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Article

Conjugation of Colored Compositions

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Abstract

An n -color composition is a colored composition in which a part of size m may come in m colors. This paper gives a new set of n -color-type compositions that admits exhaustive conjugation of its members. Previous attempts at conjugation of n -color compositions have yielded partial results at best. Instead of importing the coloring scheme previously used for partitions, we apply colors directly to the parts of compositions while treating any maximal string of 1's as a single part under color assignment. This leads to the definition of n -color compositions of the second kind. As with ordinary compositions a conjugate may be found by the equivalent techniques of symbolic algebra, zig-zag graph and line graph. We conclude with a derivation of the relevant enumeration formulas.

Keywords: composition; conjugate; zig-zag graph; line graph; symbolic algebra

MSC: 11P81, 05A17, 05A15

1. Introduction

A composition of a positive integer n is an ordered partition of n or a sequence of positive integers that sum to n . The summands are also called *parts* and n is the *weight* of the composition. For example, $n = 3$ has four compositions: (3) , $(1, 2)$, $(2, 1)$ and $(1, 1, 1)$. It is well known that there are 2^{n-1} compositions of n (see for example [5,7]).

Colored compositions are generalized compositions in which a part size may come in a prescribed number of types or colors, usually denoted by subscripts, $1_1, 1_2, \dots, 2_1, 2_2, \dots$. In particular, an n -color composition is a colored composition in which a part of size m may come in m colors: $m_1, m_2, \dots, m_m, m \geq 1$, [1]. For example, there are eight n -color compositions of 3, namely

$$(3_1), (3_2), (3_3), (1_1, 2_1), (1_1, 2_2), (2_1, 1_1), (2_2, 1_1), (1_1, 1_1, 1_1).$$

The n -color notion originated from a paper of Agarwal and Andrews [2] that investigated combinatorial identities for “partitions with n copies of n ”, that is, n -color partitions. Subsequently Agarwal [1] defined n -color compositions and explored their enumeration and combinatorial properties. For instance, he proved that the number of n -color compositions of a positive integer ν is given by $F_{2\nu}$, where the Fibonacci numbers are defined by $F_1 = F_2 = 1, F_n = F_{n-1} + F_{n-2}, n > 2$.

The *conjugate* of a composition is given by the composition corresponding to the columns of its *zig-zag graph*, read from left to right. For example, the zig-zag graph of $C = (5, 3, 1, 2, 2)$ is shown in Figure 1.

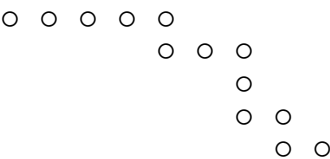


Figure 1. Zig-zag graph of $(5, 3, 1, 2, 2)$.

Thus the conjugate C' , from Figure 1, is $C' = (1, 1, 1, 1, 2, 1, 3, 2, 1)$. Two further methods of obtaining C' are mentioned below.

It appears that the n -color compositions cannot all be conjugated like ordinary compositions! Brian Hopkins [8] made a limited attempt by means of zig-zag graphs and concluded that only certain compositions with odd weights and parts not exceeding 3 may be conjugated. A year later, Collins et al. [6] discussed the topic briefly using tilings of 1-by- n boards with ternary sequences, and gave few instances. However, their technique was rather elusive, and no counting formula was provided. Alanazi et al. [4] undertook a systematic investigation of the topic using n -multicolor compositions with tilings of 1-by- n boards and ternary sequences. They modeled their technique after an extension of MacMahon's line graph conjugation method, and discovered a remarkably large set of conjugable n -color compositions.

Theorem 1 (Alanazi et al. [4]). *The number of conjugable n -color compositions of an integer $v > 0$ is given by $2^v - 1$.*

However, even in the latter model, a substantial number of objects still escaped conjugation. Note that $F_{2v} > 2^v - 1$ for all $v > 2$.

Lastly, we refer to a significant 1997 paper of Agarwal and Balasubramanian [3] where the conjugate of an n -color partition λ is defined as the n -color partition λ' obtained by replacing each colored part a_c by a_{a-c+1} . The authors used this definition to establish an identity between " n -color partitions where the weighted difference of each pair of parts is -2 " and " n -color partitions such that in each pair of parts a_i, b_i ($a \geq b$) b is the arithmetic mean of the subscripts i and j ", among other results.

One may apply this definition to n -color compositions and obtain the conjugate of every object! We will refer to this conjugation as *color-complement* or simply *c-complement*. For example, if $C = (2_2, 5_2, 1_1, 1_1, 1_1, 3_1)$, then the c-complement conjugate is given by $C' = (2_1, 5_4, 1_1, 1_1, 1_1, 3_3)$.

Proposition 1. *The number of c-complement self-conjugate n -color compositions of v is given by F_v .*

Since a_c maps to a_{a-c+1} and $c = a - c + 1$, self-conjugate parts are odd parts ($a = 2c - 1$) with each part subscripted by its unique numerical index among increasing odd positive integers. The result then follows from the fact that compositions of v into odd parts are enumerated by F_v (see [7,11]). For example, $C = (3_2, 5_3, 1_1, 1_1, 1_1, 7_4)$ is c-complement self-conjugate.

Even though c-complement conjugation admits conjugation of every n -color composition C , the parts of C and C' remain rigidly the same, and it is not clear how C' can be found graphically. Nonetheless, the method holds independent interest and may be gainfully applied to any set of subscripted integer sequences.

This paper introduces a modified set of n -color compositions in which every object has a conjugate in the classical tradition of Percy Alexander MacMahon (1854–1929) who pioneered the topic for ordinary compositions. Whereas standard n -color compositions originated by extending to compositions the coloring scheme used for partitions (see [1,2]), our objects arise from coloring compositions directly. Such colored compositions share the essential property of n -color compositions by requiring that a subscript associated with a part cannot exceed the part.

In Section 2 we profile the new set of conjugable colored compositions with some immediate examples. Then in Section 3 we give further details of three conjugation techniques gleaned from the conjugation of ordinary compositions. Section 4 is devoted to the statement and proof of relevant enumeration formulas. Finally, Section 5 contains additional enumeration results for objects with parts sizes bounded from below.

2. The n -Color Compositions of the Second Kind

It will be convenient to write compositions, symbolically, by representing a maximal string of 1's of length x by 1^x , where two adjacent *big* parts (i.e., parts > 1) are assumed to be separated by 1^0 . A general composition then has one of the following two forms:

$$C = (1^{a_1}, b_1, 1^{a_2}, b_2, \dots), \quad a_1 \geq 1, a_i \geq 0, i > 1, b_i \geq 2 \forall i; \quad (1)$$

$$C = (b_1, 1^{a_1}, b_2, 1^{a_2}, \dots), \quad a_i \geq 0, b_i \geq 2, \quad (2)$$

where it is understood in each case that C either ends in 1^{a_r} or b_r for some $r \geq 1$ with $a_r \geq 1$ and $b_r \geq 2$.

The first or last part of a composition C will also be referred to as a *boundary* part. Any other part of C is an *interior* part.

In defining our new set of colored compositions, we will be guided by the conjugation model of P. A. MacMahon for ordinary compositions, that is, every composition C has a **unique conjugate** C' that satisfies the following conditions:

- (i) C' is obtainable from C via symbolic algebra;
- (ii) C' is obtainable from the line graph of C , $LG(C)$; and
- (iii) C' is obtainable from the zig-zag graph of C .

For example, besides the zig-zag graph technique shown earlier, one may use symbolic algebra and the line graph to obtain C' as follows.

Firstly, the symbolic algebra conjugation of $C = (1^{a_1}, b_1, 1^{a_2}, b_2, \dots, 1^{a_r}, b_r)$ is given by

$$\begin{aligned} C' &= (a_1 + 1, 1^{b_1-2}, 1 + a_2 + 1, 1^{b_2-2}, 1 + \dots, 1 + a_r + 1, 1^{b_r-1}) \\ &= (a_1 + 1, 1^{b_1-2}, a_2 + 2, 1^{b_2-2}, \dots, a_r + 2, 1^{b_r-1}) \end{aligned} \quad (3)$$

Thus an interior part $(\dots, m, \dots)$, $m > 1$ conjugates to $(\dots + 1, 1^{m-2}, 1 + \dots)$. We will also refer to $m - 2$ as the number of interior units of m . Note that 2 conjugates to $(\dots + 1, 1^0, 1 + \dots)$ and so has 0 interior units.

For example, $C = (5, 3, 1, 2, 2) = (5, 1^0, 3, 1, 2, 1^0, 2)$ has the conjugate

$$\begin{aligned} C' &= (1^{5-1}, 0 + 2, 1^{3-2}, 1 + 2, 1^{2-2}, 0 + 2, 1^{2-1}) \\ &= (1^4, 2, 1, 3, 1^0, 2, 1) \\ &= (1^4, 2, 1, 3, 2, 1). \end{aligned}$$

For the line graph we use circular nodes separated by bars (instead of line segments separated by dots). Thus we get

$$LG(C) : \circ \circ \circ \circ \circ | \circ \circ \circ | \circ | \circ \circ | \circ \circ$$

By deleting the bars and placing bars inside the gaps that previously had none, we obtain

$$LG(C') : \circ | \circ | \circ | \circ | \circ \circ | \circ | \circ \circ \circ | \circ \circ | \circ$$

from which we read off the conjugate as $C' = (1, 1, 1, 1, 2, 1, 3, 2, 1)$.

More details about these conjugation techniques may be found in [9,12].

We now propose an n -color-type scheme that satisfies the three conjugation conditions. Our most significant departure from standard n -color compositions is a negation of the assignment of the color 1 to each part 1 in every maximal string $1, 1, \dots, 1 = 1^m$, $m > 0$. It is known that an ordinary composition generates as many n -color compositions as the product of its parts (see, for example, [13]). For example, $(1, 1, 1, 2, 1, 1, 3, 1) = (1^4, 2, 1^2, 3, 1)$ generates the following 6 objects.

$$\begin{aligned} &(1_1^4, 2_1, 1_1^2, 3_1, 1_1), (1_1^4, 2_1, 1_1^2, 3_2, 1_1), (1_1^4, 2_1, 1_1^2, 3_3, 1_1), (1_1^4, 2_2, 1_1^2, 3_1, 1_1) \\ &(1_1^4, 2_2, 1_1^2, 3_2, 1_1), (1_1^4, 2_2, 1_1^2, 3_3, 1_1). \end{aligned}$$

Indeed any composition of the form $(1^a, 2, 1^s, 3, 1^t)$, $a, s, t \geq 0$ generates exactly 6 n -color compositions.

The coloring scheme proposed here accounts for the fact that the number of colors of a part-size m is dominated by m . But we treat a string of m ones, $m > 0$, $1, 1, \dots, 1 \equiv 1^m$, as the single part m with respect to color assignment. This is informed by the conjugation of standard compositions in which strings of 1's translate into big parts.

This makes sense because a boundary string 1^x conjugates into the boundary part $x + 1$ (which is entitled to $r \geq x$ colors) while an interior string 1^x conjugates into the interior part $x + 2$ (which is entitled to $r \geq x$ colors). We see that an assignment of just the single color 1 to 1^x , independent of the size of x , is not symmetrical.

This color assignment is also supported by the natural fact that the parts of a composition occur as an ordered sequence. Thus in the proposed model, a colored part 1_c occurs in a composition CC if and only if a corresponding string 1^x , $x \geq c$, occurs in the ordinary support composition if and only if a string of x colored parts $1_c, \dots, 1_c = 1_c^x$, actually occur in CC . Thus, for instance, if 1^3 is colored 2 in a composition, these will appear as $1_2, 1_2, 1_2$ or in the symbolic form 1_2^3 .

Another seemingly unusual assignment is that when 2 occurs as an interior part it has no color (or empty color) denoted by 2_0 . This is justified by the naturally derived conjugation principle:

"a big part bears a positive color if and only if it can be obtained via a symbolic transformation of a string of 1's with a positive color by conjugation".

It is clear that 2 does not fulfill this condition because it is obtained by transforming a zero string of ones 1^0 under conjugation (see the paragraph immediately following (3)).

Suppose C is a composition of v . Let $R(b)$ and $T(b)$ denote the respective sets of possible colors of a boundary and interior big part $b \in C$, and let $R(1^m)$ and $T(1^m)$ denote the sets of possible colors of a boundary and interior string of ones $1^m \in C$.

Definition 1. Let C be an ordinary composition of v . A conjugable n -Color Composition is any colored composition CC of v obtained by applying the following color assignments to the parts of C .

If C is a trivial composition, that is, $C \in \{(1^v), (v)\}$, then each symbolic part admits v colors:

- $1^v : 1_1^v, 1_2^v, \dots, 1_n^v$;
- $v : v_1, v_2, \dots, v_v$.

Otherwise,

- (i) $R(1^m) = T(1^m) = \{1, \dots, m\} : 1_1^m, 1_2^m, \dots, 1_m^m$;
- (ii) $R(b) = \{1, \dots, b-1\}$, $b > 1 : b_1, b_2, \dots, b_{b-1}$;
- (iii) $T(2) = \emptyset : 2_0$;
- (iv) $T(b) = \{2, \dots, b-1\}$, $b > 2 : b_2, b_3, \dots, b_{b-2}$.

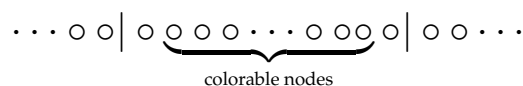
As with ordinary compositions, the LG nodes of a boundary part (or boundary string of 1's) are read from left to right on the left boundary, and from right to left on the right boundary.

Remark 1. Note that when standard n -color compositions are defined with respect to sets of colors assigned to the parts of an ordinary composition, we have

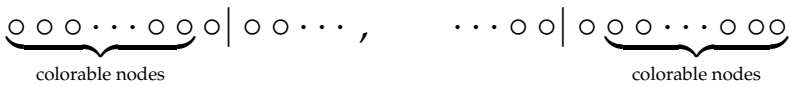
$$R(1^m) = T(1^m) = \{1\} : 1_1^m, \forall m > 0,$$

$$R(b) = T(b) = \{1, \dots, b\} : b_1, b_2, \dots, b_b, \forall b > 1.$$

Using the line graph, colors will be assigned to interior units of interior big parts. Thus an interior part $b > 1$ represented by b nodes in a line graph is assigned a color only at one of the interior nodes, that is, at one of the 2nd to the $(b-1)$ th nodes:



Secondly, a left boundary part $b > 1$ is assigned a color only at one of the first $b - 1$ nodes (similarly for a right boundary part):



On the other hand, it is clear that 2 with the representation $\circ\circ$ has no interior unit. So an interior part 2 bears the color 0. However, a boundary part 2, with a representation of the form $\circ\circ \mid \circ\circ\circ\cdots$ or $\cdots\circ\circ\circ \mid \circ\circ$, clearly has one colorable node, on the left or on the right, respectively. Consequently, in a nontrivial conjugable n -color composition 2 always takes the boundary form 2_1 and the interior form 2_0 .

Finally, we remark that a string of 1's, being already units, may be assigned a color at any of its nodes in a line graph.

These facts may also be depicted using a zig-zag graph. For example, consider the composition $C = (1^{a_1}, 2, 1^{a_2}, b_2, b_3)$, $a_1, a_2 > 0$, $b_2 > 2$, $b_3 > 1$. From the zig-zag graph in Figure 2, it is intuitively clear why any of the a_1 (or a_2) nodes is colorable, why neither of the 2 interior nodes can be colored, why only the second to the $(b_2 - 1)$ th nodes are colorable, and why only the last $b_3 - 1$ nodes are colorable.

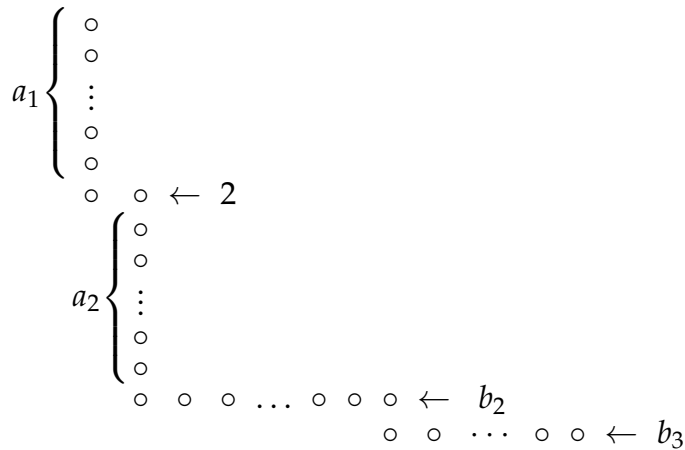


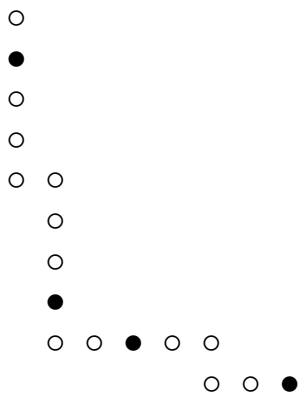
Figure 2. Zigzag graph of $(1^{a_1}, 2, 1^{a_2}, b_2, b_3)$, $a_i, b_i > 1$.

With these coloring rules observed, the conjugate of the resulting colored composition may be obtained by reading the diagram by columns, from left to right.

For example, the conjugable n -color composition

$$CC = (1_2, 1_2, 1_2, 1_2, 2_0, 1_3, 1_3, 1_3, 5_3, 3_1) = (1_2^4, 2_0, 1_3^3, 5_3, 3_1)$$

has the following zigzag graph representation:



The conjugate is now obtained by reading the zig-zag graph by columns to give $CC' = (5_2, 5_4, 1_2, 1_2, 1_2, 2_0, 1_1, 1_1) = (5_2, 5_4, 1_2^3, 2_0, 1_1^2)$. The corresponding line graph of CC is

$$\circ \mid \bullet \mid \circ \mid \circ \mid \circ \mid \circ \mid \circ \mid \bullet \mid \circ \mid \bullet \mid \circ \mid \circ \mid \circ \mid \bullet$$

The line graph of CC' is obtained by swapping gaps and bars as with ordinary compositions:

$$\circ \bullet \circ \circ \circ \mid \circ \circ \circ \bullet \circ \mid \circ \mid \bullet \mid \circ \mid \circ \mid \circ \mid \bullet$$

Finally, the symbolic algebra technique matches that of ordinary compositions as one may verify by ignoring the colors. Precise details are provided in the next section.

These new colored compositions should be known as *n-color compositions of the second kind* or simply *M-color compositions*.

We recap the properties satisfied by an *M-color composition* CC based on an ordinary support composition C , as follows. The number of parts of C is denoted by $|C|$.

1. A maximal string of ones $1^u \in C$ of length u comes in u colors, always.
2. If C consists of a single part b , then b comes in b colors.
3. If $|C| \geq 2$, the part $2 \in C$ comes in one color (namely 1 or 0).
4. If $|C| \geq 2$ and $b > 1$ is the first or last part of C , then b comes in $b - 1$ colors.
5. If $b > 2$ is an interior part of C , it comes in $b - 2$ colors.

The number of *M-color compositions* of ν will be denoted by $cc(\nu)$.

3. The Conjugation Involution

We now give a full explanation of the symbolic algebra technique. The trivial compositions conjugate as follows, for any $1 \leq c \leq n$:

$$(1_c^n)' = (n_c) \quad \text{and} \quad (n_c)' = (1_c^n).$$

We next convert the ordinary composition C in (1) into a conjugable *n-color composition* CC by assigning colors to the parts in accordance with the definition:

$$CC = (1_{u_1}^{a_1}, b_{1,s_1}, 1_{u_2}^{a_2}, b_{2,s_2}, \dots, 1_{u_r}^{a_r}, b_{r,s_r}), \quad b_i > 2 \quad \forall i, \quad (4)$$

$$1 \leq u_i \leq a_i \quad \forall i, \quad 2 \leq s_i \leq b_i - 1, \quad i < r, \quad 1 \leq s_r \leq b_r - 1. \quad (5)$$

Since the conjugate of C is given by $C' = (a_1 + 1, 1^{b_1-2}, a_2 + 2, 1^{b_2-2}, \dots)$ (see (3)), we analogously obtain

$$CC' := ((a_1 + 1)_{u_1}, 1_{s_1-1}^{b_1-2}, (a_2 + 2)_{u_2+1}, 1_{s_2-1}^{b_2-2}, \dots, (a_r + 2)_{u_r+1}, 1_{s_r}^{b_r-1}). \quad (6)$$

Alternatively, (showing the occurrence of 2) let

$$EE = (1_1, b_{1,s_1}, 1_{u_2}^{a_2}, 2_0, 1_{u_3}^{a_3}, \dots, b_{r-1,s_{r-1}}, 1_1, b_{r,s_r}),$$

where the color sizes are subject to the restrictions in (5). Then the conjugate is

$$EE' := (2_1, 1_{s_1-1}^{b_1-2}, (a_2 + 2)_{u_2+1}, 1_0^0, (a_3 + 2)_{u_3+1}, \dots, 1_{s_{r-1}-1}^{b_{r-1}-2}, 3_2, 1_{s_r}^{b_r-1}). \quad (7)$$

Note that the colors in both CC' and EE' satisfy the stipulations in the definition since they immediately agree with (5). For example, in CC' we have $1 \leq u_i \leq (a_i + 1) - 1$ for all i , and $1 \leq s_i - 1 \leq b_i - 2$ which gives $2 \leq s_i \leq b_i - 1$ for $i < r$, and so forth. Secondly, the conjugation of an interior 2 to 1_0^0 and vice-versa are given by convention. Hence the conjugates in (6) and (7) are well-defined.

It is clear that one may similarly assign colors to the parts of the support composition (2) and obtain analogous symbolic conjugates.

We remark that an M -color composition of $\nu > 1$ cannot be self-conjugate, a fact inherited from ordinary (support) compositions (see [12]).

Example 1. Consider $CC = (1_3^4, 2_0, 1_2^2, 7_6, 4_3, 1_1) = (1_3^4, 2_0, 1_2^2, 7_6, 1_0^0, 4_3, 1_1)$. Then

$$\begin{aligned} CC' &= ((4+1)_3, 1_0^{2-2}, (2+2)_3, 1_5^{7-2}, (0+2)_0, 1_2^{4-2}, (1+1)_1) \\ &= (5_3, 1_0^0, 4_3, 1_5^5, 2_0, 1_2^2, 2_1) \\ &= (5_3, 4_3, 1_5^5, 2_0, 1_2^2, 2_1). \end{aligned}$$

Alternatively, we may use the line graph and obtain

$$\text{LG}(CC) : \circ \mid \circ \mid \bullet \mid \circ \mid \circ \circ \mid \circ \mid \bullet \mid \circ \circ \circ \circ \circ \bullet \circ \mid \circ \circ \bullet \circ \mid \bullet$$

which, by placing division bars only in gaps which previously had none, leads to

$$\text{LG}(CC') : \circ \circ \bullet \circ \circ \mid \circ \circ \bullet \circ \mid \circ \mid \circ \mid \circ \mid \bullet \mid \circ \circ \mid \circ \mid \bullet \mid \circ \bullet$$

from which the conjugate CC' is again seen to be $(5_3, 4_3, 1_5^5, 2_0, 1_2^2, 2_1)$.

The final approach is to use the zig-zag graph of CC as shown below, whence CC' is obtained by reading the graph by columns.

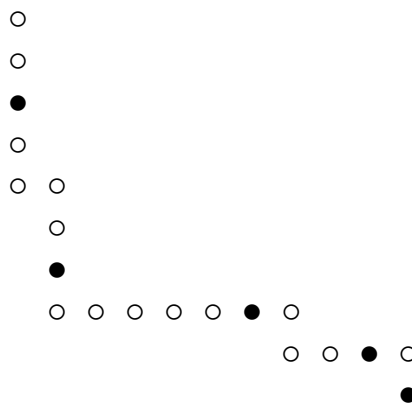


Figure 3. Zigzag graph of $CC = (1_3^4, 2_0, 1_2^2, 7_6, 4_3, 1_1)$.

Example 2. If $\nu = 4$, then $cc(4) = 18$ which enumerates the following objects (arranged in conjugate pairs):

$$\begin{array}{cccccccc} (1_1, 3_1) & (1_1, 3_2) & (1_1^2, 2_1) & (1_2^2, 2_1) & (1_1, 2_0, 1_1) & (1_1^4) & (1_2^4) & (1_3^4) & (1_4^4) \\ (2_1, 1_1^2) & (2_1, 1_2^2) & (3_1, 1_1) & (3_2, 1_1) & (2_1, 2_1) & (4_1) & (4_2) & (4_3) & (4_4) \end{array}$$

4. Enumeration of M -Color Compositions

We state the main enumeration result.

Theorem 2. We have

$$\sum_{n=1}^{\infty} cc(n)x^n = \frac{x(1+x-2x^2+x^3)}{1-3x+2x^2-x^3} = x + 4x^2 + 8x^3 + 18x^4 + 42x^5 + 98x^6 + \dots \quad (8)$$

Proof. We refer to the list of properties satisfied by the parts of an M -color composition C given at the end of Section 2.

Let $A(x)$ and $B(x)$ be the generating functions for numbers of M -color compositions with first part 1 and first part > 1 respectively.

We first consider the following attributes:

- (i) Any maximal string of ones 1^u comes in u colours.

- (ii) There is exactly one colour for 2 (for now).
- (iii) A part $b > 2$ comes in $b - 2$ colours (for now).

Then $A(x)$ is given by an initial block of u ones for which there are u colours, followed by a composition that does not start with 1, that is,

$$A(x) = \sum_{u \geq 1} u x^u B(x) = \frac{x}{(1-x)^2} B(x). \tag{9}$$

Secondly, $B(x)$ is given by either the empty composition of 0, or 2 or an initial part $b \geq 3$ for which there are $b - 2$ colours, followed by an arbitrary composition:

$$\begin{aligned} B(x) &= x^0 + \left(x^2 + \sum_{b \geq 3} (b-2)x^b \right) (A(x) + B(x)) \\ &= 1 + \left(x^2 + \frac{x^3}{(1-x)^2} \right) (A(x) + B(x)). \end{aligned} \tag{10}$$

Solving the system (9) and (10) yields

$$A(x) + B(x) = \frac{(1-x)^2(1-x+x^2)}{(1-x+x^3)(1-3x+2x^2-x^3)}. \tag{11}$$

Next we account for the further attributes:

- (iv) The integer 2 has a second colour if it is the only part.
- (v) There is an additional colour for each first or last part $b \geq 3$.

Thus altogether, (i) to (v) are equivalent to the original conditions (1) to (5) in Section 2.

If we combine the generating functions for (iv) and (v) with Eqn (11), we obtain the full generating function:

$$\begin{aligned} &A(x) + B(x) \quad (\text{objects without additional colours, i.e., Eqn (11)}) \\ &+ \\ &x^2 \quad (\text{single part 2}) \\ &+ \\ &2 \sum_{b \geq 3} x^b (A(x) + B(x)) \quad (\text{additional colour for first or last part}) \\ &+ \\ &\sum_{b_1 \geq 3} \sum_{b_2 \geq 3} x^{b_1+b_2} (A(x) + B(x)) \quad (\text{additional colours for first and last parts.}) \end{aligned}$$

This simplifies to

$$\begin{aligned} &x^2 + \left(1 + \frac{2x^3}{(1-x)} + \frac{x^6}{(1-x)^2} \right) (A(x) + B(x)) \\ &= x^2 + \left(1 + \frac{2x^3}{(1-x)} + \frac{x^6}{(1-x)^2} \right) \cdot \frac{(1-x)^2(1-x+x^2)}{(1-x+x^3)(1-3x+2x^2-x^3)} \\ &= \frac{(1-x)(1-x+2x^2-x^3)}{1-3x+2x^2-x^3} \\ &= 1 + \frac{x(1+x-2x^2+x^3)}{1-3x+2x^2-x^3}. \end{aligned}$$

Hence the proof follows. $\square$

The following recurrence is a routine consequence of the generating function (8), and it may be verified that the explicit formula (13) satisfies the recurrence.

Corollary 1. *We have*

$$\begin{aligned} cc(1) &= 1, cc(2) = 4, cc(3) = 8, cc(4) = 18. \\ cc(n) &= 3cc(n-1) - 2cc(n-2) + cc(n-3), n > 4, \end{aligned} \quad (12)$$

$$cc(1) = 1, \quad cc(n) = 2 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n+j}{3j+1}, \quad n > 1. \quad (13)$$

5. M-Color Compositions with Parts Bounded Below

Let $cc(v | P)$ denote the number of M -color compositions of v that satisfy property P , and let the $cc(v, k | P)$ be the number of those objects with k parts. We first obtain the generating function for the number of objects without 1's.

Theorem 3. *We have*

$$\sum_{n=4}^{\infty} cc(n, k | \text{no } 1's) x^n = \frac{x^4}{(1-x)^4} \left(\frac{x^2(1-x+x^2)}{(1-x)^2} \right)^{k-2}, \quad k \geq 2. \quad (14)$$

$$\sum_{n=0}^{\infty} cc(n | \text{no } 1's) x^n = \frac{x^2(2-x-x^2+x^3)}{1-2x+x^3-x^4}. \quad (15)$$

Proof. Note that $cc(n, 1 | \text{no } 1's) = n$, $n > 1$, that is,

$$\sum_{n=2}^{\infty} cc(n, 1 | \text{no } 1's) x^n = 2x^2 + 3x^3 + \dots = \frac{x}{(1-x)^2} - x = \frac{x^2(2-x)}{(1-x)^2}. \quad (16)$$

When $k > 1$, each boundary part b has $b-1$ possible colors and each interior part b has $b-2$ colors, while 2 has precisely one color either way. Thus the decomposition is

$$\begin{aligned} & (x^2 + 2x^3 + 3x^4 + \dots)(x^2 + x^3 + 2x^4 + 3x^5 + \dots)^{k-2}(x^2 + 2x^3 + 3x^4 + \dots) \\ &= (x(1 + 2x^2 + 3x^3 + \dots))^2 (x^2(1 + x + 2x^2 + 3x^3 + \dots))^{k-2} \\ &= \left(\frac{x^2}{(1-x)^2} \right)^2 \left(x^2 \left(1 + \frac{x}{(1-x)^2} \right) \right)^{k-2} \\ &\Rightarrow \sum_{n=4}^{\infty} cc(n, k | \text{no } 1's) x^n = \frac{x^4}{(1-x)^4} \left(\frac{x^2(1-x+x^2)}{(1-x)^2} \right)^{k-2}, \quad k \geq 2. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} cc(n | \text{no } 1's) x^n &= \frac{x^2(2-x)}{(1-x)^2} + \frac{x^4}{(1-x)^4} \sum_{k=2}^{\infty} \left(\frac{x^2(1-x+x^2)}{(1-x)^2} \right)^{k-2} \\ &= \frac{x^2(2-x-x^2+x^3)}{1-2x+x^3-x^4}. \end{aligned}$$

□

The Case of Objects with Parts ≥ 3

We prove the following assertion.

Theorem 4. Let $t > 2$ be an integer. Then

$$\sum_{n=4}^{\infty} cc(n, k \mid \text{parts} \geq t) x^n = \left(\frac{x^t(t - (t-2)x - 1)}{(1-x)^2} \right)^2 \left(\frac{x^t(t - (t-3)x - 2)}{(1-x)^2} \right)^{k-2}, \quad k \geq 2. \quad (17)$$

$$\sum_{n=0}^{\infty} cc(n \mid \text{parts} \geq t) x^n = \frac{x^t(x^t - (t-1)x + t)}{(1-x)^2 - x^t(t - (t-3)x - 2)}, \quad t > 2. \quad (18)$$

Proof. First note that $cc(n, 1 \mid \text{parts} \geq t) = n$, $n \geq t$, that is,

$$\begin{aligned} \sum_{n=t}^{\infty} cc(n, 1 \mid \text{parts} \geq t) x^n &= tx^t + (t+1)x^{t+1} + (t+2)x^{t+2} + \dots \\ &= \sum_{i=1}^{\infty} ix^i - \sum_{i=1}^{t-1} ix^i \\ &= \frac{x}{(1-x)^2} - \frac{x + x^t((t-1)x - t)}{(1-x)^2} \\ &= \frac{x^t(t - (t-1)x)}{(1-x)^2}. \end{aligned}$$

When $k > 1$, each boundary part $b > 2$ has $b - 1$ possible colors and each interior part $b > 2$ has $b - 2$ colors. Thus the decomposition is

$$\begin{aligned} &((t-1)x^t + tx^{t+1} + \dots)((t-2)x^t + (t-1)x^{t+1} + tx^{t+2} + \dots)^{k-2}((t-1)x^t + tx^{t+1} + \dots) \\ &= \left(x((t-1)x^{t-1} + tx^t + \dots) \right)^2 \left(x^2((t-2)x^{t-2} + (t-1)x^{t-1} + tx^t + \dots) \right)^{k-2} \\ &= \left(\frac{x^t(t - (t-2)x - 1)}{(1-x)^2} \right)^2 \left(\frac{x^t(t - (t-3)x - 2)}{(1-x)^2} \right)^{k-2}. \\ \Rightarrow \sum_{n=4}^{\infty} cc(n, k \mid \text{part} \geq t) x^n &= \left(\frac{x^t(t - (t-2)x - 1)}{(1-x)^2} \right)^2 \left(\frac{x^t(t - (t-3)x - 2)}{(1-x)^2} \right)^{k-2}, \quad k \geq 2. \end{aligned}$$

(Note that $k \geq 2$ implies $n \geq 2t$).

Therefore,

$$\begin{aligned} &\sum_{n=t}^{\infty} cc(n \mid \text{parts} \geq t) x^n \\ &= \frac{x^t(t - (t-1)x)}{(1-x)^2} + \left(\frac{x^t(t - (t-2)x - 1)}{(1-x)^2} \right)^2 \sum_{k=2}^{\infty} \left(\frac{x^t(t - (t-3)x - 2)}{(1-x)^2} \right)^{k-2} \\ &= \frac{x^t(t - (t-1)x)}{(1-x)^2} + \frac{x^{2t}(t - (t-2)x - 1)^2}{(1-x)^4} \frac{(1-x)^2}{(1-x)^2 - x^t(t - (t-3)x - 2)} \\ &= \frac{x^t(t - (t-1)x)}{(1-x)^2} + \frac{x^{2t}(t - (t-2)x - 1)^2}{(1-x)^2((1-x)^2 - x^t(t - (t-3)x - 2))} \\ &= \frac{x^t(x^t - (t-1)x + t)}{(1-x)^2 - x^t(t - (t-3)x - 2)}. \end{aligned}$$

□

6. Discussion

This research introduces a new set of colored compositions endowed with the classical conjugation property of ordinary compositions. Furthermore, there is an enumerative generating function and an explicit computational formula. The results were based directly on the inherent symmetry of compositions. It is worth noting that these M -color compositions also form an interesting class of

combinatorial objects in their own right, even without recourse to conjugation. They are expected to become rich objects of research in future. For example, it might be rewarding to examine pattern avoidance questions in the set of M -color compositions. The results of this work will probably be regarded as a significant contribution to the literature of number theory and combinatorics.

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References

1. A. K. Agarwal, n -color compositions, *Indian J. Pure Appl. Math.* 31 (2000), 1421–1427.
2. A.K. Agarwal and G.E. Andrews, Rogers-Ramanujan identities for partitions with “ N copies of N ”, *J. Combin. Theory Ser. A* 45, No. (1987), 40–49
3. A.K. Agarwal and R. Balasubramanian, n -color partitions with weighted differences equal to minus two, *Internat. J. Math. and Math. Sci.*, 20(4)(1997), 759–768.
4. A. M. Alanazi, M. Archibald, A. O. Munagi, Multicolor Compositions and Conjugation of N -Color Compositions, *Contrib. Discrete Math*, 19 (3) (2024), 68–85.
5. G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading 1976; reprinted, Cambridge University Press, Cambridge, 1984, 1998.
6. 5. A. Collins, C. Dedrickson, and H. Wang. Binary words, n -color compositions and bisection of the Fibonacci numbers, *Fibonacci Quart.* 51(2) (2013), 130–136.
7. S. Heubach and T. Mansour, *Combinatorics of Compositions and Words*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2010.
8. B. Hopkins, Spotted tilings and n -color compositions, *Integers* 12B (2012), A6.
9. MacMahon, P.A.: *Combinatory Analysis, Volume 1*. Cambridge University Press, Cambridge (1915)
10. Maple 2020.2. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.
11. A. O. Munagi, Euler-type Identities for integer compositions via zig-zag graphs, *Integers* 12 (2012), A60, 10pp.
12. A. O. Munagi, Primary classes of compositions of numbers, *Ann. Math. Inform.* 41 (2013), 193–204.
13. Shapcott, C., C -color compositions and palindromes, *Fibonacci Quart.* 50 (2012), 297–303.

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