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Article

Spectral transformations and associated linear functionals of the first kind

Juan Carlos García-Ardila [1] and Francisco Marcellán [2]

1 Departamento de Matemática Aplicada a la Ingeniería Industrial, Universidad Politécnica de Madrid, Calle José Gutierrez Abascal 2, 28006 Madrid, Spain.; juancarlos.garciaa@upm.es
2 Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida Universidad 30, 28911 Leganés, Spain; pacomarc@ing.uc3m.es
* Correspondence: juancarlos.garciaa@upm.es

Abstract: Given a quasi-definite linear functional $\mathbf{u}$ in the linear space of polynomials with complex coefficients let us consider the corresponding sequence of monic orthogonal polynomials (SMOP in short) $(P_n)_{n\geq 0}$. For the Christoffel transformation $\tilde{\mathbf{u}} = (x-c)\mathbf{u}$ with SMOP $(\widetilde{P}_n)_{n\geq 0}$, we are interested to study the relation between $\tilde{\mathbf{u}}$ and $\widetilde{\mathbf{u}^{(1)}}$, where $\mathbf{u}^{(1)}$ is the linear functional for the associated orthogonal polynomials of the first kind $(P_n^{(1)})_{n\geq 0}$ and $\widetilde{\mathbf{u}^{(1)}} = (x-c)\mathbf{u}^{(1)}$ is its Christoffel transformation. This problem is also studied for the Geronimus transformations.

Keywords: Spectral transformation; Darboux transformations; First kind orthogonal polynomials; Laguerre-Hahn linear functional.

1. Introduction and preliminaries

Let $\mathbf{u}$ be a complex-valued linear functional defined on the linear space of polynomials with complex coefficients $\mathbb{P}$, i.e. $\mathbf{u} : \mathbb{P} \to \mathbb{C}$, $p(x) \to \langle \mathbf{u}, p(x)\rangle$. The linear functional $\mathbf{u}$ is said to be quasi-definite (respectively, positive definite) if every leading principal submatrix of the Hankel matrix $H = (\mathbf{u}_{i+j})_{i,j=0}^{\infty}$ is nonsingular (respectively, positive-definite) where, by definition, $\mathbf{u}_k =: \left\langle \mathbf{u}, x^k \right\rangle, k \in \mathbb{N}$. In this case, there exists a sequence of monic polynomials $(P_n)_{n\geq 0}$ such that $\deg P_n = n$ and $\langle \mathbf{u}, P_n(x)P_m(x)\rangle = K_n\delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker symbol and $K_n \neq 0$ (see [12]). The sequence $(P_n)_{n\geq 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to $\mathbf{u}$.

Definition 1. *Let $\mathbf{u}$ be a linear functional and $q(x)$ a polynomial. Then the linear functionals $q(x)\mathbf{u}$ and $(x-c)^{-m}\mathbf{u}$ are defined, respectively, as*

$$\langle q(x)\mathbf{u}, p(x)\rangle = \langle \mathbf{u}, p(x)q(x)\rangle, \quad p \in \mathbb{P},$$

and

$$\left\langle (x-c)^{-m}\mathbf{u}, p(x)\right\rangle = \left\langle \mathbf{u}, \frac{p(x) - \sum_{k=0}^{m-1} \frac{D^k p(c)}{k!}(x-c)^k}{(x-c)^m} \right\rangle, \quad p \in \mathbb{P},$$

where D denotes the usual derivative operator.

The derivative $\mathbf{u}'$ of a linear functional $\mathbf{u}$ is the linear functional defined as

$$\left\langle \mathbf{u}', p(x)\right\rangle = -\left\langle \mathbf{u}, p'(x)\right\rangle.$$

Some times we also use the notation $D\mathbf{u}$ to denote the derivative of $\mathbf{u}$.

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Definition 2. *Given the linear application* $\theta_c : \mathbb{P} \to \mathbb{P}$ *defined by* $\theta_c p(x) = \dfrac{p(x) - p(c)}{x - c}$ *we introduce the linear functional*

$$\langle \theta_c \mathbf{u}, p(x) \rangle = \langle \mathbf{u}, \theta_c p(x) \rangle.$$

From the above Definition we get

$$\langle (x-c)^{-m}\mathbf{u}, p(x) \rangle = \langle \mathbf{u}, \theta_c^m p(x) \rangle,$$

understanding that $\theta_c^m p(x) = \theta_c(\theta_c^{m-1} p(x))$.

Definition 3. *Let* $\mathbf{u}$ *be a linear functional and* $p(x) = \sum_{k=0}^m a_k x^k$ *be a polynomial, then we define the polynomial* $(\mathbf{u} * p)(x)$ *as*

$$(\mathbf{u} * p)(x) =: \left\langle \mathbf{u}_y, \frac{xp(x) - yp(y)}{x - y} \right\rangle = \sum_{k=0}^{m} \left(\sum_{n=k}^{m} a_n \mathbf{u}_{n-k} \right) x^k$$

Here $\mathbf{u}_y$ means that the linear functional $\mathbf{u}$ acts on the variable y.

Definition 4 ([29])**.** *The product of two linear functionals* $\mathbf{u}$ *and* $\mathbf{v}$ *is defined from their moments as follows*

$$(\mathbf{uv})_n = \langle \mathbf{uv}, x^n \rangle = \sum_{k=0}^{n} \mathbf{u}_k \mathbf{v}_{n-k}, \quad n \geq 0.$$

The above product is commutative, associative and distributive with respect to the sum of linear functionals.

Let c be a complex number and let δ_c be the linear functional defined by

$$\langle \delta_c, x^n \rangle = c^n, \quad n \in \mathbb{N}.$$

Notice that for any linear functional $\mathbf{u}$, $\mathbf{u}\delta_0 = \mathbf{u}$. Moreover, if the first moment of $\mathbf{u}$ is nonzero, then there exists a unique linear functional $\mathbf{u}^{-1}$ such that $\mathbf{u}\mathbf{u}^{-1} = \delta_0$. The moments of $\mathbf{u}^{-1}$ are defined recursively by

$$(\mathbf{u}^{-1})_n = -\frac{1}{\mathbf{u}_0} \sum_{k=0}^{n-1} \mathbf{u}_{n-k} (\mathbf{u}^{-1})_k, \quad n \geq 1, \quad (\mathbf{u}^{-1})_0 = \mathbf{u}_0^{-1}.$$

Proposition 1 ([29])**.** *Let* $\mathbf{u}, \mathbf{v}$ *be linear functionals and* $p(x)$ *and* $q(x)$ *polynomials. The following properties hold.*

i) $\langle \mathbf{uv}, p(x) \rangle = \langle \mathbf{v}, (\mathbf{u} * p)(x) \rangle$.
ii) $\theta_c(pq)(x) = q(x)\theta_c p(x) + p(c)(\theta_c q(x))$.
iii) $p^2(x)\mathbf{u}^2 = (p\mathbf{u})^2 + 2xp(x)(\mathbf{u} * \theta_0 p)(x)\mathbf{u}$.
iv) $p(x)(\mathbf{uv}) = (p(x)\mathbf{v})\mathbf{u} + x(\mathbf{v} * \theta_0 p)(x)\mathbf{u}$.

From Favard's Theorem [12] we know that there exists a unique quasi-definite linear functional $\mathbf{u}$, with $(P_n)_{n\geq 0}$ its corresponding SMOP, if and only if there exist two sequences of complex numbers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$, with $a_n \neq 0, n \geq 1$, such that

$$\begin{aligned} &x\, P_n(x) = P_{n+1}(x) + b_n\, P_n(x) + a_n\, P_{n-1}(x), \quad n \geq 0, \\ &P_{-1}(x) = 0, \quad P_0(x) = 1. \end{aligned} \tag{1}$$

The above recurrence relation can be expressed in matrix form as follows. If $\mathbf{P} = (P_0, P_1, \cdots)^\top$, then $x\mathbf{P} = J\mathbf{P}$, where $A^\top$ denotes the transposed of the matrix A and J is the following tridiagonal semi-infinite matrix (monic Jacobi matrix, see [12])

$$J = \begin{pmatrix} b_0 & 1 & & \\ a_1 & b_1 & 1 & \\ & a_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Definition 5. *For $k \in \mathbb{N}$ we define the associated polynomials of the k-th kind, $(P_n^{(k)})_{n\geq 0}$, (also called k-th associated polynomials, see [12]) as the sequence of monic polynomials satisfying the recurrence relation*

$$\begin{aligned} &xP_n^{(k)}(x) = P_{n+1}^{(k)}(x) + b_{n+k}P_n^{(k)}(x) + a_{n+k}P_{n-1}^{(k)}(x), \quad n \geq 0, \\ &P_{-1}^{(k)}(x) = 0, \quad P_0^{(k)}(x) = 1. \end{aligned}$$

This means that a shift is introduced in the coefficients of the three term recurrence relation (1). According to Favard's Theorem, there exists a quasi-definite linear functional $\mathbf{u}^{(k)}$, called the k-associated transformation of $\mathbf{u}$, such that $(P_n^{(k)})_{n\geq 0}$ is its corresponding SMOP.

There is a direct representation of such polynomials as (see [6,31])

$$P_{n-k}^{(k)}(x) = \frac{1}{\left\langle \mathbf{u}, P_{k-1}^2 \right\rangle} \left\langle P_{k-1}(y)\mathbf{u}_y, \frac{P_n(x) - P_n(y)}{x-y} \right\rangle, \quad n \geq k.$$

Since $(P_n(x))_{n\geq 0}$ and $(P_{n-1}^{(1)}(x))_{n\geq 0}$ are two linearly independent solutions of the difference equation [31]

$$xw_n = w_{n+1} + b_n w_n + a_n w_{n-1}, \quad n \geq 1,$$

every solution of the above equation can be represented as a linear combination of $(P_n(x))_{n\geq 0}$ and $(P_{n-1}^{(1)}(x))_{n\geq 0}$. In particular (see [24,31])

$$P_{n-k}^{(k)}(x) = A(x,k)P_n(x) + B(x,k)P_{n-1}^{(1)}(x), \quad n \geq k, \tag{2}$$

where

$$A(x,k) = -\frac{P_{k-2}^{(1)}(x)}{\prod_{m=1}^{k-1} a_m} \quad \text{and} \quad B(x,k) = \frac{P_{k-1}(x)}{\prod_{m=1}^{k-1} a_m}.$$

Definition 6. *Let $(P_n)_{n\geq 0}$ be a SMOP with respect to $\mathbf{u}$ satisfying the recurrence relation (1). The sequence of monic polynomials $(P_n(x;\alpha))_{n\geq 0}$ is said to be co-recursive of parameter α with respect to the linear functional $\mathbf{u}$, if they also satisfy (1) but with initial conditions $P_0(x;\alpha) = 1$ and $P_1(x;\alpha) = P_1(x) - \alpha$. Notice that*

$$P_n(x;\alpha) = P_n(x) - \alpha P_{n-1}^{(1)}(x), \quad n \geq 0.$$

For co-recursive polynomials $P_n(x,\alpha)$ the following three-term recurrence relation holds

$$\begin{aligned} &x\,P_n(x;\alpha) = P_{n+1}(x;\alpha) + b_n\,P_n(x;\alpha) + a_n\,P_{n-1}(x;\alpha), \quad n \geq 1, \\ &x\,P_0(x;\alpha) = P_1(x;\alpha) + (b_0 + \alpha)\,P_0(x;\alpha). \end{aligned} \tag{3}$$

Definition 7. *Given a quasi-definite linear functional* $\mathbf{u}$, *we can define the formal series*

$$S_{\mathbf{u}}(z) =: \sum_{n=0}^{\infty} \frac{\mathbf{u}_n}{z^{n+1}}.$$

It is said to be the Stieltjes function associated with $\mathbf{u}$.

Definition 8 ([33]). *Let* $\widetilde{\mathbf{u}}$ *be a quasi-definite linear functional and* $\widetilde{S}(z)$ *its Stieltjes function.* $\widetilde{\mathbf{u}}$ *is said to be a rational spectral transformed of* $\mathbf{u}$ *if there exist polynomials* $A(z)$, $B(z)$, $C(z)$ *and* $D(z)$ *such that*

$$\widetilde{S}(z) = \frac{A(z)S_{\mathbf{u}}(z) + B(z)}{C(z)S_{\mathbf{u}}(z) + D(z)}.$$

The above mapping between two linear functionals is called rational spectral transformation. In particular if $C(z) \equiv 0$, *then* $\widetilde{\mathbf{u}}$ *is said to be a linear spectral transformed of the linear functional* $\mathbf{u}$. *In such a case the mapping between two linear functionals is called linear spectral transformation.*

Theorem 1 ([2,12,16,27,29]). *Let* $S_{\mathbf{u}}(z)$, $S_{\mathbf{u}^{-1}}(z)$, $S_{\mathbf{u}^{(1)}}(z)$ *and* $S_{\mathbf{u}^{\alpha}}(z)$ *be the Stieltjes functions associated with* $\mathbf{u}$, $\mathbf{u}^{-1}$, $\mathbf{u}^{(1)}$ *and* $\mathbf{u}^{\alpha}$, *respectively. Then the following relations hold*

i)

$$S_{\mathbf{u}}(z)S_{\mathbf{u}^{-1}}(z) = 1/z^2.$$

ii)

$$S_{\mathbf{u}^{(1)}}(z) = -\frac{\mathbf{u}_0\mathbf{u}_0^{(1)}}{a_1} z^2 S_{\mathbf{u}^{-1}}(z) + \frac{\mathbf{u}_0^{(1)}}{a_1}(z - b_0).$$

iii)

$$S_{\mathbf{u}^{\alpha}}(z)\left[\frac{-\alpha}{(\mathbf{u}^{\alpha})_0 z^2} + \frac{\mathbf{u}_0}{(\mathbf{u}^{\alpha})_0} S_{\mathbf{u}^{-1}}(z)\right] = \frac{1}{z^2}.$$

iv)

$$S_{\mathbf{u}}(z) = \frac{\mathbf{u}_0}{(z - b_0) - \dfrac{a_1}{\mathbf{u}_0^{(1)}} S_{\mathbf{u}^{(1)}}(z)}. \tag{4}$$

Moreover, from the above equations we can deduce the following relations between the corresponding linear functionals.

$$\mathbf{u}^{(1)} = -\frac{\mathbf{u}_0^{(1)}\mathbf{u}_0}{a_1} x^2 \mathbf{u}^{-1}, \qquad \mathbf{u}^{\alpha} = \frac{\mathbf{u}_0^{\alpha}}{\mathbf{u}_0}\left(\mathbf{u}^{-1} + \frac{\alpha}{\mathbf{u}_0}\delta_0'\right)^{-1}. \tag{5}$$

The analysis of perturbations of linear functionals constitutes an interesting topic in the theory of orthogonal polynomials on the real line (scalar OPRL) ([9,10,14,15,33] and references therein). Among the perturbations of linear functionals, spectral linear perturbations have attracted the interest of researchers (see [33]). Such perturbations are generated by two particular families, the so called Christoffel and Geronimus transformations.

Christoffel perturbations, that appear when considering orthogonality with respect to a new linear functional $\widetilde{\mathbf{u}} = p(x)\mathbf{u}$, where $p(x)$ is a polynomial, were studied in 1858 by E. B. Christoffel (see [13]) when $\mathbf{u}$ is the linear functional associated with the Lebesgue measure $d\mu$ supported on the interval $(-1, 1)$ and $d\widehat{\mu}(x) = p(x)d\mu(x)$, with $p(x) = (x - q_1)\cdots(x - q_N)$, a signed polynomial in the support of $d\mu$. Connection formulas between the corresponding SMOP are obtained therein. The location of their zeros as nodes of the quadrature rules is also deduced. More recently, from a numerical

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point of view in [23] the authors focus the attention on the sensitivity of Gauss–Christoffel quadrature with respect to small perturbations of the probability measure. On the other hand, the relations between the coefficients of the three term recurrence relations of the corresponding SMOP have been extensively studied, see [17], as well as the relation between the Jacobi matrices in the framework of the so-called discrete Darboux transformations. They are based on the *LU* factorization of such matrices (see [9] and [32], among others).

Notice that the zeros of orthogonal polynomials with respect to the canonical Christoffel transformation (the perturbation by a linear polynomial) of a nontrivial probability measure) are the nodes in the Gauss-Radau quadrature formula. In the case of a perturbation of the measure by a positive quadratic polynomial on the support of the measure, the zeros of the corresponding orthogonal polynomials are the nodes of the Gauss-Lobatto quadrature rule (see [19]).

Geronimus transformations appear when you deal with perturbed functionals $\hat{\mathbf{u}}$ defined by $p(x)\hat{\mathbf{u}} = \mathbf{u}$, where $p(x)$ is a polynomial and $\mathbf{u}$ is a quasi-definite linear functional. Such a kind of transformations were used by J. L. Geronimus (see [20]), in order to provide an alternative proof of a result given by W. Hahn [22] concerning to the characterization of classical orthogonal polynomials (Hermite, Laguerre, Jacobi and Bessel) as the unique families of orthogonal polynomials whose first derivatives are also orthogonal polynomials. Examples of such transformations have been done by P. Maroni [28] for a perturbation of the type $p(x) = x - c$. Examples for the quadratic and cubic case can be found in [3,5,11] and [30], respectively.

Next, we will point out the goals of our contribution. In [16] we study the following problem:

Problem 1. *(Figure 1) Let $\mathbf{u}$ be a quasi-definite functional and let $\tilde{\mathbf{u}} = (x-c)\mathbf{u}$ and $(x-c)\hat{\mathbf{u}} = \mathbf{u}$ be the canonical Christoffel and Geronimus transformation of $\mathbf{u}$, respectively. What is the relation between $\mathbf{u}^{(1)}$ and $\tilde{\mathbf{u}}^{(1)}$ (resp. $\hat{\mathbf{u}}^{(1)}$)? There, $\tilde{\mathbf{u}}^{(1)}$ (resp. $\hat{\mathbf{u}}^{(1)}$) is the associated linear functional of the first kind of $\tilde{\mathbf{u}}$ (resp. $\hat{\mathbf{u}}$).*

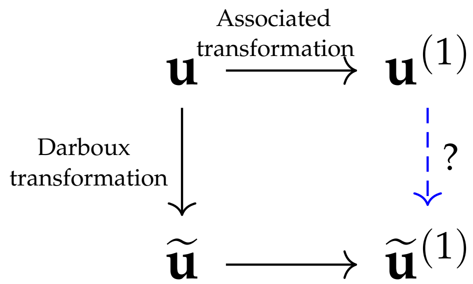

Figure 1. Structure of Problem 1.

To give a solution of the above problem we use the LU and UL factorization of the monic Jacobi matrix associated with $\mathbf{u}$, as well as the co-recursive polynomials.

In the present contribution we are interested to study the following problem.

Problem 2. *(Figure 2) Let $\mathbf{u}$ be a quasi-definite functional and let $\tilde{\mathbf{u}} = (x-c)\mathbf{u}$ and $(x-c)\hat{\mathbf{u}} = \mathbf{u}$ be the canonical Christoffel and Geronimus transformation of $\mathbf{u}$, respectively. What is the relation between $\tilde{\mathbf{u}}$ and $\widetilde{\mathbf{u}^{(1)}}$ (resp. $\widehat{\mathbf{u}^{(1)}}$)? There $\widetilde{\mathbf{u}^{(1)}}$ (resp. $\widehat{\mathbf{u}^{(1)}}$) is the Christoffel (resp. Geronimus) transformation of $\mathbf{u}^{(1)}$.*

With this in mind the structure of the manuscript is as follows. In Section 2, for the Christoffel transformation we study the connection between the linear functionals $\tilde{\mathbf{u}}$

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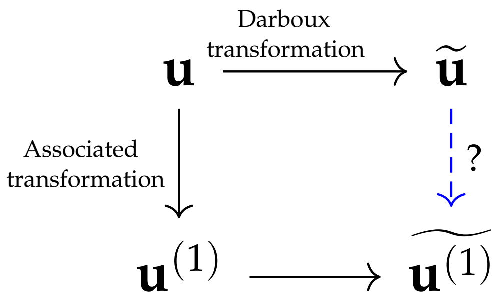

Figure 2. Structure of Problem 2.

and $\widetilde{\mathbf{u}^{(1)}}$. As consequence, we deduce the relation between the corresponding Stieltjes functions as well as the sequences of monic orthogonal polynomials. In a second step, we study the same problem when the Geronimus transformation is considered, i. e. the relation between $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{u}^{(1)}}$. In Section 3, we analyze a general family of linear functionals (the so called Laguerre-Hahn) whose Stieltjes function satisfies a Riccati equation. Once we introduce the definition of the class of such a linear functional, we study the class of a Laguerre-Hahn linear functional when either a Christoffel or a Geronimus transformation is implemented. Finally, an illustrative example about the above questions is discussed.

2. Darboux transformation and associated polynomials of the first kind.

2.1. Christoffel transformation and its associated polynomials of first kind

Let $\mathbf{u}$ be a quasi-definite linear functional and let $(P_n(x))_{n\geq 0}$ be its corresponding SMOP. If c is a complex number, the linear functional $\widetilde{\mathbf{u}} = (x-c)\mathbf{u}$ is said to be the Christoffel transformation of the functional $\mathbf{u}$. Let assume that $\widetilde{\mathbf{u}}$ is also quasi-definite (this fact is equivalent to $P_n(c) \neq 0$ for all $n \in \mathbb{N}$) and let $(\widetilde{P}_n)_{n\geq 0}$ be its SMOP. It is well known that $(P_n)_{n\geq 0}$ and $(\widetilde{P}_n)_{n\geq 0}$ are related by

$$(x-c)\widetilde{P}_n(x) = P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)}P_n(x), \quad n \geq 0.$$

We have the following relation between their Jacobi matrices

Theorem 2 ([9,32]). *Let J and $\widetilde{J}$ be the Jacobi matrices associated with $\mathbf{u}$ and $\widetilde{\mathbf{u}} = (x-c)\,\mathbf{u}$, respectively. If $P_n(c) \neq 0$, for all $n \in \mathbb{N}$, then $J - cI$ has an LU factorization, i.e.,*

$$J - cI := LU := \begin{pmatrix} 1 & & & \\ \ell_1 & 1 & & \\ & \ell_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \beta_0 & 1 & & \\ & \beta_1 & 1 & \\ & & \beta_2 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

where L is a lower bidiagonal matrix with 1's in the main diagonal and U is a upper bidiagonal matrix with $\beta_n = -P_{n+1}(c)/P_n(c)$.

$$\begin{cases} b_n - c = \ell_n + \beta_n, & b_0 - c = \beta_0, \\ a_n = \ell_n \beta_{n-1}, & n = 1, 2 \dots \end{cases}$$

Moreover, $\widetilde{J} - cI = UL$, where

$$\begin{cases} \widetilde{b}_n - c = \ell_{n+1} + \beta_n, \\ \widetilde{a}_n = \ell_n \beta_n, \end{cases} \qquad n = 0, 1, 2 \dots$$

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Proposition 2 ([33]). *Let $\mathbf{S_u}(z)$ and $\mathbf{S_{\widetilde{u}}}(z)$ be the Stieltjes functions associated with $\mathbf{u}$ and $\widetilde{\mathbf{u}}$, respectively. Then $\mathbf{S_{\widetilde{u}}}(z)$ is a linear spectral transformation of $\mathbf{S_u}(z)$. Indeed, the moments of $\widetilde{\mathbf{u}}$ and $\mathbf{u}$ satisfy the following relation*

$$\widetilde{\mathbf{u}}_n = \langle \widetilde{\mathbf{u}}, x^n \rangle = \langle \mathbf{u}, (x-c)\, x^n \rangle = \mathbf{u}_{n+1} - c\, \mathbf{u}_n.$$

From here,

$$\mathbf{S_{\widetilde{u}}}(z) = (z-c)\, \mathbf{S_u}(z) - \mathbf{u}_0. \tag{6}$$

Since $\mathbf{u}$ is a quasi-definite linear functional, then $\mathbf{u}^{(1)}$ is quasi-definite, Let $(P_n(x))_{n\geq 0}$ be the SMOP with respect to $\mathbf{u}$, and assume that $P_n(c) \neq 0$ for every $n \in \mathbb{N}$. We are interested to analyze the relation between the linear functionals $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{u}^{(1)}}$ given by $\widetilde{\mathbf{u}^{(1)}} := (x-c)\mathbf{u}^{(1)}$ and $\widetilde{\mathbf{u}} := (x-c)\mathbf{u}$, respectively.

Proposition 3. *The linear functionals $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{u}^{(1)}}$ are related as follows*

$$\widetilde{\mathbf{u}^{(1)}} = s(x-c)(x-t)\left((\widetilde{\mathbf{u}})^{-1} - \frac{1}{\mathbf{u}_0}\delta_0' \right)^{-1}, \tag{7}$$

with $t = \widetilde{a}_1 \dfrac{P_1(c)}{P_2(c)} + \widetilde{b}_0 - \dfrac{\widetilde{\mathbf{u}}_0}{\mathbf{u}_0}$ and $s = -\dfrac{P_2(c)}{P_1(c)} \dfrac{\mathbf{u}_0^{(1)}}{\widetilde{a}_1 \widetilde{\mathbf{u}}_0}$.

Proof. In [16, Proposition 16] the following relation was proved.

$$\widetilde{P}_n(x) + \frac{\widetilde{\mathbf{u}}_0}{\mathbf{u}_0} \widetilde{P}^{(1)}_{n-1}(x) = P^{(1)}_n(x) - \frac{P_{n+1}(c)}{P_n(c)} P^{(1)}_{n-1}(x), \;\; n \geq 0, \tag{8}$$

where $(\widetilde{P}^{(1)}_n)_{n\geq 0}$ is the SMOP of the first kind associated transformation of the linear functional $\widetilde{\mathbf{u}}$ (see Definition 5). Notice that the polynomials in the left hand side of (8) are co-recursive of parameter $\alpha = -\frac{\widetilde{\mathbf{u}}_0}{\mathbf{u}_0}$ with respect to the linear functional $\widetilde{\mathbf{u}}$. Let us denote

$$V_n(x) = \widetilde{P}_n(x) + \frac{\widetilde{\mathbf{u}}_0}{\mathbf{u}_0} \widetilde{P}^{(1)}_{n-1}(x), \quad n \geq 0,$$

such a monic polynomial sequence and let $\mathbf{w}$ be the linear functional such that $(V_n)_{n\geq 0}$ is the corresponding SMOP. Then from (5)

$$\mathbf{w} = \frac{\mathbf{w}_0}{\widetilde{\mathbf{u}}_0} \left((\widetilde{\mathbf{u}})^{-1} - \frac{1}{\mathbf{u}_0}\delta_0' \right)^{-1}. \tag{9}$$

If we expand the linear functional $\mathbf{u}^{(1)}$ in the dual basis

$$\left(\frac{V_n(x)\mathbf{w}}{\langle \mathbf{w}, V_n^2(x) \rangle} \right)_{n\geq 0}$$

of the polynomials $(V_n(x))_{n\geq 0}$ [29], then using the fact that $\left\langle \mathbf{u}^{(1)}, V_n(x) \right\rangle = 0$ for all $n \geq 2$, we get

$$\mathbf{u}^{(1)} = \alpha_0 \frac{\mathbf{w}}{\mathbf{w}_0} + \alpha_1 \frac{V_1(x)\mathbf{w}}{\langle \mathbf{w}, V_1^2(x) \rangle}.$$

From the orthogonal relations we obtain $\alpha_0 = \mathbf{u}_0^{(1)}$, $\alpha_1 = -\dfrac{P_2(c)}{P_1(c)} \mathbf{u}_0^{(1)}$. Thus

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$$\mathbf{u}^{(1)} = -\frac{P_2(c)}{P_1(c)} \frac{\mathbf{u}_0^{(1)}}{\langle \mathbf{w}, V_1(x) \rangle} \left(V_1(x) - \frac{P_1(c)}{P_2(c)} \frac{\langle \mathbf{w}, V_1(x) \rangle}{\mathbf{w}_0} \right) \mathbf{w}.$$

Taking into account that $\langle \mathbf{w}, V_1(x) \rangle = \widetilde{a}_1 \mathbf{w}_0$ and (9) we obtain the result. □

Let us assume that $P_n^{(1)}(c) \neq 0$ for all $n \in \mathbb{N}$. Then the SMOP $(\widetilde{P_n^{(1)}}(x))_{n \geq 0}$ with respect to $\widetilde{\mathbf{u}^{(1)}}$ satisfies

$$(x - c)\widetilde{P_n^{(1)}}(x) = P_{n+1}^{(1)}(x) - \frac{P_{n+1}^{(1)}(c)}{P_n^{(1)}(c)} P_n^{(1)}(x), \quad n \geq 0.$$

An equivalent condition is given in terms of the co-recursive polynomials $(V_n(x))_{n \geq 0}$ defined in the above Proposition. Indeed,

Corollary 1. *$\widetilde{\mathbf{u}^{(1)}}$ is quasi-definite if and only if $d_n \neq 0$ for every $n \in \mathbb{N}$, where*

$$d_n = \begin{cases} \det \begin{pmatrix} V_{n+1}(c) & V_n(c) \\ V_{n+1}(t) & V_n(t) \end{pmatrix}, & \text{if } t \neq c, \\ \\ \det \begin{pmatrix} V_{n+1}(c) & V_n(c) \\ V'_{n+1}(c) & V'_n(c) \end{pmatrix}, & \text{if } t = c, \end{cases} \quad \text{with} \quad V_n(x) = \widetilde{P}_n(x) + \frac{\widetilde{\mathbf{u}}_0}{\mathbf{u}_0} \widetilde{P}_{n-1}^{(1)}(x).$$

Moreover,

$$(x-c)(x-t)\widetilde{P_n^{(1)}}(x) = \begin{cases} \dfrac{1}{d_n} \det \begin{pmatrix} V_{n+2}(x) & V_{n+1}(x) & V_n(x) \\ V_{n+2}(c) & V_{n+1}(c) & V_n(c) \\ V_{n+2}(t) & V_{n+1}(t) & V_n(t) \end{pmatrix}, & \text{if } t \neq c, \\ \\ \dfrac{1}{d_n} \det \begin{pmatrix} V_{n+2}(x) & V_{n+1}(x) & V_n(x) \\ V_{n+2}(c) & V_{n+1}(c) & V_n(c) \\ V'_{n+2}(c) & V'_{n+1}(c) & V'_n(c) \end{pmatrix}, & \text{if } t = c, \end{cases} \quad n \geq 0,$$

where t is as in Proposition 3.

Proof. The proof is a consequence of (7). □

Proposition 4. *The Stieltjes functions $\mathbf{S}_{\widetilde{\mathbf{u}}}(z)$ and $\mathbf{S}_{\widetilde{\mathbf{u}^{(1)}}}(z)$ are related as follows.*

$$\mathbf{S}_{\widetilde{\mathbf{u}^{(1)}}}(z) = \frac{A(z)\, \mathbf{S}_{\widetilde{\mathbf{u}}}(z) + B(z)}{\mathbf{S}_{\widetilde{\mathbf{u}}}(z) + \mathbf{u}_0},$$

where

$$A(z) = \frac{\mathbf{u}_0^{(1)}}{a_1}[(z-c)(z-b_0) - a_1], \quad B(z) = \frac{\mathbf{u}_0 \mathbf{u}_0^{(1)}}{a_1}[(c-b_0)(z-c) - a_1].$$

Proof. From (4) and (6)

$$\begin{aligned} \mathbf{S}_{\widetilde{\mathbf{u}^{(1)}}}(z) &= (z-c)\mathbf{S}_{\mathbf{u}^{(1)}}(z) - \mathbf{u}_0^{(1)} \\ &= (z-c)\frac{\mathbf{u}_0^{(1)}}{a_1}\left[(z-b_0) - \mathbf{u}_0 \frac{(z-c)}{\mathbf{S}_{\widetilde{\mathbf{u}}}(z) + \mathbf{u}_0}\right] - \mathbf{u}_0^{(1)} \end{aligned}$$

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$$= \frac{(z-c)\frac{\mathbf{u}_0^{(1)}}{a_1}[(z-b_0)\mathbf{S}_{\tilde{\mathbf{u}}}(z)+(z-b_0)\mathbf{u}_0-\mathbf{u}_0(z-c)]-\mathbf{u}_0^{(1)}(\mathbf{S}_{\tilde{\mathbf{u}}}(z)+\mathbf{u}_0)}{\mathbf{S}_{\tilde{\mathbf{u}}}(z)+\mathbf{u}_0}$$
$$= \frac{A(z)\,\mathbf{S}_{\tilde{\mathbf{u}}}(z)+B(z)}{\mathbf{S}_{\tilde{\mathbf{u}}}(z)+\mathbf{u}_0}.$$

□

2.2. Geronimus transformation and its associated polynomials of first kind.

Let $\mathbf{v}$ be a quasi-definite linear functional and let $(P_n)_{n\geq 0}$ be its corresponding SMOP.

Definition 9. *Given a complex number c, a linear functional $\widehat{\mathbf{v}}$ defined by $(x-c)\widehat{\mathbf{v}} = \mathbf{v}$ is said to be the canonical Geronimus transformation of the linear functional $\mathbf{v}$.*

It is important to emphasize that $\widehat{\mathbf{v}}$ is not uniquely defined since its first moment is arbitrary. The explicit expression of $\widehat{\mathbf{v}}$ is [32]

$$\widehat{\mathbf{v}} = (x-c)^{-1}\mathbf{v} + \widehat{\mathbf{v}}_0\delta_c.$$

If we assume that $\widehat{\mathbf{v}}$ is also quasi-definite and $(\widehat{P}_n)_{n\geq 0}$ is its corresponding SMOP, then it is well known that $(P_n)_{n\geq 0}$ and $(\widehat{P}_n)_{n\geq 0}$ are related by [9,32]

$$\widehat{P}_n(x) = P_n(x) + \ell_n P_{n-1}(x), \quad n \geq 1,$$

where

$$\ell_n = -\frac{\mathbf{v}_0 P_{n-1}^{(1)}(c) + \widehat{\mathbf{v}}_0 P_n(c)}{\mathbf{v}_0 P_{n-2}^{(1)}(c) + \widehat{\mathbf{v}}_0 P_{n-1}(c)}, \quad n \geq 1. \tag{10}$$

Thus, a necessary and sufficient condition on $\widehat{\mathbf{v}}$ to be a quasi-definite linear functional is

$$\widehat{\mathbf{v}}_0 \neq -\frac{\mathbf{v}_0 P_{n-1}^{(1)}(c)}{P_n(c)}, \quad \text{for all } n \geq 1. \tag{11}$$

A second relation between $(P_n)_{n\geq 0}$ and $(\widehat{P}_n)_{n\geq 0}$ is (see [32])

$$(x-c)P_n(x) = \widehat{P}_{n+1}(x) + \beta_n \widehat{P}_n(x), \quad n \geq 0, \tag{12}$$

where $\beta_n = -\widehat{P}_{n+1}(c)/\widehat{P}_n(c)$.

Theorem 3 ([33]). *If $\mathbf{S}_{\mathbf{v}}(z)$ and $\mathbf{S}_{\widehat{\mathbf{v}}}(z)$ are the Stieltjes functions for $\mathbf{v}$ and $\widehat{\mathbf{v}}$, respectively, then $\mathbf{S}_{\widehat{\mathbf{v}}}(z)$ is a linear spectral transformation of $\mathbf{S}_{\mathbf{v}}(z)$. Indeed, taking into account that the moments of $\widehat{\mathbf{v}}$ and $\mathbf{v}$ satisfy*

$$\mathbf{v}_n = \langle \mathbf{v}, x^n \rangle = \langle \widehat{\mathbf{v}}, (x-a)\,x^n \rangle = \widehat{\mathbf{v}}_{n+1} - c\,\widehat{\mathbf{v}}_n,$$

then

$$\mathbf{S}_{\widehat{\mathbf{v}}}(z) = \frac{\mathbf{S}_{\mathbf{v}}(z) + \widehat{\mathbf{v}}_0}{(z-c)}.$$

Returning to the previous discussion, (10) and (12) imply the following relation between the corresponding monic Jacobi matrices associated with $\mathbf{v}$ and $\widehat{\mathbf{v}}$.

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Theorem 4 ([9,14,15,32]). *Let J and $\widehat{J}$ be the monic Jacobi matrices associated with $\mathbf{v}$ and $\widehat{\mathbf{v}}$, respectively. If $\widehat{\mathbf{v}}_0$ satisfies (11), then $J - cI$ has an UL factorization. Indeed,*

$$J - cI := UL := \begin{pmatrix} \beta_0 & 1 & & \\ & \beta_1 & 1 & \\ & & \beta_2 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ \ell_1 & 1 & & \\ & \ell_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix}, \tag{13}$$

or, equivalently,

$$\begin{cases} b_n - c = \ell_{n+1} + \beta_n, & n = 0, 1, \dots \\ a_n = \ell_n \beta_n, & n = 1, 2 \dots \end{cases}$$

where L is a lower bidiagonal matrix with 1's as diagonal entries and U is an upper bidiagonal matrix with $\beta_n = -\widehat{P}_{n+1}(c)/\widehat{P}_n(c)$. Moreover

$$\widehat{J} - cI = LU.$$

If we assume that $\ell_0 := 0$, then the corresponding entries satisfy

$$\begin{cases} \widehat{b}_n - c = \ell_n + \beta_n, & n = 0, 1, \dots \\ \widehat{a}_n = \ell_n \beta_{n-1}, & n = 1, 2 \dots \end{cases}$$

Observe that the UL factorization depends on the choice of $\widehat{\mathbf{v}}_0$ since $\beta_0 = \mathbf{v}_0 / \widehat{\mathbf{v}}_0$,

The matrices U and L given in (13) can be written

$$U = \left(\begin{array}{c|ccc} \beta_0 & 1 & 0 & \cdots \\ \hline 0 & & & \\ 0 & & U_1 & \\ \vdots & & & \end{array}\right), \quad L = \left(\begin{array}{c|ccc} 1 & 0 & 0 & \cdots \\ \hline \ell_1 & & & \\ 0 & & L_1 & \\ \vdots & & & \end{array}\right),$$

where U_1 and L_1 are upper and lower bidiagonal matrices, respectively. From here we deduce that the semi infinite matrix $J^{(1)} - cI$, with $J^{(1)}$ the Jacobi matrix associated with $\mathbf{v}^{(1)}$, has also an UL factorization i. e.,

$$J^{(1)} - cI := U_1 L_1 := \begin{pmatrix} \beta_1 & 1 & & \\ & \beta_2 & 1 & \\ & & \beta_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ \ell_2 & 1 & & \\ & \ell_3 & 1 & \\ & & \ddots & \ddots \end{pmatrix}.$$

Moreover, since $\beta_1 = \dfrac{a_1}{b_0 - c - \beta_0}$ then β_1 also depends on the choice of $\widehat{\mathbf{v}}_0$. Now , if we define

$$\widehat{J^{(1)}} - cI := L_1 U_1 = \begin{pmatrix} \beta_1 & 1 & & \\ \widehat{a}_2 & \widehat{b}_2 - c & 1 & \\ & \widehat{a}_3 & \widehat{b}_3 - c & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{14}$$

then $\widehat{J^{(1)}}$ is the Jacobi matrix associated with the linear functional $\widehat{\mathbf{u}^{(1)}}$ defined by

$$\widehat{\mathbf{u}^{(1)}} = (x - c)^{-1} \mathbf{u}^{(1)} + \widehat{\mathbf{u}_0^{(1)}} \delta_c, \quad \text{where} \quad \widehat{\mathbf{u}_0^{(1)}} = \mathbf{u}_0^{(1)} / \beta_1. \tag{15}$$

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Let $(\widehat{P_n^{(1)}})_{n\geq 0}$ be the SMOP with respect to $\widehat{\mathbf{u}^{(1)}}$. Then from (14) we can deduce

Proposition 5. *Let $(\widehat{P_n}^{(1)})_{n\geq 0}$ be the SMOP with respect to the first kind associated transformation of the linear functional $\widehat{\mathbf{v}}$ that will be denoted by $\mathbf{w}$. Then the polynomials $(\widehat{P_n^{(1)}})_{n\geq 0}$ are co-recursive of parameter $\alpha = \ell_1$ with respect to $\mathbf{w}$, i.e.,*

$$\widehat{P_n^{(1)}}(x) = \widehat{P_n}^{(1)}(x) - \ell_1 \widehat{P_{n-1}}^{(2)}(x), \quad n \geq 0. \tag{16}$$

Moreover, the linear functional $\widehat{\mathbf{v}^{(1)}}$ such that $(\widehat{P_n^{(1)}})_{n\geq 0}$ is the corresponding SMOP can be written as

$$\widehat{\mathbf{v}^{(1)}} = \frac{\widehat{\mathbf{v}_0^{(1)}}}{\mathbf{w}_0}\left(\mathbf{w}^{-1} - \frac{\ell_1}{\mathbf{w}_0}\delta_0'\right)^{-1}. \tag{17}$$

Proof. In [16, Proposition 24] it is proved that if the Jacobi matrix $J - cI$ has an LU factorization as in (13), then the Jacobi matrix $J_{\mathbf{w}}$ associated with $\mathbf{w}$ satisfies

$$J_{\mathbf{w}} - cI = \begin{pmatrix} \ell_1+\beta_1 & 1 & & & \\ \ell_2\beta_1 & \ell_2+\beta_2 & 1 & & \\ & \ell_3\beta_2 & \ell_3+\beta_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \widehat{b}_1 - c & 1 & & \\ \widehat{a}_2 & \widehat{b}_2 - c & 1 & \\ & \widehat{a}_3 & \widehat{b}_3 - c & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{18}$$

A comparison between the entries of the matrices (18) and (14) and taking into account (3), yield (16). (17) is a direct consequence of (5). □

Corollary 2. *Under the hypothesis of Proposition 5, we get the following relations*

(i). $\widehat{P_n^{(1)}}(x) = \left[1 - \frac{\ell_1}{\widehat{a}_1}(x - \widehat{b}_0)\right](\widehat{P_n})^{(1)}(x) + \frac{\ell_1}{\widehat{a}_1}\widehat{P}_{n+1}(x), n \geq 2.$

(ii). $(x-c)^{-1}\mathbf{v}^{(1)} + \widehat{\mathbf{v}_0^{(1)}}\delta_c = \frac{\widehat{\mathbf{v}_0^{(1)}}}{\mathbf{w}_0}\left(\mathbf{w}^{-1} - \frac{\ell_1}{\mathbf{w}_0}\delta_0'\right)^{-1}.$

Proposition 6. *The Stieltjes functions $S_{\widehat{\mathbf{v}}}$ and $S_{\widehat{\mathbf{v}^{(1)}}}$ are related as follows*

$$S_{\widehat{\mathbf{v}^{(1)}}}(z) = \frac{A(z)S_{\widehat{\mathbf{v}}}(z) + B(z)}{(z-c)^2 S_{\widehat{\mathbf{v}}}(z) + (z-c)\widehat{\mathbf{v}}_0},$$

where

$$A(z) = (z-c)\left[\frac{\mathbf{v}_0^{(1)}}{a_1}(z-b_0) + \widehat{\mathbf{v}_0^{(1)}}\right], \qquad B(z) = \frac{\mathbf{v}_0^{(1)}}{a_1}[\widehat{\mathbf{v}}_0(z-b_0) - \mathbf{v}_0] - \widehat{\mathbf{v}}_0\widehat{\mathbf{v}_0^{(1)}}.$$

Proof. It follows the guidelines of the proof of Proposition 4. □

3. Laguerre-Hahn linear functional

Definition 10 ([1,8,25]). *A linear functional $\mathbf{u}$ is said to be of the Laguerre-Hahn class if its Stieltjes function satisfies a Riccati equation*

$$\phi(z)S_{\mathbf{u}}'(z) = A(z)S_{\mathbf{u}}^2(z) + B(z)S_{\mathbf{u}}(z) + C(z), \tag{19}$$

where $\phi(z) \neq 0$, $A(z)$, $B(z)$, $C(z)$ are polynomials with

$$C(z) = (D\mathbf{u} * \theta_0\phi)(z) - (\mathbf{u} * \theta_0 B)(z) - (\mathbf{u}^2 * \theta_0^2 A)(z). \tag{20}$$

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In particular, if $A(z) \equiv 0$, then the linear functional is said to be semi-classical.

Proposition 7 ([1,8,25])**.** *Let* $\mathbf{u}$ *be a quasi-definite and normalized linear functional, i. e.* $\mathbf{u}_0 = 1$*, and let* $(P_n)_{n\geq 0}$ *be its corresponding SMOP. The following statements are equivalent*

i) $\mathbf{u}$ *is a Laguerre-Hahn functional.*

ii) $\mathbf{u}$ *satisfies the functional equation*

$$\mathcal{D}(\phi(x)\mathbf{u}) + \psi(x)\mathbf{u} - A(x)(x^{-1}\mathbf{u}^2) = 0, \tag{21}$$

where $\phi(x)$*,* $A(x)$*,* $B(x)$*,* $C(x)$ *are the polynomials in* (19) *and*

$$\psi(x) = -[\phi'(x) + B(x)].$$

iii) *Each polynomial* $P_n(x)$ *verifies the so called structure relation*

$$\phi(x)P'_{n+1}(x) + A(x)P_n^{(1)}(x) = \sum_{k=n-s}^{n+d} \lambda_{n,k}P_k(x), \quad n \geq s+1.$$

Here $\phi(x)$ *and* $A(x)$ *are the polynomials given in* (19)*,* $s = \max\{t-1, d-2\}$ *and* $d = \max\{r, m\}$*, where* $r = \deg\phi$*,* $t = \deg\psi$ *and* $m = \deg A$*.*

Remark 1. *We notice that there are changes of signs in the previous characterizations compared to the works of Maroni [27,29], Belmehdi [7], Marcellán and Prianes [25,26] and many other authors. This is because in these articles the Stieltjes function was multiplied by a negative sign.*

In characterization (ii), we must notice that you have not uniqueness in the representation. Indeed, if $\mathbf{u}$ is Laguerre-Hahn and $q(x)$ is a polynomial, then $\mathbf{u}$ also satisfies the functional equation

$$\mathcal{D}(q(x)\phi(x)\mathbf{u}) + (q(x)\psi(x) - q'(x)\phi(x))\mathbf{u} - q(x)A(x)(x^{-1}\mathbf{u}^2) = 0.$$

With this in mind we give the following definition

Definition 11 ([1,8,25])**.** *The class of a Laguerre-Hahn functional* $\mathbf{u}$ *is the nonnegative integer number defined as*

$$\mathfrak{s} := \min \max\{\deg\psi(x) - 1,\ \max\{\deg\phi(x), \deg A(x)\} - 2\},$$

where the minimum is taken among all polynomials $\phi(x)$*,* $\psi(x)$ *and* $A(x)$ *such that* $\mathbf{u}$ *satisfies* (21)*.*

Taking into account that the class of a Laguerre-Hahn linear functional is very useful in order to state a hierachy of such families, we need to give a simple way to characterize it.

Proposition 8 ([1,25])**.** *Let* $\mathbf{u}$ *be a Laguerre-Hahn linear functional and let* $\phi(x)$ *and* $\psi(x)$ *be non-zero polynomials with* $\deg\phi(x) =: r$*,* $\deg\psi(x) =: t$ *and* $\deg A(x) =: m$*, such that* (21) *holds. Let* $s := \max\{t-1, d-2\}$ *with* $d = \max\{r, m\}$*. Then* s *is the class of* $\mathbf{u}$ $(s = \mathfrak{s})$ *if and only if*

$$\prod_{a:\,\phi(a)=0} \left(|\psi(a) + \phi'(a)| + |A(a)| + \left| \langle \mathbf{u}, \theta_a\psi(x) + \theta_a^2\phi(x) - (\mathbf{u} * \theta_0[\theta_a A(x)]) \rangle \right| \right) > 0.$$

From the above Theorem, there is an alternative way to find the class in terms of the polynomials involved in the Riccati equation (19). Indeed,

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Corollary 3 ([1,25]). *Let* $\mathbf{u}$ *be a Laguerre-Hahn functional satisfying* (19) *such that* $\deg\phi(x) = r$, $\deg A(x) = m$ *and* $\deg\psi = t$ *with* $\psi(x) = -[\phi'(x) + B(x)]$. *Let* $s = \max\{t-1, d-2\}$ *with* $d = \max\{r, m\}$. *Then* $\mathfrak{s} = s$ *if and only if the polynomials* $\phi(x)$, $A(x)$, $B(x)$ *and* $C(x)$ *are coprime or, equivalently,*

$$\prod_{a:\,\phi(a)=0} \Big(|A(a)| + |B(a)| + |C(a)|\Big) > 0.$$

Theorem 5. *Let* $\mathbf{u}^{(1)}$ *be the first associated transformation of* $\mathbf{u}$ *and assume without lost of generality that* $\mathbf{u}_0 = \mathbf{u}_0^{(1)} = 1$. *If* $\mathbf{u}$ *is a Laguerre-Hahn functional of class* $\mathfrak{s}$ *satisfying* (19), *then so is* $\mathbf{u}^{(1)}$. *In this case we have that*

$$\phi_1(z)\mathbf{S}'_{\mathbf{u}^{(1)}}(z) = A_1(z)\mathbf{S}^2_{\mathbf{u}^{(1)}}(z) + B_1(z)\mathbf{S}_{\mathbf{u}^{(1)}}(z) + C_1(z), \tag{22}$$

where

$$\begin{aligned}
\phi_1(z) &= \phi(z),\\
A_1(z) &= a_1\, C(z),\\
B_1(z) &= -2(z-b_0)C(z) - B(z),\\
C_1(z) &= \frac{1}{a_1}\Big[\phi(z) + A(z) + (z-b_0)B(z) + (z-b_0)^2 C(z)\Big].
\end{aligned}$$

The above polynomials are coprime. Moreover, if $\mathfrak{s}_1$ *is the class of* $\mathbf{u}^{(1)}$, *then* $\mathfrak{s} - 2 \le \mathfrak{s}_1 \le \mathfrak{s}$.

Proof. Let a be a zero of $\phi_1(x)$. If $A_1(a) \neq 0$, we get the result. If $A_1(a) = 0$, then $B_1(a) = -B(a)$. If $B(a) \neq 0$ we stop the analysis. If $B_1(a) = 0$, then $C_1(a) = A(a) \neq 0$ necessarily, since in other case we would have a contradiction with the class of $\mathbf{u}$. Thus we have that ϕ_1, A_1, B_1 and C_1 are coprime.

Now, since $\mathbf{u}$ is of class $\mathfrak{s}$, then $\deg\phi \le \mathfrak{s}+2$, $\deg A \le \mathfrak{s}+2$ and $\deg\psi \le \mathfrak{s}+1$. Denote

$$\phi(x) = \sum_{k=0}^{\mathfrak{s}+2} \lambda_k x^k, \qquad B(x) = \sum_{k=0}^{\mathfrak{s}+1} \beta_k x^k, \qquad A(x) = \sum_{k=0}^{\mathfrak{s}+2} \alpha_k x^k. \tag{23}$$

Using (20), we have $\deg C \le \mathfrak{s}$. Moreover,

$$\begin{aligned}
C(x) := \sum_{k=0}^{\mathfrak{s}} c_k x^k &= -(\lambda_{\mathfrak{s}+2} + \beta_{\mathfrak{s}+1} + \alpha_{\mathfrak{s}+2})x^{\mathfrak{s}}\\
&\quad - (\lambda_{\mathfrak{s}+1} + 2b_0\lambda_{\mathfrak{s}+2} + \beta_{\mathfrak{s}} + \beta_{\mathfrak{s}+1}b_0 + \alpha_{\mathfrak{s}+1} + 2b_0\alpha_{\mathfrak{s}+2})x^{\mathfrak{s}-1} + \cdots.
\end{aligned}$$

On the other hand, taking into account that

$$\begin{aligned}
\psi_1(x) &= -[\phi'(x) - 2(x-b_0)C(x) - B(x)]\\
&= -[(\mathfrak{s}+2)\lambda_{\mathfrak{s}+2} - 2c_{\mathfrak{s}} - \beta_{\mathfrak{s}+1}]x^{\mathfrak{s}+1} - [(\mathfrak{s}+1)\lambda_{\mathfrak{s}+1} - 2c_{\mathfrak{s}-1} + 2b_0c_{\mathfrak{s}} - \beta_{\mathfrak{s}}]x^{\mathfrak{s}} + \cdots.
\end{aligned}$$

Then we can distinguish the following cases

(1) If $\lambda_{\mathfrak{s}+2} \neq 0$, then $\mathfrak{s}_1 = \mathfrak{s}$.

(2) If $\lambda_{\mathfrak{s}+2} = 0$ and $2c_{\mathfrak{s}} + \beta_{\mathfrak{s}+1} \neq 0$, then $\mathfrak{s}_1 = \mathfrak{s}$.

(3) If $\lambda_{\mathfrak{s}+2} = 0$ and $2c_{\mathfrak{s}} + \beta_{\mathfrak{s}+1} = 0$ we have the subcases

- **(3-1)** If $\lambda_{\mathfrak{s}+1} \neq 0$ then $\mathfrak{s}_1 = \mathfrak{s} - 1$.
- **(3-2)** If $\lambda_{\mathfrak{s}+1} = 0$ and $2c_{\mathfrak{s}-1} - 2b_0c_{\mathfrak{s}} + \beta_{\mathfrak{s}} \neq 0$ then $\mathfrak{s}_1 = \mathfrak{s} - 1$.
- **(3-3)** If $\lambda_{\mathfrak{s}+1} = 0$ and $2c_{\mathfrak{s}-1} - 2b_0c_{\mathfrak{s}} + \beta_{\mathfrak{s}} = 0$.

(3-3-1) In this case the leading coefficient of $A_1(x)$ reduces to $a_1 c_{\mathfrak{s}} = a_1(\beta_{\mathfrak{s}+1} + \alpha_{\mathfrak{s}+2})$. If zero, then from item (3) $\beta_{\mathfrak{s}+1} = 0$. In conclusion we would have $((\mathfrak{s}+2)\lambda_{\mathfrak{s}+1} + \beta_{\mathfrak{s}+1}) = \lambda_{\mathfrak{s}+2} = \alpha_{\mathfrak{s}+2} = 0$, which is contradictory with the class of $\mathbf{u}$. Thus $a_1 c_{\mathfrak{s}} \neq 0$ and $\mathfrak{s}_1 = \mathfrak{s} - 2$.

□

Corollary 4. *With the notation of (23), if $\mathfrak{s}$ and $\mathfrak{s}_1$ are the classes of $\mathbf{u}$ and $\mathbf{u}^{(1)}$, respectively, we get*

- *If $|\beta_{s+1} + \alpha_{s+2}| + |\lambda_{s+2}| \neq 0$, then $\mathfrak{s}_1 = \mathfrak{s}$.*
- *If $|\beta_{s+1} + \alpha_{s+2}| + |\lambda_{s+2}| = 0$ and $|\beta_s + 2(b_0\alpha_{s+2} + \alpha_{s+1})| + |\lambda_{s+1}| \neq 0$, then $\mathfrak{s}_1 = \mathfrak{s} - 1$.*
- *If $|\beta_{s+1} + \alpha_{s+2}| + |\lambda_{s+2}| + |\beta_s + 2(b_0\alpha_{s+2} + \alpha_{s+1})| + |\lambda_{s+1}| = 0$, then $\mathfrak{s}_1 = \mathfrak{s} - 2$.*

3.1. *Linear spectral transformation on Laguerre-Hahn functional*

Now we will deduce some results concerning the Christoffel and Geronimus transformation when the original linear functional $\mathbf{u}$ is Laguerre-Hahn.

Christoffel transformation

Theorem 6. *Let $\widetilde{\mathbf{u}} = (x-c)\mathbf{u}$. If $\mathbf{u}$ is Laguerre-Hahn functional of class $\mathfrak{s}$ with $\mathbf{u}_0 = \widetilde{\mathbf{u}}_0 = 1$ satisfying (19), then $\widetilde{\mathbf{u}}$ is also a Laguerre-Hahn functional satisfying the equation*

$$\widetilde{\phi}(z)\mathbf{S}'_{\widetilde{\mathbf{u}}}(z) = \widetilde{A}(z)\mathbf{S}^2_{\widetilde{\mathbf{u}}}(z) + \widetilde{B}(z)\mathbf{S}_{\widetilde{\mathbf{u}}}(z) + \widetilde{C}(z), \tag{24}$$

where

$$\begin{aligned}
\widetilde{\phi}(z) &= (z-c)\phi(z),\\
\widetilde{A}(z) &= A(z),\\
\widetilde{B}(z) &= \phi(z) + 2A(z) + (z-c)B(z),\\
\widetilde{C}(z) &= \phi(z) + 2A(z) + (z-c)B(z) + (z-c)^2C(z).
\end{aligned}$$

Moreover, the class of $\widetilde{\mathbf{u}}$, denoted by $\widetilde{\mathfrak{s}}$, satisfies $\mathfrak{s} - 2 \leq \widetilde{\mathfrak{s}} \leq \mathfrak{s} + 1$.

Proof. (24) is a direct consequence from the fact that you can write, by using (6), $\mathbf{S}_{\mathbf{u}}(z)$ in terms of $\mathbf{S}_{\widetilde{\mathbf{u}}}(z)$ and then you replace it in (19). Moreover, the linear functional $\widetilde{\mathbf{u}}$ satisfies the distributional equation

$$\mathcal{D}(\widetilde{\phi}(x)\widetilde{\mathbf{u}}) + \widetilde{\psi}(x)\widetilde{\mathbf{u}} - \widetilde{A}(x)(x^{-1}\widetilde{\mathbf{u}}^2) = 0,$$

with $\widetilde{\psi}(x) = (x-c)\psi(x) - 2[\phi(x) + A(x)]$ and $\psi(x) = -[\phi'(x) + B(x)]$. Thus, if $\widetilde{\mathfrak{s}}$ is the class of $\widetilde{\mathbf{u}}$, it follows from above that

$$\begin{aligned}
&\deg(\widetilde{\phi}) =: \widetilde{r} \leq \mathfrak{s} + 3, && \deg(\widetilde{\psi}) =: \widetilde{t} \leq \mathfrak{s} + 2,\\
&\widetilde{d} = \max\{\widetilde{r}, \widetilde{m}\} \leq \mathfrak{s} + 3, && \widetilde{\mathfrak{s}} \leq \max\{\widetilde{t} - 1, \widetilde{d} - 2\} \leq \mathfrak{s} + 1,
\end{aligned}$$

where $\deg \widetilde{B}(x) = \widetilde{m}$.

On the other hand, since $\widetilde{\mathbf{u}}$ is a Laguerre-Hahn functional of class $\widetilde{\mathfrak{s}}$, then there exist polynomials $\overline{\phi}(x)$, $\overline{\psi}(x)$ and $\overline{A}(x)$ such that

$$\mathcal{D}(\overline{\phi}(x)\widetilde{\mathbf{u}}) + \overline{\psi}(x)\widetilde{\mathbf{u}} + \overline{A}(x)(x^{-1}\widetilde{\mathbf{u}}^2) = 0, \tag{25}$$

and $\widetilde{\mathfrak{s}} := \max\{\deg \overline{\psi}(x) - 1,\ \max\{\deg \overline{\phi}(x), \deg \overline{A}(x)\} - 2\}$. Using (6) again, and taking into account (25), we have that $\mathbf{u}$ also satisfies the distributional equation

$$\mathcal{D}((x-c)\overline{\phi}(x)\mathbf{u}) + (x-c)\left[\overline{\psi}(x) + 2\overline{A}(x)\right]\mathbf{u} + (x-c)^2\overline{A}(x)(x^{-1}\mathbf{u}^2) = 0.$$

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With this in mind

$$\deg(\phi) = r \leq \tilde{\mathfrak{s}} + 3, \qquad \deg(\psi) = t \leq \tilde{\mathfrak{s}} + 2,$$
$$d = \max\{r, m\} \leq \tilde{\mathfrak{s}} + 4, \qquad \mathfrak{s} \leq \max\{t-1, d-2\} \leq \tilde{\mathfrak{s}} + 2.$$

As a conclusion, $\mathfrak{s} - 2 \leq \tilde{\mathfrak{s}} \leq \mathfrak{s} + 1$. □

The above gives bounds for the class of $\tilde{\mathbf{u}}$. However, the following result show that the class only depends on the value c and never take the value $\mathfrak{s} - 2$.

Theorem 7 ([21]). *Let $\tilde{\mathbf{u}} = (x-c)\mathbf{u}$ be a linear functional such that $\mathbf{u}_0 = \tilde{\mathbf{u}}_0 = 1$ and where $\mathbf{u}$ is of class $\mathfrak{s}$. If $\tilde{\mathfrak{s}}$ is the class of $\tilde{\mathbf{u}}$, then*

- $\tilde{\mathfrak{s}} = \mathfrak{s} + 1$, *if $\phi(c) \neq 0$ and $A(c) \neq 0$.*
- $\tilde{\mathfrak{s}} = \mathfrak{s}$, *if $\phi(c) = A(c) = 0$, $\psi(c) \neq 0$ and $A'(c) \neq 0$.*
- $\tilde{\mathfrak{s}} = \mathfrak{s} - 1$, *if $\phi(c) = A(c) = \psi(c) = A'(c) = 0$.*

Geronimus transformation

Theorem 8. *Let $\hat{\mathbf{u}}$ be the linear functional defined by $(x-c)\hat{\mathbf{u}} = \mathbf{u}$. Assume that $\mathbf{u}_0 = \hat{\mathbf{u}}_0 = 1$. If $\mathbf{u}$ is a Laguerre-Hahn functional of class $\mathfrak{s}$ satisfying (19), then so is $\hat{\mathbf{u}}$. In this case*

$$\hat{\phi}(z) S'_{\hat{\mathbf{u}}}(z) = \hat{A}(z) S^2_{\hat{\mathbf{u}}}(z) + \hat{B}(z) S_{\hat{\mathbf{u}}}(z) + \hat{C}(z), \tag{26}$$

where

$$\hat{\phi}(z) = (z-c)\phi(z), \tag{27}$$
$$\hat{A}(z) = (z-c)^2 A(z), \tag{28}$$
$$\hat{B}(z) = -[\phi(z) + 2(z-c)A(z)] + (z-c)B(z),$$
$$\hat{C}(z) = A(z) - B(z) + C(z).$$

The class $\hat{\mathfrak{s}}$ of $\hat{\mathbf{u}}$ depends only on the zero $x = c$. Moreover, $\mathfrak{s} - 1 \leq \hat{\mathfrak{s}} \leq \mathfrak{s} + 2$.

Proof. Let a be a zero of $\phi(x)$, then

$$\hat{\phi}'(a) + \hat{\psi}(a) = (a-c)(\phi'(a) + \psi(a) + 2A(a)), \quad \hat{A}(a) = (a-c)^2 A(a) \tag{29}$$

and

$$\langle \mathbf{u}, \theta_a \hat{\psi}(x) + \theta_a^2 \hat{\phi}(x) - (\mathbf{u} * \theta_0 \theta_a \hat{A})(x) \rangle = \langle \mathbf{u}, \theta_a \psi(x) + \theta_a^2 \phi(x) - (\mathbf{u} * \theta_0 \theta_a A)(x) \rangle + \psi(a) + \phi'(a) + A(a), \tag{30}$$

where

$$\hat{\psi}(x) = (x-c)[\psi(x) + 2A(x)]. \tag{31}$$

Observe that from (27), (28) and (31) we get (29) in a straightforward way. Now, to find (30) let us notice that

$$\theta_a \hat{\psi}(x) = (x-c)\theta_a \psi(x) + 2\hat{\mathbf{u}}_0 \theta_a A(x) + \psi(a) + 2\hat{\mathbf{u}}_0 A(a), \tag{32}$$
$$\theta_a^2 \hat{\phi}(x) = (x-c)\theta_a^2 \phi(x) + \phi'(a). \tag{33}$$

On the other hand, using (28) and Proposition 1 (ii) we get

$$(\theta_0 \theta_a A)(x) = \theta_0((x-c)^2 \theta_a A(x)) + A(a)$$
$$= (x-c)^2 (\theta_0 \theta_a A)(x) + (x-2c)(\theta_a A)(0) + A(a).$$

Thus from the above and Proposition 1 (iii) and (iv)

$$\left\langle \widehat{\mathbf{u}}^2, (x-c)^2\theta_0\theta_a A(x)\right\rangle = \left\langle \mathbf{u}^2, (\theta_0\theta_a A)(x)\right\rangle + 2\widehat{\mathbf{u}}_0\langle x\mathbf{u}, \theta_0\theta_a A(x)\rangle,$$
$$\left\langle \widehat{\mathbf{u}}^2, (x-2c)(\theta_0 A)(0)\right\rangle = \widehat{\mathbf{u}}_0((2-c)+x\widehat{\mathbf{u}}_0)(\theta_a A)(0),$$
$$\left\langle \widehat{\mathbf{u}}^2, A(a)\right\rangle = \widehat{\mathbf{u}}_0 A(a).$$

Taking into account (32), (33) and the fact that $(\theta_a A)(x) - (\theta_a A)(0) = x\theta_0\theta_a A(x)$ we get (30). Using the above we obtain that for any zero $a \neq c$ of $\phi(x)$

$$|\widehat{\psi}(a) + \widehat{\phi}'(a)| + |\widehat{A}(a)| + \left|\langle \mathbf{u}, \theta_a\widehat{\psi}(x) + \theta_a^2\widehat{\phi}(x) - (\mathbf{u} * \theta_0[\theta_a\widehat{A}(x)])\rangle\right| \neq 0.$$

The proof of the second part is essentially the same as the one given in Theorem 6 and, as a consequence, we do not deal with. However, we point out that $\widehat{\mathbf{u}}$ satisfies the distributional equation

$$\mathcal{D}((x-c)\phi(x)\widehat{\mathbf{u}}) + (x-c)[\psi(x) + 2\widehat{\mathbf{u}}_0 A(x)]\widehat{\mathbf{u}} + (x-c)^2 A(x)(x^{-1}\widehat{\mathbf{u}}^2) = 0.$$

□

Proposition 9. *Let* $(x-c)\widehat{\mathbf{u}} = \mathbf{u}$ *and let* $\mathfrak{s}$ *and* $\widehat{\mathfrak{s}}$ *be the class of* $\mathbf{u}$ *and* $\widehat{\mathbf{u}}$, *respectively. Let us define*

$$g = \begin{cases} \mathfrak{s}+2, & \deg A = \mathfrak{s}+2, \\ \mathfrak{s}+1, & \deg A < \mathfrak{s}+2. \end{cases}$$

Then

$$\phi(c) \neq 0 \Rightarrow \widehat{\mathfrak{s}} = g$$

$$\phi(c) = 0 \Rightarrow \begin{cases} A(c) - B(c) + C(c) \neq 0 \Rightarrow \widehat{\mathfrak{s}} = g, \\ A(c) - B(c) + C(c) = 0 \Rightarrow [1], \end{cases}$$

$$[1] \Rightarrow \begin{cases} -[\phi'(a) + 2A(c)] + B(c) \neq 0 \Rightarrow \widehat{\mathfrak{s}} = g-1, \\ -[\phi'(a) + 2A(c)] + B(c) = 0 \Rightarrow \begin{cases} A'(c) - B'(c) + C'(c) \neq 0 \Rightarrow \widehat{\mathfrak{s}} = g-1, \\ A'(c) - B'(c) + C'(c) = 0 \Rightarrow [2], \end{cases} \end{cases}$$

$$[2] \Rightarrow \begin{cases} \phi'(c) \neq 0 \Rightarrow \widehat{\mathfrak{s}} = g-2, \\ \phi'(c) = 0 \Rightarrow \begin{cases} A(c) \neq 0 \Rightarrow \widehat{\mathfrak{s}} = g-2, \\ A(c) = 0 \text{ is not posible.} \end{cases} \end{cases}$$

Proof. Notice that $\deg \widehat{\phi}(x) \leq \mathfrak{s}+3$, $\deg \widehat{A}(x) \leq \mathfrak{s}+4$, and $\deg \widehat{\psi}(x) \leq \max\{\mathfrak{s}+1, \mathfrak{s}+2\}$, where $\widehat{\phi}(x) = (x-c)\phi(x)$, $\widehat{A}(x) = (x-c)^2 A(x)$ and $\widehat{\psi}(x) = (x-c)(\psi(x) + 2A(x))$. Since

$$\widehat{\mathfrak{s}} = \max\left\{\deg \widehat{\psi}(x) - 1, \max\{\deg \widehat{\phi}(x), \deg \widehat{A}(x)\} - 2\right\},$$

if $\phi(c) \neq 0$, then there are two possibilities. If $\deg(A) = \mathfrak{s}+2$ then $\widehat{\mathfrak{s}} = \mathfrak{s}+2$. If $\deg(A) < \mathfrak{s}+2$, then $\widehat{\mathfrak{s}} = \mathfrak{s}+1$. Now if $\phi(c) = 0$, and $A(c) - B(c) + C(c) = 0$, then we can divide both hand sides in (26) by $(z-c)$,

$$\phi(z)S'_{\widehat{\mathbf{u}}}(z)$$

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$$= (z-c)A(z)\mathbf{S}^2_{\widehat{\mathbf{u}}}(z) + \left(-\left[\frac{\phi(z)}{(z-c)} + 2A(z)\right] + B(z)\right)\mathbf{S}_{\widehat{\mathbf{u}}}(z) + \frac{A(z)-B(z)+C(z)}{(z-c)}.$$

If $-[\phi'(a)+2A(c)]+B(c)\neq 0$, then $\widehat{\mathfrak{s}} = g-1$. On the other hand, if

$$-[\phi'(a)+2A(c)]+B(c) = 0 \tag{34}$$

and $A'(c)-B'(c)+C'(c)=0$, then in (26) we can divide both hand sides by $(z-c)^2$ and, as a consequence,

$$\frac{\phi(z)}{z-c}\mathbf{S}'_{\widehat{\mathbf{u}}}(z) = A(z)\mathbf{S}^2_{\widehat{\mathbf{u}}}(z) + \left(-\frac{\phi(z)}{(z-c)^2} + \frac{[2A(z)+B(z)]}{(z-c)}\right)\mathbf{S}_{\widehat{\mathbf{u}}}(z) + \frac{A(z)-B(z)+C(z)}{(z-c)^2}.$$

If $\phi'(c)\neq 0$, then $\widehat{\mathfrak{s}} = g-2$. Finally if $\phi'(c)=0$, then$A(c)\neq 0$ since, otherwise, $B(c)=0$ by (34). This yields the equation (21) is reducible contradicting the class of $\mathbf{u}$. □

4. Example

Example 1. *Let $(L_n^{\alpha+1})_{n\geq 0}$ be the monic Laguerre polynomials of parameter $\alpha+1$ with $\alpha > -1$, which are orthogonal with respect to the positive definite linear functional* $\mathbf{v}$

$$\langle \mathbf{v}, p(x)\rangle = \frac{1}{\Gamma(\alpha+2)}\int_0^\infty p(x)x^{\alpha+1}e^{-x}dx, \quad p(x)\in\mathbb{P}.$$

The following properties are very well known in the literature ([12])

i) *Recurrence relation.*

$$xL_n^{\alpha+1}(x) = L_{n+1}^{\alpha+1}(x) + (2n+\alpha+2)L_n^{\alpha+1}(x) + n(n+\alpha+1)L_{n-1}^{\alpha+1}(x), \quad n\geq 0,$$
$$L_0^{\alpha+1}(x) = 1, \qquad L_{-1}^{\alpha+1}(x) = 0.$$

ii) *Explicit formula as an hypergeometric function*

$$L_n^{\alpha}(x) = \frac{n!}{(-1)^n}\sum_{k=0}^{n}\frac{\Gamma(n+\alpha+1)}{(n-k)!\Gamma(\alpha+k+1)}\frac{(-x)^k}{k!}.$$

iii) $\left(\dfrac{d^i}{x^i}L_n^{\alpha+1}\right)(0) = (-1)^{n+i}\dfrac{n!\Gamma(\alpha+n+2)}{(n-i)!\Gamma(\alpha+i+2)}.$

iv) $\langle \mathbf{v}, L_n^{\alpha+1}(x)L_m^{\alpha+1}(x)\rangle = \dfrac{n!\Gamma(n+\alpha+2)}{\Gamma(\alpha+2)}\delta_{n,m}.$

The linear functional $\mathbf{v}$ satisfies the distributional equation [13] $D(x\mathbf{v}) + (x-\alpha-2)\mathbf{v} = 0$, and, as a consequence, it is a Laguerre-Hahn functional of class $\mathfrak{s}(\mathbf{v}) = 0$. Its Stieltjes function satisfies the first order linear differential equation

$$z\mathbf{S}'_{\mathbf{v}}(z) = (-z+\alpha+1)\mathbf{S}_{\mathbf{v}}(z) + 1. \tag{35}$$

In [4] the authors studied the first kind associated Laguerre polynomials which are denoted by $(L_n^{\alpha+1}(x,1))_{n\geq 0}$. In particular it was proved that these polynomials are orthogonal with respect to the positive definite functional $\mathbf{v}^{(1)}$ defined by

$$\left\langle \mathbf{v}^{(1)}, p(x)\right\rangle = \frac{1}{\Gamma(\alpha+3)}\int_0^\infty p(x)\frac{x^{\alpha+1}e^{-x}}{\left|\Psi(1,-\alpha,xe^{-\pi i})\right|^2}dx,$$

where

$$\Psi(c,a,x) = \frac{1}{\Gamma(c)}\int_0^{\infty e^{(3\pi/4)i}} t^{c-1}(1+t)^{a-c-1}e^{-xt}dt,$$

$\mathrm{Re}(c) > 0, \quad -\pi/2 < 3\pi/4 + \arg x < \pi/2.$

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The associated monic orthogonal polynomials of the first kind $(L_n^{\alpha+1}(x,1))_{n\geq 0}$ satisfy the following properties

i) *Explicit formula.*

$$L_n^{\alpha+1}(x,1) = (-1)^n(n+1)(\alpha+3)_n \times \sum_{k=0}^{n} \frac{(-n)_k x^k}{(k+1)!(\alpha+3)_k} \times {}_3F_2\left(\begin{matrix} k-n,\ 1,\ \alpha+2 \\ \alpha+k+3,\ k+2 \end{matrix} ;1 \right).$$

ii) $L_n^{\alpha+1}(0,1) = \dfrac{(-1)^n}{\alpha+1}[(\alpha+2)_{n+1} - (n+1)!]$.

iii) $\left\langle \mathbf{v}^{(1)}, L_n^{\alpha+1}(x,1)L_m^{\alpha+1}(x,1) \right\rangle = \dfrac{(n+1)!\Gamma(n+\alpha+3)}{\Gamma(\alpha+3)}\delta_{n,m}$.

Using Theorem 5 and (35) we get that the functional $\mathbf{v}^{(1)}$ is Laguerre-Hahn of class zero satisfying the distributional equation

$$D(x\mathbf{v}^{(1)}) + (x-\alpha-4)\mathbf{v}^{(1)} - (\alpha+2)\left(x^{-1}\left[\mathbf{v}^{(1)}\right]^2\right) = 0.$$

From Proposition 7 we have the structure relation

$$x\frac{d}{dx}L_{n+1}^{\alpha+1}(x,1) + (\alpha+2)L_n^{\alpha+1}(x,2) = \sum_{k=n}^{n+1} \lambda_{n,k}L_k^{\alpha+1}(x,1); \quad n \geq 1.$$

Using (2) and comparing the coefficients on both sides we get

$$\lambda_{n,n+1} = (n+1), \quad \lambda_{n,n} = (n+\alpha+3)(n+2),$$

as well as the relation

$$x\frac{d}{dx}L_{n+1}^{\alpha+1}(x,1) - L_{n+2}^{\alpha+1}(x) = (-x+\alpha+n+3)L_{n+1}^{\alpha+1}(x,1) + (n+\alpha+3)(n+2)L_n^{\alpha+1}(x,1), \quad n \geq 1.$$

Besides from (22), its Stieltjes function satisfies the differential equation

$$z\mathbf{S}'_{\mathbf{v}^{(1)}}(z) = (\alpha+2)\mathbf{S}^2_{\mathbf{v}^{(1)}}(z) + (-z+\alpha+3)\mathbf{S}_{\mathbf{v}^{(1)}}(z) + 1.$$

Next, let $\widehat{\mathbf{v}}$ be the linear functional defined by the Geronimus transformation $x\widehat{\mathbf{v}} = \mathbf{v}$ with $\widehat{\mathbf{v}}_0 = \dfrac{1}{(\alpha+1)}$. Then from (2.2) with $c = 0$ we get

$$\begin{aligned} \langle \widehat{\mathbf{v}}, p(x) \rangle &= \frac{1}{\Gamma(\alpha+2)} \int_0^\infty (p(x)-p(0))x^\alpha e^{-x}\,dx + \frac{p(0)}{\Gamma(\alpha+2)} \int_0^\infty x^\alpha e^{-x}\,dx \\ &= \frac{1}{\Gamma(\alpha+2)} \int_0^\infty p(x)x^\alpha e^{-x}dx, \quad p(x) \in \mathbb{P}. \end{aligned}$$

If $J_{\alpha+1}$ is the monic Jacobi matrix associated with $\mathbf{v}$, then $J_{\alpha+1}$ has UL factorization as in (13) with $\beta_n = \alpha+n+1$, $n \geq 0$, $\ell_n = n$, $n \geq 1$. Therefore

$$J_{\alpha+1} = UL \longmapsto \widehat{J}_{\alpha+1} =: LU = J_\alpha.$$

Notice that $\widehat{L_n^{\alpha+1}}(x) = L_n^\alpha(x)$ for all $n \in \mathbb{N}$. Now, let $\widehat{\mathbf{v}^{(1)}}$ be the Geronimus transformation of $\mathbf{v}^{(1)}$ obtained from (15), then

$$\left\langle \widehat{\mathbf{v}^{(1)}}, p(x) \right\rangle = \frac{1}{\Gamma(\alpha+3)} \int_0^\infty (p(x)-p(0)) \frac{x^\alpha e^{-x}}{|\Psi(1,-\alpha,xe^{-\pi i})|^2}dx + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+3)}p(0).$$

Let $(P_n(x,1))_{n\geq 0}$ be the SMOP with respect to $\widehat{\mathbf{v}^{(1)}}$. Then from Corollary 2 we get the three term recurrence relation

$$xP_n(x,1) = P_{n+1}(x,1)(x) + (2n+\alpha+1)P_n(x,1) + (n+1)(n+\alpha+1)P_{n-1}(x,1), \quad n \geq 1,$$
$$P_0(x,1) = 1 \qquad P_1(x,1) = x - (\alpha+2),$$

as well as the following connection formula

$$P_n(x,1) = \frac{1}{\alpha+1}\left(xL_n^{\alpha}(x,1) - L_{n+1}^{\alpha}(x)\right) \quad n \geq 2.$$

Using Theorem 8 we get that the linear functional $\mathbf{w} := \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+2)}\widehat{\mathbf{v}^{(1)}}$ is a Laguerre-Hahn functional and

$$z^2 S'_{\mathbf{w}}(z) = (\alpha+2)z^2 S^2_{\mathbf{w}}(z) - z(z+\alpha+2)S_{\mathbf{w}}(z) + z.$$

Note that it equation is reducible to

$$zS'_{\mathbf{w}}(z) = (\alpha+2)zS^2_{\mathbf{w}}(z) - (z+\alpha+2)S_{\mathbf{w}}(z) + 1.$$

Taking into account that the polynomials are coprime, then the class of $\mathbf{w}$ is also zero. Since $\widehat{\psi}_1(x) = x + (\alpha+1)$, then $\mathbf{w}$ satisfies the distributional equation

$$D(x\mathbf{w}) + (x+\alpha+1)\mathbf{w} - (\alpha+2)x\left(x^{-1}[\mathbf{w}]^2\right) = 0.$$

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