Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 August 2022 doi:10.20944/preprints202208.0422.v1

Certification of Almost Global Phase Synchronization of Coupled Oscillators

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August 22, 2022

Abstract

Phase synchronization of weakly coupled limit cycle oscillators are related to the stability of the zero solution of the reduced-order dynamics of phase differences, represented by a systems of differential equations on a hypertorus. Using Rantzer's density function, a dual form of Lyapunov function, we propose a method to certify almost global stability of an equilibrium on a hypertorus. We show that the proposed method can certify robustness of phase synchronization of all-to-all and weakly coupled limit cycle oscillators with respect to disturbances in phases. The method leverages sum of squares polynomial optimization to construct the certification function.

1 Introduction

Qualitative analysis of dynamical systems such as Lyapunov theory [25, 26] and dissipativity theory [12, 13, 14, 41] can lead to efficient algorithms for the analysis and synthesis of dynamical systems. Properties like stability, dissipativity, safety, etc. can be verified by means of the existence of certain types of functions (called certificates). A particular asymptotic property of dynamical systems, namely, the convergence of almost all solutions to an attractor, was studied by mathematicians in [29, 17], but this property drew more attention once control theorist Rantzer [37] proposed the so-called *dual Lyapunov theorem* to certify convergence of almost all solutions to a particular attractor (hereafter called *almost global stability*) using a Lyapunov-like function, called Lyapunov density. Since then, Lyapunov densities have been used for the analysis of dynamical systems [30, 31, 34, 39, 27, 11, 28, 19] and synthesis of control systems [33, 40] by means of sum of squares programming [24] (hereafter called *SOS programming*).

Rantzer claimed in [37] that the almost global stability criterion based on a Lyapunov density has "a striking convexity property related to control synthesis". Following this claim, in [33], Lyapunov densities are used to obtain a feedback controller for polynomial control systems on $\mathbb{R}^n$ leveraging SOS programming. However, due to the integrability of the Lyapunov density away from the attractor (by its definition), and its non-integrability near the attractor (see Remark 3 in [21]), a Lyapunov density for such systems is constructed as a rational polynomial. Designing both the numerator and the denominator of a Lyapunov density as well as designing the control input leads to a non-convex optimization problem. A heuristic solution to this problem is proposed in [33] where the denominator is chosen as a control Lyapunov function of the linearized system, whereas the numerator and the control input are found by a linear design inequality suggested by the dual Lyapunov theorem.

Another motivation for considering almost global stability is the fact that dynamical systems on compact manifolds (i.e. torus) cannot admit globally attracting invariant sets due to topological restrictions [2, 22]. For such systems, almost global attraction to a fixed point or a limit cycle may occur and this is usually a desired system behaviour for a control system. A particular case for a system on a compact manifold is a system of weakly coupled oscillators [15] whose phase dynamics can be approximately modelled as a dynamical system on a hypertorus[3], where each entry of the state corresponds to the phase of a particular oscillator. The case where the 2π-periodic coupling function between two oscillators consists of the first Fourier term is the well-studied Kuramoto model [1], exhibiting various dynamical behaviour such as synchronization, clustering, etc. Considering higher order Fourier terms, one can also obtain dynamical phenomena such as winnerless competition [35] and uni-directional synchronization [20]. Numerical bifurcation analysis can be used to classify possible local dynamic behaviour of the coupled system; which, for a small number of oscillators, can also provide possible (global) phase portraits of the system [4]. On the other hand, linearization based methods can provide parameter regions for local stability of the synchronized behaviour of coupled phase oscillators [8]. However, certifying almost global stability of synchronization for a given system of coupled phase oscillators remains as an open problem. Some particular coupling structures leading to almost global synchronization for Kuramoto models (with one Fourier term in the coupling function) has been shown in [6]. Further results on the almost global synchronization of the Kuramoto case can be found in [8]. A controller design leading to almost global synchronization of identical phase oscillator (not necessarily Kuramoto-type) has recently been suggested in [5].

In this work, we provide the first systematic method for constructing Lyapunov densities for systems on torus that can be used to ensure almost global attraction. We show that the non-convex optimization problem mentioned above can be circumvent when state space is a high dimensional torus. Via a stereographic projection, we transform a given dynamical system on torus to a dynamical system on Euclidean space such that the origin of the torus is mapped to infinity, converting the problem of certifying stability of the origin of the torus to the problem of certifying the divergence of almost all solutions to infinity on the Euclidean space. We then use a sufficient condition for the divergence of almost all solution on the Euclidean space to infinity based on the existence of a polynomial Lyapunov density. This means that the main motivation of proposing Lyapunov densities for certification in [36], namely to solve the control design problem via linear optimization, can actually be fulfilled for the first time in literature, up to our knowledge.

As a dynamical model on torus, we consider phase dynamics of N coupled oscillator systems, which, for example, may arise as a phase difference model for N weakly coupled limit cycle oscillators, and can be reduced to a model for $N-1$ phase difference variables. We then propose a systematic method to certify almost global stability of the origin for the reduced $(N-1)$-dimensional phase difference system. The method uses a stereographic state transformation that maps the origin of the phase difference systems to infinity, converting the problem of certifying almost global stability of the origin into the problem of certifying divergence of almost all solutions to infinity for a rational polynomial system on $\mathbb{R}^{N-1}$. The latter problem is solved by using a novel theorem based on a polynomial dual Lyapunov function, which can be approximated by the SOS method. As a result, without the need for an heuristic guess on the certification function, the method can ensure almost global stability of the phase difference model for all-to-all coupled phase oscillators.

In the sequel, we first define the problem in Section 2. Section 3 explains our main result on the divergence of almost all solutions to infinity for sytems on $\mathbb{R}^n$. In Section 4, the stability certification method for coupled oscillators obtain a dual Lyapunov theorem for the divergence (to infinity) of almost all solutions of nonlinear systems whose vector fields are polynomials on $\mathbb{R}^n$ based on a polynomial Lyapunov density that can be constructed using the SOS programming. Then, we show that under an assumption on the coupling structure, satisfied by the global coupling, almost global phase synchronization of a coupled oscillator system on $\mathbb{T}^n$ [1] can be reduced to almost global divergence of solutions of a polynomial system on $\mathbb{R}^{N-1}$. Finally, by means of the obtained dual Lyapunov theorem for divergence to infinity, we provide a systematic method for the certification of almost global attraction to the synchronization manifold for coupled systems of identical phase oscillators.

2 Problem Definition

A diffusively coupled system of N identical phase oscillators is determined by

$$\dot{\theta}_i = \omega + \sum_{j=1}^{N} c_{ij}\, g(\theta_j - \theta_i), \quad \theta_i \in \mathbb{T} = [0, 2\pi), \quad i = 1, \ldots, N, \tag{2.1}$$

where $\{c_{ij} \in \{0,1\}\}_{i,j\in\{1,\ldots,N\}}$ is the coupling matrix, ω is the natural frequency of the oscillators, and g is a 2π-periodic coupling function. Such a coupled phase oscillator system can arise as an approximate reduced model of a weakly coupled system of identical limit cycle oscillators [18]. Almost global attraction to the synchronization manifold $\{\theta_1 = \cdots = \theta_N\} \subset \mathbb{T}^N$ of the phase oscillator system implies almost global[2] phase synchronization of the corresponding coupled system of limit cycle oscillators as long as the amplitude of each oscillation is close to that of

[1]Note that $\mathbb{T}^n$ stands for $n-$dimensional torus.

[2]Here, we use the term "almost global" in the sense of almost all phase perturbations of the state along limit cycles

the limit cycle. The phase shift symmetry $(\theta_1, \dots, \theta_N) \to (\theta_1 + \epsilon, \dots, \theta_N + \epsilon)$ of system (2.1) leads to the following reduced $N-1$ dimensional phase difference systems

$$\dot{\varphi} = \mathcal{F}(\varphi), \quad \varphi = (\varphi_1, \dots, \varphi_{N-1})^{\mathrm{T}} \in \mathbb{T}^{N-1}, \tag{2.2}$$

where each φ_i is an appropriately chosen phase difference of a certain pair of oscillators such that $\varphi = (0, \dots, 0)$ if and only if $\theta_1 = \cdots = \theta_N$.

In the present paper, our aim is to certify almost global attraction to the origin for the system (2.2) using Lyapunov densities and thereby to propose a method to certify phase synchronization of weakly coupled limit cycle oscillators. In order to avoid the singularity of the Lyapunov density at the attractor and in order to be able to use the SOS method, we use the stereographic state transformation from $(0, 2\pi)^{N-1} \subset \mathbb{T}^{N-1}$ to $\mathbb{R}^{N-1}$ given by

$$\varphi_i \longrightarrow y_i = \operatorname{arccot}(\varphi_i/2), \tag{2.3}$$

which leads to the following equivalent system

$$\dot{y} = \frac{F(y)}{G(y)}, \quad y = (y_1, \dots, y_{N-1})^{\mathrm{T}} \in \mathbb{R}^{N-1}, \tag{2.4}$$

where $F(y)$ and $G(y)$ can be obtained as polynomial vector fields using Chebyshev polynomials as described in Section 3. We consider the following assumption, which is satisfied for all-to-all coupled systems as well as for some other dense coupling structures (see Assumption 4.1 and Figure 3 in Section 3).

Assumption 2.1. All subspaces $\{\varphi_i = 0\} \subset \mathbb{T}^{N-1}$ are invariant under the dynamics of (2.2).

This assumption results in the phase difference dynamics (2.2) being trapped in the open hypercube which already implies that phase dynamic oscillators in (2.1) are frequency synchronized [3].

Our aim is to characterize almost global phase synchronization in this case by showing that almost all initial conditions in $(0, 2\pi)^{N-1}$ converges to the origin $0 \in \mathbb{T}^{N-1}$; and therefore $(0, 2\pi)^{N-1}$ are invariant under (2.2). Note that the transformation (2.3) maps the subspaces $\{\varphi_i = 0\}$ to infinity. Therefore, certification of the almost global divergence to infinity of (2.4) by means of the obtained dual Lyapunov divergence theorem implies that almost all solutions of (2.2) converge to the invariant subspaces $\{\varphi_i = 0\}$. Repeating the same argument recursively for all invariant subspaces, we construct an algorithm for certification of almost global phase synchronization.

Lyapunov Density. Given the dynamical system $\dot{x}(t) = f(x(t))$, where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function such that $f(0) = 0$, assume that there is a non-negative $\rho : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}$, called Lyapunov density function, satisfying (i) $\frac{\rho(x) f(x)}{|x|}$ is integrable on $\{x \in \mathbb{R}^n : |x| \geqslant 1\}$ (integrability condition), and (ii) $[\nabla \cdot (f\rho)] > 0$ for almost all x (divergence inequality). Then the solution $x(t)$ exists for almost all initial states and tends to zero as $t \mapsto \infty$. See [37] for further details.

SOS Approach. In the SOS approximation method, non-negative polynomials are approximated by SOS polynomials, which can be written as the sum of squares. On the other hand, SOS conditions can be solved in polynomial time with the SDP method by transforming them into a matrix inequality (MATLAB SOSTOOLS package [32]).

In order to certify the almost global divergent to infinity by finding a SOS polynomial density function ρ, we solve divergence inequality

$$\nabla\left(\frac{F}{G}\rho\right) = \frac{\nabla(F\rho)G - F\rho\nabla(G)}{G^2} > 0 \tag{2.5}$$

for the system (2.4) by using SOSTOOL packages YALMIP, SEDUMI, etc. .

3 Divergence of Almost All Solutions to Infinity

In this section, we present a dual Lyapunov theorem for the divergence of almost all solutions to infinity for a system of ordinary differential equations on $\mathbb{R}^n$. The proof of Theorem 3.2 as well as its discrete-time counterpart, and a recap on transfer operators can be found in Appendix A.

[3] $\lim_{t\to\infty} \frac{\theta_i(t) - \theta_j(t)}{t} = 0$, for any $i, j \in \{1, 2, \cdots, N\}$.

Let us consider a continuous-time dynamical system of the form

$$\dot{x}(t) = F(x(t)), \quad t \in \mathbb{R} \tag{3.1}$$

on the measure space $(\mathbb{R}^n, \mathcal{B}, m)$, where $\mathcal{B}$ is the Borel σ-algebra and m is the Lebesgue measure on $\mathbb{R}^n$. In the present paper, we assume that the solutions of (3.1) exist globally.

In the sequel, we will define almost global divergence and provide sufficient condition for almost global divergence by using Lyapunov densities.

Definition 3.1. The system (3.1) is almost globally divergent to infinity if there exists a subset $N \subset \mathbb{R}^n$ satisfying $m(N) = 0$ and $T^n(x) \longrightarrow \infty$ for $\forall x$ in N^C.

As mentioned in introduction, almost global divergence is an essential tool to analyze stability properties of rotational dynamical systems. In the following part, we will provide sufficient conditions for almost globally divergence of solutions which will help us to analyze synchronization of rotational systems. Thereafter, we will use it to analyze the synchronization of systems with rotations.

Theorem 3.2. *Solutions of the system* (3.1) *are almost globally divergent to infinity if there is a positive continuous function* $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}$ *such that*

i) ρ *is integrable on* $|x| \leqslant \epsilon$ *for* $\forall\epsilon > 0$, *namely,* $\displaystyle\int_{|x|\leqslant\epsilon} \rho(x)m(dx) < \infty$, *for* $\forall\epsilon > 0$.

ii) $\nabla(F\rho)(x) > 0$.

In order to show the applicability of Theorem 3.2, we will provide the following example.

Example 3.3. Let us consider the following systems of differential equations on $\mathbb{R}^2 - \{0\}$

$$\begin{aligned} \dot{x}_1 &= \frac{ax_1x_2^2 + bx_1^3}{x_1^4 + x_2^4}, \\ \dot{x}_2 &= \frac{cx_1^2x_2 + dx_2^3}{x_1^4 + x_2^4}, \end{aligned} \tag{3.2}$$

where a, b, c and d are real numbers. Global existence of solutions follows by the boundedness of the vector field as x goes to infinity.

Divergence of almost all solutions to infinity follows from Theorem 3.2 using $\rho(x_1, x_2) = x_1^4 + x_2^4$ when $3b + c > 0$ and $a + 3d > 0$. In Figure 2, some solutions on a unit circle with radius one is drawn to illustrate that some solutions of (3.2) with constants satisfying $3b + c > 0$ and $a + 3d > 0$ are divergent.

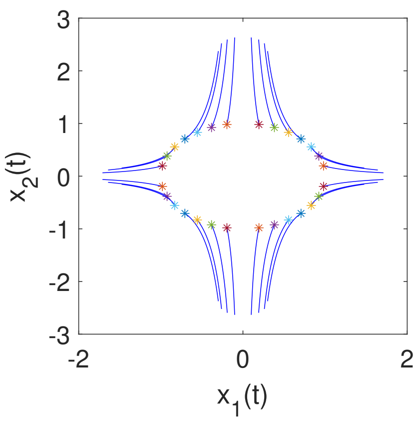

Figure 1: Some solutions of (3.2) with $a = -0.02$ $b = 0.01$ $c = -0.02$ and $d = 0.03$.The stars stands for initial states of solutions.

Moreover, we can also see from Figure 2 that some solutions of (3.2) with constants not satisfying $3b + c > 0$ and $a + 3d > 0$ are convergent.

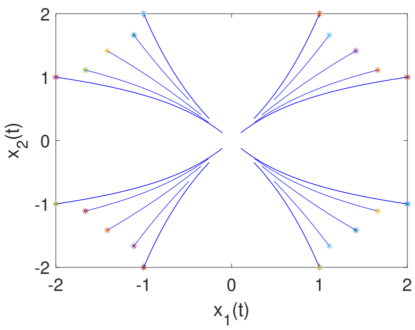

Figure 2: Some solutions of (3.2) with $a = -0.01$ $b = -0.04$ $c = -0.01$ and $d = -0.03$. The stars stands for initial states of solutions.

4 Almost Global Phase Synchronization of Coupled Phase Oscillators

Let us reproduce the system of coupled phase oscillators that has been presented in the introduction as follows:

$$\dot{\theta}_i = \omega + \sum_{j=1}^{N} c_{ij}\, g(\theta_j - \theta_i), \quad \theta_i \in \mathbb{T} = [0, 2\pi), \quad i = 1, \dots, N, \tag{4.1}$$

Here $\{c_{ij}\}$ is the coupling matrix such that $c_{ij} = 1$ if the oscillator j affects the oscillator i, and $c_{ij} = 0$ otherwise. g is a 2π-periodic coupling function modeled by its first L Fourier terms as

$$g(x) = \sum_{k=1}^{L} \alpha_k \sin(kx + \beta_k). \tag{4.2}$$

Here we assume that $\alpha_1 = 1$. This can be done via a scaling of the time variable, which also normalizes any coupling constant that usually appears in the definition of Kuramoto-type phase models.

We assume the following assumption on coupling:

Assumption 4.1. The coupling matrix $\{c_{ij}\}$ is such that the corresponding directed graph (connection graph) admits $N-1$ different balanced colorings[4] such that each coloring contains $N-1$ different colors (or equivalently only two cells have the same color) and their least upper bound is the trivial coloring (all vertices with same color).

Each coloring in Assumption 4.1 can be represented as an undirected edge connecting the two cells with the same color. Then, the $N-1$ colorings in Assumption 4.1 can be seen as a tree in the connection graph. Note that a globally (all-to-all) coupled connection graph satisfies Assumption 4.1 since any partition of the set of vertices is then balanced.

As an example, Figure 3 shows all (up to graph isomorphisms and except the all-to-all coupled graph and the trivial graph with no arrows) directed graphs with $N = 4$ vertices that satisfy Assumption 4.1.

The use of the balanced colorings is that they lead to invariant subspaces [9]. In fact if a coloring is balanced then for any coupled system of identical dynamical systems having the considered coupling structure, the subspace of the state space obtained by equating the states of cells with the same color is an invariant subspace. For instance, ...

Let us order the colorings in Assumption 4.1 and consider the ith coloring. Choose a direction for the edge representing the coloring such that i_t and i_h denote the labels of the tail and head cell of the same color in the ith coloring and define the phase difference variable as $\varphi_i := \theta_{i_t} - \theta_{i_h}$. Since the colorings, now seen as directed edges, form a tree as discussed above, and each phase difference $\theta_j - \theta_i$ can be written as a difference of certain sums of φ_k's, say

$$\theta_j - \theta_i = \sum_{k \in (j,i)^+} \varphi_k - \sum_{k' \in (j,i)^-} \varphi_{k'}, \tag{4.3}$$

where $(j,i)^+$ is the set of indices of the colorings (now directed edges) directing in reverse direction with (j,i) on the fundamental loop of (j,i)[5] and $(j,i)^-$ is the set of indices of the colorings (now directed edges) directing in the same

[4] A coloring of vertices of a directed graph is balanced if any two vertices with the same color receive equal number of inputs from any given color.

[5] The loop formed by the edge (j,i) and the edges of the tree.

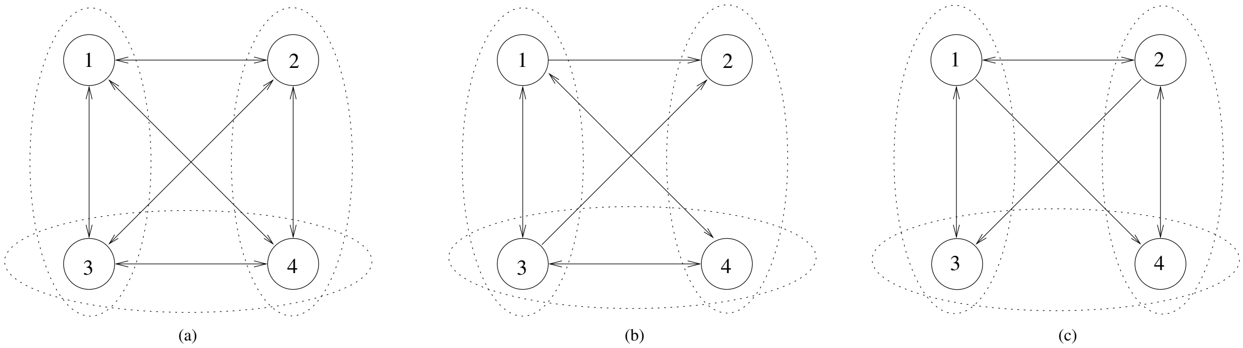

Figure 3: All directed graphs with $N = 4$ vertices satisfying Assumption 4.1 (except the all-to-all coupled graph and the trivial graph with no arrows). The $N-1 = 3$ different balanced colorings each having $N-1 = 3$ different colors are represented in Figure by three ellipses (forming a tree as discussed above), where each ellipse identifies a coloring where cells in the ellipse have the same color whereas each cell outside the ellipse have a different color (different from each other and from the color of the ellipse).

direction with (j,i) on the fundamental loop of (j,i) As a result of the above arguments, one can obtain a reduction of (4.1) to the $N-1$ dimensional phase difference systems

$$\dot{\varphi}_i = \sum_{j=1}^{N} c_{i_t j}\, g\left(\sum_{k\in(j,i_t)^+} \varphi_k - \sum_{k'\in(j,i_t)^-} \varphi_{k'}\right) - \sum_{j'=1}^{N} c_{i_h j'}\, g\left(\sum_{k\in(j',i_h)^+} \varphi_k - \sum_{k'\in(j',i_h)^-} \varphi_{k'}\right). \tag{4.4}$$

Example 4.2. The phase dynamics of the coupled phase oscillator system (4.1) with a coupling structure as in Figure... can be written as

$$\begin{aligned}
\dot{\theta}_1 &= \omega + g(\theta_2 - \theta_1) + g(\theta_3 - \theta_1)\\
\dot{\theta}_2 &= \omega + g(\theta_1 - \theta_2) + g(\theta_4 - \theta_2)\\
\dot{\theta}_3 &= \omega + g(\theta_1 - \theta_3) + g(\theta_2 - \theta_3)\\
\dot{\theta}_4 &= \omega + g(\theta_1 - \theta_4) + g(\theta_2 - \theta_4).
\end{aligned}$$

Defining $\varphi_1 := \theta_1 - \theta_3$, $\varphi_2 := \theta_2 - \theta_4$ and $\varphi_3 := \theta_3 - \theta_4$, the phase difference dynamics in (4.4) can be obtained as

$$\dot{\varphi}_1 = g(\varphi_2 - \varphi_1 - \varphi_3) + g(-\varphi_1) - g(\varphi_1) - g(\varphi_2 - \varphi_3) \tag{4.5}$$
$$\dot{\varphi}_2 = g(-\varphi_2 + \varphi_1 + \varphi_3) + g(-\varphi_2) - g(\varphi_1 + \varphi_3) - g(\varphi_2) \tag{4.6}$$
$$\dot{\varphi}_3 = g(\varphi_1) + g(\varphi_2 - \varphi_3) - g(\varphi_1 + \varphi_3) - g(\varphi_2) \tag{4.7}$$

It has been shown in [20] that (4.5) admits a locally stable heteroclinic attractor on the invariant subspaces $\{\varphi_1 = 0\}$ and $\{\varphi_2 = 0\}$. In the next section, we will show, using Theorem 3.2 that this attractor is indeed almost globally attracting.

Next, the stereographic transformation

$$\varphi_i \longrightarrow y_i = \operatorname{arccot}(\phi_i/2), \tag{4.8}$$

will be applied to the system (4.4) and show that the transformed system is a rational polynomial system on $\mathbb{R}^{N-1}$. However, first of all, for the sake of simplicity of notation, we assume the following:

Assumption 4.3. Assumption 4.1 can be satisfied with a set of colorings such that a fixed cell is always a member of the pair of cells with the same color.

It is interesting to note that the 4-cell graphs in Figure... , which satisfy Assumption 4.1, also satisfy Assumption|4.3. However, there are 5-cell graphs satisfying Assumption 4.1 but not Assumption|4.3. Therefore, in general one may need to consider the equation (4.4) and apply the transformation (4.8) directly to (4.4). Another reason for such a necessity could be that the attractor of interest is contained in a union of invariant subspaces that cannot be obtained via balanced coloring satisfying Assumption 4.3 , as we discuss in Section ....

4.1 All-to-all Coupled Oscillators.

Let us consider Assumption4.3 where the fixed cell is assumed to be the Nth cell without loss of generality and hence define

$$\begin{aligned} \varphi_1 &= \theta_1 - \theta_N, \\ \varphi_2 &= \theta_2 - \theta_N, \\ &\vdots \\ \varphi_{N-1} &= \theta_{N-1} - \theta_N. \end{aligned} \tag{4.9}$$

Using these phase difference variables, the phase difference reduction of (4.1) is obtained as

$$\dot{\varphi}_i = \sum_{j=1}^{N} c_{ij}\, g(\varphi_i - \varphi_j) - \sum_{j'=1}^{N} c_{Nj'}\, g(\varphi_{j'}), \tag{4.10}$$

Here, the parameter φ_N vanishes. Hence, the system (4.10) takes the form

$$\dot{\varphi}_i = f\left(\varphi_i; \{\varphi_k\}_{k\sim i}\right) - f\left(0; \{\varphi_k\}_{k\sim N}\right), \qquad i = 1, 2, \cdots, N-1, \tag{4.11}$$

where $\varphi_N = 0$ is assumed. The function g is defined as

$$g(x) = \lim_{L\to\infty} \sum_{j=1}^{L} \alpha_j \sin(jx + \beta_j), \tag{4.12}$$

where α_j and β_j are real numbers for each j. By employing the transformation

$$y_i = \cot\frac{\varphi_i}{2}, \qquad \text{for } i = 1, 2, \cdots, N-1,$$

and the definition of g (4.12), we arrive at the system related with variables y_i as

$$\begin{aligned} \dot{y}_i &= -\frac{1}{2}\left(1 + \cot^2\frac{\varphi_i}{2}\right)\dot{\varphi}_i \\ &= -\frac{1}{2}\left(1 + \cot^2\frac{\varphi_i}{2}\right)\left(f\left(\varphi_i; \{\varphi_k\}_{k\sim i}\right) - f\left(0; \{\varphi_k\}_{k\sim N}\right)\right) \\ &= -\frac{1}{2}\left(1 + \cot^2\frac{\varphi_i}{2}\right)\left(\sum_{k=1}^{r} g(\varphi_{i_k} - \varphi_i) - g(\varphi_{N_k})\right) \\ &= -\frac{1}{2}\left(1 + \cot^2\frac{\varphi_i}{2}\right)\left(\sum_{k=1}^{r}\sum_{j=1}^{\infty} \alpha_j\left(\sin(j(\varphi_{i_k} - \varphi_i) + \beta_j) - \sin(j(\varphi_{N_k}) + \beta_j)\right)\right) \\ &= -\frac{1}{2}\left(1 + y_i^2\right)\left(\sum_{k=1}^{r}\sum_{j=1}^{\infty} \alpha_j\Big([U_{j-1}(C(y_{i_k}))\mathcal{S}(y_{i_k})T_j(C(y_i)) \right. \\ &\quad - U_{j-1}(C(y_i))\mathcal{S}(y_i)T_j(C(y_{i_k}))]\cos\beta_j + [T_j(C(y_{i_k}))T_j(C(y_i)) \\ &\quad + U_{j-1}(C(y_{i_k}))\mathcal{S}(y_{i_k})U_{j-1}(C(y_i))\mathcal{S}(y_i)]\sin\beta_j \\ &\quad \left. - U_{j-1}(C(y_{N_k}))\mathcal{S}(y_{N_k})\cos\beta_j - T_j(C(y_{N_k}))\sin\beta_j\Big)\right) \end{aligned} \tag{4.13}$$

for $i = 1, 2, \cdots, N-1$, and $k = 1, 2, \cdots, r$, where T_n is Chebyshev polynomials of the first kind defined as

$$T_n(\cos\theta) = \cos n\theta,$$

whereas U_n is Chebyshev polynomials of the second kind given by

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

Here, we assume that $y_N = \infty$. We refer to [10] for further detail about Chebyshev polynomials of the first kind T_n and Chebyshev polynomials of the second kind U_n. Notice that φ_{i_k} is the element of the neighbor $\{\varphi_k\}_{k\sim i}$ of φ_i; and φ_{N_k} is that of the neighbor $\{\varphi_N\}_{k\sim N}$ of φ_N. We use the following notation in (4.13):

$$\begin{aligned} \mathcal{S}(y_i) &:= \sin\varphi_i = \frac{2y_i}{1+y_i^2}, \\ C(y_i) &:= \cos\varphi_i = \frac{y_i^2-1}{1+y_i^2}. \end{aligned} \tag{4.14}$$

4.2 Synchronization of All-to-all Coupled Oscillators.

In this subsection, we develop an algorithm to determine synchronization of given system by using approach of dual Lyapunav stability. Now, we introduce our approach as follows.

Step 1. Reduce the N-dimensional all-to-all coupled system (2.1) by using the phase differences to the $N-1$ dimensional system (4.10).

Step 2. Use the transformation (2.3) to obtain the system (4.13) by making use of the Chebyshev polynomials T_n, U_n and their transformations (4.14) with respect to the variables y_i for $i = 1, 2, \cdots, N-1$.

Step 3. Use SOS tools for the inequality (2.5) to find a positive polynomial function ρ in order to determine whether the system (4.14) is almost globally divergent to infinity.

If there is a proper polynomial function ρ for the system (4.14), you should continue Step 4.

Step 4. Check whether the invariant subsystems $\varphi_i = 0$ (for any fixed integer i) of the system (4.14) is almost globally divergent to infinity by finding another positive polynomial ρ_i.
See the Appendix **??** for further details of the above algorithm which is constructed by using MATLAB and Mathematica.

4.2.1 Example

In this example, we exploit the algorithm which is given in the previous Section 4.2 for the dimension there in details. We also compare our results with those in [4].

We consider the coupling function g as

$$g(x) = \alpha_1 \sin(x+\beta_1) + \alpha_2 \sin(2x+\beta_2), \tag{4.15}$$

where α_1, β_1 and β_2 are real numbers. The dynamical system is given by

$$\begin{aligned} \dot\varphi_1 &= g(\varphi_2-\varphi_1) + g(-\varphi_1) - g(\varphi_1) - g(\varphi_2), \\ \dot\varphi_2 &= g(\varphi_1-\varphi_2) + g(-\varphi_2) - g(\varphi_1) - g(\varphi_2). \end{aligned} \tag{4.16}$$

Then, we shall use transformations

$$y_i = \cot\frac{\varphi_i}{2}, \qquad i = 1, 2, \tag{4.17}$$

by which the system (4.16) is converted to

$$\begin{aligned} \dot y_1 &= \frac{-P_1(y_1,y_2)}{2(1+y_1^2)(1+y_2^2)^2}, \\ \dot y_2 &= \frac{-P_2(y_1,y_2)}{2(1+y_1^2)^2(1+y_2^2)}. \end{aligned}$$

Here, the polynomials $P_1(y_1, y_2)$ and $P_2(y_1, y_2)$ can be found in the Appendix C.

The paramater plane for the system (4.16) **for N=3.** In the paper [4], the coupling function g is given by

$$g(x) = -\sin(x-\alpha) + r\sin 2x, \tag{4.18}$$

and in order to certify local phase synchronization of the coupled oscillators, the parameter $(\alpha - r)$ plane for the system (4.16) for N=3 is indicated, see [Figure 1 in [4]]. For this system, we apply our approach to determine whether the system is almost globally phase synchronized.

In the following Figure 4, the parameter α and r is indicated by the horizontal and vertical axes, respectively.

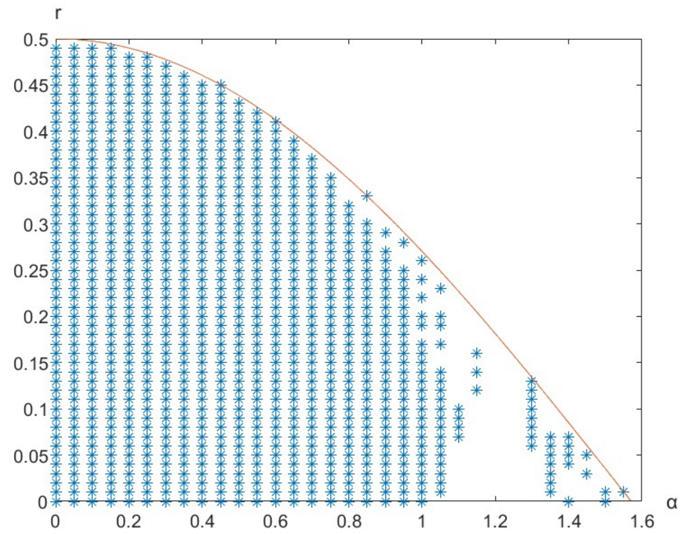

Figure 4: Parameter Plane for Almost-Global-Stability for $N = 3$

5 Conclusion

The presented results leads to certification of coupled phase oscillators for some coupling structures including the all-to-all coupling. Generalizing this approach to other coupling structure will enhance the application of the presented method. Another future direction is to implement this method for stabilization of the synchronized regime.

doi:10.20944/preprints202208.0422.v1

A Preliminaries: Almost Global Stability

In this section, we present almost global stability on an invariant set for discrete-time and continuous-time dynamical systems. For further discussions, we refer the paper [21]. In Subsection A.1, we examine discrete-time case elaborately with algebraic and geometric tools. In the second subsection, we present continuous-time case with appropriate operators.
Let $(X, \mathcal{A}, \mu)$ be a $\sigma-$finite measure space. We consider a discrete-time dynamical system

$$x(t+1) = F(x(t)), \qquad t \in \mathbb{Z}, \tag{A.1}$$

where $F : X \longrightarrow X$ is a non-singular continuously differentiable function. Let $A \subset X$ be a forward invariant set of the system (3.1), that is, $F(A) \subset A$, and consider the distance $d(A, x) = \inf$. A is said to be almost globally stable if almost all solutions converge to A, i.e., there exists a zero Lebesgue measure subset N of X such that $F^n(x) \longrightarrow A$ as $n \longrightarrow \infty$ for all x in N^c.
Assume that a measure ν is absolutely continuous with respect to μ, i.e., if $\nu(B) = 0$ whenever $\mu(B) = 0$ for a measurable set B. Then, $\nu \circ T^{-1}$ is absolutely continuous with respect to μ as well, where T is nonsingular, by using the equation

$$\mu(B) = 0 \implies \mu(T^{-1}B) = 0 \implies \nu(T^{-1}B) = 0. \tag{A.2}$$

By using Radon Nikodym theorem, for a nonnegative measurable function ρ in $\mathcal{M}^{+}(X)$, we arrive at ν as

$$\nu(T^{-1}B) = \int_{T^{-1}B} \rho d\mu. \tag{A.3}$$

Now, we can define the transfer operator called Frobenius-Perron operator as follows.

Frobenius-Perron Operator. Let the set of equivalence classes, defined as a set of functions that differs only on sets of zero Lebesgue measure, of measurable functions on X be denoted by $\mathcal{M}(X)$. The Frobenuis-Perron operator $\mathbb{P}$ corresponding to F is uniquely defined by

$$\mathbb{P} : \mathcal{M}(X) \to \mathcal{M}(X), \qquad \rho \mapsto \mathbb{P}\rho = \overline{\rho} \tag{A.4}$$

such that

$$\int_B \overline{\rho} d\mu = \int_{T^{-1}B} \rho d\mu \tag{A.5}$$

for a nonnegative measurable function ρ in $\mathcal{M}^{+}(X)$ and a measurable subset B of $\mathcal{A}$. See [23] for further details about the operator.

Koopman Operator. The Koopman operator is defined as

$$\mathbb{U} : \mathcal{M}(X) \to \mathcal{M}(X), \qquad f \mapsto \mathbb{U}f = f \circ T \tag{A.6}$$

for a nonsingular mapping T determined by a dynamical system. There is a relationship between Frobenius-Perron and Koopman operators as

$$\int_B (\mathbb{P}\rho) f d\mu = \int_B \rho(\mathbb{U}f) d\mu \tag{A.7}$$

for a nonnegative measurable functions ρ and f on X and for a measurable set B in $\mathcal{A}$. We refer [23] for more discussions about Koopman operator.

Theorem A.1 ([21])**.** *Let A be an invariant set, $F(A) \subset A$, of the system* (A.1)*. A is almost globally stable if there is a measurable function ρ on the set A^c satisfying*

$$\begin{aligned} &i) && \rho \geqslant 0, && \textit{for almost every } x \in A^C. \\ &ii) && \int_{A^C_\epsilon} \rho(x)\mu(dx) \leqslant \infty, && \textit{for } \forall \xi > 0. \\ &iii) && \mathbb{P}\rho(x) < \rho(x), && \textit{for almost every } x \in A^C. \end{aligned}$$

A.1 Discrete-Time Systems

Definition A.2. The system (A.1) is almost globally divergent to infinity if there is a subset N of X such that $\mu(N) = 0$ and $F^n(x) \to \infty$ for $\forall x$ in N^C.

Theorem A.3. *The system* (A.1) *is almost globally divergent to infinity if there is a discrete-time Lyapunov density ρ satisfying*

i) $\rho(x) \geqslant 0$, *for almost every* $x \in \mathbb{R}^n$.

ii) ρ *is integrable on the set* $\{|x| \leqslant \epsilon\}$ *for all* $\epsilon > 0$, *namely,* $\int_{|x|\leqslant\epsilon} \rho(x)\mu(dx) < \infty, \forall\epsilon > 0$.

iii) $\mathbb{P}\rho(x) < \rho(x)$, *for almost every* $x \in \mathbb{R}^n$.

The following lemmas will be used to prove Theorem A.3.

Lemma A.4. *The system* (A.1) *is almost globally divergent to infinity if and only if for any* $\epsilon > 0$, *there exists a set* $N \subset \mathbb{R}^n$ *such that that* $Leb(N) = 0$ *and* $\sum_{k=0}^{\infty} \mathbb{U}^k 1_{|x|\leqslant\epsilon}(x) < \infty$ *for* $x \in N^c$.

Proof. We use the following characterization of the divergence to infinity for a trajectory starting at x:

$$\forall\epsilon > 0, \quad \sum_{k=0}^{\infty} \mathbb{U}^k 1_{|x|\leqslant\epsilon}(x) < \infty \iff F^n(x) \to \infty, \text{ as } n \to \infty. \tag{A.8}$$

To prove necessity, assume that for any $\epsilon > 0$ there exists N_ϵ such that $\mathrm{Leb}(N) = 0$ and for $x \in N^c$

$$\infty > \sum_{k=0}^{\infty} \mathbb{U}^k 1_{|x|\leqslant\epsilon}(x) = \sum_{k=0}^{\infty} 1_{|x|\leqslant\epsilon}(F^k(x)). \tag{A.9}$$

The right-hand side of the above equation shows the number of visits of trajectories $F^n(x)$ to the closed set $|x| \leqslant \epsilon$. Thus this number is finite. Then, there exists a number $\mathcal{N}(\epsilon, x) > 0$ such that for all $n > \mathcal{N}(\epsilon, x)$, $F^n(x) \in \{x \in \mathbb{R}^n : |x| \leqslant \epsilon\}$. □

Lemma A.5. *Let ρ be a positive measurable function on X such that $\mathbb{P}\rho(x) < \rho(x)$ for almost every x in $|x| \leqslant \epsilon$. Then, the limit of $\mathbb{P}^n\rho(x)$, as n approaches infinity, exists and it is smaller than $\rho(x)$.*

Proof. We shall apply the Frobenuis-Perron operator $\mathbb{P}$ (A.4) both sides of the inequality $\mathbb{P}\rho(x) < \rho(x)$ for x in $|x| \leqslant \epsilon$, then we obtain that

$$\mathbb{P}^2\rho(x) < \mathbb{P}\rho(x) \tag{A.10}$$

for every x in the set O^2. Note that the operator $\mathbb{P}$ transforms a positive measurable function to another one. Hence $\mathbb{P}^2\rho(x)$ is a positive measurable function as well. Now, we consider the sequence

$$\begin{aligned}
\mathbb{P}\rho(x) &< \rho(x), \quad \text{for every } x \in \{x \in \mathbb{R}^n : |x| \leqslant \epsilon_1\}, \\
\mathbb{P}^2\rho(x) &< \mathbb{P}\rho(x), \text{ for every } x \in \{x \in \mathbb{R}^n : |x| \leqslant \epsilon_2\}, \\
&\vdots \\
\mathbb{P}^{n+1}\rho(x) &< \mathbb{P}^n\rho(x), \text{ for every } x \in \{x \in \mathbb{R}^n : |x| \leqslant \epsilon_{n+1}\}.
\end{aligned}$$

Thus, the sequence $\mathbb{P}^n\rho(x)$ is decreasing and it is bounded from above by $\rho(x)$ and from below by 0. By using monotone convergence theorem, we can show that the limit $\lim_{n\to\infty} \mathbb{P}\rho(x)$ exists and it is smaller than $\rho(x)$. □

Proof of Theorem A.3. Let us define a non-negative measurable function ρ_0 as

$$\rho_0 = \rho - \mathbb{P}\rho. \tag{A.11}$$

Then, we consider $\overline{\rho_0}$ defined as

$$\begin{aligned}\overline{\rho_0} &= \lim_{n\to\infty}\sum_{k=0}^{n}\mathbb{P}^k\rho_0\\ &= \lim_{n\to\infty}\sum_{k=0}^{n}\mathbb{P}^k(\rho-\mathbb{P}\rho)\\ &= \lim_{n\to\infty}\left(\sum_{k=0}^{n}\mathbb{P}^k\rho-\sum_{k=0}^{n}\mathbb{P}^{k+1}\rho\right)\\ &= \lim_{n\to\infty}\left(\rho-\mathbb{P}^{n+1}\rho\right)\\ &= \rho-\lim_{n\to\infty}\mathbb{P}^{n+1}\rho.\end{aligned} \tag{A.12}$$

By employing the Lemma A.5, the limit $\lim_{n\to\infty}\mathbb{P}^{n+1}\rho$ exists and it is smaller than ρ. Besides, if ρ is integrable on $|x|\leqslant\epsilon$ for $\forall\epsilon>0$, then $\lim_{n\to\infty}\mathbb{P}^{n+1}\rho<\rho$ is integrable as well by using Lebesgue dominated convergence theorem. We observe that $\overline{\rho_0}$ (A.12) is also integrable on $|x|\leqslant\epsilon$ for $\forall\epsilon>0$. Then, we have

$$\begin{aligned}\infty &> \int\overline{\rho_0}\;1_{|x|\leqslant\epsilon}(x)\mu(dx)\\ &= \int_{|x|\leqslant\epsilon}\overline{\rho_0}(x)\mu(dx)\\ &= \int_{|x|\leqslant\epsilon}\left(\lim_{n\to\infty}\sum_{k=0}^{n}\mathbb{P}^k\rho_0(x)\right)\mu(dx)\\ &= \lim_{n\to\infty}\int_{|x|\leqslant\epsilon}\left(\sum_{k=0}^{n}\mathbb{P}^k\rho_0(x)\right)\mu(dx)\\ &= \lim_{n\to\infty}\sum_{k=0}^{n}\int_{|x|\leqslant\epsilon}\mathbb{P}^k\rho_0(x)\mu(dx)\\ &= \lim_{n\to\infty}\sum_{k=0}^{n}\int\mathbb{P}^k\rho_0(x)\;1_{|x|\leqslant\epsilon}(x)\mu(dx)\\ &= \lim_{n\to\infty}\sum_{k=0}^{n}\int\rho_0(x)\;\mathbb{U}^k 1_{|x|\leqslant\epsilon}(x)\mu(dx)\\ &= \int\rho_0(x)\left(\lim_{n\to\infty}\sum_{k=0}^{n}\mathbb{U}^k 1_{|x|\leqslant\epsilon}(x)\right)\mu(dx)\\ &= \int\rho_0(x)\;\sum_{k=0}^{\infty}\mathbb{U}^k 1_{|x|\leqslant\epsilon}(x)\mu(dx)\end{aligned} \tag{A.13}$$

Note that, we use the Lebesgue dominated convergence theorem[38] in the third line of the equation (A.13). In the forth line of the equation (A.13), we employ the Tonelli theorem[38]. Finally, in (A.13), we obtain that

$$\int\rho_0(x)\;\sum_{k=0}^{\infty}\mathbb{U}^k 1_{|x|\leqslant\epsilon}(x)\mu(dx)<\infty.$$

Then $\sum_{k=0}^{\infty}\mathbb{U}^k 1_{|x|\leqslant\epsilon}(x)$ is finite since ρ_0 is positive almost everywhere. Hence, the system (A.1) is almost globally divergent to infinity by Lemma A.4. □

B Continous-time Systems

Theorem B.1 ([21])**.** *Let A be a compact invariant set A of the system* (3.1)*. The set A is almost globally stable if there is a continuous-time Lyapunov density for A such that*

i) $\rho(x) > 0$, *for almost* $x \in A^C$.

ii) ρ *is integrable on* A^C, *namely* $\int_{A^C} \rho(x)m(dx) < \infty$.

iii) ρ *is properly subinvariant on* A^C, *that is,* $\mathbb{A}\rho(x) < 0$ *for almost every* $x \in A^C$,

where $\mathbb{A}$ is the infinitesimal operator of the continuous semigroup $\{\mathbb{P}_t\}$, i.e., $\mathbb{A}$ is defined by $\mathbb{A}\rho = -\nabla(F\rho)$, whereas F is continuously differentiable mapping.

Lemma B.2 ([37])**.** *Consider an open subset $D \subset \mathbb{R}^n$. Assume that functions F and ρ belong to the set $C^1(D, \mathbb{R}^n)$, where ρ is integrable. Let the solution at x_0 of the system* (3.1) *be denoted by $\phi_t(x_0)$. For a measurable set Z, we assume that*

$$\phi_\tau(Z) = \{\phi_\tau(x) | x \in Z\} \subset D \tag{B.1}$$

for all $\tau \in [0, t]$. Then, we have

$$\int_{\phi_t(Z)} \rho(x)dx - \int_Z \rho(x)dx = \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (F\rho)](x)\, dx\, d\tau. \tag{B.2}$$

The proof of Lemma B.2 can be found in the paper [37].

In the sequel, we will present two lemmas on almost globally divergence of (3.1). Moreover, these lemmas are supposed to be used for the proof of Theorem 3.2.

Lemma B.3. *System* (3.1) *is almost globally divergent to infinity if and only if there is a set $N \subset \mathbb{R}^n$ satisfying $LebN = 0$ and the integral $\int_0^\infty \mathbb{U}_\tau(1_{|x|\leqslant\epsilon})d\tau < \infty$ for x in N^c and for all $\epsilon > 0$.*

Proof. First, let us prove sufficiency part. Define $\mu_x(E) := \int_0^\infty \mathbb{U}_\tau(1_{|x|\leqslant\epsilon})d\tau$. The occupation measure is used to measure the length of time that the solution curve $\phi_t(x)$ starting at x stays in the set E. In order to prove the sufficiency part, we will use contradiction. We will show that the contrapositive statement for a fixed x is true. Namely, $\mu_x(B(M)) = \infty$ for all $M > 0$ if $\lim_{t\to\infty} \neq \infty$. Then, by the second assumption, there exists $M > 0$ such that for all $t > 0$ there exists $T(t)$ such that $\phi_T(t)(x) \in B(M)$. Moreover, for all T such that $\phi_t(x) \in B(M)$, there exists $T' > T$ such that $\phi_{T'}(x) \in (B(2M))^c$. If there exists no T', then $\phi_t(x) \in B(2M)$, for all $t > T$. Hence $\mu_x(B(2M)) = \infty$. It contradict with the assumption that $\mu_x(B(M)) < \infty$ for all $M > 0$. Construct sequences $\{t_k\} \to \infty$, and $\{t'_k\} \to \infty$ such that $t'_{k+1} > t_{k+1} > t'_k > t_k$ and $\phi_{t_k}(x) \in (B(2M))^c$ and $\phi_{t'_k}(x) \in (B(M))$. By means of the continuity of solutions, construct the following time sequences $\{\bar{t}_k\} \to \infty$ and $\{\underline{t}_k\} \to \infty$ such that $t'_k > \bar{t}_k > \underline{t}_k > t_k$, $\phi_t(x) \in B(M)^c \backslash B(2M)^c$ for $t \in (\underline{t}_k, \bar{t}_k)$. See Figure 5. Then, $\|\phi_{\underline{t}_k}\| = M$ and $\|\phi_{\bar{t}_k}\| = 2M$. Since $F \in C^1(\mathbb{R})$, there exists a Lipschitz constant L in $B(2M)$. Note that $x(\bar{t}_k) = x(\underline{t}_k) + \int_{\underline{t}_k}^{\bar{t}_k} F(x(s))ds$. From Gronwall lemma, we get $\|x(\bar{t}_k)\| \leqslant \|x(\underline{t}_k)\| e^{L(\bar{t}_k - \underline{t}_k)}$. Then, $\bar{t}_k - \underline{t}_k \geqslant \frac{\ln(2)}{L} > 0$. Consequently, $\mu_x(B(2M)) \geqslant \mu_x((B(M))^c \setminus (B(2M))^c) \geqslant \sum_{k=0}^\infty (\bar{t}_k - \underline{t}_k) = \infty$. Thus, $\lim_{t\to\infty} \phi_t(x) = \infty$ if $\mu_x(B(M) < \infty$ for all $M > 0$.

Necessity part of the statement can be proven as follows. If a system is almost globally divergent then $\lim_{t\to\infty} \phi_t(x) = \infty$ for almost every x. For all $\epsilon > 0$ for almost every x there exists $T(x)$ such that $\phi_t(x) \in (B(\epsilon))^c$ for all $t \geqslant T(x)$. Thus $\mu_x(B(\epsilon)) \leqslant T(x)$. This implies that for all $\epsilon > 0$, $\mu_x(B(\epsilon)) < \infty$ for almost every x. □

Lemma B.4. *System* (3.1) *is almost globally divergent if there exists an almost everywhere positive function $\rho \in \mathcal{M}(\mathbb{R}^n)$ such that*

$$\overline{\rho} := \int_0^\infty P_\tau \rho \, d\tau$$

is integrable on B_ϵ, for every $\epsilon > 0$

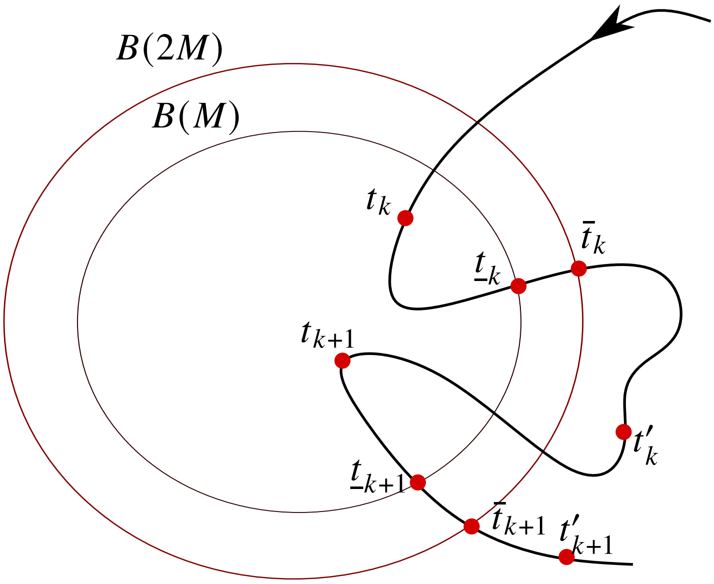

Figure 5: An illustration for the Proof of Lemma B.3

Proof. Using Tonelli's theorem with Lebesgue measure on time variable and duality between Perron and Koopman operators, we get

$$\begin{aligned} < \bar{\rho}, 1_{B(\epsilon)} > &= \int_0^\infty < \mathbb{P}_\tau \rho, 1_{B(\epsilon)}) > d\tau \\ &= \int_0^\infty < \rho, \mathbb{U}_\tau 1_{B(\epsilon)}) > d\tau \\ &=< \rho, \int_0^\infty \mathbb{U}_\tau 1_{B(\epsilon)}) d\tau >< \infty \end{aligned}$$

With the help of almost everywhere positivity of ρ, we get $\int_0^\infty \mathbb{U}_\tau 1_{B(\epsilon)}) d\tau < 0$. Thus, from Lemma B.3, we can conclude that (3.1) is almost globally divergent. □

Proof of Theorem 3.2. Let us define a measurable function ρ_0 as

$$\rho_0 = \nabla \cdot (F\rho), \tag{B.3}$$

where P_t is the continuous semigroup with respect to the system (3.1). Consider the measurable function $\overline{\rho_0}$ given as

$$\overline{\rho_0} = \int_0^\infty \mathbb{P}_\tau(\nabla \cdot (F\rho)) d\tau. \tag{B.4}$$

See [16] for further details about $\overline{\rho_0}$. $\overline{\rho_0}$ is integrable on $|x| \leqslant \epsilon$, i.e.,

$$\begin{aligned}
\int_{|x|\leqslant\epsilon} \overline{\rho_0}(x)dx &= \int_{|x|\leqslant\epsilon}\int_0^\infty \mathbb{P}_\tau(\nabla\cdot(F\rho))(x)d\tau\, dx \\
&= \int_0^\infty \int_{|x|\leqslant\epsilon} \mathbb{P}_\tau(\nabla\cdot(F\rho))(x)dx\, d\tau \\
&= \int_0^\infty \int_{T^{-\tau}(|x|\leqslant\epsilon)} \nabla\cdot(F\rho)(x)dx\, d\tau \\
&= \lim_{t\to\infty}\int_0^t \int_{T^{-\tau}(|x|\leqslant\epsilon)} \nabla\cdot(F\rho)(x)dx\, d\tau \\
&= -\lim_{t\to\infty}\int_0^{-t} \int_{T^{\tau'}(|x|\leqslant\epsilon)} \nabla\cdot(F\rho)(x)dx\, d\tau' \\
&= -\lim_{t\to\infty}\left(\int_{T^{-t}(|x|\leqslant\epsilon)} \rho(x)dx - \int_{|x|\leqslant\epsilon}\rho(x)dx\right) \\
&= -\lim_{t\to\infty}\left(\int_{|x|\leqslant\epsilon} \mathbb{P}_t\rho(x)dx - \int_{|x|\leqslant\epsilon}\rho(x)dx\right) \\
&= \int_{|x|\leqslant\epsilon}\rho(x)dx - \lim_{t\to\infty}\int_{|x|\leqslant\epsilon}\mathbb{P}_t\rho(x)dx.
\end{aligned} \tag{B.5}$$

In the fifth line of (B.5), we use the Lemma B.2. In the last line of (B.5), we refer to Theorem 7 in [7] for dominated convergence theory for nets. Next, we consider the following

$$\begin{aligned}
\infty &> \int \overline{\rho_0}(x)\ 1_{|x|\leqslant\epsilon}dx \\
&= \int_{|x|\leqslant\epsilon}\overline{\rho_0}(x)dx \\
&= \int_{|x|\leqslant\epsilon}\int_0^\infty \mathbb{P}_\tau(\nabla\cdot(F\rho))(x)d\tau dx \\
&= \int_0^\infty\int_{|x|\leqslant\epsilon} \mathbb{P}_\tau(\nabla\cdot(F\rho))(x)dxd\tau \\
&= \int_0^\infty\int \mathbb{P}_\tau(\nabla\cdot(F\rho))(x)\ 1_{|x|\leqslant\epsilon}dxd\tau \\
&= \int_0^\infty\int \nabla\cdot(F\rho)(x)\ \mathbb{U}_\tau(1_{|x|\leqslant\epsilon})dxd\tau \\
&= \int_0^\infty\int \rho_0\ \mathbb{U}_\tau(1_{|x|\leqslant\epsilon})dxd\tau \\
&= \int\rho_0\left(\int_0^\infty \mathbb{U}_\tau(1_{|x|\leqslant\epsilon})d\tau\right)dx.
\end{aligned} \tag{B.6}$$

The assumption, $\rho_0 = \nabla\cdot(F\rho)$ is positive, implies that the term $\int_0^\infty \mathbb{U}_\tau(1_{|x|\leqslant\epsilon})d\tau$ is finite. Hence, the system (3.1) is almost globally divergent to infinity from Lemma B.3. □

Lemma B.5 (Equivalence Lemma)**.** *There exists a zero Lebesgue measure set N such that almost all solutions of the system* (3.1) *converge to the invariant set A for all x in N^C if and only if for $\forall\epsilon > 0$, there are subsets N_ϵ such that $T^n(x) \to A_\epsilon$ as $n\to\infty$ for all x in N_ϵ^C, where A_ϵ is the neighborhood of A defined by*

$$A_\epsilon = \left\{x\in\mathbb{R}^n : \inf_{y\in A} d(x,y) < \epsilon\right\} \tag{B.7}$$

for the standard metric d on $\mathbb{R}^n$, whereas N is the union of subsets N_ϵ such that $m(N) = 0$.

Proof. We consider an increasing sequence $\{\epsilon_k\}$ of positive numbers such that $O_{\epsilon_i}\subset O_{\epsilon_j}$ whenever $i < j$. We

 doi:10.20944/preprints202208.0422.v1

consider

$$
\begin{aligned}
&F^n(x) \to O^c_{\epsilon_1}, && \text{for } \forall x \in N^c_{\epsilon_1} \text{ such that } \mu(N_{\epsilon_1}) = 0,\\
&F^n(x) \to O^c_{\epsilon_1} \cap O^c_{\epsilon_2}, && \text{for } \forall x \in N^c_{\epsilon_2} \text{ such that } \mu(N_{\epsilon_2}) = 0,\\
&\quad\vdots\\
&F^n(x) \to \bigcap_{k=1}^{K} O^c_{\epsilon_k}, && \text{for } \forall x \in N^c_{\epsilon_K} \text{ such that } \mu(N_{\epsilon_K}) = 0
\end{aligned}
$$

as $n \to \infty$. We observe that the trajectory $F^n(x)$ leaves each neighborhood O_{ϵ_k} as $n \to \infty$. Then there exists a positive number $\epsilon > 0$ such that $\epsilon < \epsilon_K$ and $F^n(x) \to O^c_\epsilon = \bigcap_{k=1}^{\infty} O^c_{\epsilon_k}$ for $\forall x \in N^c = \bigcap_k N^C_{\epsilon_k}$. Hence, the trajectory $F^n(x)$ diverges to infinity, as $n \to \infty$. □

C Example

In the subsection 4.2.1, then the system (4.16) is converted to

$$
\begin{aligned}
\dot{y}_1 = -0.5(1+y_1^2)\Bigg(\Bigg(&\frac{\alpha_1\cos\beta_1(2y_2)(y_1^2-1)(1+y_1^2)(1+y_2^2)}{(1+y_1^2)^2(1+y_2^2)^2} - \frac{\alpha_1\cos\beta_1(y_2^2-1)(2y_1)(1+y_1^2)(1+y_2^2)}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_1\sin\beta_1(y_2^2-1)(y_1^2-1)(1+y_1^2)(1+y_2^2)}{(1+y_1^2)^2(1+y_2^2)^2} + \frac{\alpha_1\sin\beta_1(2y_2)(2y_1)(1+y_1^2)(1+y_2^2)}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\cos\beta_2(2)(2y_2)(y_2^2-1)[2(y_1^2-1)^2-(1+y_1^2)^2]}{(1+y_1^2)^2(1+y_2^2)^2}\\
&-\frac{\alpha_2\cos\beta_2[2(y_2^2-1)^2-(1+y_2^2)^2]2(2y_1)(y_1^2-1)}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\sin\beta_2[2(y_2^2-1)^2-(1+y_2^2)^2][2(y_1^2-1)^2-(1+y_1^2)^2]}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\sin\beta_2(2)(2y_2)(y_2^2-1)2(2y_1)(y_1^2-1)}{(1+y_1^2)^2(1+y_2^2)^2}\Bigg) + \Bigg(-\frac{\alpha_1\cos\beta_1(2y_1)(1+y_1^2)(1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_1\sin\beta_1(y_1^2-1)(1+y_1^2)(1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2} - \frac{\alpha_2\cos\beta_2(4y_1)(y_1^2-1)(1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\sin\beta_2[2(y_1^2-1)^2-(1+y_1^2)^2](1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\Bigg) - \Bigg(\frac{\alpha_1\cos\beta_1(2y_1)(1+y_1^2)(1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_1\sin\beta_1(y_1^2-1)(1+y_1^2)(1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2} + \frac{\alpha_2\cos\beta_2(4y_1)(y_1^2-1)(1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\sin\beta_2[2(y_1^2-1)^2-(1+y_1^2)^2](1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\Bigg) - \Bigg(\frac{\alpha_1\cos\beta_1(2y_2)(1+y_2^2)(1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_1\sin\beta_1(y_2^2-1)(1+y_2^2)(1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2} + \frac{\alpha_2\cos\beta_2(4y_2)(y_2^2-1)(1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\sin\beta_2[2(y_2^2-1)^2-(1+y_2^2)^2](1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\Bigg)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\dot{y}_2 = -0.5(1+y_2^2)\Bigg(\Bigg(&\frac{\alpha_1\cos\beta_1(2y_1)(y_2^2-1)(1+y_1^2)(1+y_2^2)}{(1+y_1^2)^2(1+y_2^2)^2} - \frac{\alpha_1\cos\beta_1(y_1^2-1)(2y_2)(1+y_1^2)(1+y_2^2)}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_1\sin\beta_1(y_2^2-1)(y_1^2-1)(1+y_1^2)(1+y_2^2)}{(1+y_1^2)^2(1+y_2^2)^2} + \frac{\alpha_1\sin\beta_1(2y_2)(2y_1)(1+y_1^2)(1+y_2^2)}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\cos\beta_2(2)(2y_1)(y_1^2-1)[2(y_2^2-1)^2-(1+y_2^2)^2]}{(1+y_1^2)^2(1+y_2^2)^2}\\
&-\frac{\alpha_2\cos\beta_2[2(y_1^2-1)^2-(1+y_1^2)^2]2(2y_2)(y_2^2-1)}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\sin\beta_2[2(y_1^2-1)^2-(1+y_1^2)^2][2(y_2^2-1)^2-(1+y_2^2)^2]}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\sin\beta_2(2)(2y_1)(y_1^2-1)2(2y_2)(y_2^2-1)}{(1+y_1^2)^2(1+y_2^2)^2}\Bigg) + \Bigg(-\frac{\alpha_1\cos\beta_1(2y_2)(1+y_2^2)(1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_1\sin\beta_1(y_2^2-1)(1+y_2^2)(1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2} - \frac{\alpha_2\cos\beta_2(4y_2)(y_2^2-1)(1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\sin\beta_2[2(y_2^2-1)^2-(1+y_2^2)^2](1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\Bigg)\\
&-\Bigg(\frac{\alpha_1\cos\beta_1(2y_1)(1+y_1^2)(1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2} + \frac{\alpha_1\sin\beta_1(y_1^2-1)(1+y_1^2)(1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\cos\beta_2(4y_1)(y_1^2-1)(1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2} + \frac{\alpha_2\sin\beta_2[2(y_1^2-1)^2-(1+y_1^2)^2](1+y_2^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\Bigg)\\
&-\Bigg(\frac{\alpha_1\cos\beta_1(2y_2)(1+y_2^2)(1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2} + \frac{\alpha_1\sin\beta_1(y_2^2-1)(1+y_2^2)(1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\\
&+\frac{\alpha_2\cos\beta_2(4y_2)(y_2^2-1)(1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2} + \frac{\alpha_2\sin\beta_2[2(y_2^2-1)^2-(1+y_2^2)^2](1+y_1^2)^2}{(1+y_1^2)^2(1+y_2^2)^2}\Bigg)\Bigg).
\end{aligned}
$$

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