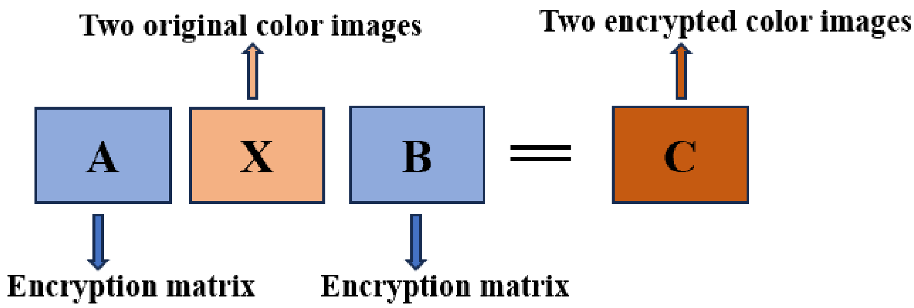

Figure 1. Scheme

Select any two color images and encrypt them according to the principles illustrated in Figure 1.

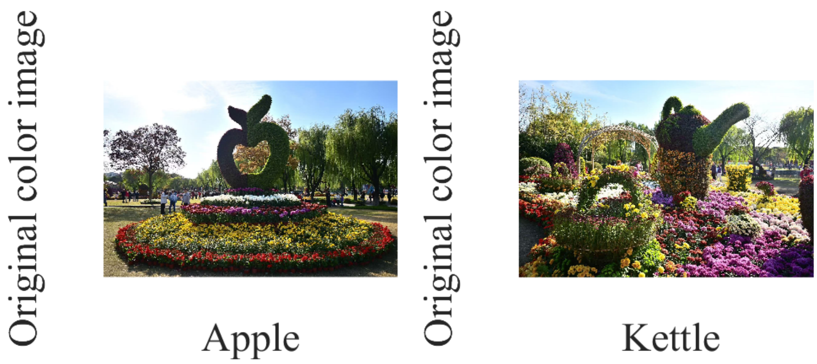

Figure 2. Original image

Encrypt the image shown in Figure 2 to obtain Figure 3.

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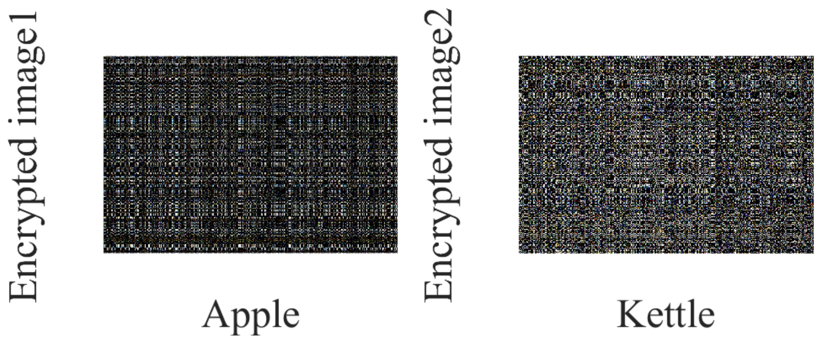

Figure 3. Encrypt image

Decrypt the encrypted image in Figure 3 using the encryption matrices A and B along with Theorem 28 to obtain Figure 4.

Figure 4. Decrypt image

From Figure 4, we can see that the decrypted image is indistinguishable from the original image. We use the Structural Similarity Index Measure (SSIM) to assess the quality of the decrypted image. The SSIM values for the images "Apple" and "Kettle" are both 1, indicating that the encryption scheme based on the dual quaternion matrix equation $AXB = C$ is highly feasible. For more details, see Table 4.

Table 4. Evaluation of Effect.

Color image name	SSIM
Apple	1
Kettle	1

6. Conclusions

This survey has provided an overview of various special solutions to matrix Equation (1) and the corresponding numerical algorithms. In the process, definitions of certain special matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{DQ}$, $\mathbb{DH}_s$, and $\mathbb{DH}_g$ have been given, along with a discussion of their related properties. The various special solutions to matrix Equation (1) have been classified, summarized, and the corresponding numerical algorithms have been explored. Furthermore, using the dual quaternion matrix equation $AXB = C$ as an example, a scheme for color image encryption and decryption has been designed, with experimental results demonstrating its feasibility. This has enriched both the theoretical and practical applications of the matrix equation in color image processing. Finally, we have found that

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most research on the special solutions of matrix Equation (1) has focused primarily on real numbers, complex numbers, and quaternions, while numerical algorithms have mostly been limited to real and complex numbers. Therefore, the following areas of work can be considered for future research:

- The exploration of special solutions to matrix Equation (1) over dual quaternions, dual split quaternions, or dual generalized commutative quaternions could be a valuable direction for future research. This includes solutions such as (anti-)symmetric solutions, (anti-)reflexive solutions, (R, S)-(skew)symmetric solutions, bisymmetric solutions, reducible solutions, and so on. Furthermore, it would be interesting to investigate whether these solutions can be considered over dual quaternion tensors, dual split quaternion tensors, or dual generalized commutative quaternion tensors.
- The study of corresponding numerical algorithms over quaternions, dual quaternions, dual split quaternions, or dual generalized commutative quaternions is another promising direction for future research. Furthermore, it would be worth exploring whether these algorithms can be extended to tensors (over the complex field), dual quaternion tensors, dual split quaternion tensors, or dual generalized commutative quaternion tensors.

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