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Exponential Income Distribution and Evolution of Unemployment Compensation in the United Kingdom

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Abstract

We show that an exponential income distribution will emerge spontaneously in a peer-to-peer economic network that shares the publicly available technology. Based on this finding, we identify the exponential income distribution as the benchmark structure of the well-functioning market economy. However, a real market economy may deviate from the well-functioning market economy. We show that the deviation is partly reflected as the invalidity of exponential distribution in describing the super-low income class that involves unemployment. In this regard, we find, theoretically, that the lower-bound μ of exponential income distribution has a linear relationship with (per capita) unemployment compensation. In this paper, we test this relationship for the United Kingdom from 2001 to 2015. Our empirical investigation confirms that the income structure of low and middle classes (about 90% of populations) in the United Kingdom exactly obeys exponential distribution, in which the lower-bound μ is exactly in line with the evolution of unemployment compensation.

Keywords: Peer-to-peer economy; Income distribution; Unemployment compensation; Technological change
JEL: D31; D51; O33

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1. Introduction

Income inequality has become a hot subject of both scholarly investigation and public discussion (Fuchs-Schündeln et al., 2010; Piketty and Saez, 2014; Autor, 2014; Ravallion, 2014; Dawid et al., 2018; Song et al., 2019; Aghion et al., 2019; Cowell et al., 2019). Over the past two decades, it has been commonly acknowledged that income distribution consists of two parts: the low- and middle-income class (larger than 90% of populations) and the top income class (less than 5% of populations); each of the two classes follows a different empirical law. For a long time, there was great interest in investigating the top income class. It has been observed widely that the top income class follows a Pareto distribution (Mandelbrot, 1960; Dragulescu and Yakovenko, 2001; Nirei and Souma, 2007; Atkinson et al., 2011; Tao et al., 2019). A large body of literature has attributed the cause of the Pareto distribution to the Matthew effect of income accumulation (Champernowne, 1953; Wold and Whittle, 1957; Dutta and Michel, 1998; Lux and Marchesi, 1999; Reed, 2001; Nirei and Souma, 2007; Benhabib et al., 2011; Malevergne et al., 2013). However, the singular focus on the top income class of households overlooks the component of earnings' inequality that is arguably most consequential for the low and middle income classes of citizens (Autor, 2014). For example, unemployed population is almost always located in the low and middle classes rather than the top class. In this paper, we focus instead on the income structure of the low and middle classes. The existing literature has shown that earnings' inequality in the low and middle classes follows an exponential distribution (Dragulescu and Yakovenko, 2001; Nirei and Souma, 2007; Newby et al., 2011; Clementi et al., 2012; Prinz, 2016; Irwin and Irwin, 2017; Rosser, 2019; Tao et al., 2019; Ma and Ruzic, 2020). Using a large sample, Tao et al. (2019) have analyzed datasets of household income from 66 countries and Hong Kong SAR, ranging from Europe to Latin America, North America, and Asia: For all the countries, the income distribution for the low and middle classes of populations uniformly follows an exponential law. Despite these empirical advances, there is scant understanding of the underlying mechanism driving the exponential income distribution. By contrast, the underlying mechanism driving the Pareto distribution is recognized commonly as the Matthew effect of income accumulation.

In recent years, many scholars, such as Jones (2015), and Nirei and Aoki (2016), have argued that it is important to analyze the income distribution in a general equilibrium model, because many variables that influence the distribution are determined endogenously. For the top income class, Aoki and Nirei (2017) constructed a tractable neoclassical growth model to generate Pareto's law of income distribution by using the dynamic general equilibrium theory. In this paper, we use a long-run Arrow-Debreu economy (ADE) in which there are many households, each of whom independently operates a firm, to simulate a peer-to-peer economic network. In particular, to eliminate opportunity inequality of market competition, we assume that this network shares the publicly available technology. Then, we show that an exponential income distribution will emerge spontaneously in such a peer-to-peer economic network. We use this finding to demonstrate that the underlying mechanism for driving the exponential income distribution is due to the equal opportunity of market competition.

Here, we outline briefly the basic idea of deriving exponential income distribution. As is well known, in the long-run competition, each firm's revenue (size) is indeterminate (Mas-Colell et al., 1995). This indicates multiple equilibria of firms' revenue allocations. In this case, we show that, if each equilibrium revenue allocation is allowed to occur with an equal opportunity, then the exponential distribution of revenue is most probable, indicating a spontaneous order (Tao, 2016). Furthermore, by introducing the property right of a firm, we demonstrate that each firm's revenue in the peer-to-peer economic network described by the long-run ADE is equivalent to a household's income. This implies that an exponential income distribution will emerge spontaneously in such a peer-to-peer economy. There is a large body of literature relating a firm's revenue to a household's income (Lucas, 1978; Rosen, 1982; Luttmer, 2007; Gabaix and Landier, 2008; Jones and Kim, 2014). By contrast, the novelty of our model is in introducing the property right of a firm into an ADE, to rigidly describe the income structure of a peer-to-peer economy. The other literature is not based on the peer-to-peer economy.

This paper has two contributions. The first is to show that the exponential income distribution emerges spontaneously in a peer-to-peer economic network that shares the publicly available technology. Because this peer-to-peer economic network is an ideal type of competitive market, we identify the exponential income distribution as the benchmark structure of the well-functioning market economy. The second contribution is to relate endogenous growth theory and unemployment theory to the ADE by using exponential income distribution. Here, we explain the second contribution simply. On the one hand, we prove theoretically that the lower-bound μ of exponential income distribution has a linear relationship with the unemployment compensation ω. This implies that, by fitting household income data to exponential income distribution, one should predict realistic unemployment compensation. In this paper, by fitting household income data in the United Kingdom to exponential income distribution, we indeed predict exactly the evolution of unemployment compensation from 2001 to 2015. This is strong evidence for the validity of the exponential income distribution. In the literature documenting search models, the unemployment compensation ω is a well-known central variable. There is a rich strand of literature dealing with this topic (Moen, 1997; Smith, 1999; Giuseppe and Postel-Vinay, 2013; Kaas and Kircher, 2015; Jerez, 2017; Wright et al., 2019). As our model provides a linear function relationship between μ and ω, this potentially bridges the search model and the ADE. On the other hand, we show theoretically that, by using the exponential income distribution, the income summation over all households leads to a neoclassical aggregate production function (or GDP), in which the technology factor emerges naturally. In our model, to guarantee that general equilibrium will occur at each time point, the emerging technology factor must be determined endogenously by the existing labor and capital stock. This bridges the endogenous growth model and the ADE. There is a huge body of literature that assumes the technology factor to be a function of either labor or capital (Arrow, 1962; Uzawa, 1965; Lucas, 1988; Romer 1994). By contrast, in our model, the relationship between the technology factor, capital, and labor arises because a standard general equilibrium mechanism has been taken into account.

The rest of the paper is organized as follows. Section 2 shows that an exponential income distribution emerges spontaneously in a peer-to-peer economic network that shares the publicly available technology. Section 3 shows that the lower-bound μ of exponential income distribution has a linear relationship with the unemployment compensation. Section 4 fits the household income data in the United Kingdom to the exponential income distribution and shows that this leads to an exact prediction for the evolution of unemployment compensation from 2001 to 2015. Section 5 shows that, to guarantee that general equilibria occur at each time point, the technology factor must be determined endogenously by the existing labor and capital stock. This indicates a path of general equilibrium growth. Section 6 introduces an index for measuring the potential deviation from general equilibrium and estimates empirically the deviation values of the United Kingdom from 2001 to 2015. Section 7 concludes. All formal proofs are collected in the Appendices.

2. Exponential income distribution in a peer-to-peer economy

In the economic literature, the relationship between exponential income distribution and equal opportunity has been implied by the generalized Pareto distribution, which is defined as (Cowell, 2000; Jenkins, 2016; Blanchet et al., 2017):

$$F_{\xi}(t \geq x) = \left(1 + \zeta\frac{x-\mu^*}{\theta^*}\right)^{-1/\zeta}, \tag{2.1}$$

where $\zeta > 0$, x denotes the income level, and $F_{\zeta}(t \geq x)$ denotes the fraction of population with the income higher than x. Here, μ^* and θ^* are two undetermined parameters.

It has been known that the generalized Pareto distribution (2.1) is a fairly general family which includes the Pareto distribution and the exponential distribution as two special cases (Singh and Maddala, 1976; Cowell, 2000). Recently, Blanchet et al (2017, 2018) proposed to use the generalized Pareto distribution (2.1) to describe the income structure of the total population that includes the low- and middle-income class and the top income class. It is easy to check that, when $\mu^* = \theta^*/\zeta$, the generalized Pareto distribution (2.1) becomes the Pareto distribution (Jenkins, 2016; Blanchet et al., 2017):

$$F_\zeta(t \geq x) = (x/\mu^*)^{-1/\zeta}, \tag{2.2}$$

where the Pareto exponent is denoted by $1/\zeta$, which measures the degree of income inequality (Jones and Kim, 2018), that is, a larger Pareto exponent is associated with lower income inequality. Intuitively, a higher income inequality hints a larger decline of equal opportunity among households. Thus, we anticipate that, as the Pareto exponent $1/\zeta \to \infty$, equation (2.1) yields a distribution being close to equal opportunity; that is,

$$F_0(t \geq x) = \lim_{\zeta \to 0}\left(1 + \zeta\frac{x-\mu^*}{\theta^*}\right)^{-1/\zeta} = e^{-\frac{x-\mu^*}{\theta^*}}, \tag{2.3}$$

which is an exponential income distribution.

Here, we show that the exponential income distribution can be generated in a peer-to-peer economy that underlines equal opportunity of market competition. A peer-to-peer economy is a decentralized model whereby two individuals interact to buy or sell goods and services directly with each other or produce goods and service, without an intermediary third-party (Einav et al., 2016; Davidson et al., 2018). In this paper, we use a long-run ADE in which there are many households, each of whom independently operates a firm, to simulate a peer-to-peer economic network. Long-run competition implies that the production function of each firm should take the constant returns form, which is the only sensible long-run production technology (Mas-Colell et al., 1995; Tao, 2016). This economic network can work in a Blockchain system (Davidson et al., 2018). In particular, to eliminate opportunity inequality of market competition, we assume that this economic network shares the publicly available technology. The property rights for the peer-to-peer economy are arranged as follows:

Definition 2.1 (Property rights for the peer-to-peer economy): *A household independently operates a firm, and the products belong to the household.*

Here, the arrangement for property rights does not eliminate the possibility that a household is employed by other households. In our setting, the employment indicates the exchange of labor[1]. The ADE is based on an abstract mathematical framework. To make it less difficult to understand our model, we introduce our idea step by step in terms of two simple examples in Appendix A. Example A.1 considers a 2-household society, which introduces what a peer-to-peer economy is. Through this example, we explain why multiple equilibrium income allocations occur in a peer-to-peer economy described by the long-run ADE. To overcome the dilemma of multiple choices, we introduce the maximum likelihood principle to seek an equilibrium income distribution that occurs with the largest probability. Example A.2 simply extends Example A.1 to an N-household society, and we show that an exponential income distribution occurs with the largest probability when[2] $N \gg 1$; that is,

$$a_i^* = \frac{1}{e^{\alpha+\beta\varepsilon_i}}, \tag{2.4}$$

$$i = 1,2, \dots, n.$$

Equation (2.4) indicates an exponential income structure in which there are a_i households, each of which obtains ε_i units of income, and i runs from 1 to n, where $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n$. In equation (2.4), α and β are two undetermined parameters.

The utility function and the production function in Examples A.1 and A.2 are assumed to be the Cobb–Douglas form so that these two examples are easily accessible. However, Tao (2016) has shown that the exponential income distribution (2.4) holds for the general case of an N-household society. In our derivation, the exponential income distribution (2.4) arises because publicly available technology and equal opportunity are allowed. The latter two are crucial features of our model. Moreover, it is important to note that equation (2.4) is the density distribution of income, where the income level takes discrete values. However, in section 4, we will show that the cumulative distribution (2.3) of income can be derived from the density distribution (2.4) when the income level takes continuous value. Before doing this, we first determine the economic meanings of α and β in equation (2.4).

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[1] The household can use a linear production function, which is a Cobb–Douglas form, to produce a unit of labor by inputting a unit of labor. In this case, the labor can be regarded as a type of product. Therefore, households can exchange their labors with each other in the peer-to-peer economy.

[2] Here, $N \gg 1$ is necessary for describing long-run competition, as described by Marshall; see Mas-Colell et al. (1995).

3. Model

Using the formula (2.4) of exponential income distribution, the aggregate production function (or GDP) can be written in the form:

$$Y = \sum_{i=1}^{n} a_i^* \varepsilon_i = \sum_{i=1}^{n} \frac{\varepsilon_i}{e^{\alpha+\beta\varepsilon_i}}. \tag{3.1}$$

The total number of households is:

$$N = \sum_{i=1}^{n} a_i^* = \sum_{i=1}^{n} \frac{1}{e^{\alpha+\beta\varepsilon_i}}. \tag{3.2}$$

Using equations (3.1) and (3.2), we can obtain:

$$dY = -\frac{\alpha}{\beta}\, dN + \frac{1}{\beta}\, d\left(N - \alpha\frac{\partial N}{\partial \alpha} - \beta\frac{\partial N}{\partial \beta}\right). \tag{3.3}$$

The derivation for equation (3.3) can be found in Appendix C.

As is well-known, the aggregate production function in the neoclassical economics can be written as:

$$Y = Y(L, K, T), \tag{3.4}$$

where, L denotes the labor, K denotes the capital, and T denotes the technology factor.

Since households are the owners of labor and capital, we rewrite equation (3.4) in the form of Hicks-neutral-like technical progress:

$$Y = Y(N(L, K), T). \tag{3.5}$$

Thus, the complete differential of equation (3.5) yields:

$$dY = \mu\, dN + \theta\, dT, \tag{3.6}$$

where $\mu = \frac{\partial Y}{\partial N}$ denotes the marginal labor–capital return[3] and $\theta = \frac{\partial Y}{\partial T}$ denotes the marginal technology return. By the law of diminishing marginal returns, the marginal labor–capital return μ should denote the lowest income level. Later, we will see that μ includes both contributions from labor wage and capital revenue [see equation

[3] The marginal labor–capital return μ denotes the increment of GDP when a new household (or firm) enters markets, while technology factor T is held constant. As households are the owners of labor and capital, μ includes both contributions from labor wage and capital revenue.

(3.15)].

Comparing equations (3.3) and (3.6), we observe that, if N and $N-\alpha\frac{\partial N}{\partial \alpha}-\beta\frac{\partial N}{\partial \beta}$ in equation (3.3) are independent, then we should have:

$$\mu=-\frac{\alpha}{\beta}, \tag{3.7}$$

$$\theta=\frac{1}{\beta}, \tag{3.8}$$

$$T=N-\alpha\frac{\partial N}{\partial \alpha}-\beta\frac{\partial N}{\partial \beta}. \tag{3.9}$$

In fact, we have the following proposition.

Proposition 3.1: *Let us order* $T=N-\alpha\frac{\partial N}{\partial \alpha}-\beta\frac{\partial N}{\partial \beta}$. N *is independent of* T *if and only if the following partial differential equation (3.10) holds and is solvable:*

$$N\frac{\partial Y(N,T)}{\partial N}+(T-N)\frac{\partial Y(N,T)}{\partial T}=Y(N,T). \tag{3.10}$$

Proof. The proof can be found in Appendix D.

Proposition 3.2: *The partial differential equation (3.10) has the general solution:*

$$\Phi\left(\frac{Y}{N},\frac{T}{N}+lnN\right)=0, \tag{3.11}$$

where $\Phi(a,b)$ *is a smooth function of* a *and* b.

Proof. The proof can be found in Appendix D.

Propositions 3.1 and 3.2 together guarantee that equations (3.7)–(3.9) hold when equation (3.10) holds. In this paper, we always assume that equation (3.10) holds. In this regard, Tao (2021) has strictly proved that, if equation (3.10) holds, then by using equation (3.9) one has:

$$T=-\ln P(\boldsymbol{s}), \tag{3.12}$$

where $P(\boldsymbol{s})$ denotes the probability of N households taking the collective equilibrum decisions $\boldsymbol{s}=(s_1,s_2,\dots,s_N)\cong(x_1^*,\dots,x_N^*;y_1^*,\dots,y_N^*)$. Here, $s_j=\left(x_j^*,y_j^*\right)$ denotes the equilibrium strategy of the household j and $j=1,\dots N$, in which x_j^* and y_j^* denote equilibrium consumption vector and equilibrium production vector of the household j, respectively, as described in Examples A.1 and A.2 in Appendix A. Based on Shannon's explanation for amount of information (Shannon, 1948), equation (3.12) indicates that the technology factor T can be interpreted as society's information stock[4] (Tao, 2021). From this sense, the technology factor T has a broader significance, e.g., interpreting Hayek's theory of knowledge (Tao, 2021). Section 5 will show further that the aggregate production function in the peer-to-peer economy is determined endogenously by equation (3.10), and it includes the Cobb–Douglas form as a special case. By equations (3.7) and (3.8), the economic meanings of the parameters α and β are determined by μ and θ.

Substituting equations (3.7) and (3.8) into equation (2.4) we obtain:

$$a_i^* = \frac{1}{e^{\frac{\varepsilon_i - \mu}{\theta}}}, \tag{3.13}$$

$$\varepsilon_i \geq \mu,$$

$$i = 1,2, \dots, n.$$

The constraint $\varepsilon_i \geq \mu$ is considered as the Rational Agent Hypothesis (Tao et al., 2019) in neoclassical economics, which states that firms (or households) enter the market if and only if they can gain the marginal labor–capital return at least to pay for the cost; otherwise they will make a loss. Here, we show further that μ can be written as a linear function of (per capita) unemployment compensation ω and interest rate r.

The complete differential of equation (3.5) can be written in the form:

$$dY = \omega \cdot dL + r \cdot dK + \theta \cdot dT. \tag{3.14}$$

where, $\omega = \frac{\partial Y}{\partial L}$ and $r = \frac{\partial Y}{\partial K}$ denote marginal labor return and marginal capital return, respectively. By the law of diminishing marginal returns, the marginal labor return ω indicates the lowest wage for a worker that, like (per capita) unemployment compensation[5], is a critical wage level at which laborers would like to either enter or exit markets. Therefore, we might as well denote ω by the (per capita) unemployment

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[4] According to Shannon's information theory (Shannon, 1948), if the probability of an event A occurring is denoted by $P(A)$, then the information content contained in the event A is equal to $-\ln P(A)$. Therefore, by equation (3.12), the technology factor T denotes the information content contained in the event of N agents taking the collective decisions $\boldsymbol{s}$.

[5] Unemployment compensation is paid by the state to unemployed workers who have lost their jobs due to layoffs or retrenchment. It is meant to provide a source of income for jobless workers until they can find employment.

compensation. Furthermore, as we assume that the capital markets are perfectly competitive, the marginal capital return r denotes the interest rate.

As Proposition 3.1 holds, by comparing equations (3.6) and (3.14) we have:

$$\mu \cdot dN = \omega \cdot dL + r \cdot dK, \tag{3.15}$$

which can be rewritten as

$$\mu = \sigma \cdot \omega - \sigma \cdot r \cdot MRTS_{LK}, \tag{3.16}$$

where, $\sigma = \frac{dL}{dN}$ denotes the marginal employment level[6] and $MRTS_{LK} = -\frac{dK}{dL}$ denotes the marginal rate of technical substitution of labor and capital. Equation (3.16) shows that μ is a linear function of (per capita) unemployment compensation ω and interest rate r. It implies that μ indeed includes both contributions from labor wage (ω) and capital revenue (r).

Consequently, by using equation (3.16), equation (3.13) can be summarized as:

$$\begin{cases} a_i^* = \frac{1}{e^{\frac{\varepsilon_i - \mu}{\theta}}} \\ \varepsilon_i \geq \mu \\ \mu = \sigma \cdot \omega - \sigma \cdot r \cdot MRTS_{LK} \end{cases}, \tag{3.17}$$

where $i = 1,2, \dots, n$.

Equation (3.17) is the basic form of the exponential income distribution in the peer-to-peer economy that shares the publicly available technology. It is the central model of this paper. Because this peer-to-peer economy is an ideal type of competitive market, we identify the exponential income distribution as the benchmark structure of the well-functioning market economy. Next, we investigate if the exponential income distribution (3.17) really describes the income structure of a typical market-economy country, i.e., the United Kingdom. Before doing this, we need to clarify that, although equation (3.17) is derived within the ADE, the values of μ and θ are unknown. In section 5, we will prove that general equilibrium occurs only when $\mu = 0$. As a real economy always deviates somewhat from general equilibrium, we should expect that the observed value of μ is non-zero. By equation (3.17), we realize that exponential income distribution is invalid in describing the income interval $[0, \mu]$, in which

[6] The marginal employment level stands for the increment of labor once a firm enters markets.

unemployed population is located so long as $\mu > 0$. From this sense, unemployment is a disequilibrium phenomenon in our model. In Figure 1, we use the household income data of the United Kingdom in 2010 to show this invalidity, where, due to the disequilibrium of economy ($\mu = 7204 > 0$), the income structure conforms to a right-skewed distribution as described by a Log-Normal function or a Gamma function. Let us denote the population located in $[0, \mu]$ by the super-low income class. If we remove the super-low income class, by observing Figure 1, the remaining income structure conforms to an exponential distribution. In next section, we confirm this observation by using empirical investigation.

[**Insert Figure 1 here**]

4. Empirical investigation and falsifiability

Equation (3.13) can be written in the continuous form:

$$f(x) = \frac{1}{\theta} e^{\frac{-(x-\mu)}{\theta}}, \tag{4.1}$$

where x denotes the income level, and $f(x)$ denotes the distribution density. The derivation for equation (4.1) can be found in Tao et al. (2019).

To conduct empirical analysis for exponential income distribution (3.17), we consider the cumulative distribution of equation (4.1); that is,

$$\mathcal{F}(t \geq x) = e^{\frac{-(x-\mu)}{\theta}}, \tag{4.2}$$

$$x \geq \mu, \tag{4.3}$$

$$\mu = \sigma \cdot \omega + c_0 \tag{4.4}$$

with[7] $\sigma \geq 0$ and $c_0 = -\sigma \cdot r \cdot MRTS_{LK} \leq 0$, where, $\mathcal{F}(t \geq x)$ is the cumulative probability distribution, i.e., the fraction of the population with an income higher than x. Thus, the exponential income distribution (3.17) is equivalent to equations (4.2)–

[7] By the definition for marginal employment level, we should have $\sigma \geq 0$. In this paper, we assume that the labor L and the capital K substitute with each other, so we have $MRTS_{LK} \geq 0$. Thus, we conclude $c_0 = -\sigma \cdot r \cdot MRTS_{LK} \leq 0$.

(4.4). Comparing equations (2.3) and (4.2), we have $\mu = \mu^*$ and $\theta = \theta^*$; therefore, we have confirmed that the cumulative distribution (2.3) of income can be derived from the density distribution (2.4) when the income level takes continuous value.

Next, we investigate carefully if the exponential income distribution (3.17) really describes the income structure of a market-economy country. The empirical investigation is divided into two steps. First, we test if the exponential law (4.2) is in accordance with the existing household income data. Second, we further test if the predicted value of the parameter μ in equation (4.2) agrees with actual data. We think that the two-step investigation will rigidly test the validity of exponential income distribution (3.17).

First, we investigate if the exponential law (4.2) is in accordance with the existing household income data. To do this, we rewrite equation (4.2) as:

$$y_i = \ln\mathcal{F}(t \geq x_i) = -\frac{1}{\theta}x_i + \frac{\mu}{\theta} \tag{4.5}$$

for $i = 1,2,\ldots,n$, where n denotes the size of sample.

As the peer-to-peer economy describes a competitive economy with equal opportunity, the exponential law (4.5) should be invalid in describing top income households who benefit from the Matthew effect of income accumulation. Therefore, when we fit equation (4.5) to the empirical data, the top income part should be removed. Furthermore, due to the constraint $x \geq \mu$, we should also remove the super-low income part that is less than μ. When the top and super-low income parts are both removed, we denote the remaining income part by the low and middle income classes. We expect that the exponential income distribution (3.17) describes the low and middle income classes.

In this paper, we conduct a rigid investigation for the United Kingdom, since it has a large size sample that involves 99 quantiles (see Data resource in Fig. 2), that is, (x_i, y_i), where $i = 1,2,\ldots,99$. The available data are from 2001 to 2015 (except 2008[8]). To eliminate the government's intervention in markets[9], we use the data of income

[8] The data was released by the GOV.UK, see Data resource in Fig. 1. Unfortunately, the GOV.UK did not announce the data in 2008. This is why we did not check the United Kingdom in 2008.

[9] Because the ADE is used to describe a well-functioning market economy, the government's

before tax to fit the exponential income distribution. According to the existing literature (Tao et al., 2019), the top income class is less than 5% of the population, so, for each year, we remove only three quantiles: (x_{97}, y_{97}), (x_{98}, y_{98}), and (x_{99}, y_{99}). In Fig. 2, the exponential income distribution fitting to household income data in the United Kingdom is plotted, where the fitting results are quite good, except for the top three quantiles. The fitting results are listed in Table 1, where all the adjusted R^2 values approach 0.99. These good results support the validity of exponential law (4.2) in describing the low and middle income classes of the United Kingdom.

[**Insert Figure 2 here**]

Second, we investigate if the predicted value of the parameter μ in equation (4.2) agrees with actual data. By fitting equation (4.5) to household income data, one can obtain the predicted value of μ. Equation (4.4) predicts a linear relation between μ and ω. If the exponential law (4.2) indeed describes actual societies, by equation (4.4) we should expect that the predicted value of μ is in line with the evolution of (per capita) unemployment compensation ω announced by the United Kingdom Government. This is the most crucial testing in the falsifiability of the exponential income distribution (3.16), which distinguishes our model from other distribution models, such as Log-Normal distribution and Gamma distribution.

As equation (4.2) is suitable only for the low and middle income classes, we have to remove both top income data and super-low income data that are less than μ. The remaining data fitting to equation (4.5) is $\{(x_g, y_g), (x_{g+1}, y_{g+1}), \dots, (x_{96}, y_{96})\}$, where $x_g \geq \mu$. This leads to difficulty in empirical analysis to find the consistent estimate of μ, as g is unknown. Regarding this, Tao et al. (2019) have proved a uniform convergence theorem, as below:

Theorem 4.1 (Uniform Convergence Theorem of μ): *For a strictly monotonic*

intervention in markets should be eliminated when possible.

increasing sequence $\{x_j\}_{j=1}^{n}$, *if there exists an integer* $g = g(n)$ *to guarantee:*

(A). $x_{i-1} < \mu < x_i$ *or* $x_i = \mu$, *where* $i = g < n$ *and* $\lim_{n\to\infty} \frac{g}{n} = 0$;

(B). $\frac{\bar{y}_g}{\hat{\beta}_g} > \delta > 0$ *for any* n;

then one has:

$$\lim_{n\to\infty} \hat{\mu}_g = \lim_{n\to\infty} \left(\bar{x}_g - \frac{\bar{y}_g}{\hat{\beta}_g}\right) = \mu,$$

where g *is uniquely determined by* n, *and* $g < \infty$. *This means:*

$$\lim_{n\to\infty} g = g^*.$$

Proof. See Tao et al. (2019). □

Theorem 4.1 guarantees that one can carry out fitting to obtain the consistent estimate value of μ only by removing the top income class and as long as the size of samples is large enough (Tao et al., 2019). The income data of the United Kingdom contain 99 quantiles, which is sufficiently large to satisfy the requirements of Theorem 4.1. Table 1 indicates that removing the top three quantiles has guaranteed a quite high adjusted R^2. In section 3, we have proposed that the households with income being less than μ can be considered as the unemployed populations. This means that, by estimating the value of μ, one may estimate the unemployment rate. As shown in Figure 1, the super-low income class approximately occupies 8% of total populations, which is in accordance with the real unemployment rate 7.9%

[Insert Table 1 here]

By fitting the household income data in the United Kingdom to equation (4.5), we have computed the estimate value of μ (refer to Table 1), where we remove only three quantiles in the top income samples. Here, we also collected the time series data of (per capita) unemployment compensation in the United Kingdom from 2001 to 2015 (except 2008[10]); refer to Data resource in Table 1, where the unemployment compensation data

[10] Because the household income data of United Kingdom in 2008 was missing, we correspondingly did not consider the unemployment compensation data in 2008.

was released since 2001. Using ordinary least squares regression, fitting the data of $\hat{\mu}(t)$ and $\omega(t)$ to equation (4.4) yields the following result:

$$\hat{\mu}(t) = 3.6166 \cdot \omega(t) - 4697 \qquad (4.6)$$

$$\begin{array}{rlll} \mathbf{se} = & (0.204) & (669) & \boldsymbol{R^2} = 0.96 \\ \mathbf{t} = & (17.677) & (-7) & \\ \mathbf{P\ value} = & (0.000) & (0.000) & \end{array}$$

Fig. 3 plots the (red) fitting line for the (blue) time series data-point $(\omega(t), \hat{\mu}(t))$, with t representing the year. As shown in equation (4.6), the fitting result is perfect, where the adjusted R^2 yields 0.96. More importantly, the present fitting result shows $\sigma = 3.6166 > 0$ with a p-value $< 10^{-9}$ and $c_0 = -4697 < 0$ with a p-value $< 10^{-4}$, which are perfectly consistent with the theoretical predictions in equation (4.4). Because the Gini coefficient of the exponential income distribution (4.2) is equal to $G = 1/[2(1+\mu/\theta)]$ (Tao et al., 2019), the empirical finding (4.6) implies that unemployment compensation has a direct relevance to policy measures intended to alleviate income inequality in the low and middle class.

[**Insert Figure 3 here**]

As equations (4.2) and (4.4) are both in high accordance with the real data, we conclude that exponential income distribution (3.17) indeed describes the income structure of the low and middle classes in the United Kingdom.

5. Endogenous technological change

In section 3, we mentioned that, when the exponential income distribution (2.4) emerges, the aggregate production function is determined endogenously by the partial differential equation (3.10), which has a general solution (3.11). Now, we show further that, if the economy stays at the general equilibrium, the aggregate production function includes the Cobb–Douglas form as a special case, where the technology factor is determined endogenously by labor and capital. Equation (3.11) can be rewritten as:

$$\frac{Y}{N} = \chi\left(\frac{T}{N} + \ln N\right), \tag{5.1}$$

where $\chi(\xi)$ is a smooth function of ξ.

To find the concrete form of the function $\chi(\xi)$, we need to introduce Proposition 5.1. Before doing this, let us first introduce Lemma 5.1 and Assumption 5.1.

Lemma 5.1: *If the economy arrives at general equilibrium, then one has* $\mu = 0$.

Proof. The proof can be found in Appendix E.

By equation (3.16), we know that μ includes contributions from both labor wage and capital revenue. $\mu = 0$ implies that the two contributions cancel each other out, so there is no profit on marginal return. This is in accordance with the existing conclusion based on general equilibrium analysis.

Assumption 5.1: *If the economy arrives at general equilibrium, then one has* $\frac{\partial \ln Y}{\partial T} \geq 0$.

Assumption 5.1 has a clear economic meaning: The growth rate from technological progress is always non-negative. Indeed, in the literature of economic growth, the technology factor T is always considered as a central power for driving economic growth. However, we infer that the disequilibrium of economy may lead to $\frac{\partial \ln Y}{\partial T} < 0$.

Proposition 5.1: *If the economy stays at general equilibrium, and if Assumption 5.1 holds, then we have:*

$$\chi(\xi) = e^{\gamma\xi}, \tag{5.2}$$

where $\gamma \geq 0$.

Proof. The proof can be found in Appendix E.

To guarantee that the economy stays at general equilibrium, by equations (5.1) and (5.2), the aggregate production function Y should be written as:

$$Y = N^{1+\gamma} e^{\gamma\frac{T}{N}}. \tag{5.3}$$

Now, we explore the relationship between N and T when the economy stays at general equilibrium. By equation (5.3), it is easy to calculate:

$$\mu = \frac{\partial Y}{\partial N} = \left(\frac{1+\gamma}{\gamma} - \frac{T}{N}\right)\gamma N^{\gamma} e^{\gamma \frac{T}{N}}. \tag{5.4}$$

By Lemma 5.1, general equilibrium implies $\mu = 0$; thus, by equation (5.4), we have:

$$T = \frac{1+\gamma}{\gamma} N. \tag{5.5}$$

By equation (3.5), the number of firms (or households), N, should be a function of L and K; that is, $N = N(L, K)$. Therefore, equation (5.5) indicates that, when the economy stays at general equilibrium, the technology factor T is determined endogenously by the labor L and capital K. In other words, to guarantee general equilibrium, the technological change should match the existing labor and capital stock[11].

Substituting equation (5.5) into equation (5.3) yields:

$$Y = e^{1+\gamma} N(L, K)^{1+\gamma}. \tag{5.6}$$

To determine the function form of $N(L, K)$, we have the following proposition.

Proposition 5.2: *If the economy stays at general equilibrium, and, if Assumption 5.1 holds, then we have*:

$$N(\lambda L, \lambda K) = \lambda^{\frac{1}{1+\gamma}} N(L, K). \tag{5.7}$$

Proof. The proof can be found in Appendix E.

Here, we simply take:

$$N = c L^a K^b, \tag{5.8}$$

where $a \geq 0$, $b \geq 0$, and $c > 0$. By Proposition 5.2, equation (5.8) implies

$$1 + \gamma = \frac{1}{a+b}. \tag{5.9}$$

By equations (5.8) and (5.9), equation (5.6) can be rewritten as:

[11] In section 3, we have interpreted the technology factor T as society's information stock. In this sense, labor and capital play a role of a "container" that stores the information or knowledge.

$$Y = (ce)^{\frac{1}{a+b}} L^{\frac{a}{a+b}} K^{\frac{b}{a+b}}. \tag{5.10}$$

Furthermore, substituting equations (5.8) and (5.9) into equation (5.5), we get:

$$T = \frac{c}{1-a-b} L^a K^b. \tag{5.11}$$

As equations (5.10) and (5.11) are a result of long-run general equilibrium, we introduce the time variable t into both of them. Thus, equations (5.10) and (5.11) can be written as evolutionary equations:

$$\begin{cases} Y(t) = (ce)^{\frac{1}{a+b}} L(t)^{\frac{a}{a+b}} K(t)^{\frac{b}{a+b}} \\ T(t) = \frac{c}{1-a-b} L(t)^a K(t)^b \end{cases}. \tag{5.12}$$

Equation (5.12) implies a Solow-like growth, while the only difference is that, for any given time t, the technology factor $T(t)$ must be based endogenously on the labor $L(t)$ and capital stock $K(t)$ at the time point t; otherwise, general equilibrium breaks down. We call the economic growth based on equation (5.12) the "equilibrium growth path", which guarantees that general equilibrium occurs at each time point. Furthermore, equation (5.12) presents a potential perspective for understanding endogenous technological change. Since the technology factor is determined endogenously by the labor and capital to guarantee general equilibria, the technological change in industries should match the existing labor and capital stock. This implies that an efficient growth path for the market-economy countries that pursue the catch-up strategy should be a gradual matching process, in which industrial developments cannot be divorced from the existing labor and capital stock (Lin, 2011; Ju et al., 2015).

6. Measuring the deviation from general equilibrium

Equation (5.12) describes the dynamic general equilibrium in an ADE. Such a dynamic equilibrium does not occur easily in the real world. Therefore, it is meaningful to seek a way of measuring the potential deviation from dynamic equilibrium. In this section, we attempt to construct an index to measure the potential deviation from general equilibrium that can be applied in a real economy.

By Lemma 5.1, we observe that $\mu = 0$ indicates general equilibrium. The proof for Lemma 5.1 is based on the equation (3.16) (see Appendix E), which yields:

$$\mu = \sigma \cdot r \left(\frac{\omega}{r} - MRTS_{LK}\right). \tag{6.1}$$

By microeconomics (Mas-Colell et al., 1995), we know that general equilibrium leads to $\frac{\omega}{r} = MRTS_{LK}$; that is $\mu = 0$. Based on this fact, we can construct an index for measuring the deviation from general equilibrium, as below:

$$I = 1 - \frac{r \cdot MRTS_{LK}}{\omega}. \tag{6.2}$$

Obviously, if general equilibrium occurs, by $\frac{\omega}{r} = MRTS_{LK}$, we have $I = 0$. In fact, we have the following proposition.

Proposition 6.1: *If* $\mu \geq 0$ *and* $MRTS_{LK} \geq 0$, *then one has* $0 \leq I \leq 1$, *where* $I = 0$ *indicates general equilibrium.*

Proof. The proof can be found in Appendix F.

Here, we explain why $\mu \geq 0$ and $MRTS_{LK} \geq 0$ should hold in a well-functioning market-economy country. $MRTS_{LK} \geq 0$ implies that labor L and capital K substitute for each other. This is a natural assumption in neoclassical economics. $\mu \geq 0$ is a corollary of the Rational Agent Hypothesis $\varepsilon_i \geq \mu$ for $i = 1,2,\ldots,n$ (see section 3). To see this, we observe that $\mu < 0$ implies a negative income level. Obviously, a rational agent will not accept negative income if there exists unemployment compensation. In particular, by empirically investigating household income data from 66 countries and Hong Kong SAR, Tao et al. (2019) have shown that $\mu > 0$ does hold.

To account for the dynamic general equilibrium, equation (6.2) can be written in a dynamic form:

$$I(t) = 1 - \frac{r(t) \cdot MRTS_{LK}(t)}{\omega(t)}. \tag{6.3}$$

To estimate $r(t) \cdot MRTS_{LK}(t)$, we can consider the dynamic form of equation (3.16):

$$\mu(t) = \sigma(t) \cdot \omega(t) - \sigma(t) \cdot r(t) \cdot MRTS_{LK}(t). \tag{6.4}$$

Generally, it is hard to collect the data $r(t)$ because it contains the contributions from all potential capital forms. Therefore, here, we assume approximately that $\sigma(t) \cdot r(t) \cdot MRTS_{LK}(t)$ does not vary with respect to the time t; that is,

$$\sigma(t) \cdot r(t) \cdot MRTS_{LK}(t) \approx \sigma \cdot r \cdot MRTS_{LK}. \tag{6.5}$$

Thus, equation (6.4) can be rewritten as:

$$\mu(t) = \sigma \cdot \omega(t) - \sigma \cdot r \cdot MRTS_{LK}, \tag{6.6}$$

which is exactly the equation (4.6) that has indicated that the approximate assumption (6.5) is valid for 2001–2015.

Based on equation (6.5), equation (6.3) can be replaced by

$$I(t) = 1 - \frac{r \cdot MRTS_{LK}}{\omega(t)}. \tag{6.7}$$

If one can collect the time series data of $\mu(t)$ and $\omega(t)$, by equations (6.6) and (6.7), one can estimate $I(t)$ for each time point t. In section 4, we have collected the time series data of $\omega(t)$ and $\mu(t)$ in the United Kingdom from 2001 to 2015 (except 2008); see Table 1. Therefore, we can measure the deviation from general equilibrium for each year in terms of the United Kingdom. We have listed the estimated value of $I(t)$ from 2001 to 2015 (except 2008) in Table 1. To show the evolution of $I(t)$, we also plot the points $\left(t, I(t)\right)$ in Fig. 4, where there is a significant deviation from general equilibrium around 2008; that is, $I(t)$ jumps from 0.5 interval in $t = 2007$ to 0.6 interval in $t = 2009$. This aggravates deterioration of market efficiency. By equation (6.7) we propose that reducing unemployment compensation ω and increasing interest rate r are two alternative potential measures to ease the deterioration.

[Insert Figure 3 here]

7. Conclusion

By introducing the property right of a firm, we show that a long-run ADE can be used to simulate a peer-to-peer economic network that shares the publicly available technology in which there are many households, each of whom independently operates a firm. We have shown that an exponential income distribution will emerge spontaneously in such an economic network. Based on this finding, we identify the exponential income distribution as the benchmark structure of the well-functioning market economy. However, a real market economy may deviate from the well-functioning market economy. We show that the deviation is partly reflected as the invalidity of exponential distribution in describing the super-low income class that involves unemployment. In this regard, we find, theoretically, that the lower-bound μ of exponential income distribution has a linear relationship with (per capita) unemployment compensation. By fitting the household income data in the United Kingdom (from 2001 to 2015) to the exponential income distribution, we observe that the income structure of the United Kingdom consists of three parts: super-low income class (unemployed population), low and middle income classes, and top income class. The low and middle income classes occupy about 90% of populations. By contrast, top income class and super-low income class only occupy a small proportion of population (about 10%). In particular, we find a high accordance between exponential distribution and empirical data for the low and middle income classes, in which the fitting parameter μ predicts exactly the evolution of unemployment compensation in the United Kingdom from 2001 to 2015.

To explore macroeconomic behaviors of a well-functioning market-economy country, we show that, by using the exponential income distribution, the income summation over all households leads to an aggregate production function with Hicks-neutral-like technical progress, in which the technology factor emerges naturally. In particular, we show that, to guarantee general equilibrium, the emerging technology factor must be determined endogenously by the existing labor and capital stock. This is to say, at each time point of economic evolution, the technological change in industries should match the existing labor and capital stock; otherwise, general equilibrium breaks down. This finding provides a novel perspective for understanding endogenous technological change. To conduct empirical observation, we propose an index for measuring the potential deviation from general equilibrium and apply it to the United

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Kingdom. Consequently, we observe empirically a significant deviation from general equilibrium around 2008. We propose that adjusting unemployment compensation and interest rate are two possible alternative measures to reduce the deviation.

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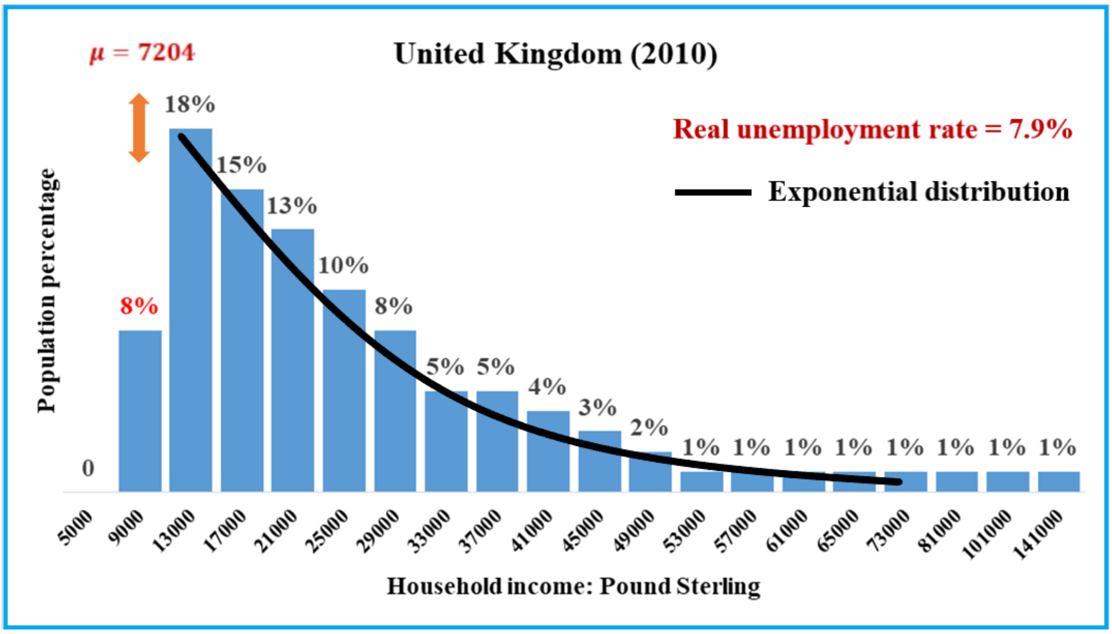
Figure 1. The density distribution of household income in the United Kingdom[12] in 2010



Note: The estimated value $\mu = 7204$ can be found in Table 1.

[12] Data resource: https://www.gov.uk/government/statistics/percentile-points-from-1-to-99-for-total-income-before-and-after-tax

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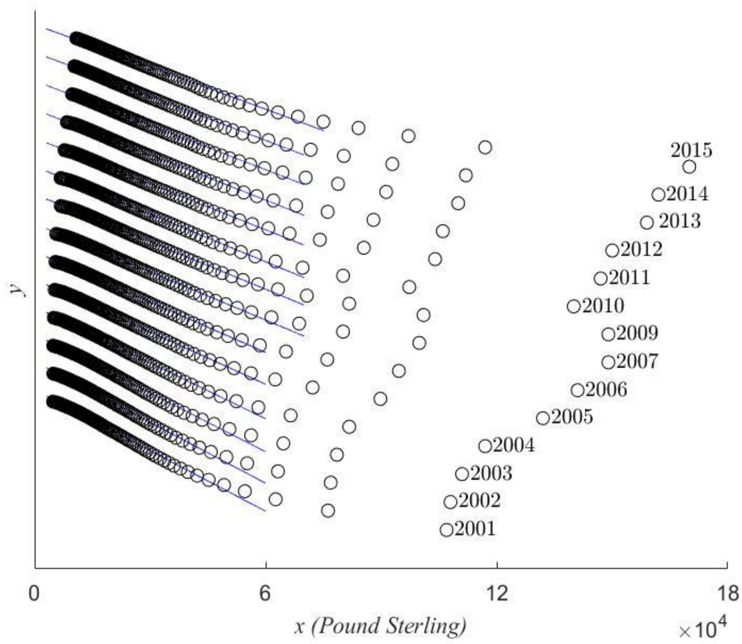
Figure 2. Exponential income distribution fitting to household income data in the United Kingdom



Note[13]: Household income data in the United Kingdom are from 2001 to 2015 (except 2008), and they are plotted as the black circles. Exponential income distribution (4.5) is plotted as the blue line for each year. In the data resource, the household income data in 2008 are missing. To eliminate the government's intervention in markets, we use the data of income before tax to fit the exponential income distribution.

[13] Data resource: https://www.gov.uk/government/statistics/percentile-points-from-1-to-99-for-total-income-before-and-after-tax

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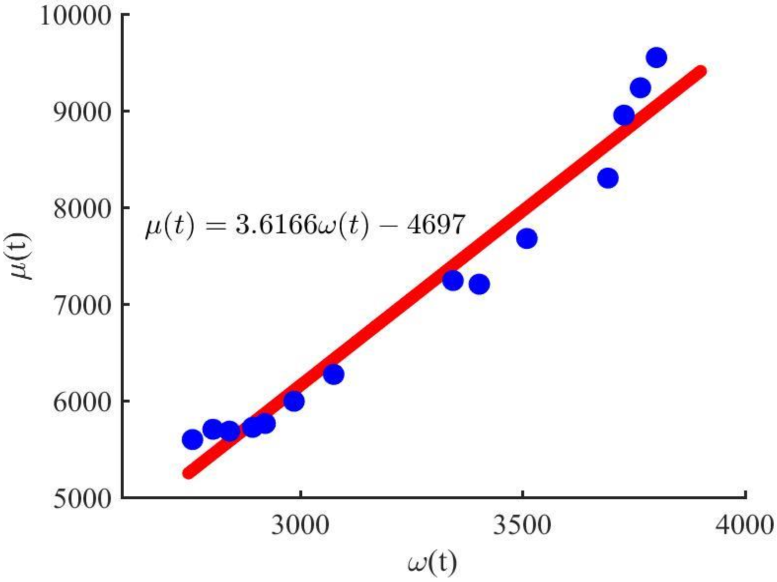
Figure 3. Statistical fit between the marginal labor–capital return $\hat{\mu}$ and the (per capita) unemployment compensation ω.



Note: The local currency unit is Pound Sterling.

Figure 4. The evolution of $I(t)$ in the United Kingdom from 2001 to 2015.

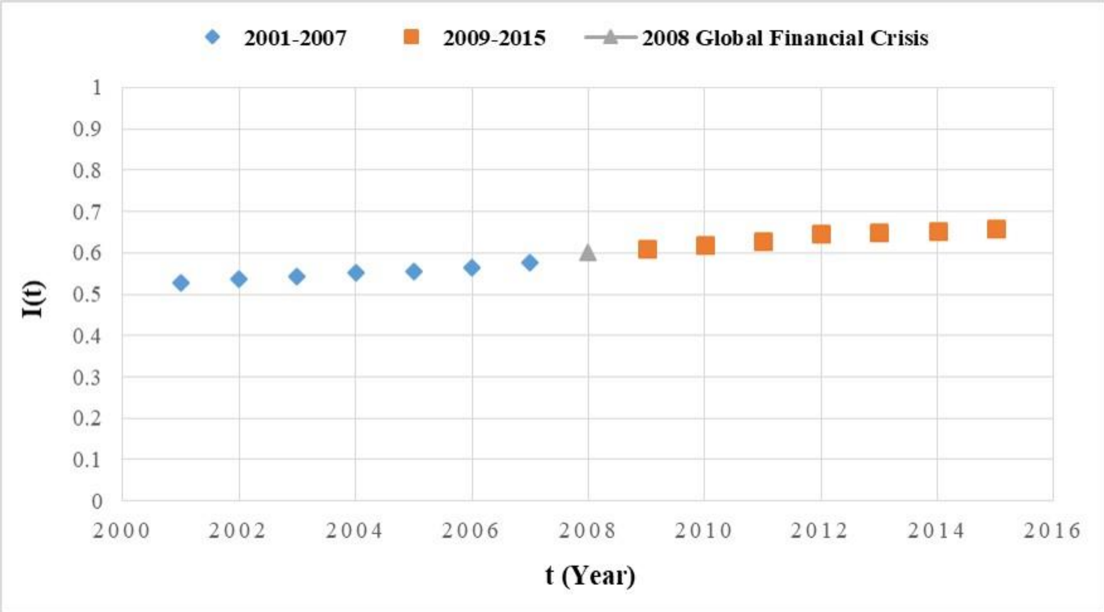


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Table 1. Empirical results for the United Kingdom

YEAR (t)	ADJUSTED R^2	$\hat{\mu}(t)$	$\omega(t)$	$I(t)$
2001	0.995	5597	2759	0.529
2002	0.995	5703	2805	0.536
2003	0.995	5684	2842	0.542
2004	0.996	5723	2894	0.551
2005	0.996	5763	2922	0.555
2006	0.995	5993	2987	0.565
2007	0.994	6271	3076	0.577
2009	0.995	7242	3344	0.611
2010	0.996	7204	3403	0.618
2011	0.993	7677	3510	0.629
2012	0.992	8302	3692	0.648
2013	0.990	8953	3728	0.651
2014	0.989	9236	3765	0.654
2015	0.986	9549	3801	0.658

Note: The local currency unit is Pound Sterling. The adjusted R^2 and $\hat{\mu}$ are obtained uniformly by removing three quantiles in the top income samples for each year. The official data of (per capita) unemployment compensation ω are found at OECD. Stat,[14] where the data before 2001 are missing.

[14] Data resource: https://stats.oecd.org/Index.aspx?DataSetCode=FIXINCLSA

Appendices

Appendix A

Example A.1: 2-household peer-to-peer economy

We simply consider an agricultural society, where there are two households, each of which independently operates a firm, so there are two firms. We assume that there are two types of goods[15]*: labor and rice. The consumption vector and initial endowment vector of household 1 are* $x_1 = (x_{11}, \quad x_{21})$ *and* $\omega_1 = (1/3, \quad 0)$*, respectively, where* x_{11} *denotes the amount of labor and* x_{21} *denotes the amount of rice. The consumption vector and initial endowment vector of household 2 are* $x_2 = (x_{12}, \quad x_{22})$ *and* $\omega_2 = (2/3, \quad 0)$*, respectively. In such a society, firms produce rice by using labor, and, due to the property right arrangement, the products of firm* i *belong to the household* i*, where* $i = 1,2$*; meanwhile, households will consume rice. For simplicity, we assume that the utility functions of households 1 and 2 are* $u_1(x_{11}, x_{21}) = x_{21}$ *and* $u_2(x_{12}, x_{22}) = x_{22}$*, respectively. Because there is not any monopolistic technology in the ADE, we assume that firms 1 and 2 employ the same Cobb–Douglas production technology* $\{(-x, \quad y)|y = x\}$*; thus, the production vectors of firms 1 and 2 are* $y_1 = (-h_1, \quad h_1)$ *and* $y_2 = (-h_2, \quad h_2)$*, respectively. We further assume that the price of labor is* $p_1 = 1$ *and the price of rice is* p_2*; thus, the price vector is* $p = (p_1 = 1, \quad p_2)$*.*

Question: Solve the general equilibrium allocation $\{p^*; x_1^*, x_2^*; y_1^*, y_2^*\}$.

Solution:

It is easy to list the following equations.

Maximizing firms' profit follows:

$$\max_{y_1} p \cdot y_1. \tag{A.1}$$

[15] The assumption of two types of goods (labor and rice) indicates a kind of schizophrenic society, which seems to be unrealistic. However, the assumption is made solely to guarantee that Example A.1 is easily accessible. In this regard, Tao (2016) has considered any types of goods.

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$$\max_{y_2} p \cdot y_2. \tag{A.2}$$

Maximizing households' utility follows:

$$\begin{cases} \max\limits_{x_{11},x_{21}} u_1(x_{11}, x_{21}) = x_{21} \\ p \cdot x_1 \le p \cdot \omega_1 + p \cdot y_1 \end{cases}. \tag{A.3}$$

$$\begin{cases} \max\limits_{x_{12},x_{22}} u_2(x_{12}, x_{22}) = x_{22} \\ p \cdot x_2 \le p \cdot \omega_2 + p \cdot y_2 \end{cases}. \tag{A.4}$$

Market clear follows:

$$x_1 + x_2 = \omega_1 + \omega_2 + y_1 + y_2. \tag{A.5}$$

Due to the arrangement for property rights, the profit $p \cdot y_1$ in equation (A.3) uniquely belongs to household 1, and likewise the profit $p \cdot y_2$ in equation (A.4) uniquely belongs to household 2. However, in a general ADE, the profit $p \cdot y_1$ (and $p \cdot y_2$) will be shared by households 1 and 2 (Arrow and Debreu, 1954). Therefore, the peer-to-peer economy is only a special case of the general ADE.

Using equations (A.1)–(A.5), it is easy to calculate the general equilibrium allocation $\{p^*; x_1^*, x_2^*; y_1^*, y_2^*\}$, as below:

$$p^* = (1, \quad 1), \tag{A.6}$$

$$x_1^* = (0, \quad 1/3), \tag{A.7}$$

$$x_2^* = (0, \quad 2/3), \tag{A.8}$$

$$y_1^* = (-\tau \quad \tau), \tag{A.9}$$

$$y_2^* = (\tau - 1, \quad 1 - \tau), \tag{A.10}$$

where τ denotes an arbitrary number satisfying $0 \le \tau \le 1$.

Equations (A.6)–(A.10) show that, although p^*, x_1^*, and x_2^* are unique, y_1^* and y_2^* will change as τ changes. This means that $\{p^*; x_1^*, x_2^*; y_1^*, y_2^*\}$ are multiple equilibria. Because p^*, x_1^*, and x_2^* are fixed, the multiplicity of equilibria is due to the uncertainty of y_1^* and y_2^*. For simplicity, we might as well denote $\{p^*; x_1^*, x_2^*; y_1^*, y_2^*\}$ by $\{y_1^*, y_2^*\}$. According to the first fundamental theorem of welfare economics, general equilibrium is always Pareto optimal. Regarding multiple equilibria, welfare economists propose that the best outcome can be selected through a "social welfare function" (Mas-Colell et al., 1995). Unfortunately, such a social welfare

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function has been refuted by Arrow's Impossibility Theorem (Arrow, 1963). For multiple equilibria, this leads to a dilemma of social choice. Next, we show that the dilemma can be eliminated if one introduces the maximum likelihood principle.

Before doing this, let us first introduce the definition of household income.

Definition A.1 (Household income): *Based on the property rights for the peer-to-peer economy, the income of a household equals the total sale of the products that its firm has produced.*

Definition A.1 indicates that the income of a household is determined by the revenue (or size) of its firm. There has been a large body of literature relating a firm's revenue to a household's income (Lucas, 1978; Rosen, 1982; Luttmer, 2007; Gabaix and Landier, 2008; Jones and Kim, 2014). Here, Definition A.1 differs from the definition for wealth. For instance, the wealth of the household i should be defined by $p^* \cdot \omega_i + p^* \cdot y_i^*$, where $i = 1{,}2$. However, in this paper, we only investigate the income distribution of households.

Using Definition A.1, we can interpret the equilibrium production allocation, $\{y_1^*,\ y_2^*\}$, as the equilibrium income allocation between households 1 and 2, $(I_1,\quad I_2)$. Using equations (A.6), (A.9), and (A.10) it is easy to ascertain that the equilibrium revenues of firm 1 and 2 are τ and $1-\tau$, respectively. Since the revenue of a firm equals the sale of products that it has produced, by Definition A.1, the equilibrium income allocation between households 1 and 2 is as below:

$$(I_1,\quad I_2) = (\tau,\quad 1-\tau). \qquad \text{(A.11)}$$

where I_i denotes the income of household i and $i = 1{,}2$. Therefore, the GDP of this society equals $Y = \tau + (1-\tau) = 1$.

Now, we introduce the maximum likelihood principle. It is a basic principle of statistics to infer unknown probability distributions under given constrained conditions (Foley, 1994; Martin et al., 2012). For example, Gauss (1857) first used this principle to determine the normal distribution (Tao et al., 2017), which was also known as the Gauss distribution. In the context of evolutionary game, this principle is referred to as "stochastically stable" (Young, 1998).

Definition A.2 (Maximum likelihood principle): Among all possible distribution functions satisfying the given constrained conditions, a distribution function is the maximum likelihood distribution if it occurs with the highest probability.

To see how the maximum likelihood principle is applied to Example A.1, we assume simply that the minimal unit of money is $1/2$. In Example A.2, we will consider the general case of a money unit. Based on the setting for the minimal unit of money, according to allocation (A.11), there are only three possible equilibrium allocations in Example A.1:

$$(0,\quad 1)\ ,\ (1/2,\quad 1/2),\ (1,\quad 0). \tag{A.12}$$

According to previous discussion, these three allocations are all Pareto optimal, so one cannot choose the best one by using the theory of social choice. To apply the maximum likelihood principle, let us first introduce the notion of income distribution. If we assume that the number of households whose income is 0 equals a_1, the number of households whose income is $1/2$ equals a_2, and the number of households whose income is 1 equals a_3, then $\{a_i\}_{i=1}^{3} = \{a_1 \quad a_2 \quad a_3\}$ is called an income distribution. In Example A.2, we will give the general definition for income distribution.

Obviously, according to the definition of income distribution above, the multiple allocations (A.12) only lead to two income distributions, as below:

$$\{a_1 = 0 \quad a_2 = 2 \quad a_3 = 0\} = \{(1/2,\quad 1/2)\}. \tag{A.13}$$

$$\{a_1 = 1 \quad a_2 = 0 \quad a_3 = 1\} = \{(0,\quad 1), (1,\quad 0)\}. \tag{A.14}$$

By using the maximum likelihood principle we choose the likeliest one from income distributions (A.13) and (A.14). To this end, we observe that the ADE is a decentralized system with procedural justice, in which there is no criterion for what constitutes a just outcome (equilibrium) other than the procedure itself. Rawls (1999) has proposed that each outcome produced by a system with procedural justice is fair and, hence, can be selected with an equal opportunity. Following Rawls' proposal, we make the axiom of equal opportunity as follows:

Axiom A.1 (Equal opportunity): *If an Arrow-Debreu economy (ADE) generates multiple general equilibria, then each equilibrium is selected with an equal opportunity by social members, i.e., each equilibrium occurs with an equal probability.*

The validity of Axiom A.1 is supported by Arrow's Impossibility Theorem. If the equal opportunity among competitive equilibria does not hold, this means that there are some competitive equilibria that are better than are others from a perspective of social choice, contradicting Arrow's Impossibility Theorem [16] (Tao, 2016). Since each allocation in multiple equilibria (A.12) is produced by ADE, by Axiom A.1, these allocations should occur with an equal probability of $1/3$; that is,

$$P\{(1/2,\quad 1/2)\} = P\{(0,\quad 1)\} = P\{(1,\quad 0)\} = 1/3. \tag{A.15}$$

Using equation (A.15), the probabilities of income distributions (A.13) and (A.14) occurring are

$$P\{a_1 = 0 \quad a_2 = 2 \quad a_3 = 0\} = 1/3. \tag{A.16}$$

$$P\{a_1 = 1 \quad a_2 = 0 \quad a_3 = 1\} = 2/3. \tag{A.17}$$

Thus, we have: $P\{a_1 = 1 \quad a_2 = 0 \quad a_3 = 1\} > P\{a_1 = 0 \quad a_2 = 2 \quad a_3 = 0\}$.

According to Definition A.2, the income distribution (A.14) is the maximum likelihood distribution (MLD) because it occurs with the highest probability. If we denote the MLD by $\{a_i^*\}_{i=1}^3$, then we have

$$\{a_i^*\}_{i=1}^3 = \{a_1^* = 1 \quad a_2^* = 0 \quad a_3^* = 1\}. \tag{A.18}$$

According to the maximum likelihood principle, the MLD can be taken as a real occurrence (i.e., spontaneous order). The main purpose of this section is to find the general function form of the MLD in an ADE. To obtain such a general function form, we have to extend a 2-household economy to an N-household economy.

Example A.1 can be easily extended to the case of an N-household, as below.

Example A.2: *N*-household peer-to-peer economy

We consider an N-household agricultural society, in which each household

[16] Here, we have considered non-dictatorship as an axiom of defining an Arrow-Debreu economy.

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independently operates a firm. We assume that there are two types of goods: labor and rice. The consumption vector and initial endowment vector of the household i *are* $x_i = (x_{1i}, \quad x_{2i})$ *and* $\omega_i = (\pi_i, \quad 0)$*, respectively, where* x_{1i} *denotes the amount of labor,* x_{2i} *denotes the amount of rice, and* $i = 1,2,\ldots,N$*. Here* $\sum_{i=1}^{N}\pi_i = 1$ *and* $\pi_i > 0$ *for* $i = 1,2,\ldots,N$*. Firms produce rice by using labor, and, due to the property right arrangement, the products of firm* i *belong to the household* i*; meanwhile, households will consume rice. For simplicity, we assume that the utility function of the household* i *is* $u_i(x_{1i}, x_{2i}) = x_{2i}$*. Like Example A.1, we assume that firm* i *employs the Cobb–Douglas production technology* $\{(-x, \quad y)|y = x\}$*; thus, the production vector of the firm* i *is* $y_i = (-h_i, \quad h_i)$*. We further assume that the price of labor is* $p_1 = 1$ *and the price of rice is* p_2*; thus, the price vector is* $p = (p_1 = 1, \quad p_2)$*.*

Like Example A.1, it is easy to compute the general equilibrium solution of an N-household society:

$$p^* = (1, \quad 1), \tag{A.19}$$

$$x_i^* = (0, \quad \pi_i), \tag{A.20}$$

$$y_i^* = (-\tau_i, \quad \tau_i), \tag{A.21}$$

for $i = 1,2,\ldots,N$

where each τ_i denotes an arbitrary number satisfying $0 \leq \tau_i \leq 1$ and $\sum_{i=1}^{N}\tau_i = 1$.

By using the same technique of deriving equation (A.11), equilibrium income allocation among N households can be written as:

$$(I_1 \quad I_2 \quad \cdots \quad I_N) = (\tau_1 \quad \tau_2 \quad \cdots \quad \tau_N). \tag{A.22}$$

where I_i denotes the income of household i. By equation (A.22), the GDP of this society equals $Y = \sum_{i=1}^{N}\tau_i = 1$.

For the case of N households, the number of equilibrium income allocation in the income distribution $\{a_i\}_{i=1}^{3}$ can be summarized as (Tao, 2016):

$$\Omega(\{a_i\}_{i=1}^{3}) = \frac{N!}{\prod_{i=1}^{3} a_i!}. \tag{A.23}$$

To check the validity of equation (A.23), we apply it to Example A.1, where $N = 2$. Using equation (A.23), we can compute:

$$\Omega(\{a_1 = 0 \quad a_2 = 2 \quad a_3 = 0\}) = \frac{2!}{0!\ 2!\ 0!} = 1. \tag{A.24}$$

$$\Omega(\{a_1 = 1 \quad a_2 = 0 \quad a_3 = 1\}) = \frac{2!}{1!\ 0!\ 1!} = 2. \tag{A.25}$$

Obviously, equations (A.24) and (A.25) agree with equations (A.13) and (A.14), respectively.

Next, we seek the MLD $\{a_i^*\}_{i=1}^3$ for N households by using equation (A.23), where $N \gg 1$. To this end, we need to observe that, by Axiom A.1, each equilibrium income allocation occurs with an equal probability and that, by Definition A.2, the MLD occurs with the highest probability. This means that $\{a_i^*\}_{i=1}^3$ should contain the most equilibrium income allocations; that is, $\Omega(\{a_i^*\}_{i=1}^3)$ takes the largest value. Therefore, seeking the MLD $\{a_i^*\}_{i=1}^3$ is equivalent to solving the functional extremum problem:

$$\begin{cases} \max\limits_{\{a_i\}_{i=1}^3} \Omega(\{a_i\}_{i=1}^3) \\ Y = a_1 \times 0 + a_2 \times \frac{1}{2} + a_3 \times 1 \cdot \\ N = a_1 + a_2 + a_3 \end{cases} \tag{A.26}$$

When $N \gg 1$, the extremum solution $\{a_i^*\}_{i=1}^3$ is:

$$a_i^* = \frac{1}{e^{\alpha+\beta\varepsilon_i}}, \tag{A.27}$$

$$i = 1,2,3.$$

where $\varepsilon_1 = 0$, $\varepsilon_2 = \frac{1}{2}$, $\varepsilon_3 = 1$. Here, α and β are Lagrange multipliers. The derivation for equation (A.27) can be found in Appendix B.

To find the general form of the MLD, we further consider the general case of money unit; that is, $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n$. Thus, we can present the general definition for income distribution as below:

Definition A.3 (Income distribution): *A sequence of non-negative numbers,* $\{a_i\}_{i=1}^n$, *is called an income distribution if it obeys the following two conventions:*

(1). There are a total of n *possible income levels:* $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n$;

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(2). There are a_i households, each of which obtains ε_i units of income, and i runs from 1 to n.

By using Definition A.3, the equation (A.23) can be extended easily to:

$$\Omega(\{a_i\}_{i=1}^{n}) = \frac{N!}{\prod_{i=1}^{n} a_i!}. \tag{A.28}$$

Applying the technique described in Appendix B into equation (A.28), it is easy to compute the MLD $\{a_i^*\}_{i=1}^{n}$, as below:

$$a_i^* = \frac{1}{e^{\alpha+\beta\varepsilon_i}}, \tag{A.29}$$

$$i = 1,2, \dots, n.$$

Equation (A.29) is the general form of MLD in the ADE, which is an exponential income distribution. Tao (2016) has verified that equation (A.29) holds for the general case of ADE.

Appendix B

To guarantee $N \gg 1$, here we assume $a_i \gg 1$ for $i = 1,2,3$. Because x and $\ln x$ have the same monotonicity, the extremum problem (A.26) can be written as:

$$\begin{cases} \max\limits_{\{a_i\}_{i=1}^{3}} \ln\Omega(\{a_i\}_{i=1}^{3}) \\ Y = a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3, \\ N = a_1 + a_2 + a_3 \end{cases} \tag{B.1}$$

where $\varepsilon_1 = 0$, $\varepsilon_2 = \frac{1}{2}$, $\varepsilon_3 = 1$.

Let us write down the Lagrange function of equation (B.1):

$$L(\{a_i\}_{i=1}^{3}) = \ln\Omega(\{a_i\}_{i=1}^{3}) - \alpha \sum_{i=1}^{3} a_i - \beta \sum_{i=1}^{3} a_i\, \varepsilon_i, \tag{B.2}$$

where α and β are Lagrange multipliers.

Using Stiring's formula

$$\ln m! = \mathrm{m}(\ln m - 1), \tag{B.3}$$

equation (B.2) can be written in the form:

$$L(\{a_i\}_{i=1}^{3}) = \ln N! - \sum_{i=1}^{3} a_i \ln a_i + \sum_{i=1}^{3} a_i - \alpha \sum_{i=1}^{3} a_i - \beta \sum_{i=1}^{3} a_i\, \varepsilon_i, \tag{B.4}$$

where $m \gg 1$.

Obviously, the extremum solution $\{a_i^*\}_{i=1}^{n}$ should satisfy:

$$\frac{\partial L(\{a_i\}_{i=1}^3)}{\partial a_i}|_{a_i=a_i^*} = 0 \tag{B.5}$$

for $i = 1,2,3$.

Substituting equation (B.4) into (B.5) we get:

$$a_i^* = \frac{1}{e^{\alpha+\beta\varepsilon_i}}, \tag{B.6}$$

$i = 1,2,3$.

Appendix C

By equations (3.1) and (3.2), one has:

$$\frac{\partial N}{\partial \alpha} = -N, \tag{C.1}$$

$$\frac{\partial N}{\partial \beta} = -Y. \tag{C.2}$$

The differential of equation (C.2) yields:

$$dY = -d\left(\frac{\partial N}{\partial \beta}\right) = -\frac{1}{\beta}d\left(\beta\frac{\partial N}{\partial \beta}\right) + \frac{1}{\beta}\frac{\partial N}{\partial \beta}d\beta. \tag{C.3}$$

By equation (3.2), N is a function of α and β; therefore, the complete differential of N is:

$$dN = \frac{\partial N}{\partial \alpha}d\alpha + \frac{\partial N}{\partial \beta}d\beta, \tag{C.4}$$

which leads to:

$$\frac{\partial N}{\partial \beta}d\beta = dN - \frac{\partial N}{\partial \alpha}d\alpha. \tag{C.5}$$

Substituting equation (C.5) into equation (C.3) yields:

$$dY = -\frac{1}{\beta}d\left(\beta\frac{\partial N}{\partial \beta}\right) + \frac{1}{\beta}dN - \frac{1}{\beta}\frac{\partial N}{\partial \alpha}d\alpha. \tag{C.6}$$

On the other hand, we have:

$$d\left(\alpha\frac{\partial N}{\partial \alpha}\right) = \alpha d\left(\frac{\partial N}{\partial \alpha}\right) + \frac{\partial N}{\partial \alpha}d\alpha, \tag{C.7}$$

which leads to:

$$\frac{\partial N}{\partial \alpha}d\alpha = d\left(\alpha\frac{\partial N}{\partial \alpha}\right) - \alpha d\left(\frac{\partial N}{\partial \alpha}\right). \tag{C.8}$$

Substituting equation (C.8) into equation (C.6) yields:

$$dY = -\frac{1}{\beta}d\left(\beta\frac{\partial N}{\partial \beta}\right) + \frac{1}{\beta}dN - \frac{1}{\beta}d\left(\alpha\frac{\partial N}{\partial \alpha}\right) + \frac{\alpha}{\beta}d\left(\frac{\partial N}{\partial \alpha}\right)$$

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$$= \frac{\alpha}{\beta} d\left(\frac{\partial N}{\partial \alpha}\right) + \frac{1}{\beta} d\left(N - \alpha\frac{\partial N}{\partial \alpha} - \beta\frac{\partial N}{\partial \beta}\right). \quad (C.9)$$

By equation (C.1), one can rewrite equation (C.9) in the form:

$$dY = -\frac{\alpha}{\beta} dN + \frac{1}{\beta} d\left(N - \alpha\frac{\partial N}{\partial \alpha} - \beta\frac{\partial N}{\partial \beta}\right). \quad (C.10)$$

Appendix D

Proof of Proposition 3.1

Proof. We first verify the sufficiency. Substituting equations (C.1) and (C.2) into equation (3.9) yields:

$$T = N + \alpha N + \beta Y. \quad (D.1)$$

Since T is independent of N, by equation (3.3), we get:

$$\frac{\partial Y}{\partial N} = -\frac{\alpha}{\beta}, \quad (D.2)$$

$$\frac{\partial Y}{\partial T} = \frac{1}{\beta}. \quad (D.3)$$

Substituting equations (D.2) and (D.3) into equation (D.1) yields equation (3.10).

Now, we verify the necessity. If equation (3.10) holds and is solvable, then N is obviously independent of T. □

Proof of Proposition 3.2

Proof. The characteristic equation of the partial differential equation (3.10) is

$$\frac{dN}{N} = \frac{dT}{T-N} = \frac{dY}{Y}. \quad (D.4)$$

By $\frac{dN}{N} = \frac{dY}{Y}$, one can get a first integral:

$$\frac{Y}{N} = C_1, \quad (D.5)$$

where C_1 is a constant.

On the other hand, it is easy to calculate:

$$TdN - NdT = -N^2 d\left(\frac{T}{N}\right). \quad (D.6)$$

Substituting equation (D.6) into $\frac{dN}{N} = \frac{dT}{T-N}$ yields:

$$d\left(\frac{T}{N}\right) = -d(\ln N). \quad (D.7)$$

Thus, we get another first integral:

$$\frac{T}{N} + \ln N = C_2, \tag{D.8}$$

where C_2 is a constant.

Combining equations (D.5) and (D.8) we obtain the general solution (3.11). □

Appendix E

Proof of Lemma 5.1.

Proof. Equation (3.16) can be rewritten as:

$$\mu = \sigma \cdot r\left(\frac{\omega}{r} - MRTS_{LK}\right).$$

Through microeconomics (Mas-Colell et al., 1995), we have established that, when the economy stays at general equilibrium, one has $\frac{\omega}{r} = MRTS_{LK}$. □

Proof of Proposition 5.1

Proof. By equation (5.1), it is easy to calculate:

$$\frac{\partial Y}{\partial N} = \chi(\xi) - \frac{T-N}{N}\frac{d\chi(\xi)}{d\xi}, \tag{E.1}$$

$$\frac{\partial Y}{\partial T} = \frac{d\chi(\xi)}{d\xi}, \tag{E.2}$$

where $\xi = \frac{T}{N} + \ln N$.

As the economy stays at general equilibrium, by Lemma 5.1 and equation (E.1) we have:

$$\chi(\xi) = \left(\frac{T}{N} - 1\right)\frac{d\chi(\xi)}{d\xi}. \tag{E.3}$$

By equation (5.1), one has:

$$\frac{d\ln\chi(\xi)}{d\xi} = N\frac{\partial \ln Y}{\partial T}. \tag{E.4}$$

By Assumption 5.1 and equation (E.4), we conclude $\frac{d\ln\chi(\xi)}{d\xi} \geq 0$. Thus, by equation (E.3), we have:

$$\lim_{N\to\infty}\left(\frac{T}{N} - 1\right)^{-1} \geq 0. \tag{E.5}$$

Inequality (E.5) implies that T should be a function of N; otherwise, $\lim_{N\to\infty}\left(\frac{T}{N} -\right.$

$1\Big)^{-1} = -1 < 0$.

Furthermore, the limit value $\lim_{N\to\infty}\left(\frac{T}{N}-1\right)^{-1}$ is unique; otherwise, by equation (E.3), we conclude that $\frac{d\ln\chi(\xi)}{d\xi}$ is a multivalued function around $N \to \infty$. This contradicts the smoothness of the function $\chi(\xi)$.

If we order $\gamma = \lim_{N\to\infty}\left(\frac{T}{N}-1\right)^{-1}$, then we have $\gamma \geq 0$. Substituting $\gamma = \lim_{N\to\infty}\left(\frac{T}{N}-1\right)^{-1}$ into equation (E.3), we further obtain:

$$\chi(\xi) = e^{\gamma\xi} \text{ for } N \to \infty,$$

where we have abandoned the constant of integration.

As exponential income distribution (2.4) emerges when $N \gg 1$, which indicates $N \to \infty$, we complete the proof. □

Proof of Proposition 5.2

Proof. By the setting in section 2, the long-run production technology exhibits the constant return, so we have:

$$Y(\lambda L, \lambda K) = \lambda Y(L, K). \tag{E.6}$$

As the economy stays at general equilibrium and Assumption 5.1 holds, by Proposition 5.1, we conclude that equation (5.6) holds. Combining equations (5.6) and (E.6), one has:

$$Y(\lambda L, \lambda K) = e^{1+\gamma}N(\lambda L, \lambda K)^{1+\gamma} = \lambda e^{1+\gamma}N(L, K)^{1+\gamma}, \tag{E.7}$$

which indicates:

$$N(\lambda L, \lambda K) = \lambda^{\frac{1}{1+\gamma}}N(L, K). \quad \square$$

Appendix F

Proof of Proposition 6.1

Proof. Equation (3.16) can be rewritten as:

$$\frac{\mu}{\sigma\cdot\omega} = 1 - \frac{r\cdot MRTS_{LK}}{\omega}. \tag{F.1}$$

As $\mu \geq 0$, by equation (F.1), we have:

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$$1 - \frac{r \cdot MRTS_{LK}}{\omega} \geq 0, \tag{F.2}$$

where we have considered $\omega \geq 0$ and $\sigma \geq 0$.

Furthermore, as $MRTS_{LK} \geq 0$, by equation (3.16), we should have:

$$\mu \leq \sigma \cdot \omega,$$

which, by equation (F.1), leads to:

$$1 - \frac{r \cdot MRTS_{LK}}{\omega} \leq 1. \tag{F.3}$$

Let us order $I = 1 - \frac{r \cdot MRTS_{LK}}{\omega}$. By inequalities (F.2) and (F.3), we get $0 \leq I \leq 1$.

□