Article

Not peer-reviewed version

K-g-Fusion Frames on Cartesian Products of Two Hilbert C^*-Modules

Sanae Touaiher , Maryam G. Alshehri * , Mohamed Rossafi

Posted Date: 6 October 2025

doi: 10.20944/preprints202510.0440.v1

Keywords: Hilbert C^*-module; fusion frame; generalized fusion frame; cartesian product; perturbation

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 October 2025 doi:10.20944/preprints202510.0440.v1

Article

K-g-Fusion Frames on Cartesian Products of Two Hilbert C^*-Modules

Sanae Touaiher [1], Maryam G. Alshehri [2,*] and Mohamed Rossafi [3]

1 Laboratory Analysis, Geometry and Applications, University of Ibn Tofail, Kenitra, Morocco
2 Department of Mathematics, Faculty of Science, University of Tabuk, P.O.Box741, Tabuk 71491, Saudi Arabia
3 Laboratory Analysis, Geometry and Applications, Higher School of Education and Training, University of Ibn Tofail, Kenitra, Morocco
* Correspondence: mgalshehri@ut.edu.sa

Abstract

In this paper, we introduce and investigate the concept of K-g-fusion frames in the Cartesian product of two Hilbert C^*-modules over the same unital C^*-algebra. Our main result establishes that the Cartesian product of two K-g-fusion frames remains a K-g-fusion frame for the direct-sum module. we give explicit formulae for the associated synthesis, analysis and frame operators and prove natural relations (direct-sum decomposition of the frame operator). Furthermore, we prove a perturbation theorem showing that small perturbations of the component families, measured in the operator or norm sense, still yield a K-g-fusion frame for the product module, with explicit new frame bounds obtained.

Keywords: Hilbert C^*-module; fusion frame; generalized fusion frame; cartesian product; perturbation

MSC: Primary 46L08; Secondary 42C15; 46L05

---

1. Introduction

The concept of frames, first introduced by Duffin and Schaeffer [3], provides stable yet redundant representations of vectors in Hilbert spaces. Since its inception, frame theory has become a fundamental tool with wide-ranging applications in harmonic analysis, wavelet theory, signal processing, sampling theory, and operator theory see [1].

Several extensions of frame theory have been proposed to address increasingly sophisticated settings, including g-frames [11], and fusion frames [2], among others [6,9,10]. Each of these generalizations enhances the flexibility of frame representations while preserving their fundamental stability properties. In this context, the notion of *K-g-fusion frames*, which unifies the features of K-frames, g-frames, and fusion frames, offers a powerful framework for studying operator-related decompositions in Hilbert spaces and beyond.

A natural direction of research has been the extension of frame theory to Hilbert C^*-modules, initiated by Frank and Larson [4]. In contrast to Hilbert spaces, Hilbert C^*-modules present significant challenges, arising from the absence of projections onto arbitrary closed submodules and the presence of a C^*-algebra-valued inner product. Despite these difficulties, frame concepts have been successfully adapted, leading to a variety of results in this setting, see [12–14].

The aim of this paper is to advance the theory of K-g-fusion frames on *Cartesian products* of Hilbert C^*-modules. Such products naturally emerge in operator algebras, module decompositions, and block-matrix methods, and hence provide a rich framework for our study.

The paper is organized as follows. Section 2 reviews the fundamental notions of Hilbert C^*-modules and adjointable operators, and introduces the concept of K-g-fusion frames together with their operator-theoretic features. Section 3 contains the main result concerning Cartesian products of

doi:10.20944/preprints202510.0440.v1

K-g-fusion frames. In Section 4, we establish perturbation results, while the final section is devoted to concluding remarks and illustrative examples.

2. Preliminaries

We briefly recall the basic definitions and facts about Hilbert C^*-modules needed in the sequel. Standard references are [5,7].

Definition 1. *Let $\mathcal{A}$ be a unital C^*-algebra. A left Hilbert C^*-module over $\mathcal{A}$ is a left $\mathcal{A}$-module $\mathcal{H}$ equipped with a map*

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$$

called the $\mathcal{A}$-valued inner product*, satisfying:*

1. $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ *for all* $a \in \mathcal{A}$, $x, y, z \in \mathcal{H}$*;*
2. $\langle x, y \rangle = \langle y, x \rangle^*$ *for all* $x, y \in \mathcal{H}$*;*
3. $\langle x, x \rangle \geq 0$ *in* $\mathcal{A}$*, and* $\langle x, x \rangle = 0 \iff x = 0$.

The associated norm is defined by $\|x\| := \|\langle x, x \rangle\|^{1/2}$*, and completeness with respect to this norm is assumed.*

For Hilbert $\mathcal{A}$-modules $\mathcal{H}, \mathcal{K}$, we denote by $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ the set of adjointable operators from $\mathcal{H}$ into $\mathcal{K}$, i.e. those operators $T : \mathcal{H} \to \mathcal{K}$ for which there exists an adjoin $T^* : \mathcal{K} \to \mathcal{H}$ satisfying

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \text{for all } x \in \mathcal{H},\ y \in \mathcal{K}.$$

If $\mathcal{K} = \mathcal{H}$, then we simply write $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ instead of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ is called *positive*, written $T \geq 0$, if

$$\langle Tx, x \rangle \geq 0 \quad \text{for all } x \in \mathcal{H}.$$

The partial order on self-adjoint operators is determined by this cone.

For a closed submodule $W \subset \mathcal{H}$, an *orthogonal projection* $P_W \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ satisfies

$$P_W^2 = P_W = P_W^* \quad \text{and} \quad \mathrm{Ran}(P_W) = W.$$

Unlike the Hilbert space case, not every closed submodule is complemented in $\mathcal{H}$.

In this work we restrict attention to orthogonally complemented submodules.

In what follows, all sums indexed by a countable set I are assumed to converge in norm in $\mathcal{A}$ whenever convergence is asserted.

These preliminaries allow us to introduce K-g-fusion frames in Hilbert C^*-modules in the next.

Given two Hilbert $\mathcal{A}$-modules $\mathcal{H}_1$ and $\mathcal{H}_2$, their external direct sum (or product) is defined as

$$\mathcal{H}_1 \oplus \mathcal{H}_2 := \{(x_1, x_2) : x_1 \in \mathcal{H}_1,\ x_2 \in \mathcal{H}_2\},$$

with the natural left $\mathcal{A}$-module action

$$a \cdot (x_1, x_2) := (ax_1, ax_2), \qquad a \in \mathcal{A},$$

and $\mathcal{A}$-valued inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle := \langle x_1, y_1 \rangle_{\mathcal{H}_1} + \langle x_2, y_2 \rangle_{\mathcal{H}_2}.$$

With this structure, $\mathcal{H}_1 \oplus \mathcal{H}_2$ is a Hilbert $\mathcal{A}$-module.

Moreover, if $V \subset \mathcal{H}_1$ and $W \subset \mathcal{H}_2$ are orthogonally complemented submodules, then their direct sum $V \oplus W$ is an orthogonally complemented submodule of $\mathcal{H}_1 \oplus \mathcal{H}_2$, with projection operator

$$P_{V \oplus W} = P_V \oplus P_W.$$

This observation will be essential in constructing product families of frames and proving stability under perturbations.

Definition 2. *[8]*
Let $\mathcal{H}$ and $\mathcal{K}$ be countably generated Hilbert $\mathcal{A}$-modules. Suppose that:

- $\{v_i\}_{i\in I}$ *is a family of positive invertible elements from the center of* $\mathcal{A}$*;*
- $\{W_i\}_{i\in I}$ *is a family of orthogonally complemented closed submodules of* $\mathcal{H}$*;*
- $\{\mathcal{H}_i\}_{i\in I}$ *is a family of closed submodules of* $\mathcal{K}$*;*
- *for each* $i \in I$, $\Lambda_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)$*;*
- $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$.

We say that $\Lambda = (W_i, \Lambda_i, v_i)_{i\in I}$ *is a* K-g-fusion frame *for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in I}$ *if there exist scalars* $0 < A \le B < \infty$ *such that*

$$A\,\langle K^* f, K^* f\rangle \;\le\; \sum_{i\in I} v_i^2\,\langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle \;\le\; B\langle f, f\rangle, \quad \forall f \in \mathcal{H}.$$

The constants A and B are called the lower *and* upper bounds *of the K-g-fusion frame. In addition:*

- *If the inequalities hold with $K = I_{\mathcal{H}}$, then Λ is a g-*fusion frame*, i.e.*

$$A\langle f, f\rangle \;\le\; \sum_{i\in I} v_i^2\,\langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle \;\le\; B\langle f, f\rangle, \quad \forall f \in \mathcal{H}.$$

- *If, in addition, $K = I_{\mathcal{H}}$ and $\Lambda_i = P_{W_i}$ for all $i \in I$, then Λ reduces to a* fusion frame *for $\mathcal{H}$.*

Now, for a K-g-fusion frame $\Lambda = (W_i, \Lambda_i, v_i)_{i\in I}$ of $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$,

- The *analysis operator*

$$T^*_\Lambda : \mathcal{H} \longrightarrow \ell^2(\{\mathcal{H}_i\}_{i\in I}),$$

is defined by

$$T^*_\Lambda f = \left(v_i \Lambda_i P_{W_i} f\right)_{i\in I}, \quad f \in \mathcal{H}.$$

- The *synthesis operator*

$$T_\Lambda : \ell^2(\{\mathcal{H}_i\}_{i\in I}) \longrightarrow \mathcal{H},$$

is the adjoint of T^*_Λ and is given by

$$T_\Lambda\big((f_i)_{i\in I}\big) = \sum_{i\in I} v_i P_{W_i} \Lambda_i^* f_i, \quad (f_i)_{i\in I} \in \ell^2(\{\mathcal{H}_i\}_{i\in I}).$$

- The *frame operator*

$$S_\Lambda : \mathcal{H} \longrightarrow \mathcal{H}$$

is defined by

$$S_\Lambda f = T_\Lambda T^*_\Lambda f = \sum_{i\in I} v_i^2\, P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} f, \quad f \in \mathcal{H}.$$

3. Product K-g-Fusion Frames and Main Theorem

Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert $\mathcal{A}$-modules. For each $i \in I$ let $W_i \subset \mathcal{H}_1$ and $V_i \subset \mathcal{H}_2$ be orthogonally complemented closed submodules with projections $P_{W_i} \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $P_{V_i} \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$. Let $\mathcal{H}_{1,i}, \mathcal{H}_{2,i}$ be Hilbert $\mathcal{A}$-modules and let $\Lambda_i \in \mathrm{End}^*_{\mathcal{A}}(W_i, \mathcal{H}_{1,i}), \Gamma_i \in \mathrm{End}^*_{\mathcal{A}}(V_i, \mathcal{H}_{2,i})$ be adjointable maps. Assume that $(W_i, \Lambda_i, v_i)_{i\in I}$ is a K_1-g-fusion frame for $\mathcal{H}_1$ with bounds $A_1, B_1 > 0$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 October 2025 doi:10.20944/preprints202510.0440.v1

and that $(V_i, \Gamma_i, v_i)_{i\in I}$ is a K_2-g-fusion frame for $\mathcal{H}_2$ with bounds $A_2, B_2 > 0$, where $K_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_i)$ $(i = 1, 2)$. Define, for each $i \in I$,

$$\Theta_i : \mathcal{H}_1 \oplus \mathcal{H}_2 \longrightarrow \mathcal{H}_{1,i} \oplus \mathcal{H}_{2,i}, \qquad \Theta_i(x, y) \;=\; \left(\Lambda_i P_{W_i} x,\ \Gamma_i P_{V_i} y\right).$$

Then we have the following theorem:

Theorem 1. *Assume that $(W_i, \Lambda_i, v_i)_{i\in I}$ is a K_1-g-fusion frame for $\mathcal{H}_1$ with bounds $A_1, B_1 > 0$ and that $(V_i, \Gamma_i, v_i)_{i\in I}$ is a K_2-g-fusion frame for $\mathcal{H}_2$ with bounds $A_2, B_2 > 0$, then the family $\{(W_i \oplus V_i,\ \Theta_i,\ v_i)\}_{i\in I}$ is a $(K_1 \oplus K_2)$-g-fusion frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$ with bounds $A = \min\{A_1, A_2\}$ and $B = \max\{B_1, B_2\}$. Furthermore, if $T^{(1)}$ and $T^{(2)}$ are respectively the synthesis operators of $(W_i, \Lambda_i, v_i)_{i\in I}$ and $(V_i, \Gamma_i, v_i)_{i\in I}$ and $S^{(1)}$ and $S^{(2)}$ their frame operators, then the synthesis operator T of the product satisfies $T = T^{(1)} \oplus T^{(2)}$, and the frame operator satisfies $S = S^{(1)} \oplus S^{(2)}$.*

Proof. Since $(W_i, \Lambda_i, v_i)_{i\in}$ is a K_1-g-fusion frame for $\mathcal{H}_1$, it is in particular a Bessel family. Thus, there exists a scalar $B_1 > 0$ such that for all $x \in \mathcal{H}_1$

$$\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \le B_1 \langle x, x\rangle,$$

where the series converges in norm in $\mathcal{A}$. Similarly, for (V_i, Γ_i, v_i) there is $B_2 > 0$ with analogous norm-convergent series

$$\sum_{i\in I} v_i^2 \langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y\rangle \le B_2 \langle y, y\rangle,$$

for all $y \in \mathcal{H}_2$. Therefore for any $(x, y) \in \mathcal{H}_1 \oplus \mathcal{H}_2$,

$$\begin{aligned}\sum_{i\in I} v_i^2 \langle \Theta_i(x, y), \Theta_i(x, y)\rangle &= \sum_{i\in I} v_i^2 \Big(\langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle + \langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y\rangle\Big) \\ &= \Big(\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\Big) + \Big(\sum_{i\in I} v_i^2 \langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y\rangle\Big),\end{aligned}$$

each summand being norm convergent in $\mathcal{A}$; hence the whole sum converges in norm. Moreover

$$\sum_{i\in I} v_i^2 \langle \Theta_i(x, y), \Theta_i(x, y)\rangle \le B_1 \langle x, x\rangle + B_2 \langle y, y\rangle \le \max\{B_1, B_2\}\ \langle (x, y), (x, y)\rangle$$

shows the desired uniform Bessel bound on the product.

By the K_1-g-fusion inequality on $\mathcal{H}_1$ we have the $\mathcal{A}$-valued inequality

$$A_1\ \langle K_1^* x, K_1^* x\rangle \le \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \le B_1\ \langle x, x\rangle,$$

and similarly for $\mathcal{H}_2$:

$$A_2\ \langle K_2^* y, K_2^* y\rangle \le \sum_{i\in I} v_i^2 \langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y\rangle \le B_2\ \langle y, y\rangle.$$

We can see that these two $\mathcal{A}$-valued inequalities yields

$$A_1 \langle K_1^* x, K_1^* x\rangle + A_2 \langle K_2^* y, K_2^* y\rangle \ \le\ \sum_{i\in I} v_i^2 \langle \Theta_i(x, y), \Theta_i(x, y)\rangle \ \le\ B_1 \langle x, x\rangle + B_2 \langle y, y\rangle.$$

Since A_1, A_2 are positive scalars,

$$A_1 \langle K_1^* x, K_1^* x\rangle + A_2 \langle K_2^* y, K_2^* y\rangle \ge \min\{A_1, A_2\}\big(\langle K_1^* x, K_1^* x\rangle + \langle K_2^* y, K_2^* y\rangle\big),$$

and likewise

$$B_1 \langle x, x\rangle + B_2 \langle y, y\rangle \le \max\{B_1, B_2\}\big(\langle x, x\rangle + \langle y, y\rangle\big).$$

Observing that $(K_1 \oplus K_2)^*(x,y) = (K_1^* x, K_2^* y)$ and that $\langle (x,y),(x,y)\rangle = \langle x,x\rangle + \langle y,y\rangle$, we obtain the claimed inequalities with $A = \min\{A_1, A_2\}$ and $B = \max\{B_1, B_2\}$, that is

$$A\langle (K_1 \oplus K_2)^*(x,y), (K_1 \oplus K_2)^*(x,y)\rangle \;\le\; \sum_{i\in I} v_i^2 \langle \Theta_i(x,y), \Theta_i(x,y)\rangle \;\le\; B\langle (x,y),(x,y)\rangle.$$

Now, compute the adjoint $\Theta_i^* : \mathcal{H}_{1,i} \oplus \mathcal{H}_{2,i} \to \mathcal{H}_1 \oplus \mathcal{H}_2$. For $(u,v) \in \mathcal{H}_{1,i} \oplus \mathcal{H}_{2,i}$ and $(x,y) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ one has

$$\langle \Theta_i(x,y),(u,v)\rangle = \langle \Lambda_i P_{W_i} x, u\rangle_{\mathcal{H}_{1,i}} + \langle \Gamma_i P_{V_i} y, v\rangle_{\mathcal{H}_{2,i}} = \langle (x,y), (P_{W_i}\Lambda_i^* u,\ P_{V_i}\Gamma_i^* v)\rangle,$$

hence

$$\Theta_i^*(u,v) = \left(P_{W_i}\Lambda_i^* u,\ P_{V_i}\Gamma_i^* v\right).$$

Therefore the operator $\Theta_i^*\Theta_i$ acts on (x,y) by

$$\Theta_i^*\Theta_i(x,y) = \left(P_{W_i}\Lambda_i^*\Lambda_i P_{W_i} x,\ P_{V_i}\Gamma_i^*\Gamma_i P_{V_i} y\right).$$

Multiplying by the scalar weight v_i^2 and summing over i gives the frame operator on the product:

$$S(x,y) \;=\; \sum_{i\in I} v_i^2\, \Theta_i^*\Theta_i(x,y) \;=\; \Big(\sum_{i\in I} v_i^2 P_{W_i}\Lambda_i^*\Lambda_i P_{W_i} x,\ \sum_{i\in I} v_i^2 P_{V_i}\Gamma_i^*\Gamma_i P_{V_i} y\Big).$$

The right-hand side is precisely $(S^{(1)}x,\ S^{(2)}y)$ where

$$S^{(1)} = \sum_i v_i^2 P_{W_i}\Lambda_i^*\Lambda_i P_{W_i} \text{ and } S^{(2)} = \sum_i v_i^2 P_{V_i}\Gamma_i^*\Gamma_i P_{V_i}$$

are the frame operators of the component families. Thus $S = S^{(1)} \oplus S^{(2)}$. In particular S is positive and the operator inequalities $A(K_1K_1^* \oplus K_2K_2^*) \le S \le BI_{\mathcal{H}_1\oplus\mathcal{H}_2}$ hold in $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. This completes the proof. □

Example 1. *Let $\mathcal{A} = \mathbb{C}^2$ with coordinate-wise operations and the usual involution. Consider the left $\mathcal{A}$-modules $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{A}^6$. Any element of $\mathcal{H}_1$ or $\mathcal{H}_2$, can be written as $x = (x_1, \dots, x_6)$, where $x_m = (a_m, b_m) \in \mathcal{A} = \mathbb{C}^2$, for $m = 1, ..., 6$. The $\mathcal{A}$-valued inner product is given by*

$$\langle x, y\rangle = \Big(\sum_{m=1}^{6} a_m\overline{c_m},\ \sum_{m=1}^{6} b_m\overline{d_m}\Big),$$

for $y = (y_1, \dots, y_6)$ with $y_m = (c_m, d_m)$. In particular,

$$\langle x, x\rangle = \Big(\sum_{m=1}^{6} |a_m|^2,\ \sum_{m=1}^{6} |b_m|^2\Big).$$

Define two diagonal adjointable operators K_1 and K_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively by:

$$K_1(x_1, \dots, x_6) = (x_1,\ 2x_2,\ x_3,\ 2x_4,\ x_5,\ 2x_6),$$

$$K_2(y_1, \dots, y_6) = (y_1,\ 3y_2,\ y_3,\ 3y_4,\ y_5,\ 3y_6).$$

Both are self-adjoint, so $K_i^* = K_i \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_i)$ for $i = 1, 2$.

Denote $e_1 = (1,0,0,0,0,0)$, ... , $e_6 = (0,0,0,0,0,1)$ the canonical elements of $\mathcal{H}_j$, $j = 1, 2$. For the Hilbert C^*-module $\mathcal{H}_1$ define

$$W_1 = \mathrm{span}_{\mathcal{A}}\{e_1, e_2\}, \quad W_2 = \mathrm{span}_{\mathcal{A}}\{e_3, e_4\}, \quad W_3 = \mathrm{span}_{\mathcal{A}}\{e_5, e_6\},$$

and let

$$\Lambda_i : W_i \to \mathcal{A}^2, \qquad \Lambda_i(x_{2i-1}, x_{2i}) = (x_{2i-1}, x_{2i}).$$

For $\mathcal{H}_2$ choose the submodules:

$$V_1 = \text{span}_{\mathcal{A}}\{e_1, e_3\}, \quad V_2 = \text{span}_{\mathcal{A}}\{e_2, e_5\}, \quad V_3 = \text{span}_{\mathcal{A}}\{e_4, e_6\},$$

and define

$$\Gamma_i : V_i \to \mathcal{A}^2, \qquad \Gamma_i(x_j, x_k) = (x_j, x_k).$$

All weights v_i, $i = 1,2,3$ are choosing equal to 1.
Now observe that for all $x \in \mathcal{H}_1$ and all $y \in \mathcal{H}_2$ we have

$$\begin{aligned} \frac{1}{4}\langle K_1^* x, K_1^* x\rangle &\leq \sum_{i=1}^{3}\langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle &= \langle x, x\rangle, \\ \frac{1}{9}\langle K_2^* y, K_2^* y\rangle &\leq \sum_{i=1}^{3}\langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y\rangle &= \langle y, y\rangle. \end{aligned}$$

So (V_i, Γ_i) is a K_2-g-fusion frame with bounds $A_2 = \frac{1}{9}$, $B_2 = 1$.
The product family

$$\{(W_i \oplus V_i, \Theta_i, 1)\}_{i=1}^{3}, \qquad \Theta_i(x, y) = (\Lambda_i P_{W_i} x,\ \Gamma_i P_{V_i} y),$$

is a $(K_1 \oplus K_2)$-g-fusion frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$ with bounds

$$A = \min\{A_1, A_2\} = \tfrac{1}{9}, \qquad B = \max\{B_1, B_2\} = 1.$$

That is, for all $(x, y) \in \mathcal{H}_1 \oplus \mathcal{H}_2$,

$$\frac{1}{9}\langle (K_1 \oplus K_2)^*(x, y),\ (K_1 \oplus K_2)^*(x, y)\rangle \leq \sum_{i=1}^{3}\langle \Theta_i(x, y), \Theta_i(x, y)\rangle \leq \langle (x, y), (x, y)\rangle.$$

4. Perturbation Theorem

Let $K_i \in \text{End}_{\mathcal{A}}^*(\mathcal{H}_i)$ $(i = 1, 2)$, and assume that $\{(W_i, \Lambda_i, v_i)\}_{i \in I}$ is a K_1-g-fusion frame for $\mathcal{H}_1$ with frame bounds $0 < A_1 \leq B_1 < \infty$, and that $\{(V_i, \Gamma_i, v_i)\}_{i \in I}$ is a K_2-g-fusion frame for $\mathcal{H}_2$ with frame bounds $0 < A_2 \leq B_2 < \infty$. Denote their product frame by $\mathcal{F} = \{(W_i \oplus V_i, \Theta_i, v_i)\}_{i \in I}$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, by taking the common weights v_i $(i \in I)$. The following theorem gives a perturbation result saying that if each component family is a K-g-fusion frame and each component perturbation is small, then the perturbed product family is again a K-g-fusion frame on the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$.

Theorem 2. *Let $\mathcal{F}' = \{(W_i', V_i', \Lambda_i', \Gamma_i', v_i)\}_{i \in I}$ be a perturbed family with the same weights v_i, where $W_i' \subset \mathcal{H}_1$ and $V_i' \subset \mathcal{H}_2$ are orthogonally complemented submodules, and Λ_i', Γ_i' are adjointable operators. Assume there exist scalars $r_1, r_2 > 0$ such that, for all $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$,*

$$\begin{cases} \sum_{i \in I} v_i^2 \langle (\Lambda_i P_{W_i} - \Lambda_i' P_{W_i'}) x_1,\ (\Lambda_i P_{W_i} - \Lambda_i' P_{W_i'}) x_1\rangle \leq r_1 \langle K_1^* x_1, K_1^* x_1\rangle, \\ \\ \sum_{i \in I} v_i^2 \langle (\Gamma_i P_{V_i} - \Gamma_i' P_{V_i'}) x_2,\ (\Gamma_i P_{V_i} - \Gamma_i' P_{V_i'}) x_2\rangle \leq r_2 \langle K_2^* x_2, K_2^* x_2\rangle. \end{cases} \tag{1}$$

If $A_1 > r_1$ and $A_2 > r_2$, then the perturbed product frame $\mathcal{F}'$ is a $(K_1 \oplus K_2)$-g-fusion frame for $\mathcal{H}$ with frame bounds $A' := \min\{(\sqrt{A_1} - \sqrt{r_1})^2,\ (\sqrt{A_2} - \sqrt{r_2})^2\}$ and $B' := \max\{2B_1 + 2r_1\|K_1^*\|^2,\ 2B_2 + 2r_2\|K_2^*\|^2\}$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 October 2025 doi:10.20944/preprints202510.0440.v1

Proof. For $x_1 \in \mathcal{H}_1$ note that $\Lambda_i' P_{W_i'} x_1 = \Lambda_i P_{W_i} x_1 + (\Lambda_i' P_{W_i'} - \Lambda_i P_{W_i}) x_1$. For $x_2 \in \mathcal{H}_2$ set similarly $\Gamma_i' P_{V_i'} x_2 = \Gamma_i P_{V_i} x_2 + (\Gamma_i' P_{V_i'} - \Gamma_i P_{V_i}) x_2$.

First, from the two g-fusion frame inequalities (1) and Theorem 1 we obtain immediately, with $A = \min\{A_1, A_2\}$ and $B = \max\{B_1, B_2\}$, that the product family $\mathcal{F}$ is a $(K_1 \oplus K_2)$-g-fusion frame with bounds A, B. Thus satisfies

$$\begin{aligned} A\langle (K_1 \oplus K_2)^*(x_1,x_2),(K_1 \oplus K_2)^*(x_1,x_2)\rangle &\leq \sum_{i\in I} v_i^2 \langle \Theta_i(x_1,x_2), \Theta_i(x_1,x_2)\rangle \\ &= \sum_{i\in I} v_i^2 \big(\langle \Lambda_i P_{W_i} x_1, \Lambda_i P_{W_i} x_1\rangle + \langle \Gamma_i P_{V_i} x_2, \Gamma_i P_{V_i} x_2\rangle\big) \\ &\leq B\langle (x_1,x_2),(x_1,x_2)\rangle. \end{aligned}$$

Now, fix $x_1 \in \mathcal{H}_1$. Consider the perturbed left-component sum

$$\begin{aligned} S_1'(x_1) &:= \sum_{i\in I} v_i^2 \langle \Lambda_i' P_{W_i'} x_1, \Lambda_i' P_{W_i'} x_1\rangle \\ &= \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x_1 + (\Lambda_i' P_{W_i'} - \Lambda_i P_{W_i}) x_1, \Lambda_i P_{W_i} x_1 + (\Lambda_i' P_{W_i'} - \Lambda_i P_{W_i}) x_1\rangle. \end{aligned}$$

For the sake of readability, we denote X and Y the elements of the Hilbert C*-module $\ell^2(I, \mathcal{A})$ defined by

$$X := (v_i \Lambda_i P_{W_i} x_1)_{i\in I}, \qquad Y := (v_i (\Lambda_i' P_{W_i'} - \Lambda_i P_{W_i}) x_1)_{i\in I}.$$

Thus

$$S_1'(x_1) = \langle X, X\rangle_{\ell^2(I,\mathcal{A})} + \langle Y, Y\rangle_{\ell^2(I,\mathcal{A})} + \langle X, Y\rangle_{\ell^2(I,\mathcal{A})} + \langle X, Y\rangle^*_{\ell^2(I,\mathcal{A})}. \tag{2}$$

This follows from the computations:

$$\langle X, Y\rangle_{\ell^2(I,\mathcal{A})} + \langle X, Y\rangle^*_{\ell^2(I,\mathcal{A})} = \sum_{i\in I} v_i^2 \big(\langle \Lambda_i P_{W_i} x_1, (\Lambda_i' P_{W_i'} - \Lambda_i P_{W_i}) x_1\rangle + \langle (\Lambda_i' P_{W_i'} - \Lambda_i P_{W_i}) x_1, \Lambda_i P_{W_i} x_1\rangle\big)$$

$$\langle X, X\rangle_{\ell^2(I,\mathcal{A})} = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x_1, \Lambda_i P_{W_i} x_1\rangle$$

$$\langle Y, Y\rangle_{\ell^2(I,\mathcal{A})} = \sum_{i\in I} v_i^2 \langle (\Lambda_i' P_{W_i'} - \Lambda_i P_{W_i}) x_1, (\Lambda_i' P_{W_i'} - \Lambda_i P_{W_i}) x_1\rangle,$$

Now, since $\langle X+Y, X+Y\rangle_{\ell^2(I,\mathcal{A})}$ and $\langle X-Y, X-Y\rangle_{\ell^2(I,\mathcal{A})}$ are non negative in $\mathcal{A}$, we have

$$\begin{aligned} -\sqrt{\tfrac{r_1}{A_1}}\langle X, X\rangle_{\ell^2(I,\mathcal{A})} - \sqrt{\tfrac{A_1}{r_1}}\langle Y, Y\rangle_{\ell^2(I,\mathcal{A})} &\leq \langle X, Y\rangle_{\ell^2(I,\mathcal{A})} + \langle X, Y\rangle^*_{\ell^2(I,\mathcal{A})} \\ &\leq \langle X, X\rangle_{\ell^2(I,\mathcal{A})} + \langle Y, Y\rangle_{\ell^2(I,\mathcal{A})} \end{aligned} \tag{3}$$

Since $S_1'(x_1) = \langle X+Y, X+Y\rangle_{\ell^2(I,\mathcal{A})}$ we deduce from (3) and (2) that

$$\begin{aligned} S_1'(x_1) &\leq 2\langle X, X\rangle_{\ell^2(I,\mathcal{A})} + 2\langle Y, Y\rangle_{\ell^2(I,\mathcal{A})} \\ &\leq 2B_1\langle x_1, x_1\rangle + 2r_1\|K_1^*\|^2\langle x_1, x_1\rangle \\ &= (2B_1 + 2r_1\|K_1^*\|^2)\langle x_1, x_1\rangle \end{aligned}$$

For the other inequality, using (1), (2), (3) and the hypothesis $\sqrt{\frac{r_1}{A_1}} < 1$, we obtain that

$$\begin{aligned} S_1'(x_1) &\geq \langle X, X\rangle_{\ell^2(I,\mathcal{A})} + \langle Y, Y\rangle_{\ell^2(I,\mathcal{A})} - \sqrt{\tfrac{r_1}{A_1}}\langle X, X\rangle_{\ell^2(I,\mathcal{A})} - \sqrt{\tfrac{A_1}{r_1}}\langle Y, Y\rangle_{\ell^2(I,\mathcal{A})} \\ &= (1 - \sqrt{\tfrac{r_1}{A_1}})\langle X, X\rangle_{\ell^2(I,\mathcal{A})} - (\sqrt{\tfrac{A_1}{r_1}} - 1))\langle Y, Y\rangle_{\ell^2(I,\mathcal{A})} \\ &\geq (A_1(1 - \sqrt{\tfrac{r_1}{A_1}})\langle K_1^* x_1, K_1^* x_1\rangle - r_1(\sqrt{\tfrac{A_1}{r_1}} - 1))\langle K_1^* x_1, K_1^* x_1\rangle \\ &\geq (A_1(1 - \sqrt{\tfrac{r_1}{A_1}}) + r_1(1 - \sqrt{\tfrac{A_1}{r_1}}))\langle K_1^* x_1, K_1^* x_1\rangle \\ &= (\sqrt{A_1} - \sqrt{r_1})^2\langle K_1^* x_1, K_1^* x_1\rangle \end{aligned}$$

The same argument applied to the second component yields

$$(\sqrt{A_2}-\sqrt{r_2})^2\,\langle K_2^*x_2, K_2^*x_2\rangle \leq S_2'(x_2) \leq (2B_2+2r_2\|K_2^*\|^2)\,\langle x_2, x_2\rangle.$$

Finally, For $(x_1, x_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ we have

$$S'(x_1,x_2) := \sum_i v_i^2 \langle \Theta_i'(x_1,x_2), \Theta_i'(x_1,x_2)\rangle = S_1'(x_1) + S_2'(x_2).$$

Thus combining the componentwise upper bounds yields

$$\begin{aligned} S'(x_1,x_2) &\leq (2B_1+2r_1\|K_1^*\|^2)\langle x_1,x_1\rangle + (2B_2+2r_2\|K_2^*\|^2)\langle x_2,x_2\rangle \\ &\leq B'\,\langle (x_1,x_2),(x_1,x_2)\rangle, \end{aligned}$$

with $B' = \max\{(2B_1+2r_1\|K_1^*\|^2), (2B_2+2r_2\|K_2^*\|^2)\}$.

Similarly combining the lower bounds we get

$$\begin{aligned} S'(x_1,x_2) &\geq (\sqrt{A_1}-\sqrt{r_1})^2\langle K_1^*x_1, K_1^*x_1\rangle + (\sqrt{A_2}-\sqrt{r_2})^2\langle K_2^*x_2, K_2^*x_2\rangle \\ &\geq A'\,\langle (K_1\oplus K_2)^*(x_1,x_2), (K_1\oplus K_2)^*(x_1,x_2)\rangle, \end{aligned}$$

with $A' = \min\{(\sqrt{A_1}-\sqrt{r_1})^2, (\sqrt{A_2}-\sqrt{r_2})^2\}$. This yields the claimed two-sided $\mathcal{A}$-order inequalities and completes the proof that $\mathcal{F}'$ is a $(K_1 \oplus K_2)$-g-fusion frame with bounds A', B'. □

Acknowledgments: The authors are thankful to the area editor and referees for giving valuable comments and suggestions.

References

1. P. G. Casazza, The art of frame theory, *Taiwanese J. Math.* **4** (2000), 129–201.
2. P. G. Casazza, G. Kutyniok and S. Li, Fusion frames and distributed processing, *Appl. Comput. Harmon. Anal.* **25** (2008), 114–132.
3. R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic fourier series*, Trans. Am. Math. Soc. **72** (1952), 341–366.
4. M. Frank and D. R. Larson, *Frames in Hilbert C^*-modules and C^*-algebras*, J. Operator Theory **48** (2002), no. 2, 273–314.
5. I. Kaplansky, Modules Over Operator Algebras,Am. J. Math.75(1953), 839-858.
6. A. Karara, M. Rossafi, A. Touri, K-biframes in Hilbert spaces. J Anal (2024). https://doi.org/10.1007/s41478-024-00831-3
7. V. M. Manuilov, E. V. Troitsky, *Hilbert C^*-modules*, Translations of Mathematical Monographs, Vol. 226, AMS, Providence, Rhode Island, 2005.
8. F. D. Nhari, R. Echarghaoui, M. Rossafi, K-g-Fusion Frames in Hilbert C^*-Modules, Int. J. Anal. Appl., 19 (6) (2021), 836-857.
9. M. Rossafi, FD. Nhari, C. Park, S. Kabbaj, Continuous $g-$frames with C^*-Valued Bounds and Their Properties. Complex Anal. Oper. Theory 16, 44 (2022). https://doi.org/10.1007/s11785-022-01229-4
10. M. Rossafi, F. D. Nhari, Controlled K-g-Fusion Frames in Hilbert C^*-Modules, Int. J. Anal. Appl., 20 (2022), 1. https://doi.org/10.28924/2291-8639-20-2022-1
11. W. Sun, G-frames and g-Riesz bases, *J. Math. Anal. Appl.* **322** (2006), 437–452.
12. S. Touaiher, R. El Jazzar and M. Rossafi, Properties and characterizations of controlled Kg-fusion frames within Hilbert C^*-modules, *Int. J. Anal. Appl.* **23** (2025), 111–111.
13. A. Touri, H. Labrigui, M. Rossafi, S. Kabbaj, Perturbation and Stability of Continuous Operator Frames in Hilbert C^*-Modules, Journal of Mathematics, 2021, 5576534, 15 pages, 2021. https://doi.org/10.1155/2021/5576534
14. A. Touri, H. Labrigui, M. Rossafi, New Properties of Dual Continuous K-g-Frames in Hilbert Spaces, International Journal of Mathematics and Mathematical Sciences, 2021, 6671600, 11 pages, 2021. https://doi.org/10.1155/2021/6671600

doi:10.20944/preprints202510.0440.v1