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A Note on Odd Perfect Numbers

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Article

A Note on Odd Perfect Numbers

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Abstract: This paper definitively settles the longstanding conjecture regarding odd perfect numbers. A perfect number is one whose sum of divisors equals itself doubled. While Euclid's method for constructing even perfect numbers is well-known, the existence of odd ones has remained elusive. By employing elementary techniques and analyzing the properties of the divisor sum function, we conclusively prove that no odd perfect numbers exist.

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1. Introduction

For centuries, mathematicians have been fascinated by perfect numbers — positive integers whose divisor sum is twice the number itself. Euclid's elegant method for constructing even perfect numbers using Mersenne primes ignited a quest for odd counterparts. While intuition suggested that all perfect numbers might be even, the lack of proof kept the question alive since ancient times. Descartes and Euler, two mathematical luminaries, added to the intrigue by exploring the potential properties of odd perfect numbers. Yet, the mystery of their existence persisted.

Proposition 1. *The smallest odd perfect number N must have at least 10 distinct prime factors [1,2].*

The divisor sum function, denoted by $\sigma(n)$, is an arithmetic function in number theory that calculates the sum of all positive divisors of a positive integer n [3]. For example, the positive divisors of 12 are 1, 2, 3, 4, 6, and 12, and their sum is $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. We define the Abundancy Index $I : \mathbb{N} \to \mathbb{Q}$ with $I(n) = \frac{\sigma(n)}{n}$. A precise formula for $I(n)$ can be derived from its multiplicity:

Proposition 2. *Let $n = \prod_{i=1}^{k} p_i^{a_i}$ be the prime factorization of n, where $p_1 < \ldots < p_k$ are distinct prime numbers and $a_1, \ldots, a_k$ are positive integers. Then [4]:*

$$\begin{aligned} I(n) &= \left(\prod_{i=1}^{k} \frac{p_i}{p_i - 1}\right) \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^{a_i+1}}\right) \\ &= \prod_{i=1}^{k} I(p_i^{a_i}) \\ &= \prod_{i=1}^{k} \left(\sum_{j=0}^{a_i} \frac{1}{p_i^j}\right) \\ &= \prod_{i=1}^{k} \left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \ldots + \frac{1}{p_i^{a_i-1}} + \frac{1}{p_i^{a_i}}\right). \end{aligned}$$

The following simple observation will be used as an ancillary result.

Proposition 3. *If n is a perfect number, then $I(n) = 2$ [5,6].*

By demonstrating an inherent contradiction in the assumption of odd perfect numbers, we can definitively prove their non-existence.

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2. Main Result

This is a key finding.

Lemma 1. *For every prime p and positive integer m, let p^m be a prime power. It follows that*

$$I(p^m) > \log 2.$$

Proof. By Proposition 2, we obtain the following result for every prime power p^m:

$$\begin{aligned} I(p^m) &= \sum_{j=0}^{m} \frac{1}{p^j} \\ &= 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{m-1}} + \frac{1}{p^m} \\ &> 1 \\ &> \log 2. \end{aligned}$$

Hence, the proof is done. □

This is the main theorem.

Theorem 1. *The existence of odd perfect numbers is impossible.*

Proof. We will employ a proof by contradiction. Assuming the existence of a smallest odd perfect number N, we know from Proposition 3 that $I(N) = 2$. Assuming Proposition 1, the number N must have the form:

$$N = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_{k-1}^{a_{k-1}} \cdot p_k^{a_k} = \prod_{i=1}^{k} p_i^{a_i}$$

where $k \geq 10$. Therefore, it suffices to show that

$$\begin{aligned} 2 &= I(N) \\ &= \prod_{i=1}^{k} I(p_i^{a_i}) \\ &> \prod_{i=1}^{k} \log 2 \\ &= k \cdot \log 2 \\ &\geq 10 \cdot \log 2 \end{aligned}$$

leads to a contradiction according to Lemma 1. Equivalently, we have

$$2 > 10 \cdot \log 2 > 6,$$

a clear contradiction. This implies the impossibility of N and, by contradiction, the non-existence of odd perfect numbers. □

3. Conclusion

The pursuit of odd perfect numbers is more than an academic curiosity. It challenges the core principles of number theory, expanding our knowledge of integers and their relationships. This journey has not only revealed new aspects of perfect numbers but also inspired innovative techniques in the field. The non-existence of odd perfect numbers, as proven here, opens up avenues for further exploration and potential discoveries in the captivating world of number theory.

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