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# The Abu-Ghuwaleh Hyper-Residue Calculus for Improper Integrals Phase-Lifted Spectral Solvers, Finite-Part Regularization, Branch Kernels, and Continuous Spectra

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**Abstract**

We develop a new calculus for improper integrals in which the decisive object is not the meromorphic structure of the integrand in the physical variable, but a spectral measure $\mu$ generating the kernel through a Laplace orbit and a Mellin symbol $\Gamma(z)\mathcal{Z}_\mu(z;\omega)$. The first main result is an exact phase-lifted solver for oscillatory and non-oscillatory improper integrals. The second is the Abu-Ghuwaleh hyper-residue theorem, which yields meromorphic continuation for gapped spectra and explicit pole coefficients at the negative integers. The third is a finite-part regularization theorem that turns divergent endpoint integrals into regular values of the same symbol. The framework then extends to discrete entire kernels, branch and logarithmic kernels, multiscale analytic germs, fractional lattices, continuous spectral densities, and annihilator theorems that convert spectral support constraints into ordinary or infinite-order differential equations. A benchmark section evaluates hard examples one by one, including essential-singularity kernels, oscillatory $u^{-1}$ problems, finite-part $u^{-2}$ integrals, branch kernels, nonlinear phases, Mittag–Leffler and Airy kernels, polylogarithmic kernels, and algebraic continuous-spectrum kernels. The resulting method is exact, constructive, and specifically adapted to a large family of improper integrals for which contour geometry is not the natural language.

**Keywords:** improper integrals; Mellin methods; finite-part regularization; spectral measures; generalized Dirichlet series; branch kernels; integral transforms; differential equations

**MSC:** 30E20; 33E20; 44A20; 44A35; 34A05

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## 1. Introduction

The residue theorem is one of the deepest and most beautiful tools in analysis. When an improper integral can be embedded into a contour integral of a meromorphic function with manageable poles, residues often produce a closed form with remarkable elegance. But the classical contour language is not always the natural one. Many difficult integrals that arise in modern analysis are built from entire kernels, branch kernels, multiscale exponentials, logarithmic barriers, Mittag–Leffler kernels, Airy kernels, or finite-part divergent combinations. In such situations the integrand may fail to be meromorphic in any useful sense, or the contour geometry may become artificially complicated even though the integral itself has hidden structure.

The present paper proposes a different backbone. Instead of asking for poles of the physical-variable integrand, we ask for a *spectral measure* that generates the kernel as a Laplace superposition of elementary modes. This changes the geometry completely. The integral is no longer read from contour closure in the $t$-plane, but from a Mellin–Gamma symbol attached to the spectral measure. The residues that matter are no longer residues of the original integrand. They are residues of the spectral symbol after Mellin lifting. That is why we call them *hyper-residues*.

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The classical residue theorem, Mellin–Barnes methods, divergent-series regularization, and asymptotic integral theory remain foundational for definite and improper integrals; see, for example, [8,10,11,13,15–17]. The point of the present work is not to displace that classical body, but to construct a native solving geometry for a different family of integrals—those generated by spectral Laplace orbits, nonlinear phase lifts, and endpoint regularization.

The new theory has four structural strengths.

1. **It is an exact solver, not a formal umbrella.** For every admissible spectral measure $\mu$ and every admissible phase lift $\phi$, the improper integral is computed exactly by one symbol,
$$\Gamma(z)\mathcal{Z}_\mu(z;\omega),$$
without approximation.
2. **It canonically regularizes divergent endpoint behaviour.** For gapped spectra the Mellin symbol extends meromorphically to the full $z$-plane, and Hadamard finite-part values at negative integers are simply regular values of $\Gamma(z)\mathcal{Z}_\mu(z;\omega)$. The associated pole coefficients are the hyper-residues.
3. **It treats discrete and continuous spectra in the same language.** Entire and branch kernels lead to discrete measures, while algebraic and Stieltjes-type kernels arise from continuous spectral densities. Thus the same backbone controls both exponential-scale and algebraic-scale phenomena.
4. **It yields operational consequences beyond evaluation.** The spectral symbol gives exact recovery by Mellin inversion, convolution laws for products of kernels, polynomial or entire annihilators that generate differential equations, and phase-lift invariance for nonlinear transformations such as $t \mapsto t^r$ or $t \mapsto \log(1+t)$.

This paper is connected to the Abu-Ghuwaleh line on improper integrals, fractional definitions, differential-equation generators, and entire-kernel transforms [1–7]. The 2022 papers introduced theorem families for improper integrals; the 2023 paper developed new fractional definitions for complex analytic functions; the 2024 master-generator paper connected symbolic kernels to ordinary differential equations; and the 2025 Master Integral Transform paper furnished a broader entire-kernel framework. The present manuscript takes a different step: it isolates a new solving theory whose backbone is the spectral measure and whose computational engine is the hyper-residue calculus.

The key message can be said very simply. If one can write the kernel as
$$K(u) = \int_0^\infty e^{-au}\, \mathrm{d}\mu(a),$$
then the integral
$$\int_0^\infty u^{z-1} e^{i\omega u} K(u)\, \mathrm{d}u$$
is controlled by
$$\Gamma(z) \int_0^\infty (a - i\omega)^{-z}\, \mathrm{d}\mu(a).$$
Everything else in this paper is a strengthening of this fact: phase lifting, meromorphic continuation, finite-part regularization, branch and logarithmic kernels, continuous spectra, annihilators, and benchmark evaluations.

## 2. Materials and Methods: Spectral Measures, Phase Lifts, and Mellin Symbols

### 2.1. Admissible Spectral Measures

**Definition 1.** *;[Spectral-admissible measure] A complex Borel measure $\mu$ on $[0,\infty)$ is called* spectral-admissible *if the following two conditions hold:*

*(i)* *for every integer $N \geq 0$,*
$$\int_0^\infty (1+a)^N\, \mathrm{d}|\mu|(a) < \infty;$$

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*(ii)* *the negative-moment abscissa*

$$\sigma_\mu := \sup\left\{\sigma > 0 : \int_0^\infty a^{-\sigma} \, \mathrm{d}|\mu|(a) < \infty\right\}$$

*is positive.*

*For such a measure we define the* Laplace orbit kernel

$$K_\mu(u) := \int_0^\infty e^{-au} \, \mathrm{d}\mu(a), \qquad u > 0,$$

*and, for $\omega \in \mathbb{R}$, the* spectral zeta symbol

$$\mathcal{Z}_\mu(z;\omega) := \int_0^\infty (a - i\omega)^{-z} \, \mathrm{d}\mu(a),$$

*whenever the integral converges absolutely. The principal branch is used for the logarithm in $(a - i\omega)^{-z}$.*

;

**Remark 1.** *If $\omega \neq 0$, then $a - i\omega$ never vanishes on $[0, \infty)$, and the principal branch is unambiguous. If $\omega = 0$, the definition reduces to $a^{-z}$ and the convergence strip is governed by $\sigma_\mu$.*

**Definition 2.** *;[Admissible phase lift] A map $\phi : (0, \infty) \to (0, \infty)$ is called an* admissible phase lift *if $\phi$ is $C^1$, strictly increasing, satisfies $\phi'(t) > 0$ for all $t > 0$, and*

$$\phi(0^+) = 0, \qquad \phi(+\infty) = +\infty.$$

;

**Proposition 1** (Basic properties of $K_\mu$)**.** *If $\mu$ is spectral-admissible, then $K_\mu \in C^\infty(0, \infty)$, and for every $m \geq 0$,*

$$K_\mu^{(m)}(u) = (-1)^m \int_0^\infty a^m e^{-au} \, \mathrm{d}\mu(a).$$

*Moreover, for every $\delta > 0$ and $m \geq 0$, $K_\mu^{(m)}$ is bounded on $[\delta, \infty)$.*

**Proof.** Differentiation under the integral sign is justified by

$$\int_0^\infty a^m e^{-au} \, \mathrm{d}|\mu|(a) \leq \int_0^\infty a^m e^{-a\delta} \, \mathrm{d}|\mu|(a) < \infty$$

for $u \geq \delta$, using the moment condition and the elementary inequality $a^m e^{-a\delta} \leq C_{m,\delta}(1 + a)^{m+1}$. □

*2.2. The Phase-Lifted Exact Solver*

**Theorem 1** (The Abu-Ghuwaleh spectral solver)**.** *Let $\mu$ be spectral-admissible, let $\phi$ be an admissible phase lift, let $\omega \in \mathbb{R}$, and let $k \in \mathbb{N} \cup \{0\}$. If*

$$0 < \operatorname{Re} z < \sigma_\mu,$$

*then*

$$\mathcal{I}_{\mu,\phi}^{(k)}(z,\omega) := \int_0^\infty \phi(t)^{z-1} (\log\phi(t))^k e^{i\omega\phi(t)} K_\mu(\phi(t)) \phi'(t) \, \mathrm{d}t = (-\partial_z)^k \big[\Gamma(z) \mathcal{Z}_\mu(z;\omega)\big]. \tag{1}$$

*In particular, for $k = 0$,*

$$\mathcal{I}_{\mu,\phi}^{(0)}(z,\omega) = \Gamma(z) \mathcal{Z}_\mu(z;\omega).$$

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**Proof.** Make the substitution $u = \phi(t)$. Since $\phi$ is an admissible phase lift, it is a $C^1$ bijection from $(0,\infty)$ onto $(0,\infty)$, and therefore

$$\mathcal{I}^{(k)}_{\mu,\phi}(z,\omega) = \int_0^\infty u^{z-1}(\log u)^k e^{i\omega u} K_\mu(u)\ \mathrm{d}u.$$

Thus it suffices to prove the formula for $\phi(t) = t$.

Fix $0 < \sigma < \sigma_\mu$ and choose $\varepsilon > 0$ so small that $\sigma + \varepsilon < \sigma_\mu$. The elementary bound

$$|\log u|^k \le C_{k,\varepsilon}(u^{-\varepsilon} + u^{\varepsilon}), \qquad u > 0,$$

gives

$$\int_0^\infty u^{\sigma-1}|\log u|^k e^{-au}\ \mathrm{d}u \le C_{k,\varepsilon}\Gamma(\sigma+\varepsilon)a^{-\sigma-\varepsilon} + C_{k,\varepsilon}\Gamma(\sigma-\varepsilon)a^{-\sigma+\varepsilon}.$$

Integrating in $a$ against $\mathrm{d}|\mu|(a)$ yields absolute convergence because $\sigma + \varepsilon < \sigma_\mu$ and the positive moments are finite. Therefore Fubini's theorem applies:

$$\int_0^\infty u^{z-1}(\log u)^k e^{i\omega u} K_\mu(u)\ \mathrm{d}u = \int_0^\infty \left(\int_0^\infty u^{z-1}(\log u)^k e^{-(a-i\omega)u}\ \mathrm{d}u\right) \mathrm{d}\mu(a).$$

For $\operatorname{Re} z > 0$,

$$\int_0^\infty u^{z-1}(\log u)^k e^{-\lambda u}\ \mathrm{d}u = (-\partial_z)^k\big[\Gamma(z)\lambda^{-z}\big], \qquad \operatorname{Re}\lambda > 0.$$

Taking $\lambda = a - i\omega$ and integrating in $a$ gives (1). □

**Corollary 1** (Real and imaginary parts)**.** *Under the assumptions of Theorem 1, for $0 < \operatorname{Re} z < \sigma_\mu$,*

$$\int_0^\infty \phi(t)^{z-1}\cos(\omega\phi(t))K_\mu(\phi(t))\phi'(t)\ \mathrm{d}t = \operatorname{Re}\big[\Gamma(z)\mathcal{Z}_\mu(z;\omega)\big],$$

*and*

$$\int_0^\infty \phi(t)^{z-1}\sin(\omega\phi(t))K_\mu(\phi(t))\phi'(t)\ \mathrm{d}t = \operatorname{Im}\big[\Gamma(z)\mathcal{Z}_\mu(z;\omega)\big].$$

**Corollary 2** (Exact Mellin recovery for gapped spectra)**.** *Let $\mu$ be gapped, fix $\omega \in \mathbb{R}$, and choose $c > 0$. Then for every $u > 0$,*

$$e^{i\omega u}K_\mu(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u^{-z}\Gamma(z)\mathcal{Z}_\mu(z;\omega)\ \mathrm{d}z.$$

**Proof.** By Theorem 1 with $\phi(t) = t$, the Mellin transform of $u \mapsto e^{i\omega u}K_\mu(u)$ equals $\Gamma(z)\mathcal{Z}_\mu(z;\omega)$ on each line $\operatorname{Re} z = c > 0$. Because $\mu$ is gapped, there exists $a_* > 0$ with $\operatorname{supp}\mu \subset [a_*,\infty)$. Hence $|\arg(a - i\omega)| \le \theta < \pi/2$ uniformly on the support of $\mu$, so on $\operatorname{Re} z = c$,

$$|\mathcal{Z}_\mu(z;\omega)| \le C_c e^{\theta|\Im z|}$$

for some $C_c > 0$. Stirling's formula gives

$$|\Gamma(c+it)| \le C_c'(1+|t|)^{c-1/2}e^{-\pi|t|/2},$$

therefore $\Gamma(c+it)\mathcal{Z}_\mu(c+it;\omega)$ is absolutely integrable in $t$. Mellin inversion yields the claimed formula. □

**Proposition 2** (Spectral convolution law)**.** *Let $\mu$ and $\nu$ be spectral-admissible. Write $\mu * \nu$ for their additive convolution on $[0,\infty)$. Then*

$$K_{\mu*\nu}(u) = K_\mu(u)K_\nu(u), \qquad u > 0.$$

*Consequently,*
$$\int_0^\infty u^{z-1}e^{i\omega u}K_\mu(u)K_\nu(u)\,\mathrm{d}u = \Gamma(z)\int_0^\infty (a-i\omega)^{-z}\,\mathrm{d}(\mu*\nu)(a)$$
*whenever* $0 < \operatorname{Re} z < \sigma_{\mu*\nu}$.

**Proof.** By definition of additive convolution,
$$K_{\mu*\nu}(u) = \int_0^\infty e^{-au}\,\mathrm{d}(\mu*\nu)(a) = \int_0^\infty\int_0^\infty e^{-(x+y)u}\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) = K_\mu(u)K_\nu(u).$$
The integral identity follows from Theorem 1. □

**Remark 2** (Operational reading). *Theorem 1 is the actual computation procedure:*

*Step 1:* *Choose a phase variable* $u = \phi(t)$ *that linearizes the oscillation or decay.*
*Step 2:* *Identify the kernel as a Laplace orbit kernel* $K_\mu(u)$.
*Step 3:* *Compute the spectral symbol* $\mathcal{Z}_\mu(z;\omega)$.
*Step 4:* *Evaluate* $\Gamma(z)\mathcal{Z}_\mu(z;\omega)$ *at the desired* $z$, *or take a regular value if the target point is a pole.*

*This is the hyper-residue method in practice.*

## 3. Main Results: The Abu-Ghuwaleh Spectral Solver, Hyper-Residues, and Finite-Part Regularization

### 3.1. Gapped Measures and Taylor Jets

**Definition 3.** *;[Gapped spectral measure] A spectral-admissible measure* $\mu$ *is called* gapped *if there exists* $a_* > 0$ *such that*
$$\operatorname{supp}\mu \subset [a_*,\infty).$$

;

For gapped measures the low-frequency obstruction disappears entirely. The symbol $\mathcal{Z}_\mu(z;\omega)$ becomes entire in $z$, and all singularities of $\Gamma(z)\mathcal{Z}_\mu(z;\omega)$ come from the Mellin–Gamma factor.

**Proposition 3** (Spectral Taylor jet). *Let* $\mu$ *be gapped and let*
$$m_n(\mu,\omega) := \int_0^\infty (a-i\omega)^n\,\mathrm{d}\mu(a), \qquad n\in\mathbb{N}\cup\{0\}.$$
*Then for every integer* $N \ge 1$,
$$e^{i\omega u}K_\mu(u) = \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}m_n(\mu,\omega)u^n + R_N(u,\omega), \qquad u>0,$$
*where*
$$R_N(u,\omega) = O(u^N) \qquad (u\to 0^+).$$

**Proof.** Since $\mu$ is gapped, every moment of $(a-i\omega)$ is finite. We write
$$e^{i\omega u}K_\mu(u) = \int_0^\infty e^{-(a-i\omega)u}\,\mathrm{d}\mu(a).$$
Taylor's theorem with integral remainder gives
$$e^{-\lambda u} = \sum_{n=0}^{N-1}\frac{(-\lambda u)^n}{n!} + \frac{(-\lambda)^N u^N}{(N-1)!}\int_0^1 (1-\theta)^{N-1}e^{-\theta\lambda u}\,\mathrm{d}\theta, \qquad \operatorname{Re}\lambda \ge a_* > 0.$$

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Integrating in $\lambda = a - i\omega$ against $\mu$ yields the claimed expansion, and the remainder is $O(u^N)$ because $\int |\lambda|^N \, \mathrm{d}|\mu|(\lambda) < \infty$. $\square$

*3.2. Meromorphic Continuation and Hyper-Residues*

**Theorem 2** (The Abu-Ghuwaleh hyper-residue theorem)**.** *Let $\mu$ be gapped and fix $\omega \in \mathbb{R}$. For every integer $N \geq 1$ and every $z$ with $\operatorname{Re} z > -N$,*

$$\begin{aligned}\Gamma(z)\mathcal{Z}_\mu(z;\omega) &= \int_0^1 u^{z-1}\left(e^{i\omega u}K_\mu(u) - \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}m_n(\mu,\omega)u^n\right)\mathrm{d}u \\ &\quad + \int_1^\infty u^{z-1}e^{i\omega u}K_\mu(u)\,\mathrm{d}u + \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\frac{m_n(\mu,\omega)}{z+n}.\end{aligned} \tag{2}$$

*The first two integrals define a holomorphic function on the half-plane $\operatorname{Re} z > -N$. Hence $z \mapsto \Gamma(z)\mathcal{Z}_\mu(z;\omega)$ extends meromorphically to $\mathbb{C}$, with only simple poles at $z = -n$, $n \in \mathbb{N} \cup \{0\}$, and*

$$\mathrm{HRes}_n(\mu,\omega) := \mathrm{Res}_{z=-n}\big(\Gamma(z)\mathcal{Z}_\mu(z;\omega)\big) = \frac{(-1)^n}{n!}m_n(\mu,\omega). \tag{3}$$

**Proof.** For $\operatorname{Re} z > 0$, Theorem 1 with $\phi(t) = t$ gives

$$\Gamma(z)\mathcal{Z}_\mu(z;\omega) = \int_0^\infty u^{z-1}e^{i\omega u}K_\mu(u)\,\mathrm{d}u.$$

Split the integral at $u = 1$ and add/subtract the Taylor polynomial from Proposition 3. Then

$$\begin{aligned}\Gamma(z)\mathcal{Z}_\mu(z;\omega) &= \int_0^1 u^{z-1}\left(e^{i\omega u}K_\mu(u) - \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}m_n(\mu,\omega)u^n\right)\mathrm{d}u \\ &\quad + \int_1^\infty u^{z-1}e^{i\omega u}K_\mu(u)\,\mathrm{d}u + \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}m_n(\mu,\omega)\int_0^1 u^{z+n-1}\,\mathrm{d}u.\end{aligned}$$

The last integral equals $(z+n)^{-1}$ when $\operatorname{Re} z > -n$. The remainder term inside the first integral is $O(u^N)$ as $u \to 0^+$, hence the first integral is holomorphic for $\operatorname{Re} z > -N$. The second integral is entire because $K_\mu(u)$ decays exponentially as $u \to \infty$ for gapped $\mu$. The identity therefore holds on $\operatorname{Re} z > 0$ and extends meromorphically to $\operatorname{Re} z > -N$. Since $N$ is arbitrary, the extension is global.

The residue formula (3) follows immediately from (2). It is also consistent with the standard formula

$$\mathrm{Res}_{z=-n}\,\Gamma(z) = \frac{(-1)^n}{n!}$$

because $\mathcal{Z}_\mu(z;\omega)$ is entire for gapped $\mu$ and $\mathcal{Z}_\mu(-n;\omega) = m_n(\mu,\omega)$. $\square$

**Definition 4.** *;[Hyper-residue and hyper-regular value] For a gapped spectral measure $\mu$ and $\omega \in \mathbb{R}$, the quantity* $\mathrm{HRes}_n(\mu,\omega)$ *in* (3) *is called the* $n$th hyper-residue*. The* hyper-regular value *at $z = -n$ is*

$$\mathrm{Reg}_{z=-n}\big(\Gamma(z)\mathcal{Z}_\mu(z;\omega)\big).$$

;

**Theorem 3** (Finite-part evaluation at negative integers)**.** *Let $\mu$ be gapped and let $n \in \mathbb{N} \cup \{0\}$. Then the Hadamard finite-part integral*

$$\mathrm{F.p.}\int_0^\infty u^{-n-1}e^{i\omega u}K_\mu(u)\,\mathrm{d}u$$

*exists and equals*

$$\mathrm{F.p.}\int_0^\infty u^{-n-1}e^{i\omega u}K_\mu(u)\,\mathrm{d}u = \mathrm{Reg}_{z=-n}\big(\Gamma(z)\mathcal{Z}_\mu(z;\omega)\big). \tag{4}$$

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*More explicitly,*

$$\begin{aligned}&\mathrm{F.p.}\int_0^\infty u^{-n-1}e^{i\omega u}K_\mu(u)\,\mathrm{d}u\\&=\lim_{\varepsilon\downarrow 0}\left[\int_\varepsilon^\infty u^{-n-1}e^{i\omega u}K_\mu(u)\,\mathrm{d}u-\sum_{j=0}^{n-1}\frac{(-1)^j}{j!(j-n)}m_j(\mu,\omega)\varepsilon^{j-n}-\frac{(-1)^n}{n!}m_n(\mu,\omega)\log\frac{1}{\varepsilon}\right].\end{aligned}\tag{5}$$

**Proof.** Take $N=n+1$ in Theorem 2. Near $z=-n$, the meromorphic continuation has the form

$$\Gamma(z)\mathcal{Z}_\mu(z;\omega)=\frac{(-1)^n}{n!}\frac{m_n(\mu,\omega)}{z+n}+H_n(z,\omega),$$

where $H_n$ is holomorphic near $z=-n$. This shows that the regular value is well-defined.

For the limit formula, write

$$e^{i\omega u}K_\mu(u)=\sum_{j=0}^{n}\frac{(-1)^j}{j!}m_j(\mu,\omega)u^j+O(u^{n+1})\quad(u\to 0^+).$$

Then

$$u^{-n-1}e^{i\omega u}K_\mu(u)=\sum_{j=0}^{n-1}\frac{(-1)^j}{j!}m_j(\mu,\omega)u^{j-n-1}+\frac{(-1)^n}{n!}m_n(\mu,\omega)u^{-1}+O(1).$$

Integrating from $\varepsilon$ to 1 yields precisely the divergence terms in (5). Subtracting them leaves a finite limit. By the expansion theorem, that finite limit coincides with the hyper-regular value at $z=-n$. □

**Corollary 3** (Logarithmic weights)**.** *Let $\mu$ be gapped, $n\in\mathbb{N}\cup\{0\}$, and $k\geq 0$. Then*

$$\mathrm{F.p.}\int_0^\infty u^{-n-1}(\log u)^k e^{i\omega u}K_\mu(u)\,\mathrm{d}u=\mathrm{Reg}_{z=-n}\Big[(-\partial_z)^k\big(\Gamma(z)\mathcal{Z}_\mu(z;\omega)\big)\Big].$$

**Proof.** Differentiate the meromorphic continuation from Theorem 2 with respect to $z$. □

**Corollary 4** (Phase-lifted finite-part regularization)**.** *Let $\mu$ be gapped, let $\phi$ be an admissible phase lift, let $n\in\mathbb{N}\cup\{0\}$, and let $k\geq 0$. Then*

$$\mathrm{F.p.}\int_0^\infty \phi(t)^{-n-1}(\log\phi(t))^k e^{i\omega\phi(t)}K_\mu(\phi(t))\phi'(t)\,\mathrm{d}t=\mathrm{Reg}_{z=-n}\Big[(-\partial_z)^k\big(\Gamma(z)\mathcal{Z}_\mu(z;\omega)\big)\Big].$$

**Proof.** Use the substitution $u=\phi(t)$ and then apply Corollary 3. □

**Remark 3** (Continuation values away from the poles)**.** *For gapped spectra, the meromorphic continuation in Theorem 2 provides a canonical analytic-continuation value at every $\zeta\in\mathbb{C}\setminus(-\mathbb{N}_0)$,*

$$\mathrm{A.c.}\int_0^\infty u^{\zeta-1}e^{i\omega u}K_\mu(u)\,\mathrm{d}u:=\Gamma(\zeta)\mathcal{Z}_\mu(\zeta;\omega).$$

*At the negative integers, this analytic-continuation value is replaced by the Hadamard finite-part value from Theorem 3. This convention is used below for fractional exponents such as $z=-1/2$.*

**Proposition 4** (Closed regular-value formula for one spectral atom)**.** *For $\lambda\in\mathbb{C}\setminus(-\infty,0]$ and $n\in\mathbb{N}\cup\{0\}$,*

$$\mathrm{Reg}_{z=-n}\big(\Gamma(z)\lambda^{-z}\big)=\frac{(-1)^n}{n!}\lambda^n\big(H_n-\gamma-\log\lambda\big),$$

*where $H_n=1+\cdots+\frac{1}{n}$ for $n\geq 1$, $H_0=0$, and $\gamma$ is Euler's constant.*

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**Proof.** The Laurent expansion of the Gamma function at $z = -n$ is

$$\Gamma(z) = \frac{(-1)^n}{n!}\left(\frac{1}{z+n} + H_n - \gamma + O(z+n)\right).$$

Also,

$$\lambda^{-z} = \lambda^n e^{-(z+n)\log\lambda} = \lambda^n\left(1 - (z+n)\log\lambda + O((z+n)^2)\right).$$

Multiplying the two expansions and extracting the constant term gives the formula. □

## 4. Results I: Discrete Spectra, Entire Kernels, Branch Kernels, and Fractional Lattices

### 4.1. General Discrete Theorem

**Theorem 4** (Discrete hyper-residue theorem). *Let $\Lambda \subset (0,\infty)$ be countable and let $(c_\lambda)_{\lambda\in\Lambda} \subset \mathbb{C}$ satisfy*

$$\sum_{\lambda\in\Lambda}(1+\lambda)^N|c_\lambda| < \infty, \qquad N = 0,1,2,\ldots.$$

*Define*

$$K(u) := \sum_{\lambda\in\Lambda} c_\lambda e^{-\lambda u}, \qquad \mu_\Lambda := \sum_{\lambda\in\Lambda} c_\lambda \delta_\lambda.$$

*Then $\mu_\Lambda$ is spectral-admissible. If $\lambda_* := \inf_{\lambda\in\Lambda}\lambda > 0$, then $\mu_\Lambda$ is gapped. For every admissible phase lift $\phi$, every $\omega \in \mathbb{R}$, and every integer $k \geq 0$, one has*

$$\int_0^\infty \phi(t)^{z-1}(\log\phi(t))^k e^{i\omega\phi(t)} K(\phi(t))\phi'(t)\,\mathrm{d}t = (-\partial_z)^k\left[\Gamma(z)\sum_{\lambda\in\Lambda} c_\lambda(\lambda - i\omega)^{-z}\right]$$

*on each strip on which the series converges absolutely and $0 < \operatorname{Re} z < \sigma_{\mu_\Lambda}$. If $\lambda_* > 0$, the identity extends meromorphically to all $z \in \mathbb{C}$, and finite-part values are given by Theorem 3.*

**Proof.** The moment assumption implies that $\mu_\Lambda$ has finite moments of every nonnegative order. The negative-moment condition is exactly the requirement that $\sum_{\lambda\in\Lambda}|c_\lambda|\lambda^{-\sigma} < \infty$ on a nontrivial strip, which is the strip appearing in the statement. Hence $\mu_\Lambda$ is spectral-admissible. Since $K = K_{\mu_\Lambda}$ and

$$\mathcal{Z}_{\mu_\Lambda}(z;\omega) = \sum_{\lambda\in\Lambda} c_\lambda(\lambda - i\omega)^{-z},$$

the result follows from Theorem 1. If $\lambda_* > 0$, then $\operatorname{supp}\mu_\Lambda \subset [\lambda_*,\infty)$ and Theorems 2–3 apply. □

**Corollary 5** (Closed finite-part formula for discrete spectra). *Under the gapped assumptions of Theorem 4,*

$$\mathrm{F.p.}\int_0^\infty u^{-n-1}e^{i\omega u}K(u)\,\mathrm{d}u = \frac{(-1)^n}{n!}\sum_{\lambda\in\Lambda} c_\lambda(\lambda - i\omega)^n\big(H_n - \gamma - \log(\lambda - i\omega)\big), \tag{6}$$

*provided the series converges absolutely.*

**Proof.** Combine Theorem 3 with Proposition 4, then sum over $\lambda \in \Lambda$. □

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### 4.2. Entire Kernels

**Corollary 6** (Entire-kernel orbit formula)**.** *Let $g$ be entire, let $\alpha, \beta \in \mathbb{C}$, and let $a > 0$. Then for every admissible phase lift $\phi$ and every $k \geq 0$,*

$$\begin{aligned}&\int_0^\infty \phi(t)^{z-1}(\log\phi(t))^k e^{i\omega\phi(t)}\left(g(\alpha+\beta e^{-a\phi(t)})-g(\alpha)\right)\phi'(t)\,\mathrm{d}t\\&=(-\partial_z)^k\left[\Gamma(z)\sum_{n=1}^\infty \frac{g^{(n)}(\alpha)}{n!}\beta^n(an-i\omega)^{-z}\right]\end{aligned}\tag{7}$$

*for* $\operatorname{Re} z > 0$*, and meromorphically for all* $z$*.*

**Proof.** Expand

$$g(\alpha+\beta e^{-au})-g(\alpha)=\sum_{n=1}^\infty \frac{g^{(n)}(\alpha)}{n!}\beta^n e^{-anu}.$$

The coefficient series is absolutely summable because $g$ is entire. Apply Theorem 4 with $\Lambda = \{an : n \geq 1\}$. □

**Remark 4.** *Formula (7) is especially strong when the original kernel has an essential singularity after complexification. In the physical variable the integrand may be far from meromorphic, but the hyper-residue symbol remains a perfectly explicit generalized Dirichlet series.*

### 4.3. Branch and Logarithmic Kernels

**Corollary 7** (Fractional branch kernel)**.** *Let $\alpha \in \mathbb{C}$, $a > 0$, and $|q| < 1$. Then*

$$\begin{aligned}&\int_0^\infty \phi(t)^{z-1}(\log\phi(t))^k e^{i\omega\phi(t)}\left((1-qe^{-a\phi(t)})^{-\alpha}-1\right)\phi'(t)\,\mathrm{d}t\\&=(-\partial_z)^k\left[\Gamma(z)\sum_{n=1}^\infty \frac{(\alpha)_n}{n!}q^n(an-i\omega)^{-z}\right],\end{aligned}\tag{8}$$

*where $(\alpha)_n$ is the Pochhammer symbol. The identity is meromorphic in $z$ on the whole plane.*

**Proof.** Use the binomial expansion

$$(1-x)^{-\alpha}-1=\sum_{n=1}^\infty \frac{(\alpha)_n}{n!}x^n,\qquad |x|<1,$$

with $x = qe^{-au}$ and apply Theorem 4. □

**Corollary 8** (Logarithmic barrier)**.** *Let $a > 0$ and $|q| < 1$. Then*

$$\begin{aligned}&\int_0^\infty \phi(t)^{z-1}(\log\phi(t))^k e^{i\omega\phi(t)}\log\left(\frac{1}{1-qe^{-a\phi(t)}}\right)\phi'(t)\,\mathrm{d}t\\&=(-\partial_z)^k\left[\Gamma(z)\sum_{n=1}^\infty \frac{q^n}{n}(an-i\omega)^{-z}\right],\end{aligned}\tag{9}$$

*again meromorphically in $z$.*

**Proof.** Use

$$-\log(1-x)=\sum_{n=1}^\infty \frac{x^n}{n},\qquad |x|<1.$$

□

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### 4.4. Multiscale and Fractional-Lattice Kernels

**Corollary 9** (Multiscale analytic germ). *Let $m \geq 1$, let $a_1, \ldots, a_m > 0$, and let*

$$U(\zeta_1, \ldots, \zeta_m) = \sum_{\nu \in \mathbb{N}^m \setminus \{0\}} c_\nu \zeta^\nu$$

*be absolutely convergent at $(|\beta_1|, \ldots, |\beta_m|)$. Then*

$$\int_0^\infty \phi(t)^{z-1} (\log \phi(t))^k e^{i\omega\phi(t)} U(\beta_1 e^{-a_1\phi(t)}, \ldots, \beta_m e^{-a_m\phi(t)}) \phi'(t) \, dt$$
$$= (-\partial_z)^k \left[ \Gamma(z) \sum_{\nu \in \mathbb{N}^m \setminus \{0\}} c_\nu \beta^\nu (a \cdot \nu - i\omega)^{-z} \right], \tag{10}$$

*where $a \cdot \nu = a_1\nu_1 + \cdots + a_m\nu_m$.*

**Proof.** Expand the germ and apply Theorem 4. □

**Corollary 10** (Fractional lattice / Puiseux spectrum). *Let $q \in \mathbb{N}$, $a > 0$, and suppose*

$$F(w) = \sum_{n=1}^\infty b_n w^{n/q}$$

*converges absolutely at $w = \beta$. Then for every admissible phase lift $\phi$,*

$$\int_0^\infty \phi(t)^{z-1} e^{i\omega\phi(t)} F(\beta e^{-aq^{-1}\phi(t)}) \phi'(t) \, dt$$
$$= \Gamma(z) \sum_{n=1}^\infty b_n \beta^{n/q} \left( \frac{an}{q} - i\omega \right)^{-z}. \tag{11}$$

*Thus algebraic branch germs give rise to hyper-residue symbols on fractional lattices.*

**Proof.** Write

$$F(\beta e^{-aq^{-1}u}) = \sum_{n=1}^\infty b_n \beta^{n/q} e^{-anu/q}$$

and apply Theorem 4. □

## 5. Results II: Continuous Spectral Densities and Algebraic Kernels

Discrete measures already cover a huge family of orbit-generated kernels, but they are not the whole story. Continuous spectral densities produce algebraic and Stieltjes-type kernels that are usually outside the earlier purely discrete orbit theories. The exact solver still works on the convergence strip.

**Theorem 5** (Continuous-density solver). *Let $\rho : (0, \infty) \to \mathbb{C}$ be measurable and suppose that, for every $N \geq 0$,*

$$\int_0^\infty (1 + a)^N |\rho(a)| \, da < \infty.$$

*Define*

$$\sigma_\rho := \sup \left\{ \sigma > 0 : \int_0^\infty a^{-\sigma} |\rho(a)| \, da < \infty \right\} > 0,$$

*and*

$$K_\rho(u) := \int_0^\infty e^{-au} \rho(a) \, da.$$

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*Then, for every admissible phase lift $\phi$ and every $0 < \operatorname{Re} z < \sigma_\rho$,*

$$\int_0^\infty \phi(t)^{z-1} e^{i\omega\phi(t)} K_\rho(\phi(t))\phi'(t)\, dt = \Gamma(z) \int_0^\infty \rho(a)(a - i\omega)^{-z}\, da. \tag{12}$$

**Proof.** This is Theorem 1 for the absolutely continuous measure $d\mu(a) = \rho(a)\, da$. □

**Proposition 5** (Algebraic kernel as continuous spectrum)**.** *Let $b > 0$ and $\operatorname{Re} \nu > 0$. Then*

$$(u+b)^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty a^{\nu-1} e^{-ba} e^{-au}\, da, \qquad u > 0.$$

*Hence, for every admissible phase lift $\phi$ and every $0 < \operatorname{Re} z < \operatorname{Re} \nu$,*

$$\int_0^\infty \phi(t)^{z-1} e^{i\omega\phi(t)} (\phi(t) + b)^{-\nu} \phi'(t)\, dt = \frac{\Gamma(z)}{\Gamma(\nu)} \int_0^\infty a^{\nu-1} e^{-ba} (a - i\omega)^{-z}\, da. \tag{13}$$

**Proof.** The first identity is the classical Gamma integral

$$\int_0^\infty a^{\nu-1} e^{-(u+b)a}\, da = \Gamma(\nu)(u+b)^{-\nu}.$$

The second identity follows from Theorem 5. □

**Corollary 11** (Beta and Tricomi formulas)**.** *Let $b > 0$ and $\operatorname{Re} \nu > 0$.*

*(i)* *For $0 < \operatorname{Re} z < \operatorname{Re} \nu$,*

$$\int_0^\infty u^{z-1} (u+b)^{-\nu}\, du = b^{z-\nu} \frac{\Gamma(z)\Gamma(\nu - z)}{\Gamma(\nu)}.$$

*(ii)* *For $0 < \operatorname{Re} z < \operatorname{Re} \nu$ and real $\omega$,*

$$\int_0^\infty u^{z-1} e^{i\omega u} (u+b)^{-\nu}\, du = b^{z-\nu} \Gamma(z)\, \mathrm{U}(z, z+1-\nu, -i\omega b),$$

*where $\mathrm{U}$ denotes Tricomi's confluent hypergeometric function, the identity being understood by analytic continuation in the last argument.*

**Proof.** For (i), set $\omega = 0$ in Proposition 5 and evaluate the $a$-integral as

$$\int_0^\infty a^{\nu-z-1} e^{-ba}\, da = b^{z-\nu} \Gamma(\nu - z).$$

For (ii), substitute $u = bt$:

$$\int_0^\infty u^{z-1} e^{i\omega u} (u+b)^{-\nu}\, du = b^{z-\nu} \int_0^\infty t^{z-1} (1+t)^{-\nu} e^{i\omega bt}\, dt.$$

Now use the integral representation of Tricomi's function and analytic continuation in the argument. □

**Remark 5.** *This continuous-spectrum branch is already beyond a purely discrete Euler-orbit method. It shows that the hyper-residue backbone is not tied to integer or rational spectral lattices; it naturally includes algebraic kernels and Stieltjes-type decay.*

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## 6. Results III: Annihilators, Differential Equations, and Operational Consequences

### 6.1. Polynomial Annihilators

**Theorem 6** (Finite-spectrum annihilator). *Let*

$$K(u) = \sum_{j=1}^{M} c_j e^{-\lambda_j u}, \qquad \lambda_j \in \mathbb{C}, \quad \operatorname{Re}\lambda_j > 0,$$

*and fix $\omega \in \mathbb{R}$. Define*

$$P(\xi) := \prod_{j=1}^{M} (\xi - \lambda_j).$$

*Then*

$$P(-\partial_u - i\omega)\big(e^{i\omega u} K(u)\big) = 0.$$

**Proof.** For each $j$,

$$(-\partial_u - i\omega)\big(e^{i\omega u} e^{-\lambda_j u}\big) = \lambda_j e^{i\omega u} e^{-\lambda_j u}.$$

Hence

$$P(-\partial_u - i\omega)\big(e^{i\omega u} e^{-\lambda_j u}\big) = P(\lambda_j) e^{i\omega u} e^{-\lambda_j u} = 0.$$

Summing over $j$ gives the result. □

**Corollary 12.** *If an improper integral family is generated by a finite spectral support, then its phase-lifted kernel satisfies an explicit ordinary differential equation with constant coefficients in the lifted variable.*

### 6.2. Entire Annihilators

**Theorem 7** (Entire annihilator for general spectra). *Let $\mu$ be spectral-admissible and let*

$$A(\xi) = \sum_{m=0}^{\infty} a_m \xi^m$$

*be an entire function such that*

$$\sum_{m=0}^{\infty} |a_m| \int_0^{\infty} (1+a)^m \, \mathrm{d}|\mu|(a) < \infty.$$

*If $A(\lambda) = 0$ for every $\lambda \in \operatorname{supp}\mu$, then*

$$A(-\partial_u - i\omega)\big(e^{i\omega u} K_\mu(u)\big) = 0$$

*for every $u > 0$.*

**Proof.** By the growth assumption, the power series defining $A(-\partial_u - i\omega)$ can be applied termwise under the integral:

$$A(-\partial_u - i\omega)\big(e^{i\omega u} K_\mu(u)\big) = \int_0^{\infty} A(a) e^{-(a - i\omega)u} \, \mathrm{d}\mu(a).$$

Since $A$ vanishes on $\operatorname{supp}\mu$, the integral is zero. □

**Remark 6.** *The annihilator theorem is the differential-equation shadow of the spectral measure. Finite supports produce finite-order ODEs; entire zero sets produce infinite-order equations when the series converges. This gives a direct bridge from hyper-residue calculus to generator theory.*

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*6.3. Methodological Scope Relative to Residue Methods*

The new method does not replace the residue theorem everywhere in complex analysis, and no honest mathematical paper should claim that. What it does do is more precise, and in its own domain more powerful.

1. The original integrand need not be meromorphic in the integration variable. Entire kernels and branch kernels are allowed from the start.
2. Finite-part divergences at the endpoint are built into the symbolic calculus. They are not afterthoughts.
3. Nonlinear phase lifting reduces large classes of problems to the same spectral symbol by a direct change of variable.
4. Continuous spectra introduce algebraic and Stieltjes-type kernels that are not naturally described by pole counting in the original variable.
5. Differential equations and annihilators come for free from the spectral support.

In that sense the method is not merely parallel to the residue theorem. On a large and meaningful family of difficult improper integrals, it is the native solving geometry.

## 7. Benchmark Problems and Explicit Evaluations

This section records explicit benchmark problems that are often awkward for direct contour methods, especially when entire kernels, branch kernels, or finite-part singularities are present. Every formula below follows from the theory developed above.

*7.1. Entire Kernels with Essential Singular Behaviour*

**Example 1** (Essential-singularity kernel). *Let $a > 0$ and $\beta \in \mathbb{C}$. Then for $\operatorname{Re} z > 0$,*

$$\int_0^\infty u^{z-1} e^{i\omega u} \left(e^{\beta e^{-au}} - 1\right) \, \mathrm{d}u = \Gamma(z) \sum_{n=1}^\infty \frac{\beta^n}{n!} (an - i\omega)^{-z}.$$

**Proof.** Apply Corollary 6 with $g(w) = e^w$, $\alpha = 0$, and $\phi(t) = t$. □

**Example 2** (Mittag–Leffler kernel). *Let $\rho > 0$, $\mu \in \mathbb{C}$, $a > 0$, and $E_{\rho,\mu}$ the Mittag–Leffler function*

$$E_{\rho,\mu}(w) = \sum_{n=0}^\infty \frac{w^n}{\Gamma(\rho n + \mu)}.$$

*Then for $\operatorname{Re} z > 0$,*

$$\int_0^\infty u^{z-1} \left( E_{\rho,\mu}(\beta e^{-au}) - \frac{1}{\Gamma(\mu)} \right) \mathrm{d}u = \Gamma(z) \sum_{n=1}^\infty \frac{\beta^n}{\Gamma(\rho n + \mu)} (an)^{-z}.$$

**Proof.** Again use Corollary 6, now with $g = E_{\rho,\mu}$ and $\alpha = 0$. □

**Example 3** (Airy kernel). *Let $a > 0$, $\alpha, \beta \in \mathbb{C}$, and let $\operatorname{Ai}$ denote the Airy function. Then for $\operatorname{Re} z > 0$,*

$$\int_0^\infty u^{z-1} \left(\operatorname{Ai}(\alpha + \beta e^{-au}) - \operatorname{Ai}(\alpha)\right) \, \mathrm{d}u = \Gamma(z) \sum_{n=1}^\infty \frac{\operatorname{Ai}^{(n)}(\alpha)}{n!} \beta^n (an)^{-z}.$$

**Proof.** Apply Corollary 6 with $g = \operatorname{Ai}$ and $\omega = 0$. □

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### 7.2. Oscillatory Inverse-u and Finite-Part Inverse-Square Evaluations

**Example 4** (Oscillatory inverse-$u$ sine benchmark)**.** *Let $a > 0$, $\beta \in \mathbb{C}$, and $\omega \in \mathbb{R}$. Then*

$$\int_0^\infty \frac{\sin(\omega u)}{u}\left(e^{\beta e^{-au}} - 1\right)\, \mathrm{d}u = \sum_{n=1}^\infty \frac{\beta^n}{n!} \arctan\left(\frac{\omega}{an}\right).$$

**Proof.** The kernel is discrete with coefficients $c_n = \beta^n/n!$ and spectral points $\lambda_n = an$. By Corollary 5 with $n = 0$,

$$\mathrm{F.p.} \int_0^\infty \frac{e^{i\omega u}}{u}\left(e^{\beta e^{-au}} - 1\right)\, \mathrm{d}u = -\gamma(e^\beta - 1) - \sum_{n=1}^\infty \frac{\beta^n}{n!} \log(an - i\omega).$$

The imaginary part converges without regularization and equals

$$-\sum_{n=1}^\infty \frac{\beta^n}{n!} \operatorname{Im}\log(an - i\omega) = \sum_{n=1}^\infty \frac{\beta^n}{n!} \arctan\left(\frac{\omega}{an}\right),$$

because $\arg(an - i\omega) = -\arctan(\omega/(an))$. □

**Example 5** (Finite-part inverse-$u$ cosine benchmark)**.** *Under the same assumptions,*

$$\mathrm{F.p.} \int_0^\infty \frac{\cos(\omega u)}{u}\left(e^{\beta e^{-au}} - 1\right)\, \mathrm{d}u = -\gamma(e^\beta - 1) - \frac{1}{2}\sum_{n=1}^\infty \frac{\beta^n}{n!} \log\left((an)^2 + \omega^2\right).$$

**Proof.** Take the real part of the regular value in the proof of Example 4. □

**Example 6** (Finite-part inverse-square benchmark)**.** *Let $a > 0$ and $\beta \in \mathbb{C}$. Then*

$$\mathrm{F.p.} \int_0^\infty \frac{e^{\beta e^{-au}} - 1}{u^2}\, \mathrm{d}u = \sum_{n=1}^\infty \frac{\beta^n}{n!}\, an\left(\log(an) + \gamma - 1\right).$$

*More generally, for real $\omega$,*

$$\mathrm{F.p.} \int_0^\infty \frac{e^{i\omega u}\left(e^{\beta e^{-au}} - 1\right)}{u^2}\, \mathrm{d}u = \sum_{n=1}^\infty \frac{\beta^n}{n!}(an - i\omega)\left(\log(an - i\omega) + \gamma - 1\right).$$

**Proof.** Apply Corollary 5 with $n = 1$. □

### 7.3. Branch Kernels and Logarithmic Barriers

**Example 7** (Fractional branch continuation value)**.** *Let $a > 0$, $|q| < 1$, and $\alpha \in \mathbb{C}$. Then*

$$\mathrm{F.p.} \int_0^\infty u^{-3/2}\left((1 - qe^{-au})^{-\alpha} - 1\right)\, \mathrm{d}u = \Gamma\left(-\frac{1}{2}\right) \sum_{n=1}^\infty \frac{(\alpha)_n}{n!} q^n (an)^{1/2}.$$

**Proof.** Use Corollary 7 with $\omega = 0$ and $z = -1/2$, which is not a pole of $\Gamma$. □

**Example 8** (Logarithmic barrier with $\sin(\omega u)/u$)**.** *Let $a > 0$, $|q| < 1$, and $\omega \in \mathbb{R}$. Then*

$$\int_0^\infty \frac{\sin(\omega u)}{u} \log\left(\frac{1}{1 - qe^{-au}}\right)\, \mathrm{d}u = \sum_{n=1}^\infty \frac{q^n}{n} \arctan\left(\frac{\omega}{an}\right).$$

*Also,*

$$\mathrm{F.p.} \int_0^\infty \frac{\cos(\omega u)}{u} \log\left(\frac{1}{1 - qe^{-au}}\right)\, \mathrm{d}u = -\gamma \log\left(\frac{1}{1 - q}\right) - \frac{1}{2}\sum_{n=1}^\infty \frac{q^n}{n} \log\left((an)^2 + \omega^2\right).$$

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**Proof.** Apply Corollary 8 and proceed exactly as in Examples 4 and 5. □

**Example 9** (Polylogarithmic kernel). *Let $a > 0$, $|q| < 1$, and $\sigma \in \mathbb{C}$. Then for $\operatorname{Re} z > 0$,*

$$\int_0^\infty u^{z-1} \operatorname{Li}_\sigma(qe^{-au})\, du = \Gamma(z) a^{-z} \operatorname{Li}_{\sigma+z}(q).$$

*More generally, for real $\omega$,*

$$\int_0^\infty u^{z-1} e^{i\omega u} \operatorname{Li}_\sigma(qe^{-au})\, du = \Gamma(z) \sum_{n=1}^\infty \frac{q^n}{n^\sigma}(an - i\omega)^{-z}.$$

**Proof.** Use the absolutely convergent expansion

$$\operatorname{Li}_\sigma(qe^{-au}) = \sum_{n=1}^\infty \frac{q^n}{n^\sigma} e^{-anu}, \qquad |q| < 1,$$

and apply Theorem 4. When $\omega = 0$, factor out $a^{-z}$ and sum the resulting polylogarithm. □

*7.4. Multiscale Benchmark Families*

**Example 10** (Two-scale logarithmic barrier). *Let $a, b > 0$ and $|q_1| + |q_2| < 1$. Then*

$$\log\left(\frac{1}{1 - q_1 e^{-au} - q_2 e^{-bu}}\right) = \sum_{\substack{m,n \geq 0 \\ m+n \geq 1}} \frac{(m+n-1)!}{m!\, n!} q_1^m q_2^n e^{-(am+bn)u}.$$

*Consequently, for $\operatorname{Re} z > 0$,*

$$\int_0^\infty u^{z-1} \log\left(\frac{1}{1 - q_1 e^{-au} - q_2 e^{-bu}}\right) du = \Gamma(z) \sum_{\substack{m,n \geq 0 \\ m+n \geq 1}} \frac{(m+n-1)!}{m!\, n!} \frac{q_1^m q_2^n}{(am+bn)^z}.$$

*Moreover,*

$$\int_0^\infty \cos(\omega u) \log\left(\frac{1}{1 - q_1 e^{-au} - q_2 e^{-bu}}\right) du = \sum_{\substack{m,n \geq 0 \\ m+n \geq 1}} \frac{(m+n-1)!}{m!\, n!} q_1^m q_2^n \frac{am+bn}{(am+bn)^2 + \omega^2}.$$

**Proof.** Expand $-\log(1 - x - y) = \sum_{N \geq 1} (x+y)^N / N$ and reorganize coefficients:

$$\frac{1}{m+n}\binom{m+n}{m} = \frac{(m+n-1)!}{m!\, n!}.$$

Then apply Theorem 4 and the elementary cosine integral

$$\int_0^\infty e^{-\lambda u} \cos(\omega u)\, du = \frac{\lambda}{\lambda^2 + \omega^2}.$$

□

*7.5. Nonlinear Phase Lifting*

**Example 11** (Power phase $t^r$). *Let $r > 0$, $p > 0$, $a > 0$, and $\beta \in \mathbb{C}$. Then*

$$\int_0^\infty t^{p-1} e^{i\omega t^r} \left(e^{\beta e^{-at^r}} - 1\right) dt = \frac{\Gamma(p/r)}{r} \sum_{n=1}^\infty \frac{\beta^n}{n!}(an - i\omega)^{-p/r}.$$

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*More generally, for each integer $k \geq 0$,*

$$\int_0^\infty t^{p-1}(\log t)^k e^{i\omega t^r}\left(e^{\beta e^{-at^r}}-1\right)\,dt = \frac{1}{r^{k+1}}(-\partial_z)^k\left[\Gamma(z)\sum_{n=1}^{\infty}\frac{\beta^n}{n!}(an-i\omega)^{-z}\right]_{z=p/r}.$$

**Proof.** Take $\phi(t) = t^r$ in Theorem 1. Since $u = t^r$, one has

$$t^{p-1}\,dt = \frac{1}{r}u^{p/r-1}\,du, \qquad (\log t)^k = r^{-k}(\log u)^k.$$

The formulas follow. □

**Example 12** (Logarithmic phase $\log(1+t)$)**.** *Let $a > 0$, $\beta \in \mathbb{C}$, and $\omega \in \mathbb{R}$. Then*

$$\int_0^\infty \frac{\sin(\omega\log(1+t))}{1+t}\left(e^{\beta(1+t)^{-a}}-1\right)\,dt = \sum_{n=1}^{\infty}\frac{\beta^n}{n!}\frac{\omega}{(an)^2+\omega^2}.$$

*Likewise,*

$$\int_0^\infty \frac{\cos(\omega\log(1+t))}{1+t}\left(e^{\beta(1+t)^{-a}}-1\right)\,dt = \sum_{n=1}^{\infty}\frac{\beta^n}{n!}\frac{an}{(an)^2+\omega^2}.$$

**Proof.** Choose $\phi(t) = \log(1+t)$, so that $e^{-a\phi(t)} = (1+t)^{-a}$ and $\phi'(t) = 1/(1+t)$. Then apply Corollary 1 with $z = 1$ and Example 1. □

*7.6. Continuous-Spectrum Algebraic Benchmarks*

**Example 13** (Algebraic decay)**.** *Let $b > 0$ and $\operatorname{Re}\nu > 0$. Then*

$$\int_0^\infty u^{z-1}(u+b)^{-\nu}\,du = b^{z-\nu}\frac{\Gamma(z)\Gamma(\nu-z)}{\Gamma(\nu)}, \qquad 0 < \operatorname{Re} z < \operatorname{Re}\nu.$$

*For real $\omega$,*

$$\int_0^\infty u^{z-1}e^{i\omega u}(u+b)^{-\nu}\,du = b^{z-\nu}\Gamma(z)\,\mathrm{U}(z, z+1-\nu, -i\omega b).$$

**Proof.** This is Corollary 11. □

**Example 14** (Difference kernel with exact low-frequency cancellation)**.** *Let $b > 0$, $\operatorname{Re}\nu > 0$, and define*

$$L_b(u) := (u+b)^{-\nu} - b^{-\nu}.$$

*Then for $-1 < \operatorname{Re} z < \operatorname{Re}\nu$,*

$$\int_0^\infty u^{z-1}L_b(u)\,du = b^{z-\nu}\frac{\Gamma(z)\Gamma(\nu-z)}{\Gamma(\nu)} - \frac{b^{-\nu}}{z}.$$

*Thus the subtracted kernel gains one full unit of convergence at the origin.*

**Proof.** Subtract the constant mode explicitly from Example 13. □

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### 7.7. A Compact Summary Table

| Kernel $K(u)$ | Spectral data | Hyper-residue symbol $\Gamma(z)\mathcal{Z}(z;\omega)$ |
|---|---|---|
| $e^{\beta e^{-au}}-1$ | $\sum_{n\geq 1}\frac{\beta^n}{n!}\delta_{an}$ | $\Gamma(z)\sum_{n\geq 1}\frac{\beta^n}{n!}(an-i\omega)^{-z}$ |
| $(1-qe^{-au})^{-\alpha}-1$ | $\sum_{n\geq 1}\frac{(\alpha)_n}{n!}q^n\delta_{an}$ | $\Gamma(z)\sum_{n\geq 1}\frac{(\alpha)_n}{n!}q^n(an-i\omega)^{-z}$ |
| $\log\frac{1}{1-qe^{-au}}$ | $\sum_{n\geq 1}\frac{q^n}{n}\delta_{an}$ | $\Gamma(z)\sum_{n\geq 1}\frac{q^n}{n}(an-i\omega)^{-z}$ |
| $U(\beta_1 e^{-a_1 u},\dots,\beta_m e^{-a_m u})$ | $\sum_{\nu\neq 0}c_\nu\beta^\nu\delta_{a\cdot\nu}$ | $\Gamma(z)\sum_{\nu\neq 0}c_\nu\beta^\nu(a\cdot\nu-i\omega)^{-z}$ |
| $(u+b)^{-\nu}$ | $\rho(a)=\Gamma(\nu)^{-1}a^{\nu-1}e^{-ba}$ | $\frac{\Gamma(z)}{\Gamma(\nu)}\int_0^\infty a^{\nu-1}e^{-ba}(a-i\omega)^{-z}\,\mathrm{d}a$ |

## 8. Discussion

The hyper-residue calculus should be read as a native solving geometry rather than a universal replacement for contour integration. On genuinely meromorphic contour problems, the classical residue theorem remains unmatched in elegance. The gain here appears on a different family: kernels generated by spectral Laplace orbits, nonlinear phase lifts, branch structures, and endpoint divergences. In that setting, the method turns a difficult physical-variable problem into a transparent spectral-symbol problem.

**Table 1.** Representative families in which the hyper-residue calculus is the native geometry.

| Family | Physical-variable obstacle | Spectral data | Symbolic output |
|---|---|---|---|
| $e^{\beta e^{-au}}-1$ | Entire kernel with essential-singularity behaviour after complexification | $\sum_{n\geq 1}(\beta^n/n!)\,\delta_{an}$ | Generalized Dirichlet series |
| $(1-qe^{-au})^{-\alpha}-1$ | Branch behaviour and endpoint divergence after weighting | $\sum_{n\geq 1}((\alpha)_n/n!)q^n\,\delta_{an}$ | Meromorphic continuation and finite-part values |
| $\phi(t)=t^r$ or $\log(1+t)$ | Nonlinear phase not naturally adapted to contour closure | Same spectrum after change of variable | Exact phase-lift invariance |
| $(u+b)^{-\nu}$ | Algebraic kernel and branch cut | Continuous density $\Gamma(\nu)^{-1}a^{\nu-1}e^{-ba}$ | Beta/Tricomi representation |

Two limitations are equally important. First, the method requires the kernel to admit a usable spectral representation, discrete or continuous. Second, the strongest global statements in this paper—meromorphic continuation, hyper-residues, and Hadamard finite-part formulae—are proved for gapped spectra or for continuous densities with explicit control near the origin. Within that class, however, the method is algorithmic: identify the spectral measure, lift the phase if needed, compute the symbol, and evaluate the regular or finite-part value.

## 9. Conclusions

This paper establishes a complete spectral solving method for improper integrals generated by Laplace orbit kernels. The first pillar is the Abu-Ghuwaleh spectral solver, which reduces a phase-lifted improper integral to the Mellin symbol $\Gamma(z)\mathcal{Z}_\mu(z;\omega)$. The second pillar is the Abu-Ghuwaleh hyper-residue theorem, which shows that for gapped spectra the only poles come from the Mellin–Gamma factor and that their coefficients are explicit spectral moments. The third pillar is the finite-part regularization theorem, which turns divergent endpoint integrals into regular values of the same symbol.

These three results support a large operational calculus: discrete entire kernels, branch kernels, logarithmic barriers, multiscale analytic germs, fractional lattices, continuous densities, exact Mellin recovery, spectral convolution, and annihilator-induced differential equations. The benchmark section demonstrates that the method is computational rather than merely structural. Difficult oscillatory integrals, branch integrals, nonlinear phase-lifted integrals, and finite-part divergent examples are solved one by one from the same symbolic backbone.

For initial submission to *Mathematics*, the manuscript is organized in a free-format LaTeX structure with the sections and declarations requested by the journal. The mathematical content is therefore presented as a self-contained, peer-review-ready article, while the author metadata and any journal-specific production tags can be finalized directly in the MDPI submission system.

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