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Posted Date: 4 June 2025

doi: 10.20944/preprints202506.0303.v1

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Article

An Approximation to Riemann Hypothesis

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Abstract: In this paper, we will give an approximate result to Riemann hypothesis.

Keywords: Riemann; zeta-function; Riemann hypothesis

1. Introduction

Riemann zeta-function $\zeta(s)$ is originally defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for } \operatorname{Re} s > 1.$$

and it also can be expressed as the product form

$$\zeta(s) = \prod_p \frac{1}{1 - 1/p^s}, \quad \text{for } \operatorname{Re} s > 1.$$

This formula is called Euler's product formula, which indicates the relation between $\zeta(s)$ and prime numbers. About $\zeta(s)$ there is a well-known Riemann hypothesis, states that all the non-trivial zeros of $\zeta(s)$ are on the critical line $\operatorname{Re} s = 1/2$. The researches on the conjecture are no doubt a most time-consuming one in mathematics, refer to see the survey paper [3].

The so-called trivial zeros of $\zeta(s)$ are $s = -2, -4, \dots$, and nontrivial zeros of $\zeta(s)$ are known all in the critical strip $0 \leq \operatorname{Re} s \leq 1$.

Denote by $N(T)$ and $N_0(T)$ respectively as the numbers of zeros of $\zeta(\sigma + it)$ in the region $0 \leq \sigma \leq 1, 0 \leq t \leq T$, and on the critical line $\sigma = 1/2, 0 \leq t \leq T$. Riemann hypothesis is that

$$N_0(T) = N(T).$$

For $N(T)$, it is known that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1.1}$$

And for $N_0(T)$, Hardy firstly shown that there are infinity many zeros on the critical line, and then he and Littlewood [5] and Selberg [8] proved that

$$\kappa = \frac{N_0(T)}{N(T)} > 0.$$

Levinson [6] proved

$$\kappa \geq \frac{1}{3}.$$

and then this result has been improved successively. Conrey [2], Feng [4] proved respectively

$$\kappa \geq 0.407, \quad \kappa \geq 0.412.$$

In this paper, we will prove that

Theorem 1.

$$N(T) = N_0(T) + \mathcal{O}(\dots) \tag{1.2}$$

where $\mathcal{E} \ll T^{7/9} \log^2 T$.

The main arguments in this paper are based the papers [1,6,7], but instead of using Riemann-Siegel formula, it will be applied an auxiliary function $\omega(s, T_1, T_2)$ defined in Lemma 1, which will play a role of mollifier and ferry, it firstly used in [7] but here with a small modification.

2. Some Lemmas

Lemma 1. Suppose that $T \leq T_1 \leq T_2 \leq 2T$, $\lambda = T^{14/9}$, $s_1 = \lambda + c + iu$, ($c > 0$), $s = v + it$, define

$$\omega(s, T_1, T_2) = \frac{e^\lambda}{2\pi i} \int_{T_1}^{T_2} \Gamma(s_1 - s) \lambda^{s-s_1} ds_1. \quad (2.1)$$

Let $\sigma = \lambda + c - v$, there is

$$|\omega(s, T_1, T_2)| \leq O\left(e^{-(c-v)^2/\sigma}\right) \quad (2.2)$$

Let $\Delta = (20\lambda \log T)^{1/2}$, if $t \in [T_1 + \Delta, T_2 - \Delta]$, then

$$|\omega(c + it, T_1, T_2) - 1| \ll T^{-10}. \quad (2.3)$$

And if $t \leq T_1 - \Delta$, or $t \geq T_2 + \Delta$, then

$$|\omega(c + it, T_1, T_2)| \ll T^{-10}. \quad (2.4)$$

If $|t - u| = o(\sigma^{2/3})$, then

$$\arg(\Gamma(s_1 - s) \lambda^{s-s_1}) = \frac{t - u}{2\sigma} - \frac{(t - u)^3}{6\sigma^2} + \epsilon. \quad (2.5)$$

Proof. By Stirling's formula, it has

$$\begin{aligned} \operatorname{Re}(\log \Gamma(\sigma + (u - t)i)) &= \left(\sigma - \frac{1}{2}\right) \frac{\log(\sigma^2 + (u - t)^2)}{2} - \sigma + \frac{1}{2} \log(2\pi) \\ &\quad - (u - t) \arctan\left(\frac{u - t}{\sigma}\right) + O(1/\sigma) \end{aligned}$$

And

$$\begin{aligned} &\operatorname{Re}(\log(\Gamma(\sigma + (u - t)i))) + \operatorname{Re}(\log(\lambda^{-(\sigma + (u - t)i)})) + \lambda \\ &= -\frac{(c - v)^2}{\sigma} - \frac{(u - t)^2}{4\sigma^2} - \frac{(u - t)^2}{2\sigma} - \frac{1}{2} \log \sigma + \frac{1}{2} \log(2\pi) + O(1/\sigma) \end{aligned}$$

Hence,

$$|\omega(s, T_1, T_2)| \leq e^{-(c-v)^2/\sigma} \frac{\sigma^{-1/2}}{\sqrt{2\pi}} \int_{T_1}^{T_2} \exp(-(u - t)^2/\sigma) du \ll e^{-(c-v)^2/\sigma}.$$

Besides, it is familiar that

$$e^{-\lambda} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s_1 - s) \lambda^{s-s_1} ds.$$

Hence,

$$1 - \omega(c + it, T_1, T_2) = R_1 + R_2,$$

where

$$R_1 = \frac{e^\lambda}{2\pi} \int_{T_2}^{\infty} \Gamma(\lambda + (u - t)i) \lambda^{-(\lambda + (u - t)i)} du,$$

$$R_2 = \frac{e^\lambda}{2\pi} \int_{-\infty}^{T_1} \Gamma(\lambda + (u-t)i) \lambda^{-(\lambda+(u-t)i)} du,$$

Hence, if $t \in [T_1 + \Delta, T_2 - \Delta]$, then

$$\begin{aligned} |R_1| &\ll \int_{T_2}^{\infty} \left| e^\lambda \Gamma(\lambda + (u-t)i) \lambda^{-(\lambda+(u-t)i)} \right| du \\ &\ll \lambda^{-1/2} \int_{T_2}^{\infty} \exp(-(u-t)^2/2\lambda) du \\ &\ll T^{-10}. \end{aligned}$$

and similarly

$$|R_2| \ll \lambda^{-1/2} \int_{-\infty}^{T_1} \exp(-(u-t)^2/2\lambda) du \ll T^{-10}.$$

if $t \leq T_1 - \Delta$, or $t \geq T_2 + \Delta$, then

$$|\omega(c+it, T_1, T_2)| \ll \lambda^{-1/2} \int_{T_1}^{T_2} \exp(-(u-t)^2/2\lambda) du \ll T^{-10}.$$

If $|t-u| = o(\sigma^{2/3})$, then

$$\begin{aligned} \operatorname{Im}(\log(\blacksquare(s_1-s)^{-s-s_1})) &= \frac{t-u}{2\sigma} + \frac{(t-u)^3}{3\sigma^2} - \frac{(t-u)^3}{2\sigma^2} \\ &\quad - \frac{(t-u)^5}{5\sigma^4} + \frac{(t-u)^5}{4\sigma^4} + \epsilon \\ &= \frac{t-u}{2\sigma} - \frac{(t-u)^3}{6\sigma^2} + \frac{(t-u)^5}{20\sigma^4} + \epsilon. \end{aligned}$$

□

Lemma 2. Let $\zeta_1(s), \zeta_2(s)$ be $\zeta(s)$ or $\zeta'(s)$ and

$$\zeta_1(s) = \sum_n a_n n^{-s}, \quad \text{for } \operatorname{Re} s > 1,$$

$$\zeta_2(s) = \sum_n b_n n^{-s}, \quad \text{for } \operatorname{Re} s > 1.$$

Let $0 < a < 1/2$, λ same as in Lemma 1, let $s_1 = \lambda + c + iu$, $c \geq 0$, define

$$g(u) = \frac{e^\lambda}{2\pi i} \int_{(a)} \Gamma(s_1 - s) \lambda^{-s_1+s} \zeta_1(s) \bar{\zeta}_2(s) ds \quad (2.6)$$

Then for $T \leq u \leq 2T$, $1 < \beta < \lambda + c$, there is

$$g(u) = e^\lambda \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_k \bar{b}_h}{h^{2\beta}} \left(\frac{h}{k} \right)^{s_1} e^{-\lambda h/k} + O(T^{-10}). \quad (2.7)$$

Proof. We move the integral path from (a) to (β), the residue at the pole $s = 1$ is

$$R \ll e^\lambda \Gamma(\lambda + c - 1 + iu) \lambda^{-(\lambda+c-1+iu)} \ll T^{-10}$$

Hence,

$$\begin{aligned}
g(u) &= \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_k \bar{b}_h}{k^{\beta} h^{\beta}} \frac{e^{\lambda}}{2\pi i} \int_{(\beta)} \Gamma(s_1 - s) \lambda^{-s_1+s} \left(\frac{h}{k}\right)^{it} ds + O(T^{-10}) \\
&= \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_k \bar{b}_h}{h^{2\beta}} \frac{e^{\lambda}}{2\pi i} \int_{(\beta)} \Gamma(s_1 - s) \lambda^{-s_1+s} \left(\frac{h}{k}\right)^s ds + O(T^{-10}) \\
&= \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_k \bar{b}_h}{h^{2\beta}} \frac{e^{\lambda}}{2\pi i} \int_{(\lambda+c-\beta)} \Gamma(w) \lambda^{-w} \left(\frac{h}{k}\right)^{s_1-w} dw + O(T^{-10}) \\
&= e^{\lambda} \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_k \bar{b}_h}{h^{2\beta}} \left(\frac{h}{k}\right)^{s_1} e^{-\lambda h/k} + O(T^{-10}).
\end{aligned}$$

□

Lemma 3. Let $\theta = (20 \log T / \lambda)^{1/2}$, λ as before, $0 \leq c \leq \Delta/10$, then

$$J_1 = \int_{1+\theta}^{\infty} e^{\lambda} v^{\lambda+c} e^{-\lambda v} dv \ll T^{-8} \lambda^{-1/2}, \quad (2.8)$$

$$J_2 = \int_0^{1-\theta} e^{\lambda} v^{\lambda+c} e^{-\lambda v} dv \ll T^{-10}. \quad (2.9)$$

Proof. For (2.6), by the integration by parts

$$\begin{aligned}
J_1 &= -\frac{e^{\lambda} v^{\lambda+c} e^{-\lambda v}}{\lambda} \Big|_{1+\theta}^{\infty} + \frac{\lambda+c}{\lambda} \int_{1+\theta}^{\infty} e^{\lambda} v^{\lambda+c-1} e^{-\lambda v} dv \\
&\leq \frac{\exp(-2\theta^2 \lambda/5)}{\lambda} + (1+\theta/10) \frac{J_1}{1+\theta}
\end{aligned}$$

That is,

$$\frac{9}{10} \frac{\theta \lambda}{1+\theta} J_1 \leq \exp(-2\theta^2 \lambda/5) \leq T^{-8}$$

and

$$J_1 \leq T^{-8} \lambda^{-1/2}.$$

For (2.7),

$$J_2 \leq e^{\lambda} (1-\theta)^{\lambda} e^{-\lambda(1-\theta)} \leq \exp(-\theta^2 \lambda/2) \leq T^{-10}.$$

□

3. The Proof of Theorem 1

Proof. Let $h(s) = \pi^{-s/2} \Gamma(s/2)$, then the functional equation of $\zeta(s)$ can be written as

$$h(s) \zeta(s) = h(1-s) \zeta(1-s) \quad (3.1)$$

By Stirling's formula, it has

$$\log h(s) = \frac{1}{2}(s-1) \log \frac{s}{2\pi} - \frac{s}{2} + C_0 + O\left(\frac{1}{s}\right) \quad (3.2)$$

Let $f(s) = \log h(s)$, then

$$f'(s) = \frac{h'(s)}{h(s)} = \frac{1}{2} \log \frac{s}{2\pi} + O\left(\frac{1}{s}\right) \quad (3.3)$$

and for larger t

$$f'(s) + f'(1-s) = \log \frac{t}{2\pi} + O\left(\frac{1}{s}\right) \quad (3.4)$$

Taking logarithm of Equation (3.1), and then derivative, it follows

$$h(s)\zeta(s)(f'(s) + f'(1-s)) = -h(s)\zeta'(s) - h(1-s)\zeta'(1-s) \quad (3.5)$$

We note that the right side of (3.5) is a sum of two conjugative complex numbers as $s = 1/2 + it$, so the zeros of the right side of (3.5) occur if and only if

$$\arg(h(s)\zeta'(s)) \equiv \pi/2 \pmod{\pi} \quad (3.6)$$

On the left side of (3.5), clearly, $h(s)$ is never zero, and by (3.4), so these zeros are just the zeros of $\zeta(1/2 + it)$.

Moreover, let $\chi(s) = h(1-s)/h(s)$, then $\zeta(s) = \chi(s)\zeta(1-s)$, and

$$\zeta'(s) = -\chi(s)\{(f'(s) + f'(1-s))\zeta(1-s) + \zeta'(1-s)\} \quad (3.7)$$

By (3.6), the zeros of $\zeta(1/2 + it)$ are the ones

$$\arg(h(1-s)\{(f'(s) + f'(1-s))\zeta(1-s) + \zeta'(1-s)\}) \equiv \pi/2 \pmod{\pi}$$

on $\sigma = 1/2$, equivalently,

$$\arg(h(s)\{(f'(s) + f'(1-s))\zeta(s) + \zeta'(s)\}) \equiv \pi/2 \pmod{\pi} \quad (3.8)$$

on $\sigma = 1/2$. Write $\mathcal{L}(s) = f'(s) + f'(1-s)$, and denote by

$$G(s) = \zeta(s) + \zeta'(s)/\mathcal{L}(s) \quad (3.9)$$

The investigation above means

$$N_0(T) = \frac{1}{\pi} \Delta_0^T \arg(hG(1/2 + it)) \quad (3.10)$$

By (3.2), it can be known that

$$\Delta_0^T \arg(h(1/2 + it)) = \frac{T}{2} \log \frac{T}{2\pi} - \frac{T}{2} + O(\log T) \quad (3.11)$$

So, the main task to determine $N_0(T)$ is to evaluate $\Delta_0^T \arg(G(1/2 + it))$.

Take $L = \log(T/2\pi)$, $U \leq T$.

Let D be the rectangle with the vertices $1/2 + iT$, $3 + iT$, $3 + i(T + U)$, $1/2 + i(T + U)$. First of all, we might as well assume there are no zeros of $G(s)$ on the boundary of D , then by the theory of complex function, the change of $\arg G(s)$ around D is equal to 2π times $N_G(D)$, the number of zeros of $G(s)$ in D .

On the right side of D

$$|G(3 + it) - 1| \leq \sum_{n \geq 2} n^3 + O(1/L) \leq 1/3$$

so, $\arg G(s)$ change less than π . Moreover, by a known result [9, §9.4], a extension of Jessen's theorem, we can know that on the lower bound and the upper bound $\arg G(s) = O(L)$. Hence

$$\Delta_0^T \arg(G(1/2 + it)) = -2\pi N_G(D) + O(\log T) \quad (3.12)$$

Now the work is turned into to evaluate $N_G(D)$.

Take $a = 1/2 - 1/L$, and let $\mathcal{D}$ be the rectangle with vertices $a + iT$, $c + iT$, $c + i(T + U)$, $a + i(T + U)$, taking the integral $\int_{\mathcal{D}} \log G(s) ds$, by the Littlewood's Lemma [9, §9.9], it has

$$\begin{aligned} & \int_T^{T+U} \log |G(a + it)| dt - \int_T^{T+U} \log |G(c + it)| dt + \int_a^c \arg G(\sigma + i(T + U)) d\sigma \\ & - \int_a^c \arg G(\sigma + iT) d\sigma = 2\pi \sum \text{dist} \end{aligned} \quad (3.13)$$

where $\sum \text{dist}$ is the sum of the distances of the zeros of $G(s)$ from the left. As in the situation of D and take account on the order of $G(s)$, we can know

$$\int_a^c \arg G(\sigma + iT) d\sigma, \int_a^c \arg G(\sigma + i(T + U)) d\sigma \ll O(L) \quad (3.14)$$

In addition, by (3.9), it is easy to know

$$\int_T^{T+U} \log G(c + it) dt = \int_T^{T+U} \log \zeta(c + it) dt + O(1/L)$$

and it is familiar that

$$\log \zeta(s) = \sum_n \frac{-\Lambda(n)}{n^s \log n}$$

So

$$\int_T^{T+U} \log |G(c + it)| dt \ll 1.$$

The rest is to calculate the first integral of (3.13).

By the concavity of logarithm, it has

$$\begin{aligned} \int_T^{T+U} \log |G(a + it)| dt &= \frac{1}{2} \int_T^{T+U} \log |G(a + it)|^2 dt \\ &\leq \frac{1}{2} U \log \left(\frac{1}{U} \int_T^{T+U} |G(a + it)|^2 dt \right) \end{aligned} \quad (3.15)$$

At first, we simplify $G(s)$ as

$$G_0(s) = \zeta(s) + \frac{\zeta'(s)}{L}. \quad (3.16)$$

Then

$$\begin{aligned} G(s) &= G_0(s) + E(s). \\ E(s) &= \left(\frac{1}{\mathcal{L}(s)} - \frac{1}{L} \right) \zeta'(s) \ll \frac{1}{L^3} \zeta'(s). \end{aligned}$$

And

$$\begin{aligned} \int_{T_1}^{T_2} |G(a + it)|^2 dt &= \int_{T_1}^{T_2} |G_0(a + it)|^2 dt + 2\text{Re} \int_{T_1}^{T_2} G_0(a + it) E(a - it) dt \\ &\quad + \int_{T_1}^{T_2} |E(a + it)|^2 dt \end{aligned}$$

By Cauchy inequality

$$\int_{T_1}^{T_2} G_0(a + it) E(a - it) dt \leq \left(\int_{T_1}^{T_2} |G_0(a + it)|^2 dt \int_{T_1}^{T_2} |E(a + it)|^2 dt \right)^{1/2}$$

The third integral in the right side of (3.16) is much smaller than the first one, which will be actually calculated later, hence

$$\int_{T_1}^{T_2} |G(a+it)|^2 dt = (1+\epsilon) \int_{T_1}^{T_2} |G_0(a+it)|^2 dt.$$

Moreover, let

$$\omega_1(s, T_1, T_2) = \omega^{1/2}(s, T_1, T_2).$$

By (2.5), we can know that on the upper bound and the lower bound of $\mathcal{D}$, there is

$$\arg(\omega_1(s, T_1 - \Delta, T_2 + \Delta)) \leq O(T^3/\lambda^2) \quad (3.17)$$

and

$$\int_a^c \arg(\omega_1(v + T_j i, T_1 - \Delta, T_2 + \Delta)) dv \leq O(cT^{-1/9}), \quad (j = 1, 2). \quad (3.18)$$

It is assumed that $c \leq \Delta/10$.

Moreover, by Lemma 1,

$$\int_{T_1}^{T_2} \log |\omega_1(c+it, T_1 - \Delta, T_2 + \Delta)| dt \ll T^{-9}. \quad (3.19)$$

This means that the function $\omega_1(s, T_1 - \Delta, T_2 + \Delta)$ may be viewed as a mollifier. Let

$$\mathcal{G}(s) = G_0(s)\omega_1(s, T_1 - \Delta, T_2 + \Delta).$$

By (2.5),

$$\begin{aligned} \int_{T_1}^{T_2} |\mathcal{G}(s)|^2 dt &= \int_{T_1}^{T_2} |G_0(s)|^2 |\omega_1(s, T_1 - \Delta, T_2 + \Delta)|^2 dt \\ &= \int_{T_1}^{T_2} |G_0(s)|^2 |\omega(s, T_1 - \Delta, T_2 + \Delta)| dt \\ &\leq (1 + O(T^{-2/9})) \int_{T_1}^{T_2} |G_0(s)|^2 \omega(s, T_1 - \Delta, T_2 + \Delta) dt \end{aligned} \quad (3.20)$$

In the next is mainly to calculate the last integral.

By Lemma 1, it has

$$\begin{aligned} &\int_{T_1}^{T_2} \omega(a+it, T_1 - \Delta, T_2 + \Delta) |G_0(a+it)|^2 dt \\ &= \frac{e^\lambda}{2\pi} \int_{T_1}^{T_2} \int_{T_1 - \Delta}^{T_2 + \Delta} \Gamma(s_1 - (a+it)) \lambda^{-(s_1 - (a+it))} du |G_0(a+it)|^2 dt \\ &= \frac{e^\lambda}{2\pi} \int_{T_1 - \Delta}^{T_2 + \Delta} \int_{T_1}^{T_2} \Gamma(s_1 - (a+it)) \lambda^{-(s_1 - (a+it))} |G_0(a+it)|^2 dt du \\ &\leq \frac{e^\lambda}{2\pi} \int_{T_1 - \Delta}^{T_2 + \Delta} \int_{-\infty}^{\infty} \Gamma(s_1 - (a+it)) \lambda^{-(s_1 - (a+it))} |G_0(a+it)|^2 dt du + o(1) \\ &= I_{11} + I_{12} + I_{21} + I_{22} + o(1). \end{aligned}$$

where

$$\begin{aligned} I_{11} &= \frac{e^\lambda}{2\pi} \int_{T_1-\Delta}^{T_2+\Delta} \int_{-\infty}^{\infty} \Gamma(s_1 - (a+it)) \lambda^{-(s_1-(a+it))} |\zeta(a+it)|^2 dt du \\ I_{12} &= \frac{e^\lambda}{2\pi L} \int_{T_1-\Delta}^{T_2+\Delta} \int_{-\infty}^{\infty} \Gamma(s_1 - (a+it)) \lambda^{-(s_1-(a+it))} \zeta(a+it) \zeta'(a-it) dt du \\ I_{21} &= \frac{e^\lambda}{2\pi L} \int_{T_1-\Delta}^{T_2+\Delta} \int_{-\infty}^{\infty} \Gamma(s_1 - (a+it)) \lambda^{-(s_1-(a+it))} \zeta(a-it) \zeta'(a+it) dt du \\ I_{22} &= \frac{e^\lambda}{2\pi L^2} \int_{T_1-\Delta}^{T_2+\Delta} \int_{-\infty}^{\infty} \Gamma(s_1 - (a+it)) \lambda^{-(s_1-(a+it))} |\zeta'(a+it)|^2 dt du. \end{aligned}$$

In the following specify $T_1 = T, T_2 = T + U$. We first calculate I_{11} , by Lemma 2

$$\begin{aligned} I_{11} &= e^\lambda \int_{T_1-\Delta}^{T_2+\Delta} \sum_{j_1, j_2} \frac{1}{j_2^{2\beta}} \rho^{s_1} \exp(-\lambda \rho) du, \quad \rho = j_2/j_1. \\ &= I_{11,0} + I_{11,1} + I_{11,2} + I_{11,3}. \end{aligned}$$

where

$$\begin{aligned} I_{11,0} &= \sum_{j_1=j_2} \frac{e^\lambda}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du, \\ I_{11,1} &= \sum_{\rho \geq 1+\theta} \frac{e^\lambda}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du \\ I_{11,2} &= \sum_{\rho \leq 1-\theta} \frac{e^\lambda}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du \\ I_{11,3} &= \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} \frac{e^\lambda}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du. \end{aligned}$$

Clearly

$$I_{11,0} = (U + 2\Delta) \sum_{j_2} \frac{1}{j_2^{2\beta}} = c_{11}(U + 2\Delta).$$

By Lemma 3,

$$\begin{aligned} I_{11,1} &\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \sum_{j_2/j_1 \geq 1+\theta} e^\lambda \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\ &\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_1^{j_2/(1+\theta)} e^\lambda \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx \\ &\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_{1+\theta}^{j_2} e^\lambda v^{\lambda+c-2} \exp(-\lambda v) dv \\ &\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_{1+\theta}^{\infty} e^\lambda v^{\lambda+c-2} \exp(-\lambda v) dv \\ &\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} T^{-8} \lambda^{-1/2} \ll T^{-8}. \end{aligned}$$

$$\begin{aligned}
I_{11,2} &\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \sum_{j_2/j_1 \leq 1-\theta} e^{\lambda} \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
&\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_{j_2/(1-\theta)}^{\infty} e^{\lambda} \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx \\
&\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_0^{1-\theta} e^{\lambda} v^{\lambda+c-2} \exp(-\lambda v) dv \\
&\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} e^{\lambda} (1-\theta)^{\lambda+c-2} \exp(-\lambda(1-\theta)). \\
&\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} T^{-10} \ll T^{-9}
\end{aligned}$$

And

$$\begin{aligned}
I_{11,3} &\ll (U+2\Delta) \sum_{j_2} \frac{1}{j_2^{2\beta}} \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} e^{\lambda} \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
&\ll (U+2\Delta) \sum_{j_2 \geq 1/\theta} \frac{1}{j_2^{2\beta}} \left(\frac{j_2}{1-\theta} - \frac{j_2}{1+\theta}\right) T^{1/5} \\
&\ll (U+2\Delta) \sum_{j_2 \geq 1/\theta} \frac{2\theta}{j_2^{2\beta-1}(1-\theta^2)} T^{1/5} \\
&\ll (U+2\Delta) \theta^{2\beta-1} T^{1/5} \ll T^{-10}, \quad (\beta \geq 10 \log T).
\end{aligned}$$

For I_{12} , by Lemma 2

$$\begin{aligned}
I_{12} &= \frac{1}{L} e^{\lambda} \int_{T_1-\Delta}^{T_2+\Delta} \sum_{j_1, j_2} \frac{-\log j_2}{j_2^{2\beta}} \rho^{s_1} \exp(-\lambda \rho) du \\
&= I_{12,0} + I_{12,1} + I_{12,2} + I_{12,3}.
\end{aligned}$$

where

$$\begin{aligned}
I_{12,0} &= \frac{1}{L} \sum_{j_1=j_2} \frac{-e^{\lambda} \log j_2}{j_2^{2c}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du, \\
I_{12,1} &= \frac{1}{L} \sum_{\rho \geq 1+\theta} \frac{-e^{\lambda} \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du \\
I_{12,2} &= \frac{1}{L} \sum_{\rho \leq 1-\theta} \frac{-e^{\lambda} \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du \\
I_{12,3} &= \frac{1}{L} \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} \frac{-e^{\lambda} \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du.
\end{aligned}$$

Clearly,

$$I_{12,0} = (U+2\Delta) \frac{1}{L} \sum_{j_2} \frac{-\log j_2}{j_2^{2\beta}} = c_{12}(U+2\Delta).$$

By Lemma 3

$$\begin{aligned}
 I_{12,1} &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \sum_{j_2/j_1 \geq 1+\theta} e^{\lambda} \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
 &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_1^{j_2/(1+\theta)} e^{\lambda} \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx \\
 &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_{1+\theta}^{j_2} e^{\lambda} v^{\lambda+c-2} \exp(-\lambda v) dv \\
 &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_{1+\theta}^{\infty} e^{\lambda} v^{\lambda+c-2} \exp(-\lambda v) dv \\
 &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} T^{-8} \lambda^{-1/2} \ll T^{-8}.
 \end{aligned}$$

and

$$\begin{aligned}
 I_{12,2} &\ll \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \sum_{j_2/j_1 \leq 1-\theta} e^{\lambda} \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
 &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_{j_2/(1-\theta)}^{\infty} e^{\lambda} \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx \\
 &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_0^{1-\theta} e^{\lambda} v^{\lambda+c-2} \exp(-\lambda v) dv \\
 &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} e^{\lambda} (1-\theta)^{\lambda+c-2} \exp(-\lambda(1-\theta)) \\
 &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} T^{-10} \ll T^{-9}.
 \end{aligned}$$

and

$$\begin{aligned}
 I_{12,3} &\ll \frac{1}{L} (U+2\Delta) \sum_{j_2} \frac{1}{j_2^{2\beta}} \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} e^{\lambda} \log j_2 \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
 &\ll \frac{1}{L} (U+2\Delta) \sum_{j_2 \geq 1/\theta} \frac{\log j_2}{j_2^{2\beta}} \left(\frac{j_2}{1-\theta} - \frac{j_2}{1+\theta}\right) T^{1/5} \\
 &\ll \frac{1}{L} (U+2\Delta) \sum_{j_2 \geq 1/\theta} \frac{2\theta \log j_2}{j_2^{2\beta-1}(1-\theta^2)} T^{1/5} \\
 &\ll (U+2\Delta) \theta^{2\beta-1} T^{1/5} \ll T^{-10}.
 \end{aligned}$$

For I_{21} , by Lemma 2

$$\begin{aligned}
 I_{21} &= \frac{1}{L} e^{\lambda} \int_{T_1-\Delta}^{T_2+\Delta} \sum_{j_1, j_2} \frac{-\log j_1}{j_2^{2\beta}} \rho^{s_1} \exp(-\lambda \rho) du \\
 &= I_{21,0} + I_{21,1} + I_{21,2} + I_{21,3},
 \end{aligned}$$

where

$$\begin{aligned} I_{21,0} &= \frac{1}{L} \sum_{j_1=j_2} \frac{-e^\lambda \log j_1}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du, \\ I_{21,1} &= \frac{1}{L} \sum_{\rho \geq 1+\theta} \frac{-e^\lambda \log j_1}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du \\ I_{21,2} &= \frac{1}{L} \sum_{\rho \leq 1-\theta} \frac{-e^\lambda \log j_1}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du \\ I_{21,3} &= \frac{1}{L} \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} \frac{-e^\lambda \log j_1}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du. \end{aligned}$$

Clearly,

$$I_{21,0} = (U + 2\Delta) \frac{1}{L} \sum_{j_2} \frac{-\log j_2}{j_2^{2\beta}} = c_{21}(U + 2\Delta)$$

By Lemma 3

$$\begin{aligned} I_{21,1} &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \sum_{j_2/j_1 \geq 1+\theta} e^\lambda \log j_1 \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\ &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_1^{j_2/(1+\theta)} e^\lambda \log x \left(\frac{j_2}{x}\right)^{\lambda+a} \exp(-\lambda j_2/x) dx \\ &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_{1+\theta}^{j_2} e^\lambda \log(j_2/v) v^{\lambda+c-2} \exp(-\lambda v) dv \\ &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_{1+\theta}^{\infty} e^\lambda v^{\lambda+c-2} \exp(-\lambda v) dv \\ &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} T^{-8} \lambda^{-1/2} \ll T^{-8}. \end{aligned}$$

and

$$\begin{aligned} I_{21,2} &\ll \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \sum_{j_2/j_1 \leq 1-\theta} e^\lambda \log j_1 \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\ &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_{j_2/(1-\theta)}^{\infty} e^\lambda \log x \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx \\ &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1}}{j_2^{2\beta}} \int_0^{1-\theta} e^\lambda \log(j_2/v) v^{\lambda+c-2} \exp(-\lambda v) dv \\ &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} e^\lambda (1-\theta)^{\lambda+c-2} \exp(-\lambda(1-\theta)) \\ &\ll \frac{1}{L} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} T^{-10} \ll T^{-9}. \end{aligned}$$

and

$$\begin{aligned}
 I_{21,3} &\ll \frac{1}{L}(U+2\Delta) \sum_{j_2} \frac{1}{j_2^{2\beta}} \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} e^\lambda \log j_1 \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
 &\ll \frac{1}{L}(U+2\Delta) \sum_{j_2 \geq 1/\theta} \frac{\log j_2}{j_2^{2\beta}} \left(\frac{j_2}{1-\theta} - \frac{j_2}{1+\theta}\right) T^{1/5} \\
 &\ll \frac{1}{L}(U+2\Delta) \sum_{j_2 \geq 1/\theta} \frac{2\theta \log j_2}{j_2^{2\beta-1}(1-\theta^2)} T^{1/5} \\
 &\ll (U+2\Delta)\theta^{2\beta-1} T^{1/5} \ll T^{-10}.
 \end{aligned}$$

For I_{22} , by Lemma 2

$$\begin{aligned}
 I_{22} &= \frac{1}{L^2} e^\lambda \int_{T_1-\Delta}^{T_2+\Delta} \sum_{j_1, j_2} \frac{\log j_1 \log j_2}{j_2^{2\beta}} \rho^{s_1} \exp(-\lambda \rho) du \\
 &= I_{22,0} + I_{22,1} + I_{22,2} + I_{22,3},
 \end{aligned}$$

where

$$\begin{aligned}
 I_{22,0} &= \frac{1}{L^2} \sum_{j_1=j_2} \frac{e^\lambda \log j_1 \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du, \\
 I_{22,1} &= \frac{1}{L^2} \sum_{\rho \geq 1+\theta} \frac{e^\lambda \log j_1 \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du \\
 I_{22,2} &= \frac{1}{L^2} \sum_{\rho \leq 1-\theta} \frac{e^\lambda \log j_1 \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du \\
 I_{22,3} &= \frac{1}{L^2} \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} \frac{e^\lambda \log j_1 \log j_2}{j_2^{2\beta}} \int_{T_1-\Delta}^{T_2+\Delta} \rho^{s_1} \exp(-\lambda \rho) du.
 \end{aligned}$$

Clearly,

$$I_{22,0} = (U+2\Delta) \frac{1}{L^2} \sum_{j_2} \frac{\log^2 j_2}{j_2^{2\beta}} = c_{22}(U+2\Delta).$$

By Lemma 3

$$\begin{aligned}
 I_{22,1} &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \sum_{j_2/j_1 \geq 1+\theta} e^\lambda \log j_1 \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
 &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_1^{j_2/(1+\theta)} e^\lambda \log x \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx \\
 &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_{1+\theta}^{j_2} e^\lambda \log(j_2/v) v^{\lambda+c-2} \exp(-\lambda v) dv \\
 &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log^2 j_2}{j_2^{2\beta}} \int_{1+\theta}^{\infty} e^\lambda v^{\lambda+c-2} \exp(-\lambda v) dv \\
 &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log^2 j_2}{j_2^{2\beta}} T^{-8} \lambda^{-1/2} \ll T^{-8}.
 \end{aligned}$$

and

$$\begin{aligned}
 I_{22,2} &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \sum_{j_2/j_1 \leq 1-\theta} e^{\lambda \log j_1} \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
 &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_{j_2/(1-\theta)}^{\infty} e^{\lambda \log x} \left(\frac{j_2}{x}\right)^{\lambda+c} \exp(-\lambda j_2/x) dx \\
 &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log j_2}{j_2^{2\beta}} \int_0^{1-\theta} e^{\lambda \log(j_2/v)} v^{\lambda+c-2} \exp(-\lambda v) dv \\
 &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log^2 j_2}{j_2^{2\beta}} e^{\lambda(1-\theta)^{\lambda+c-2}} \exp(-\lambda(1-\theta)) \\
 &\ll \frac{1}{L^2} \sum_{j_2} \frac{\theta^{-1} \log^2 j_2}{j_2^{2\beta}} T^{-10} \ll T^{-9}.
 \end{aligned}$$

and

$$\begin{aligned}
 I_{22,3} &\ll \frac{1}{L^2} (U+2\Delta) \sum_{j_2} \frac{\log j_2}{j_2^{2\beta}} \sum_{\substack{1-\theta \leq \rho \leq 1+\theta \\ j_1 \neq j_2}} e^{\lambda \log j_1} \left(\frac{j_2}{j_1}\right)^{\lambda+c} \exp(-\lambda j_2/j_1) \\
 &\ll \frac{1}{L^2} (U+2\Delta) \sum_{j_2 \geq 1/\theta} \frac{\log^2 j_2}{j_2^{2\beta}} \left(\frac{j_2}{1-\theta} - \frac{j_2}{1+\theta}\right) T^{1/5} \\
 &\ll \frac{1}{L^2} (U+2\Delta) \sum_{j_2 \geq 1/\theta} \frac{2\theta \log^2 j_2}{j_2^{2\beta-1} (1-\theta^2)} T^{1/5} \\
 &\ll (U+2\Delta) \theta^{2\beta-1} T^{1/5} \ll T^{-10}.
 \end{aligned}$$

Combining all the evaluations above, and recall (3.20), it follows

$$\int_T^{T+U} |\mathcal{G}(a+it)|^2 dt = c_0(U+2\Delta)(1+O(T^{-2/9})) + o(1).$$

where

$$\begin{aligned}
 c_0 &= c_{11} + c_{12} + c_{21} + c_{22} \\
 &= 1 + \sum_{j=2}^{\infty} \frac{(1 - \log j/L)^2}{j^{2\beta}} \\
 &= 1 + O(T^{-10}). \quad (\beta \geq 10 \log T)
 \end{aligned}$$

Let $U = T$, by (3.15),

$$\begin{aligned}
 \int_T^{T+U} \log |\mathcal{G}(a+it)| dt &\leq \frac{T}{2} \log \left(1 + \frac{2\Delta}{T}\right) + \frac{T}{2} \log(1 + O(T^{-2/9})) + \epsilon \\
 &\leq \Delta + O(T^{7/9}).
 \end{aligned} \tag{3.21}$$

By (3.13), (3.14), (3.18), (3.19) and (3.21), and recall that $a = 1/2 - 1/L$, it follows

$$2\pi N_G(D) \leq \frac{\Delta + O(T^{7/9}) + O(cT^{-1/9}) + O(L^2)}{1/2 - a} \ll T^{7/9} \log^2 T.$$

i.e.

$$\Delta_{2T}^T \arg G(1/2 + it) \leq O(T^{7/9} \log^2 T).$$

and

$$(N(2T) - N(T)) - (N_0(2T) - N_0(T)) \leq O(T^{7/9} \log^2 T).$$

Then let T be $T/2^k$, $1 \leq k \leq \log_2(T)$, and summing. This proves Theorem 1 in the case that there are no zeros of $G(s)$ on the boundary of D .

For the rest case, provided modifying the left side of D as the indented one with semicircles around the zeros of $G(s)$, and notice that, by (3.7), on the side $\sigma = 1/2$, a zero of $G(s)$ is also a zero of $\zeta'(s)$, and so a zero of $\zeta(s)$, with multiplicity one greater, with a similar argument as [6], (1.2) also can be followed, the detail refer to [6], and Theorem 1 is proved. $\square$

Besides, we know that on the critical line a zero of $G(s)$ is also a zero of $\zeta'(s)$, and so a zero of $\zeta(s)$, with multiplicity one greater. Hence

$$\sum (m - 1) \leq N_G(D).$$

where sum is over the distinct zeros of $\zeta(s)$ on the left side of D , m is the multiplicity of a zero.

And so,

$$\sum_{m \geq 2} m \leq 2N_G(D) \leq O(T^{7/9} \log^2 T). \quad (3.22)$$

This means that the zeros of $\zeta(s)$ on the critical line are almost all simple.

Finally, we would like to mention that, with Lemma 1, it is possible to give further improvement by to extend the left side of the rectangle $\mathcal{D}$ to the left half plane, provided to modify the related parameters appropriately.

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