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Parametric Identifiability of Dynamic Systems Based on Adaptive Systems

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Parametric Identifiability of Dynamic Systems Based on Adaptive Systems

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Abstract. Many publications have been devoted to the problem of parametric identifiability (PI). The major focus is on a priori identifiability. The parametric identifiability problem using experimental data (the so-called practical identifiability (PID)) is less studied. This is a parametric identification task. PI has not been studied using current data (in adaptive systems). We propose an approach to estimating PI based on the application of Lyapunov functions. The role of the excitation constancy is shown. Conditions of local parametric identifiability (LPI) for a class of linear dynamical systems are got based on current experimental data. The case is considered when both the state vector and the input-output set are measured. Estimates are obtained for the parametric residual. Case of limiting LPI on the set of current data is studied. Influence of initial conditions on PI is analysed. The case of m-parametric identifiability is studied. Approach to estimating the PI of linear dynamical systems and systems with periodic coefficients based on the application of Lyapunov exponents is proposed. The LPI of decentralised systems is analysed. Examples are given.

Keywords: parametric identifiability; periodic dynamical system; lyapunov function; adaptive algorithm; decentralized system; nonlinearity; quadratic condition; Lyapunov exponent

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1. Introduction

Estimation of the model parameters is possible if the conditions guaranteeing their receipt are met. Many publications have been devoted to the issues of identifiability (see, for example, [1-10]). Much attention is paid to the analysis of a priori identifiability (AI) (in the literature, it is structural identifiability). AI conditions often have an algebraic form. To obtain them, such approaches are used as differential algebra [11], time series analysis [12] and some others [4,13-15]. The observability role [16] in identifiability problems is noted.

Some authors study the identifiability problem based on experimental data (see review [4]). This is practical identifiability. PID is based on obtaining a mathematical model and verifying it. This approach gives good results for systems with a known structure. In [17], low-order models are used to solve the problem of unidentifiable parameters. This approach is based on performing many adjustments.

Statistical hypotheses and criteria are used to solve the problem of estimating unidentifiable parameters. The probability profile parameter is used in [18]. Markov chains based on the Monte Carlo method [19] are used to estimate unidentifiable parameters. The apply of these approaches is associated with certain difficulties.

The Fisher information matrix is used to solve of PID problems [20]. Other statistical approaches are discussed in [4]. The result of solving the PID problem is the model with an accurate forecast. If this is not true, then the structural identification problem is solved. A more complete analysis of the state of the PID problem is given in the review [4]. Note that the PID problem interpretation does not accurately reflect the problem. This is the parametric identification problem with decision-making elements.

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As follows from the presented analysis, the emphasis is on the study AI problem. Practical identifiability has not been sufficiently investigated. The focus is on synthesizing a mathematical model using various methods and evaluating its predictive properties. Various statistics, methods, and criteria are used to decision-making about the PRI. If the parametric identifiability condition is not met, various multistep procedures are proposed. These approaches are not always effective. For a more complex class of systems (multidimensional, decentralized, and interconnected), this problem requires further investigation. PI issues were not considered in adaptive systems.

In this paper, we study the PI problem for a class of adaptive models. The approach is proposed for obtaining conditions of local PI based on a class of adaptive algorithms. Conditions for limiting LPI are obtained. We show the dependence of adaptive identification system (ASI) properties on the initial conditions. A generalization of the results is given for the case of m-parametric identifiability. The linear system case with periodic parameters is considered. The PI problem solution is reduced to the application of Lyapunov exponents.

2. Problem Statement

Consider the system

$$\begin{aligned} \dot{X} &= \bar{A}X + Bu, \\ y &= C^T X, \end{aligned} \qquad (1)$$

where $u \in \mathbb{R}$, $y \in \mathbb{R}$ are input and output, $X \in \mathbb{R}^n$ is the state vector, $C = [1\ 0\ ....\ 0]^T$, $B = [0\ 0\ ....\ 0\ b]^T$, $\bar{A} \in \mathbb{R}^{n \times n}$.

Set of experimental data

$$\mathbb{I}(t) = \{u(t),\ y(t),\ t \in \mathbb{J} = [t_0, t_k]\}. \qquad (2)$$

Assumption 1. $\bar{A}$ is Hurwitz matrix.

Problem. Evaluate the system (1) parametric identifiability using the of the set $\mathbb{I}(t)$ analysis.

3. Approach to PI Estimation

The representation is valid for the system (1) in space (u, y):

$$\dot{y} = A^T P, \qquad (4)$$

where $A \in \mathbb{R}^{2n}$ is the vector of parameters, $P \in \mathbb{R}^{2n}$ is the generalised input vector, which is got based on the processing (u, y) by a system of auxiliary filters.

To evaluate elements of vector A, introduce the model based on the set $\mathbb{I}_t = \{u(t),\ y(t)\}$ for each t:

$$\dot{\hat{y}} = -k(y - y) + \hat{A}^T P, \qquad (5)$$

where $\hat{A} \in \mathbb{R}^{2n}$ is the vector of model parameters, $k > 0$ is the parameter setting the properties of the model.

Equation for identification error (prediction) $e = \hat{y} - y$:

$$\dot{e} = -\kappa e + \Delta A^T P, \qquad (6)$$

where $\Delta A=\hat{A}-A$.

Let the elements of the vector $p_i \in P$ by constantly excited (CE):

$$\mathscr{CE}_{p_i}: \underline{\alpha}_i \le p_i^2(t) \le \bar{\alpha}_i \quad \forall t \in [t_0, T], \tag{7}$$

where $\underline{\alpha}_i > 0,\ \bar{\alpha}_i > 0$.

Notation:

(i) $\mathscr{S}_X(\bar{A})$ is a class of systems (1);

(ii) $\mathscr{S}_y(A)$ is a congruent representation (4) on the set $\mathbb{I}_t=\{u(u),\ y(t),\ t>t_0\}$;

(iii) Ω_i is the frequency spectrum of the element $p_i \in P$;

(iv) $p_i \in \mathscr{CE}_{p_i}$ or $p_i \in \mathscr{CE}$;

(v) $p_i \notin \mathscr{CE}$ if the CE condition is not held true.

(vi) the variable is $u(t) \in \mathscr{CE}$ if it has a non-degenerate frequency spectrum for all $t \in \mathbb{J}$.

Definition 1. The system (1) of class $\mathscr{S}_X(\bar{A})$ is locally parametrically identifiable on the set $\mathbb{I}_t$ if the condition

$$A \in \mathbb{G}_A=\left\{A \in \mathbb{R}^{2n}: \left\|A-A^*\right\| \le \varepsilon_A \ \forall t \ge t^* > t_0 \ \&\ \mathbb{I}_t\right\}, \tag{8}$$

is fulfilled for its representation (4) in class $\mathscr{S}_y(A)$, where A^* is some reference vector of system parameters (4), $\varepsilon_A \ge 0$.

We see if there is an identification algorithm vector $\hat{A}$ of the system (5) on $\mathbb{I}(t)=\{\mathbb{I}_t,\ t>t_0\}$ for some $\{A(t_{0,i})\}$, then starting from the moment t^*, the condition (8) will be fulfilled for the estimates of vector A.

Consider the Lyapunov function $V_e(t)=0.5e^2(t)$.

Theorem 1. Let 1) assumption 1 holds, i.e. the matrix in (1) $\bar{A} \in \mathscr{H}$; 2) the system (1) represents how (4) to set $\mathbb{I}(t)$; 3) the identification system is described by equation (6); 4) $u \in \mathscr{CE}_u$; $y \in \mathscr{CE}_y$, $P \in \mathscr{CE}_P$. Then the system (4) is locally parametrically identifiable in the region $\mathbb{G}_A$ if $\Delta A=0$ follows from $\Delta A^T P=0$ and the condition is satisfied:

$$\left\|\Delta A(t)\right\|^2 \le \frac{2k^2}{\bar{\alpha}_P} V_e(t), \tag{9}$$

where $P(t)P^T(t) \le \bar{\alpha}_P I_n,\ \bar{\alpha}_P > 0$, $I_n \in \mathbb{R}^{n \times n}$ *is the unit matrix.*

The proof of Theorem 1 is given in Appendix A.

Remark 1. The vector $X(t)$ reconstruction in (1), based on (4) and schemes proposed in the literature, gives estimates that do not correspond to components $X(t)$. This follows directly from (4). Therefore, adaptive control laws based on the use of vector P elements are applied in control systems. Estimation $x_2=\dot{y}$, $x_2 \in X$ can be obtained directly from (4). The remaining components $X(t)$ are determined based on the symbolic differentiation operation.

Corollary from Theorem 1. *If the adaptive algorithm*

$$\Delta \dot{A}=-\Gamma e P, \tag{10}$$

is used to evaluate the vector A in (4), then the local parametric identifiability of the system (1) follows from the estimation

$$V(t) \le V(t_0) - 2k\int_{t_0}^{t} V_e(\tau)d\tau,$$

where $V(t)=V_e(t)+V_\Delta(t)$, $V_\Delta(t)=0.5\Delta A^T \Gamma^{-1} \Delta A$, $\Gamma=\Gamma^T>0$ *is a diagonal matrix.*

Let vector $P(t)$ elements be measurable for each t. Here, the system (4) is detectable. Then observability, detectability and recoverability of the system (1) follow from properties of system (4).

The proof corollary from Theorem 1 is given in Appendix B.

Consider the Lyapunov (LF) function $V_E = 0.5E^T RE$, where $R = R^T > 0$.

Structures of classes $\mathcal{S}_X(\bar{A})$ and $\mathcal{S}_y(A)$ are congruent, so the following statement is valid for system (1).

Theorem 2. *Let 1) the conditions of Theorem 1 are fulfilled for the system (4) of class* $\mathcal{S}_y(A)$; *2) classes* $\mathcal{S}_X(\bar{A})$ *and* $\mathcal{S}_y(A)$ *are congruent; 3)* $u \in \mathcal{CE}_u$, $X(t) \in \mathcal{CE}_X$. *Then system (1) is locally parametrically identifiable on class* $\mathcal{S}_X(\bar{A})$ *if*

$$\left\|\Delta\bar{A}\right\|^2 \bar{\alpha}_X + \left\|\Delta B\right\|^2 \bar{\alpha}_u \leq 4\mu^2 V_E,$$

where $E^T QE \geq \mu E^T RE$, $\mu > 0$, $RK + K^T R = -Q$, $Q = Q^T > 0$, $\left\|\Delta\bar{A}\right\| = \mathrm{tr}\left(\Delta\bar{A}^T \Delta\bar{A}\right)$, tr is *the matrix trace.*

The proof of Theorem 2 is given in Appendix C.

Consider the adaptive model

$$\dot{\hat{X}} = -KE + \hat{\bar{A}}X + Bu, \tag{11}$$

and apply the integral algorithm class

$$\begin{aligned} \dot{\hat{\bar{A}}} &= -\Gamma_{\bar{A}} ERX^T \\ \Delta\dot{\hat{B}} &= -\Gamma_B REu \end{aligned} \tag{12}$$

to tuning of matrix $\hat{\bar{A}}$, $\hat{B}$.

The identification system is described by the equation:

$$\dot{E} = KE + \Delta AX + \Delta Bu, \tag{13}$$

where $E = \hat{X} - X$, $\hat{X} \in \mathbb{R}^n$ is model (11) state vector, $K \in \mathcal{H}$ is the matrix of dimension $n \times n$, $\hat{\bar{A}}, \hat{B}$ are tuning matrices.

Corollary from Theorem 2. If the conditions of Theorem 2 are fulfilled, and the class of algorithms (12) is used to tuning the model (11) parameters, then the local parametric identifiability of the system (1) follows from the estimation

$$V_W(t) \leq V_W(t_0) - 2\mu \int_{t_0}^{t} V_E(\tau) d\tau. \tag{14}$$

The proof corollary from Theorem 1 is given in Appendix D.

We see that the local PI depends on the choice of initial conditions, and fulfilment of the requirements for variables and system input.

Remark 2. Presented results differ from the results [4] based on the application of AI methods. If the decision is made based on experimental data, then various statistics [4] are used. In this paper, we apply the approach to the PI analysis based on the current data analysis. This approach has not been used in PI tasks.

If conditions of Theorem 2 are fulfilled, then the class of algorithms (12) will be called locally identifying.

In the future, for the convenience of reference, the adaptive algorithm (10) will be related to class $\mathcal{I}_{IA}^{y}$, and the law (12) to class $\mathcal{I}_{IA}^{X}$.

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Definition 2. A system (1) of class $\mathcal{S}_X(\bar{A})$ is extremely locally parametrically identifiable (ELPI) on the set $\mathbb{I}_t$ if the condition

$$A \in \tilde{\mathbb{G}}_A = \left\{ A \in \mathbb{R}^{2n} : \lim_{t\to\infty} \left\| A - A^* \right\| \to \tilde{\varepsilon}_0\ \tilde{\varepsilon} \in \mathbb{O}(0)\ \forall t \geq t^* > t_0\ \&\ \mathbb{I}_t \right\}, \quad (15)$$

is satisfied for its representation (4) in class $\mathcal{S}_y(A)$, where A^* is some reference vector of system (5) parameters, $\mathbb{O}(0)$ is an area of zero.

Here, the vector A identifiability is understood as the limit proximity to A^*. Under certain conditions, the global PI of the vector A follows from (15).

Consider again the system (6) and LF $V_\Delta(t) = \Delta A^T(t)\Gamma^{-1}\Delta A(t)$.

Theorem 3. Let the conditions of Theorem 1 be fulfilled and (i) there is a Lyapunov function V_Δ admitting an infinitesimal upper limit; (ii) there is $\vartheta > 0$ such that the condition $e\Delta A^T P = \vartheta\left(\left\|\Delta A\right\|^2 + e^2\right)$ is satisfied for sufficiently large t in some area of zero; (iii) $P \in \mathcal{CE}_P$ with parameters $\underline{\alpha}_P, \bar{\alpha}_P$; (iv) The inequality

$$\dot{V}_\Delta \leq -\frac{3}{4}\vartheta\underline{\alpha}_P\underline{\lambda}_\Gamma V_\Delta + \frac{4}{3\underline{\alpha}_P}\vartheta\bar{\alpha}_P V_e, \quad (16)$$

is valid for the trajectories of the adaptive system (6) and (10), where $\underline{\lambda}_\Gamma$ is the minimum eigenvalue of the matrix Γ. Then the system (6), (10) is locally parametrically identifiable on the set $\mathbb{I}_t$ with estimating $V_\Delta(t) \leq S_\Delta(t)$ if the functional condition

$$\frac{16}{9\underline{\alpha}_P^2\underline{\lambda}_\Gamma}\bar{\alpha}_P V_e \leq V_\Delta, \quad (17)$$

is satisfied, where:

$$S_\Delta(t) = e^{-\sigma(t-t_0)}S_\Delta(t_0) + \pi\int_{t_0}^{t} e^{-\sigma(t-\tau)}V_e(\tau)d\tau \quad (18)$$

is the upper solution of the comparison system $\dot{S}_\Delta = -\sigma S_\Delta + \pi V_e$ *for (16) if* $S_\Delta(t_0) \geq V_\Delta(t_0)$,

$\pi = \frac{4}{3\underline{\alpha}_P}\vartheta\bar{\alpha}_P$, $\sigma = 0.75\vartheta\underline{\alpha}_P\underline{\lambda}_\Gamma$.

The proof of Theorem 3 is given in Appendix E.

We see that the PI in the class of algorithms (10) or (12) depends on the initial conditions and properties of the information set. LPI is guaranteed for systems of class $\mathcal{S}_X(\bar{A})$ and $\mathcal{S}_y(A)$ with asymptotic stability by error. However, estimate elements of the matrices will belong to the domain $\mathbb{G}_A$. This is a typical state of adaptive identification systems based on the class of algorithms (10), (12).

Remark 3. The region $\mathbb{G}_A$ can be compressed to $\tilde{\mathbb{G}}_A$ and limiting conditions for LPI can be got if conditions (9) or (17) for ASI are fulfilled. In real-world conditions, ASI guarantees almost extremely local parametric identifiability.

4. On ELPI

The ELPI fulfilment guarantees the transition to global PI (GPI). For static procedures (least squares method, maximum likelihood method), ELPI is ensured by the properties of the information matrix. For methods based on the class $\mathscr{K}_{IA}$, properties of the information matrix are not directly applicable, because the processes are complex in ASI.

With GPI, we understand the condition fulfilment:

$$A \in \breve{\mathbb{G}}_A = \left\{ A \in \mathbb{R}^{2n} : \left\| A - A^* \right\| = 0 \ \forall t \geq t^* > t_0 \ \& \ \mathbb{I}_t \right\}. \tag{19}$$

The proposed interpretation of GPI as belonging the parameters of model (6) to the set $\breve{\mathbb{G}}_A$ is linked to the absolute stability of an adaptive system.

Global parametric identifiability follows from Theorem 4 for systems of class $\mathscr{S}_y(A)$.

Theorem 4. *Let: 1) the conditions of Theorems 1 and 3 are fulfilled; 2) The system of inequalities is valid for processes in the system (6), (10)*

$$\begin{bmatrix} \dot{V}_e \\ \dot{V}_\Delta \end{bmatrix} \leq \underbrace{\begin{bmatrix} -k & \frac{1}{k}\bar{\alpha}_P \bar{\lambda}_\Gamma \\ \frac{4}{3\underline{\alpha}_P}\vartheta \bar{a}_P & -\frac{3}{4}\vartheta \underline{\alpha}_P \underline{\lambda}_\Gamma \end{bmatrix}}_{A_G} \underbrace{\begin{bmatrix} V_e \\ V_\Delta \end{bmatrix}}_{W_G}, \tag{20}$$

where $\bar{\lambda}_\Gamma, \underline{\lambda}_\Gamma$ *are the maximum and minimum eigenvalues of the matrix* Γ. *Then (a) the system (4) is globally parametrically identifiable on the class* $\mathscr{S}_y(A)$, *(b) the system (6), (10) is exponentially stable with the estimate:*

$$W_G(t) \leq e^{A_G(t-t_0)} S_G(t_0), \tag{21}$$

if

$$9k^2 \underline{\alpha}_P^2 \underline{\lambda}_\Gamma \geq 16 \bar{a}_P^2 \bar{\lambda}_\Gamma, \tag{22}$$

where $S_G \in \mathbb{R}^2$ *is the state vector of the comparison system* $\dot{S}_G(t) = A_G S_G(t)$, $S_G(t_0) \geq W_G(t_0)$.

The proof of Theorem 4 is given in Appendix F.

From Theorem 4, we obtain GPI on the set of initial conditions and ELPI. Since the systems are congruent, this condition is also valid for systems (1) of class $\mathscr{S}_X(\bar{A})$. To substantiate this statement, apply the approach [22].

5. About m-Parametric Identifiability

Let the CE condition not be fulfilled. The problem of identifiability, and identification, must be solved. Consider the approach to solving this problem using the example of the class $\mathscr{S}_y(A)$ system.

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Let the system and the model have the form (4) and (5). We assume that $u \notin \mathcal{CE}$. The term $\Delta A^T P$ in (6) is represented as

$$\Delta A^T P = \Delta A^T P = \left[\Delta \tilde{A}^T \ \delta A^T\right]\left[\tilde{P}^T \ \bar{P}^T\right]^T,$$

where $\tilde{P}(t) \in \mathcal{CE}_{\tilde{P}}$, $\bar{P}(t) \notin \mathcal{CE}$; $\left[\Delta \tilde{A}^T \ \delta A^T\right]$ is the representation ΔA corresponding to the vector $P(t)$.

Transform the equation for error (6) to the form

$$\dot{e} = -\kappa e + \Delta \tilde{A}^T P + \omega(A, P), \tag{23}$$

where $\omega = \delta A^T \bar{P}$, $\omega(A,P) \in \mathbb{R}$ is uncertainty caused by non-fulfilment of the condition $u \in \mathcal{CE}$, $\Delta \tilde{A} = \hat{\tilde{A}} - A$, $\tilde{A} \subset \hat{A}$ is the part of the vector $\hat{A}$ evaluated on the class $\mathcal{S}_{IA}^{y}$, $\hat{\tilde{A}} \in \mathbb{R}^{2m}$, $m < n$.

Let $\left|\omega(A,P)\right| \le \varepsilon_\omega$, where $\varepsilon_\omega \ge 0$.

Definition 3. A system of class $\mathcal{S}_y(A)$ is m-locally parametrically identifiable on the set $\tilde{\mathbb{I}}_t = \{P(t), u(t) \notin \mathcal{CE}, t > t_0\}$ if the condition

$$\tilde{A} \in \breve{\mathbb{G}}_A = \left\{\tilde{A} \in \mathbb{R}^{2m} : \left\|\tilde{A} - A^*\right\| \le \varepsilon_m,\ \varepsilon_m > 0\ \forall t \ge t^* > t_0\ \&\ \tilde{\mathbb{I}}_t\right\}. \tag{24}$$

is satisfied with its representation (4) in class $\mathcal{S}_y(A)$.

Theorem 5. *Let (i) the system (1) be stable; (ii) the Lyapunov function $\tilde{V}_\Delta = 0.5\Delta \tilde{A}^T \Gamma_{\tilde{A}}^{-1} \Delta \tilde{A}$ admits an infinitesimal limit, where $\Gamma_{\tilde{A}}^T = \Gamma_{\tilde{A}} > 0$ is the diagonal matrix; (iii) $u(t) \notin \mathcal{CE}$. Then the system (4) is locally parametrically identifiable in the domain $\breve{\mathbb{G}}_A$ if*

$$0.5\upsilon_P \left\|\Delta \tilde{A}\right\|^2 + 0.5\varepsilon_\omega^2 \le 2k_m V_e \tag{25}$$

and all trajectories of the system (4) belong the area

$$\mathbb{G}_{\Xi} = \left\{e(t) \in \mathbb{R}, \Delta A \in \mathbb{R}^{2n} : V_v(t) \le V_v(t_0) - 2k_m \eta^{-1} V_e + 0.5\eta^{-1}\varepsilon_\omega^2\right\}, \tag{26}$$

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where $V_v = V_e + \tilde{V}_\Delta$, $\eta = \min(1, 2\underline{\lambda}_\Gamma)$.

The proof of Theorem 5 is given in Appendix G.

From Theorem 5, we see that the PI domain depends on the CE fulfilment of the information set of the system. If the CE condition is not fulfilled, the parameter ε_m increases because of the effect of parametric uncertainty ω. Here, estimate (14) is more realistic and, under certain conditions, ELPI is possible with estimate (18).

Remark 4. In biological systems, structural identifiability issues are considered. Most times, lineal systems with numerous parameters are studied. Various algorithms are proposed and identifiability conditions are investigated to reduce the number of estimated parameters. In ASI, a multiplicative approach is used to identify a system with various parameters [23]. Here, PI is understood as parametric identifiability in some parametric domain $\mathrm{G}(\mathbb{G}_A)$, depending on the vector of multiplicative parameters (MPV). As a rule, MPV estimates belong to a certain limited area, which is formed based on of a priori information and analysis of the information set. This identifiability applies to systems satisfying specified quality requirements.

6. Lyapunov Exponents in PI Problem

6.1. Stationary System of Class $\mathscr{S}_X(\overline{A})$

Lyapunov characteristic exponents (LE) are the characteristic of a dynamical system. LE is an indirect PI estimate of the system. This approach to PI has not been considered in the literature. The LE application has its own peculiarities in the proposed paradigm of PI. In particular, it is necessary to consider the issue of detectability, recoverability and identifiability of LE based on the information set of the system. Identifiability is understood as the detectability of Lyapunov exponents. Known approaches allow us to estimate only the maximum (largest) LE [24]. A more promising approach is based on the analysis of geometric frameworks (GF) reflecting the change in LE [24]. Issues of LE detectability based on GF analysis are presented in [24]. Therefore, they are not considered here. Detectability is the important issue for evaluating LE.

In [24], the criteria for $\mathscr{LP}$-detectability of Lyapunov exponents are presented. $\mathscr{LP}$-detectability and recoverability we understood as the ability to the LE estimate. $\mathscr{LP}$-detectability imposes certain requirements on experimental data. The approach allows us to obtain the full range LE.

Let $m = n - 1 - \upsilon$, where υ is the number of non-recoverable LE.

Definition 4. The system (1) is called m-detectable with a υ-non-recoverability level if the υ lineal (LE) has an insignificant level.

As follows from definition 4, that if the system of class $\mathscr{S}_X(\overline{A})$ is m-detectable with a level of υ-non-recoverability, then this is a sufficient condition for m-parametric identifiability of the system. The CE requirement plays an important role, as it guarantees the S-synchronizability and structural identifiability of the nonlinear system.

Remark 5. The definition 4 provides sufficient conditions for evaluating PI systems of class $\mathscr{O}_X(\overline{A})$. This issue requires further study. Note that LE (for the classes $\mathscr{O}_X(\overline{A})$ under consideration, the Lyapunov exponents are the eigenvalues of the matrix) depend on system parameters.

Analysing nonstationary (periodic) systems is more difficult, since it is difficult to isolate the parametric space here.

Using LE translates the PI problem into the space of Lyapunov exponent [25] for periodic systems (PS).

6.2. PS for Class $\mathscr{O}_X(\overline{A})$

Consider the system (1) with the matrix $\overline{A}(t)$. For convenience, the system will be denoted by S_{per}.

Assumptions.

A1. $\overline{A}(t)$ is a bounded continuous Frobenius matrix

$$\left\|\overline{A}(t)\right\| \leq \alpha_A, \tag{27}$$

where $\alpha_A > 0$, $\|\cdot\|$ is matrix norm.

A2. $\overline{A}(t)$ is almost periodic, i.e. a subsequence can be selected from any sequence

$$\overline{A}(t) = \overline{A}(t - \tau_i) \tag{28}$$

converging uniformly along the entire axis to some almost periodic matrix $\overline{A}(t)$.

A3. $\overline{A}(t)$ is the Hurwitz matrix for almost all $t \geq 0$

Let $\mathscr{R}_{\overline{A}} = \left\{\chi_i(t), i = \overline{1,n}, t > t_0\right\}$ is a spectrum of LE χ_i $\left(i = \overline{1,n}\right)$.

Definition 5. The function $\chi_i(t)$ is almost periodic in the Bohr sense [26] or the $\mathscr{BF}$ - function [25], if such a positive number $l = l(\delta)$ exists, that any segment $[a, a+l]$ contains at least one number T_f, for which it is hold

$\left|f(x+T_f) - f(x)\right| < \delta$ and $t \in [0, \infty)$. (29)

If $\chi_i(t)$ is a $\mathscr{BF}$ -function, then it is $\alpha\pi$ -almost periodic [26], where α, π are positive numbers.

Let the order of the system S_{per} be known. Apply the geometric structure $\mathscr{SK}_{\Delta k'_{s,\rho}}$ to decide on the spectrum $\mathscr{R}_{\overline{A}}$ [26]. Here $k_s(t,\rho) = \rho\left(\hat{y}_g\right)/t$, $\rho\left(\hat{y}_g\right) = \rho_g = \ln\left|y_g(t)\right|$, $\hat{y}_g(t)$ is an evaluation of the general solution of the system (1).

$\mathscr{SK}_{\Delta k'_{s,\rho}}$ described by the function $f_{sk}(t): k_s \to \Delta k_s$, where Δk_s. $k_s(t)$ is $\mathscr{BF}_{\alpha\pi}$ -function, $f_{sk}(t)$ contains areas $\mathscr{D}_{sk}$, where a drastic change is taking place.

Theorem 6. *If the system S_{per} is stable and recoverable, and the function $f_{sk}(t)$ contains at the interval $\left[t_0, t^*\right] \subset \mathbb{J}_g$ $\left(t^* \leq \overline{t}\right)$ at least regions $\mathscr{D}_{sk}$, then the system S_{per} has an order m and is $\mathscr{LP}$ -identifiable ($\mathscr{LP}$ -detectable).*

In terms of Theorem 6, $\mathbb{J}_g$ is a time interval in which an estimate of the general solution of the system S_{per} are obtained.

It follows from Theorem 6 that the system S_{per} is identifiable on the set $\left\{\mathcal{SK}_{k^i_{s,\rho}}\right\}$. As shown in [25], the location of local minima on $\mathcal{SK}_{k^i_{s,\rho}}$ coincides with regions $\mathcal{D}_{sk}$ of the structure $\mathcal{SK}_{\Delta k'_{s,\rho}}$. This result allows us to obtain the set $\mathcal{M}_{LE}$ containing estimates LE of the system S_{per}. Cardinal $\mathcal{M}_{LE}$ may not match the LE number of the system. $\mathcal{M}_{LE}$ characterizes the set of system S_{per} lineals.

The detectability (identifiability) of the periodic system (1) with the matrix $\overline{A}(t)$ follows from [25].

Theorem 7. Let 1) assumptions A1-A3 are fulfilled for the system; 2) the system S_{per} is recoverable; 3) $u(t)\in\mathcal{CE}_u$; 4) set $\left\{k^i_{s,\rho}(t)\right\}$ elements are $\mathcal{BF}_{\alpha\pi}$-functions; 5) the structure $\mathcal{SK}_{\Delta k^i_{s,\rho}}$ contains at least v regions of $\mathcal{D}^v_{sk}$, which to local minima correspond to the structure $\mathcal{SK}_{k^i_{s,\rho}}$. Then the set $\mathcal{M}_{LE}$ is $\mathcal{LP}$-detectable or fully detectable.

Corollary from Theorem 7. *If structures $\mathcal{SL}_{\Delta k^i_{s,\rho}}$ contain only m of regions $\mathcal{D}^v_{sk}$, which to local minima correspond on $\mathcal{SK}_{k^i_{s,\rho}}$, then the system S_{per} is m-detectable with an υ-non-recoverable level.*

Remark 6. Eigenvalues $\lambda_i(t)$ of the matrix $\overline{A}(t)$ are periodic functions of time. Therefore, lineals $\mathcal{L}^i(t)$ and $\mathcal{L}^{i+1}(t)$ corresponding to these functions may overlap. This can generate an infinite range of LE. Determine the acceptable range for $\mathcal{M}_{LE}$ and the number that determines the mobility of the largest Lyapunov exponent. The set $\mathcal{M}_{LE}$ upper bound is determined by the allowable mobility limit of the largest LE χ_1. The estimate is fair for χ_1 [25]:

$$\chi_1 \le \sup \mathbb{J}^1_{k^i_{s,\rho}}, \quad (30)$$

where $\mathbb{J}^1_{k^i_{s,\rho}}$ is the interval of the change of the ith indicator $k^i_{s,\rho}$. The region $\mathcal{M}_{LE}$ lower boundary is bounded by the smallest LE κ_n [25].

Definition 6. The system S_{per} of class $\mathcal{S}_X\left(\overline{A}\right)$ with matrix $\overline{A}(t)$ satisfying assumptions A1–A3 is locally $\mathcal{M}_{LE}$-identifiable on the set $\mathbb{I}_t^o=\left\{X(t),\,u(t),\,t>t_0\right\}$, if spectrum $\mathcal{R}_{\overline{A}}$ of LE belonging to the class $\mathcal{BF}$-function exists such that

$$\chi_i\left(\overline{A}\right)\in\mathcal{M}_{LE}=\left\{\chi_i(t)\in\mathcal{R}_{\overline{A}}:\left|\chi_i-\chi_i^*\right|\le\varepsilon_\chi \quad \forall t\in\left[t^*,\,t^{**}\right],\,t^*>t_0\ \&\,\mathbb{I}(t)\right\},\quad i=\overline{1,n}, \quad (31)$$

where $\varepsilon_\chi\ge 0$.

The problem of assessing the LE adequacy has its own specifics. Let $\mathcal{S}_{\hat{y}_g,\hat{x}_g}$ is phase portrait of the system S_{per}.

Definition 7 [25]**.** Estimates of Lyapunov exponents χ_i are χ-adequate in the $\mathbb{R}$ space, if areas of their definition coincide with $\alpha\pi$-almost-periodicity regions of structure $S_{\hat{y}_g,\hat{x}_g}$.

Theorem 8 [25]. *Let:(i) the S_{per}-system is stable and recoverable; (ii) the set $\mathcal{M}_{LE}$ is $\mathcal{LP}$-detectable; (iii) definition regions $\mathcal{D}^j_{sl}$ on the $\mathcal{SL}_{\Delta k^i_{s,\rho}}$ structure coincide with $\alpha\pi$-almost-periodicity regions for the G5 structure $S_{\hat{y}_g,\hat{x}_g}$. Then estimates of elements for the set $\mathcal{M}_{LE}$ are χ-adequate to the regions $\alpha\pi$-almost periodicity $S_{\hat{y}_g,\hat{x}_g}$.*

Remark 8. We have considered only one approach to assessing the PI of periodic systems. The PS can be considered as a system with an interval parametric domain and identifiability can be estimated within the specified limits. Here, the approaches described above in sections 4 and 5 are applied.

So, the problem of estimating LPI is reduced to a more adequate task of estimating LE for these systems. The PI conditions in a special space are got and the methods of its estimation are given. The

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adequacy concept of LE estimates is introduced and the area for the LE location is highlighted. We have shown the existence of a LE set for the system S_{per}.

7. About LPI for Decentralized Systems

Consider a decentralized system (DS)

$$S_i : \begin{cases} \dot{X}_i = A_i X_i + B_i u_i + \sum_{j=1, j\neq i}^{m} \bar{A}_{ij} X_j + F_i(X_i), \\ Y_i = C_i X_i, \end{cases} \tag{32}$$

where $X_i \in \mathbb{R}^{n_i}$, $Y_i \in \mathbb{R}^{q_i}$ are state and output vectors of the S_i-subsystem, $u_i \in \mathbb{R}$ is control, $i = \overline{1,m}$, $\sum_{i=1}^{m} n_i = n$. The elements of the matrices $A_i \in {}^{n_i \times n_i}, B_i \in {}^{n_i}, \bar{A}_{ij} \in ?^{n_i \times n_j}$ are unknown; $C_i \in \mathbb{R}^{q_i \times n_i}$. The matrix $\bar{A}_{ij}$ reflects the mutual influence of the subsystem S_j. $F_i(X_i) \in \mathbb{R}^{n_i}$ considers the nonlinear state of subsystem 2 S_i, and the $A_i \in \mathscr{H}$ is the Hurwitz matrix (stable).

Assumption 2. $F_i(X_i)$ belongs to the class

$$\mathscr{N}_F(\pi_1, \pi_2) = \{F(X) \in {}^n : \pi_1 X \le F(X) \le \pi_2 X,\ F(0) = 0\} \tag{33}$$

and satisfies the quadratic condition

$$(\pi_2 X - F(X))^T (F(X) - \pi_1 X) \ge 0, \tag{34}$$

where $\pi_1 > 0, \pi_2 > 0$.

The information set of measurements for the S_i-subsystem has the form

$$\mathbb{I}_{o,i} = \{X_i(t),\ u_i(t),\ X_j(t),\ t \in \mathbb{J} = [t_0, t_k]\}.$$

Mathematical model

$$\dot{\hat{X}}_i = K_i(\hat{X}_i - X_i) + \hat{A}_i X_i + \hat{B}_i u_i + \sum_{j=1, j\neq i}^{m} \hat{\bar{A}}_{ij} X_j + F_i(X_i), \tag{35}$$

where $K_i \in \mathscr{H}$ is a matrix with known elements; $\hat{A}_i$, $\hat{B}_i$, $\hat{\bar{A}}_{ij}$ are tuning matrices, $\hat{F}_i$ is a priori defined nonlinear vector function.

Problem. Obtain PI estimates for the system (32) based on the set $\mathbb{I}_{o,i}$ analysis.

DS (32) is nonlinear, so the condition CE (7) is represented as:

$$\mathscr{EC}^{\Sigma}_{\underline{\alpha}_{u_i}, \bar{\alpha}_{u_i}} : (\underline{\alpha}_{u_i} \le u_i^2(\tau) \le \bar{\alpha}_{u_i}) \& (\Omega_{u_i}(\omega) \subseteq \Omega_S(\omega)),$$

Where $\Omega_{u_i}(\omega)$ is the set of frequencies for u_i; $\Omega_S(\omega)$ is the set of acceptable frequencies of input u_i, ensuring S-synchronizability of the system.

Get the equation for the error $E_i = \hat{X}_i - X_i$:

$$\dot{E}_i = K_i E_i + \Delta A_i X_i + \Delta B_i u_i + \sum_{j=1, j\neq i}^{M} \Delta \bar{A}_{ij} X_j + \Delta F_i(X_i), \quad (37)$$

where $\Delta A_i = \hat{A}_i - A_i, \Delta B_i = \hat{B}_i - B_i, \Delta \bar{A}_{ij} = \hat{\bar{A}}_{ij} - \bar{A}_{ij}, \Delta F_i = F_i - F_i$ are parametric residuals.

Consider the system (37) and LF $V_i(E_i) = 0.5 E_i^T R_i E_i$, where $R_i = R_i^T > 0$ is a positive symmetric matrix.

Let $\|\Delta A_i\| = \sqrt{\mathrm{tr}(\Delta A_i^T \Delta A_i)}$, $\|\Delta \bar{A}_{ij}\| = \sqrt{\mathrm{tr}(\Delta A_{ij}^T \Delta A_{ij})}$ are the norm of matrixes ΔA_i, $\Delta \bar{A}_{ij}$, $\mathrm{tr}(\cdot)$ is the trace of the matrix.

The following modification of Theorem 1 [27] is true.

Theorem 9. *Let: 1) the matrix* $A_i \in \mathscr{H}$ *; 2)* $X_i(t) \in \mathscr{CE}_{\underline{\alpha}_{X_i}, \bar{\alpha}_{X_i}}$, $X_j(t) \in \mathscr{PE}_{\underline{\alpha}_{X_j}, \bar{\alpha}_{X_j}}$ $u_i(t) \in \mathscr{PE}_{\underline{\alpha}_{u_i}, \bar{\alpha}_{u_i}}$ *; 3)* $F_i(X_i) \in \mathscr{N}_F(\pi_1, \pi_2)$ *and*

$$\|F_i(X_i)\|^2 \leq \eta \bar{\alpha}_{X_i}, \quad \Delta F_i^T \Delta F_i \leq 2\eta \bar{\alpha}_{X_i} + \delta_{F_i},$$

where $\pi_1 > 0, \pi_2 > 0,$ $\eta = \eta(\pi_1, \pi_2) > 0, \bar{\alpha}_{\Xi} = \bar{\alpha}_{X_i}(X_i) > 0$, $\eta = 2\bar{\pi} + \pi^2$, $\bar{\pi} = \pi_1 \pi_2$, $\pi = \pi_1 + \pi_2$, $\delta_{F_i} > 0$. *Then subsystem (32) is locally parametrically identifiable on the set* $\mathbb{I}_{o,i}$ *if*

$$2\left(\bar{\alpha}_{X_i} \|\Delta A_i\|^2 + \bar{\alpha}_{u_i} \|\Delta B_i\|^2 + \sum_{j=1, j\neq i}^{m} \bar{\alpha}_{X_j} \|\Delta \bar{A}_{ij}\|^2 + 2\eta \bar{\alpha}_{X_i} + \delta_{F_i}\right) \leq \bar{\lambda}_{Q_i} V_i, \quad (38)$$

where $\bar{\lambda}_{Q_i} = \lambda_{Q_i} - k_i$, $\lambda_i > 0$ *is the minimum eigenvalue of the matrix* Q_i; $k_i > 0$, $K_i R_i + K_i^T R_i = -Q_i$, $Q_i = Q_i^T > 0$.

The proof of Theorem 9 is given in Appendix H.

Corollary 1 of Theorem 9. Let conditions of Theorem 9 be fulfilled. Then the nonlinearity $F_i(X_i)$ is locally structurally identifiable in the parametric sector

$$\mathscr{O}_{X_i}(\pi_1, \pi_2) = \left\{X_i \in \mathbb{R}^{n_i} : \pi_1 X_i \leq F_i(X_i) \leq \pi_2 X_i\right\},$$

if

$$\|\Delta F_i\|^2 \leq 0.25 \bar{\lambda}_i V_i + z_i, \quad (39)$$

where $z_i \geq 0$.

The proof corollary 1 from Theorem 9 is given in Appendix I.

Consider the system (37) and class $\mathscr{H}_{IA}^{S_i}$ algorithms to tune its parameters:

$$\Delta \dot{A}_i = -\Gamma_{A_i} E_i R_i X_i^T,$$
$$\Delta \dot{\bar{A}}_{ij} = -\Gamma_{\bar{A}_{ij}} E_j R_i X_j^T,$$
$$\Delta \dot{B}_i = -\Gamma_{B_i} E_i^T R_i u_i, \quad (40)$$

where Γ_{A_i}, $\Gamma_{\bar{A}_{ij}}$, Γ_{B_i} are diagonal matrices of the corresponding dimensions.

Lyapunov function for analysis of system (37) and (4): $W_{S_i} = V_i + V_{\Delta,i}$:

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$$V_{\Delta,i} = 0.5\mathrm{tr}\left(\Delta A_i^T \Gamma_{A_i}^{-1} \Delta A_i\right) + 0.5 \sum_{j=1, j\neq i}^{m} \mathrm{tr}\left(\Delta \bar{A}_{ij}^T \Gamma_{A_{ij}}^{-1} \Delta \bar{A}_{ij}\right) + 0.5 \Delta B_i^T \Gamma_{B_i}^{-1} \Delta B_i, \quad V_i(E_i) = 0.5 E_i^T R_i E_i .$$

Corollary 2 of Theorem 9. Let 1) conditions of Theorem 9 be fulfilled; 2) the class $\mathscr{A}_{IA}^{S_i}$ of algorithms is used to tuning parameters of the model (35). Then the system (32) is locally parametrically identifiable if

$$\sigma_i \chi_i (t - t_0) \le 2\tilde{\mu}_i \sigma_i \int_{t_0}^{t} V(\tau) d\tau , \quad (41)$$

where $\chi_i = 2\eta \bar{\alpha}_{X_i} + \delta_{F_i}$, $\gamma_i^{-1} = \min\left(1, 2\underline{\lambda}_{R_i}\right), \sigma = \mu_i - \bar{\lambda}_{R_i} > 0$; $\underline{\lambda}_{R_i}, \bar{\lambda}_{R_i}$ *are The minimum and maximum eigenvalues of the matrix* R_i, *and the estimate is held*

$$W_{S_i}(t) \le W_{S_i}(t_0) - 2\tilde{\mu}_i \sigma_i \int_{t_0}^{t} V(\tau) d\tau + \sigma_i \chi_i (t - t_0) . \quad (42)$$

The proof corollary 2 from Theorem 9 is given in Appendix J.

As follows from Theorem 9, system (32) is LPI and structurally identifiable with nonlinearity $F_i(X_i)$ in the parametric sector $\mathscr{S}_{X_i}(\pi_1, \pi_2)$ on the set of initial conditions and $\mathbb{I}_{o,i}$.

If we perform nonlinearity factorization (see, for example, [27])

$$\hat{F}_i(X_i) = \tilde{F}_i\left(X_i, N_{i,1}\right) \hat{N}_{i,2} , \quad (43)$$

where $\hat{N}_{i,1} \in \mathbb{R}^{n_{i,1}}$ is a priori estimation of known parameters, $\hat{N}_{i,2} \in \mathbb{R}^{n_{i,2}}$ is vector of tuning parameters, the structure $\tilde{F}_i\left(X_i, N_{i,1}\right)$ is formed a priori considering the known vector $N_{i,1}$, and apply the algorithm

$$\dot{\hat{N}}_{i,2} = -\Gamma_{F_i} \tilde{F}_i R_i E_i\left(X_i, N_{i,1}\right) , \quad (44)$$

where Γ_{F_i} is a diagonal matrix with positive diagonal elements, then we obtain the conditions for global parametric identifiability for DS on the class of algorithms $\mathscr{A}_{IA}^{S_i}$ and (44). They are based on the modernization of results [27].

8. Examples

1. Consider an engine control system with the Bouc–Wen hysteresis

$$m\ddot{x} + c\dot{x} + F(x,z,t) = f(t), \quad (45)$$

$$F(x,z,t) = \alpha k x(t) + (1-\alpha) k d z(t) , \quad (46)$$

$$\dot{z} = d^{-1}\left(a\dot{x} - \beta|\dot{x}||z|^n sign(z) - \gamma \dot{x} |z|^n\right) , \quad (47)$$

where $m > 0$ is mass, $c > 0$ is damp, $F(x,z,t)$ is the recovering force, $d > 0$, $n > 0$, $k > 0$, $\alpha \in (0,1)$, $f(t)$ is exciting force, a, β, γ are some numbers. Set of experimental data $\mathbb{I}_o = \{f(t), y(t), t \in \mathbb{J}\}$. Vector of parameters $A = [m, c, a, k, \alpha, \beta, \gamma, n]^T$.

To estimate PI on the set $\mathbb{I}_o$, equation (45) is transformed to the form [28]

$$\dot{x} = a_1 x + a_2 p_x + a_3 p_z + b p_f , \quad (48)$$

$$\dot p_x = -\mu p_x + x, \dot p_f = -\mu p_f + f, \dot p_z = -\mu p_z + z, \quad \mu > 0,$$

where $a_1 = -(c-\mu m)/m$, $a_2 = -(\alpha k - \mu(c-\mu m))/m$, $a_3 = -((1-\alpha)k)/m$.

Model for system identification (48)

$$\dot{\hat x} = -k_x(\hat x - x) + \hat a_1 \hat x + a_2 p_x + \hat a_3 p_z + \hat b p_f, \tag{49}$$

where $k_x > 0$; $\hat a_i(t)$, $i = 1,2,3$, $\hat b(t)$ are adjustable model parameters. Let $e = \hat x - x$. From (48) and (49), we obtain the equation for the identification error:

$$\dot e = -k_x e + \Delta a_1 x + \Delta a_2 p_x + \Delta a_3 p_z + \Delta b p_f, \tag{50}$$

where $\Delta a_1 = \hat a_1(t) - a_1$, $\Delta a_2 = \hat a_2(t) - a_2$, $\Delta a_3 = a_3(t) - a_3$, $\Delta b = \hat b(t) - b$.

The variable z is not measured. Apply the model to estimate z:

$$\dot{\hat x}_z = -k_x(\hat x_z - x) + \hat a_1 \hat x + a_2 p_x + \hat b p_f. \tag{51}$$

and introduce a residual $\varepsilon_z = x - \hat x_z$. Let ε_z is current estimate z. Then we get the model to evaluate z

$$\dot{\hat z} = -k_z(z - \varepsilon_z) + \dot{\tilde x} - \hat\beta|\dot{\tilde x}||\hat z|^n sign(z) - \hat\gamma \dot{\tilde x}|\hat z|^n \tag{52}$$

where $k_z > 0$; $\hat\beta$, $\hat\gamma$ are estimates of hysteresis parameters (47); $\dot{\tilde x} = (x(t+\tau) - x(t))/\tau$, τ is the integration step.

Introduce a residual $\varepsilon = \hat z - \varepsilon_z$, satisfying equations

$$\dot\varepsilon = -k_z\varepsilon + \Delta\dot x + \Delta\beta|\dot{\tilde x}||\hat z|^n sign(z) + \beta\eta_\beta + \Delta\gamma\dot{\tilde x}|\hat z|^n + \gamma\eta_\gamma, \tag{53}$$

$$\eta_\beta = |\dot x||z|^n sign(z) - |\dot{\tilde x}||\hat z|^n sign(z), \quad \eta_\gamma = \dot x|z|^n - \dot{\tilde x}|\hat z|^n,$$

where $\Delta\dot x = \dot{\tilde x} - \dot x$, $\Delta\beta = \beta - \hat\beta$, $\Delta\gamma = \gamma - \hat\gamma$, $\Delta\beta = \beta - \hat\beta$, $\Delta\gamma = \gamma - \hat\gamma$. Present (49) as:

$$\dot{\hat x} = -k_x(\hat x - x) + \hat a_1 \hat x + a_2 p_x + \hat a_3 p_{\hat z} + \hat b p_f, \tag{54}$$

$$\dot p_{\hat z} = -\mu p_z + \hat z, \tag{55}$$

and (50) is written as:

$$\dot e = -k_x e + \Delta a_1 x + \Delta a_2 p_x + \Delta a_3 p_{\hat z} + \Delta b p_f. \tag{56}$$

Evaluate the identification quality using the Lyapunov function. $V_\varepsilon(t) = 0.5\varepsilon^2(t)$. Get adaptive algorithms from $\dot V_\varepsilon < 0$:

$$\Delta\dot{\beta} = -\chi_{\beta}\varepsilon\left|\dot{x}\tilde{z}\right|^{n} sign(z),\ \ \Delta\dot{\gamma} = -\chi_{\gamma}\varepsilon\dot{x}\left|\tilde{z}\right|^{n}, \quad (57)$$

where $\chi_{\beta} > 0, \chi_{\gamma} > 0$ are parameters ensuring the stability of algorithms (57).

Consider functionals:

$$\Delta D_{a}(t) = \left(\Delta a_{1}^{2}(t) + \Delta a_{2}^{2}(t) + \Delta a_{3}^{2}(t) + \Delta b^{2}(t)\right)^{0.5},\ \ \Delta D_{\gamma,\beta}(t) = \left(\Delta\gamma^{2}(t) + \Delta\beta^{2}(t)\right)^{0.5}. \quad (58)$$

Figures 1 and 2 represent PI evaluations of the system (45) – (47). The ASI has two loops: the main one (variable e) and the auxiliary one (variable ε). Figure 1 shows the structure $\mathcal{I}_{|e|,\Delta D_{a}}$ described by the function $\varphi_{a} :|e| \to \Delta D_{a}$, and Figure 2 presents the structure $\mathcal{I}_{|\varepsilon|,\Delta D_{\gamma,\beta}}$ described by the function $\varphi_{\gamma,\beta} :|\varepsilon| \to \Delta D_{\gamma,\beta}$.

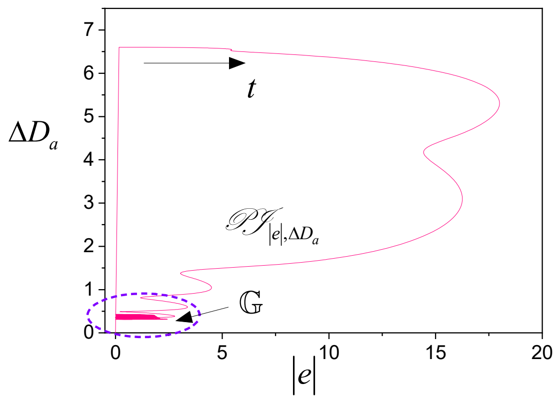

Figure 1. Structure $\mathcal{I}_{|e|,\Delta D_{a}}$

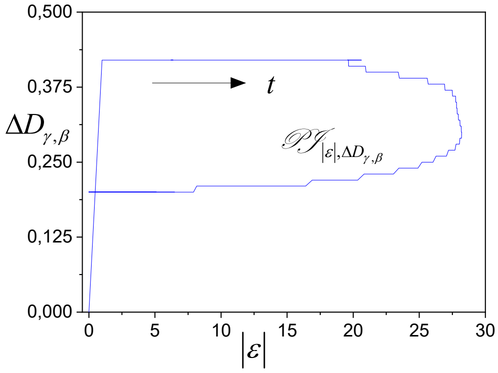

Figure 2. Structure $\mathcal{I}_{|\varepsilon|,\Delta D_{\gamma,\beta}}$

Presented structures confirm the fulfilment of the Theorem 1 conditions, since trajectories of the system for sufficiently large t get into the region $\mathbb{G}$.

2. Consider the system, the phase portrait of which is shown in Figure 3. The set of experimental data $\mathbb{I}(t)$ is known. Input $u(t) = 5 + 2\sin(0.2\pi t)$. Figure 3 shows the presence of oscillations in the system, the frequency of which differs from the frequency of the input. Therefore, the system is the system with periodic coefficients.

To determine LE, we apply the approach [25] and obtain estimates of the general solution and its derivative.

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$$\hat{y}_q(t) = [0.75; 0.07; -0.22][1\ u(t)\ \dot{u}(t)]^T, \quad \hat{\dot{y}}_q(t) = [-0.394; -0.059; 0,078][1\ u(t)\ \dot{u}(t)]^T. \tag{59}$$

The coefficients of determination for these models are equal 0.99. Next, we determine estimates of the free movement for the system. The evaluation of order for the system follows from Figure 4.

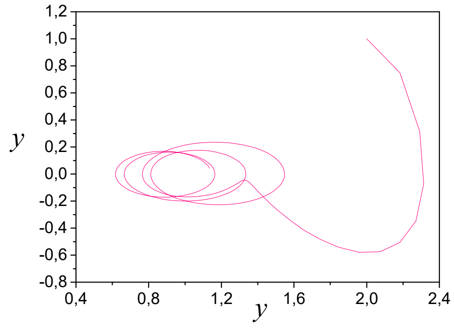

Figure 3. Phase portrait

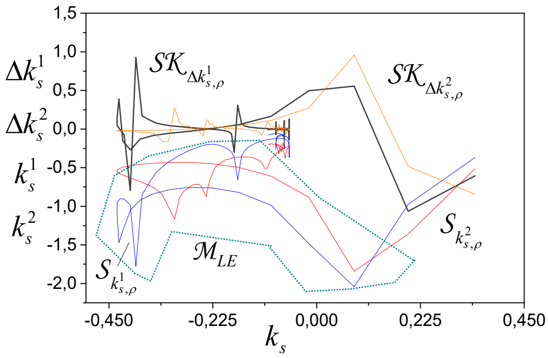

Figure 4. LE set

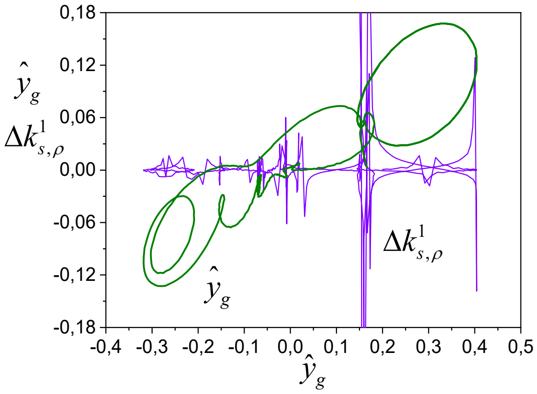

Figure 5. Estimate of χ -adequacy of LE

It follows from $\mathcal{SK}_{\Delta k^1_{s,\rho}}$ that the system has a third order. From $\mathcal{SK}_{\Delta k^1_{s,\rho}}$ and $\mathcal{SK}_{\Delta k^2_{s,\rho}}$, we get the set LE (Figure 4).

The upper estimate for κ_m is -2.04. Mobility limit for χ_1 is -0.8. χ -adequacy confirmation of LE estimates is shown in Figure 5. The eigenvalues of the state matrix of the system S_{per} are:

$$\mathcal{M}_{LE} = \{-2.04; -1.842; -1.77; -1.167; -0.878\}.$$

So, we see that the set $\mathcal{M}_{LE}$ is $\mathcal{LP}$ -detectable, and the system has the third order. Since the elements of the set $\mathcal{M}_{LE}$ are recoverable and detectable, the system S_{per} is LPI.

9. Conclusions

The problem of estimating parametric identifiability based on current experimental data is considered. Methods of the priori identifiability based on the analysis of the information matrix are not applicable in this case. We consider the approach based on the application of the second Lyapunov method to the PI study. LPI conditions are got based on the adaptive identification application to the linear dynamical system. We analyse data on state vector and current information on input and output in the problem PI. Conditions and estimates have been obtained that guarantee PI and LPI.

The m-parametric identifiability case is considered when the condition of constant excitation is not fulfilled. PI estimates are got for decentralised nonlinear systems and systems with periodic parameters. We show that Lyapunov exponents should be used to PI analyses of the system with periodic parameters.

Modelling examples are presented that confirm the efficiency of the proposed approach.

Appendix A

Proof of Theorem 1. Consider the LF $V_e(t)=0.5e^2(t)$. For $\dot{V}_e$, we get:

$$\dot{V}_e=-ke^2+e\Delta A^T P \tag{A.1}$$

or, applying condition 4 of Theorem 1,

$$\dot{V}_e\leq -kV+\frac{1}{2k}\bar{\alpha}_P\left\|\Delta A\right\|^2, \tag{A.2}$$

where $\bar{\alpha}_P\leq PP^T\leq \underline{\alpha}_P$, $\underline{\alpha}_P>0,\ \bar{\alpha}_P>0$.

Let $u(t)$ and $y(t)$ correspond to Fourier series with multiple frequencies Ω_u, Ω_y, where Ω_y depends on the spectrum $u(t)$. System (1) is a frequency filter and $A\in\mathcal{Y}$. Therefore, the frequency spectra of the elements of the vector P will vary. Therefore, sets Ω_{p_i}, where $p_i\in P$, will not have common areas. Then $\Delta A=0$ follows from the identifiability condition $\Delta A^T P=0$.

So, condition 4 is necessary for the local identifiability of the system (6). As condition 4 is fulfilled, for the limited trajectories (identifiability) of the system (4), it is necessary that:

$$\left\|\Delta A(t)\right\|^2\leq\frac{2\kappa^2}{\rho}V(t). \tag{A.3}$$

Appendix B

Proof of Corollary 1 from Theorem 1. For $\dot{V}_e$, we get:

$$\begin{aligned}\dot{V}_e&=-ke^2+e\Delta A^T P=-ke^2-\Delta\dot{A}^T\Gamma^{-1}\Delta A,\\ \dot{V}_e&=-2kV_e-2\dot{V}_\Delta,\end{aligned} \tag{B.1}$$

where $V_e = 0.5e^2$, $V_\Delta = 0.5\Delta A^T \Gamma^{-1} \Delta A$. Let $V = V_e + V_\Delta$. Present (B.1) as:

$$\begin{aligned} &\dot{V}_e \le -2kV_e - 2\dot{V}_\Delta, \\ &0.5\dot{V} \le -kV_e, \end{aligned} \tag{B.2}$$

where from

$$V(t) \le V(t_0) - 2k\int_{t_0}^{t} V_e(\tau)d\tau \tag{B.3}$$

or $V(t) \le V(t_0)$.■

Appendix C

Proof of Theorem 2. The derivative LF $V_E = 0.5E^T RE$ has the form

$$\dot{V}_E = -E^T QE + E^T R\left(\Delta AX + \Delta Bu\right) \tag{C.1}$$

or

$$\dot{V}_E \le -\mu E^T RE + \left|E^T R\left(\Delta AX + \Delta Bu\right)\right|, \tag{C.2}$$

where $E^T QE \ge \mu E^T RE$, $\mu > 0$, $Q = Q^T > 0$ is a positive definite matrix satisfying the equation $RK + K^T R = -Q$, $R = R^T > 0$. Then (C.2)

$$\dot{V}_E \le -\mu E^T RE + \left|E^T R\left(\Delta AX + \Delta Bu\right)\right| \le -2\mu V_E + 0.5\mu^{-1}\left(\bar{\alpha}_X \left\|\Delta A\right\|^2 + \bar{\alpha}_u \left\|\Delta B\right\|^2\right), \tag{C.3}$$

where $\left\|\Delta A\right\|^2 = \mathrm{tr}\left(\Delta A^T \Delta A\right)$, $\left\|\Delta B\right\|^2 = \Delta B^T \Delta B$, $\underline{\alpha}_X I_n \le X(t)X^T(t) \le \bar{\alpha}_X I_n$, I_n is the identity matrix.

From (C.3), we obtain the condition of LPI:

$$\left\|\Delta A\right\|^2 \bar{\alpha}_X + \left\|\Delta B\right\|^2 \bar{\alpha}_u \le 4\mu^2 V_E .■ \tag{C.4}$$

Appendix D

Proof of Corollary 1 from Theorem 2. Consider LF $V_W = V_E + V_{A,B}$, where:

$$V_{A,B} = 0.5\mathrm{tr}\left(\Delta A^T \Gamma_A^{-1} \Delta A\right) + 0.5\Delta B^T \Gamma_B^{-1} \Delta B,$$

Γ_A, Γ_B are diagonal matrices with positive diagonal elements.

If we consider (12), then (C.1) is written as:

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$$\begin{aligned}&\dot{V}_E = -E^T QE + E^T R\left(\Delta AX + \Delta Bu\right) = \\ &= -E^T QE - \underbrace{\mathrm{tr}\left(\Delta \dot{A}^T \Gamma_A^{-1} R\Delta A\right) - \Delta \dot{B}^T \Gamma_B^{-1} R\Delta B}_{\dot{V}_{A,B}}.\end{aligned} \tag{D.1}$$

Obtain

$$\dot{V}_W \le -2\mu V_E \Rightarrow V_W(t) \le V_W(t_0) - 2\mu \int_{t_0}^{t} V_E(\tau)d\tau \,.\blacksquare \tag{D.2}$$

Appendix E

Proof of Theorem 3. Apply algorithm (10) and represent the derivative $V_\Delta = \Delta A^T \Gamma^{-1} \Delta A$ as:

$$\dot{V}_\Delta = -e\Delta A^T P\,. \tag{E.1}$$

Let $\vartheta \ge 0$ exist such that in some region 0 the condition $e\Delta A^T P = \vartheta\left(\left\|\Delta A\right\|^2 + e^2\right)$ is satisfied.

Then (E.1)

$$\begin{aligned}&\dot{V}_\Delta = -\vartheta\left(\left\|\Delta A\right\|^2 \left\|P\right\|^2 + e^2\right) = -\frac{3}{4}\vartheta\left\|\Delta A\right\|^2 \left\|P\right\|^2 - \frac{1}{4}\vartheta\left\|\Delta A\right\|^2 \left\|P\right\|^2 - \vartheta e^2 \le \\ &\le -\frac{3}{4}\vartheta\left\|\Delta A\right\|^2 \left\|P\right\|^2 + 2\left|e\right|\vartheta\left\|\Delta A\right\|\left\|P\right\|.\end{aligned} \tag{E.2}$$

As $P \in \mathscr{CE}_P$ и $\underline{\alpha}_P \le \left\|P(t)\right\|^2 \le \overline{\alpha}_P$, then

$$\dot{V}_\Delta \le -\frac{3}{4}\vartheta\underline{\alpha}_P\left\|\Delta A\right\|^2 + 2\left|e\right|\vartheta\sqrt{\overline{\alpha}_P}\left\|\Delta A\right\|. \tag{E.3}$$

Apply the inequality [29]

$$-az^2 + bz \le -\frac{a}{2}z^2 + \frac{1}{2a}b^2\,,\quad a > 0, b > 0, z > 0\,.$$

Then

$$\dot{V}_\Delta \le -\frac{3}{8}\vartheta\underline{\alpha}_P\left\|\Delta A\right\|^2 + \frac{2}{3\underline{\alpha}_P}\vartheta\overline{\alpha}_P e^2\,. \tag{E.4}$$

As $\left\|\Delta A\right\|^2 \ge 2\underline{\lambda}_\Gamma V_\Delta$, where $\underline{\lambda}_\Gamma$ is the minimum eigenvalue of the matrix Γ. Then:

$$\dot{V}_\Delta \le -\frac{3}{4}\vartheta\underline{\alpha}_P\underline{\lambda}_\Gamma V_\Delta + \frac{4}{3\underline{\alpha}_P}\vartheta\overline{\alpha}_P V_e\,. \tag{E.5}$$

It follows from (E.5) that PI is guaranteed on a certain set $\left\{\hat{A}\left(t_0\right)\right\}$ and on the set $\mathbb{I}_t$ if

$$V_\Delta \geq \frac{16}{9\underline{\alpha}_P^2 \underline{\lambda}_\Gamma} \bar{a}_P V_e$$

and fair evaluation $V_\Delta(t) \leq S_\Delta(t)$, where:

$$S_\Delta(t) = e^{-\sigma(t-t_0)} S_\Delta\left(t_0\right) + \frac{4}{3\underline{\alpha}_P} \vartheta \bar{a}_P \int_{t_0}^{t} e^{-\sigma(t-\tau)} V_e(\tau) d\tau$$

$\sigma = 0.75\vartheta \underline{\alpha}_P \underline{\lambda}_\Gamma$, $S(t)$ is the upper solution of the comparison system $\dot{S} = -\sigma S + \pi V_e$ for $V_\Delta(t)$ (E.5), if $S_\Delta\left(t_0\right) \geq V_\Delta\left(t_0\right)$, $\pi = \frac{4}{3\underline{\alpha}_P} \vartheta \bar{a}_P$.■

Appendix F

Proof of Theorem 4. From the proofs of the corollary of Theorem 1 and Theorem 3, we obtain

$$\begin{aligned} \dot{V}_e &\leq -kV + \frac{1}{2k} \bar{\alpha}_P \|\Delta A\|^2, \\ \dot{V}_\Delta &\leq -\frac{3}{4} \vartheta \underline{\alpha}_P \underline{\lambda}_\Gamma V_\Delta + \frac{4}{3\underline{\alpha}_P} \vartheta \bar{a}_P V_e. \end{aligned} \tag{F.1}$$

As $\|\Delta A\|^2 = \Delta A^T \Gamma \Gamma^{-1} \Delta A \leq \bar{\lambda}_\Gamma \Delta A^T \Gamma^{-1} \Delta A \leq 2\bar{\lambda}_\Gamma V_\Delta$, then (F.1)

$$\begin{bmatrix} \dot{V}_e \\ \dot{V}_\Delta \end{bmatrix} \leq \underbrace{\begin{bmatrix} -k & \frac{1}{k} \bar{\alpha}_P \bar{\lambda}_\Gamma \\ \frac{4}{3\underline{\alpha}_P} \vartheta \bar{a}_P & -\frac{3}{4} \vartheta \underline{\alpha}_P \underline{\lambda}_\Gamma \end{bmatrix}}_{A_G} \underbrace{\begin{bmatrix} V_e \\ V_\Delta \end{bmatrix}}_{W_G}. \tag{F.2}$$

The matrix A_G is an M-matrix [30] if conditions $(-1)^i \Delta_i\left(A_G\right) > 0$ are fulfilled for the major minors. Obtain

$$k > 0, \quad 9k^2 \underline{\alpha}_P^2 \underline{\lambda}_\Gamma \geq 16 \bar{a}_P^2 \bar{\lambda}_\Gamma. \tag{F.3}$$

If the conditions (F.3) are fulfilled, then the adaptive system (6), (10) is exponentially stable (ES). As follows from the ES, estimates of the vector A in (4) are extremely locally parametrically identifiable under given initial conditions. The estimate (21) is got using the approach described in the proof of Theorem 3.■

Appendix G

Proof of Theorem 5. Consider (23) and represent the derivative of LF V_e as:

$$\dot{V}_e = -ke^2 + e\Delta\tilde{A}P + e\omega \tag{G.1}$$

or

$$\begin{aligned} \dot{V}_e &\le -kV + \frac{1}{2k}\bar{\alpha}_P\|\Delta A\|^2, \\ \dot{V}_e &\le -ke^2 + e^2 + 0.5\upsilon_P\|\Delta\tilde{A}\|^2 + 0.5\varepsilon_\omega^2, \end{aligned} \tag{G.2}$$

where $k_m = k - 1 > 0$, $\|P(t)\|^2 \le \upsilon_P$, $\upsilon_P \ne \bar{\alpha}_P$. From (G.2), we obtain the condition m-local PI:

$$0.5\upsilon_P\|\Delta\tilde{A}\|^2 + 0.5\varepsilon_\omega^2 \le 2k_mV_e. \tag{G.3}$$

Represent (G.1) in the form

$$\dot{V}_e = -ke^2 + e\left(\Delta\tilde{A}\tilde{P} + \delta A^T\bar{P}\right) = -ke^2 - \Delta\tilde{A}\Gamma^{-1}\Gamma\Delta\tilde{A} + e\omega. \tag{G.4}$$

Then (G.4)

$$\dot{V}_e \le -2kV_e - 2\underline{\lambda}_\Gamma\dot{\tilde{V}}_\Delta + 0.5e^2 + 0.5\omega^2. \tag{G.5}$$

Transform (G.5)

$$\begin{aligned} \dot{V}_e &\le -2\underbrace{k_m}_{k-1}V_e - 2\underline{\lambda}_\Gamma\dot{\tilde{V}}_\Delta + 0.5\omega^2, \\ \dot{V}_e + 2\underline{\lambda}_\Gamma\dot{\tilde{V}}_\Delta &\le -2k_mV_e + 0.5\omega^2. \end{aligned} \tag{G.6}$$

Let $\eta = \min\left(1, 2\underline{\lambda}_\Gamma\right)$, $V_v = V_e + \tilde{V}_\Delta$ and $k_m = k - 1$. Then:

$$\dot{V}_v \le -2k_m\eta^{-1}V_e + 0.5\eta^{-1}\varepsilon_\omega^2. \tag{G.7}$$

The estimate for V_v (see (26)) follows from (G.7). ■

Appendix H

Proof of Theorem 9. $\dot{V}_i$ has the form:

$$\dot{V}_i = -E_i^TQ_iE_i + E_i^TR_i\left(\Delta A_iX_i + \Delta B_iu_i + \sum_{j=1, j\ne i}^{m}\Delta\bar{A}_{ij}X_j + \Delta F_i\left(X_i\right)\right) \tag{H.1}$$

or

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$$\begin{aligned}\dot{V}_i &\le \underbrace{-E_i^T Q_i E_i}_{-\lambda V_i} + \underbrace{\left|E_i^T R_i\right|}_{\sqrt{k_i V_i}} \left| \Delta A_i X_i + \Delta B_i u_i + \sum_{j=1, j\ne i}^{m} \Delta \bar{A}_{ij} X_j + \Delta F_i\left(X_i\right) \right| \le \\ &\le -\lambda_i V_i + k_i V_i + 0.5\left(\Delta A_i X_i + \Delta B_i u_i + \sum_{j=1, j\ne i}^{m} \Delta \bar{A}_{ij} X_j + \Delta F_i\left(X_i\right) \right)^2,\end{aligned} \tag{H.2}$$

where $\lambda_i > 0$ is the minimum eigenvalue of the matrix Q_i.

Apply the Cauchy-Bunyakovsky-Schwarz inequality and Titu's lemma to the last term in (H.2) and get

$$\begin{aligned}&0.5\left(\Delta A_i X_i + \Delta B_i u_i + \sum_{j=1, j\ne i}^{m} \Delta \bar{A}_{ij} X_j + \Delta F_i\left(X_i\right) \right)^2 \le \\ &\le 2\left(\left\|\Delta A_i\right\|^2 \left\|X_i\right\|^2 + \left\|\Delta B_i\right\|^2 \left|u_i\right|^2 + \sum_{j=1, j\ne i}^{m} \left\|\Delta \bar{A}_{ij}\right\|^2 \left\|X_j\right\|^2 + \left\|\Delta F_i\left(X_i\right)\right\|^2 \right).\end{aligned} \tag{H.3}$$

Consider condition 1) of Theorems 9 and $\dot{V}_i$ write as:

$$\dot{V}_i \le -\bar{\lambda}_i V_i + 2\left(\bar{\alpha}_{X_i} \left\|\Delta A_i\right\|^2 + \bar{\alpha}_{u_i} \left\|\Delta B_i\right\|^2 + \sum_{j=1, j\ne i}^{m} \bar{\alpha}_{X_j} \left\|\Delta \bar{A}_{ij}\right\|^2 + \left\|\Delta F_i\left(X_i\right)\right\|^2 \right), \tag{H.4}$$

where $\bar{\lambda}_i = \lambda_i - k_i$. Apply Lemmas 1, 2 [26] and get for $\left\|\Delta F_i\right\|^2$

$$\Delta F_i^T \Delta F_i = \left\|\Delta F_i\right\|^2 \le 2\eta \bar{\alpha}_{X_i} + \delta_{F_i}, \tag{H.5}$$

where $\eta = 2\bar{\pi} + \pi^2$, $\pi = \pi_1 + \pi_2$, $\bar{\pi} = \pi_1 \pi_2$, $\delta_{F_i} \ge 0$.

Then (H.4)

$$\dot{V}_i \le -\bar{\lambda}_i V_i + 2\left(\bar{\alpha}_{X_i} \left\|\Delta A_i\right\|^2 + \bar{\alpha}_{u_i} \left\|\Delta B_i\right\|^2 + \sum_{j=1, j\ne i}^{m} \bar{\alpha}_{X_j} \left\|\Delta \bar{A}_{ij}\right\|^2 + 2\eta \bar{\alpha}_{X_i} + \delta_{F_i} \right). \tag{H.6}$$

If state variables are CE and the condition (38) is fulfilled, then the system (32) is the LPI on the set $\mathbb{I}_{o,i}$.■

Appendix I

Proof of Corollary 1 from Theorem 9. As follows from Theorem 9, DS is locally parametrically identifiable if the condition (38) is satisfied. Apply Lemmas 1, 2 [26] to the last terms in (H.6) and get:

$$2\eta \bar{\alpha}_{X_i} + \delta_{F_i} = 2\left(\eta \bar{\alpha}_{X_i} + \delta_{F_i}\right) - \delta_{F_i} = 2\left\|\Delta F_i\right\|^2 - \delta_{F_i}, \tag{I.1}$$

Therefore,

$$\left\|\Delta F_i\right\|^2 \le 0.25 \bar{\lambda}_i V_i + 0.5\left(\delta_{F_i} - \chi_i \theta_i - \zeta_{\alpha,j} \zeta_j\right), \tag{I.2}$$

where $\theta_i = \min_i\left(\|\Delta A_i\|^2 + \|\Delta B_i\|^2\right)$, $\zeta_j = \min_j \sum_{j=1, j\neq i}^{m} \|\Delta \bar{A}_{ij}\|^2$, $\zeta_{\alpha,j} = \min_j \sum_{j=1, j\neq i}^{m} \bar{\alpha}_{X_j}$.

As $\zeta_{\alpha,j}\zeta_j \leq \delta_{F_i} - \chi_i\theta_i$, then $\|\Delta F_i\|^2 \leq 0.25\bar{\lambda}_i V_i + z_i$. ■

Appendix J

Proof of Corollary 2 from Theorem 9. Represent $\dot{V}_i$ (H.2) as:

$$\dot{V}_i = -E_i^T Q_i E_i + E_i^T R_i\left(\Delta A_i X_i + \Delta B_i u_i + \sum_{j=1, j\neq i}^{m} \Delta\bar{A}_{ij} X_j + \Delta F_i\left(X_i\right) + \Delta F_i\left(X_i\right)\right) =$$

$$= -E_i^T Q_i E_i - \mathrm{tr}\left(\Delta \dot{A}_i^T \Gamma_{A_i}^{-1} R_i \Delta A_i\right) - \sum_{j=1, j\neq i}^{m} \mathrm{tr}\left(\Delta \dot{\bar{A}}_{ij}^T \Gamma_{\bar{A}_{ij}}^{-1} R_i \Delta\bar{A}_{ij}\right) - \quad \text{(J.1)}$$

$$-\Delta \dot{B}_i^T \Gamma_{B_i}^{-1} R_i \Delta B_i + E_i^T R_i \Delta F_i\left(X_i\right).$$

Let $E_i^T Q_i E_i \geq \mu_i E_i^T R_i E_i \geq 2\mu_i V_i$, where $\mu_i \geq 0$. Then:

$$\dot{V}_i \leq -2\mu_i V_i - 2\underline{\lambda}_{R_i} \dot{V}_{\Delta,i} + \left\|E_i^T R_i\right\|\left\|\Delta F_i\right\| \leq$$
$$\leq -2\mu_i V_i - 2\underline{\lambda}_{R_i} \dot{V}_{\Delta,i} + \bar{\lambda}_{R_i} V_i + 0.5\left\|\Delta F_i\right\|^2, \quad \text{(J.2)}$$

where $\underline{\lambda}_{R_i}$, $\bar{\lambda}_{R_i}$ are minimum and maximum eigenvalues of the matrix R_i.

The estimate (H.5) is fair for ΔF_i. Therefore, (J.2) is represented as:

$$\dot{V}_i \leq -2\Big(\underbrace{\mu_i - \bar{\lambda}_{R_i}}_{\sigma_i}\Big)V_i - 2\underline{\lambda}_{R_i}\dot{V}_{\Delta,i} + \underbrace{\eta\bar{\alpha}_{X_i} + \delta_{F_i}}_{\chi_i},$$
$$\dot{V}_i + 2\underline{\lambda}_{R_i}\dot{V}_{\Delta,i} \leq -2\sigma_i V_i + \chi_i, \quad \text{(J.3)}$$

where $\chi_i = 2\eta\bar{\alpha}_{X_i} + \delta_{F_i}$. Let $\gamma_i^{-1} = \min\left(1, 2\underline{\lambda}_{R_i}\right), \sigma = \mu_i - \bar{\lambda}_{R_i} > 0$. Transform (J.3):

$$\dot{W}_{S_i} \leq -2\sigma_i\gamma_i V_i + \gamma_i\chi_i, \quad \text{(J.4)}$$

Then

$$W_{S_i}(t) \leq W_{S_i}(t_0) - 2\tilde{\mu}_i\sigma_i \int_{t_0}^{t} V(\tau)d\tau + \sigma_i\chi_i\left(t - t_0\right) \quad \text{(J.5)}$$

if $\sigma_i\chi_i\left(t - t_0\right) \leq 2\tilde{\mu}_i\sigma_i \int_{t_0}^{t} V(\tau)d\tau$. ■

References

1. Villaverde A. F. Observability and structural identifiability of nonlinear biological systems. arXiv:1812.04525v1 [q-bio.QM] 11 Dec 2018, 2022.
2. Renardy M., Kirschner D., and Eisenberg M. Structural identifiability analysis of age-structured PDE epidemic models. *Journal of Mathematical Biology*, 2022, 84(1-2); 2022.

3. Weerts, H. H. M., Dankers, A. G., &, Van den Hof, P. M. J. Identifiability in dynamic network identification. *IFAC-PapersOnLine*, 2015, 48(28), pp. 1409-1414. https://doi.org/10.1016/j.ifacol.2015.12.330
4. Wieland F.-G., Hauber A. L., Rosenblatt M., Tönsing C. and Timmer J. On structural and practical identifiability. *Current opinion in systems biology*, 2021, 2, pp. 60–69.
5. Miao H., Xia X., Perelson A. S., and Wu H. On identifiability of nonlinear ode models and applications in viral dynamics. *SIAM Rev Soc Ind Appl Math*, 2011; 53(1), pp. 3–39. doi:10.1137/090757009
6. Bellman R, Åström K: On structural identifiability. *Math Biosci*. 1970, 7, pp. 329–339, https://doi.org/10.1016/0025-5564(70)90132-X.
7. Anstett-Collina F., Denis-Vidalc L., Millérioux G. A priori identifiability: An overview on definitions and approaches. *Annual reviews in control*, 2021, 50, pp.139-149. doi.10.1016/j.arcontrol.2020.10.006. hal-03003563
8. Hong H., Ovchinnikov A., Pogudin G., Yap C. Global identifiability of differential models. *Communications on Pure and Applied Mathematics*, 2020, 73 (9). 18311879.
9. Denis-Vidal L., Joly-Blanchard G., Noiret C., System identifiability (symbolic computation) and parameter estimation (numerical computation). *Numerical Algor*ithms, 2003, 34(2-4), pp. 283–292.
10. Boubaker O., Fourati A., Structural identifiability of nonlinear systems: an overview. In: *Proc. of the IEEE International Conference on Industrial Technology, ICIT'04, December 8-10*, 2004, pp. 1244–1248.
11. S. Audoly, G. Bellu, L. D'Angi `o, M. P. Saccomani, and C. Cobelli. Global identifiability of nonlinear models of biological systems. *IEEE Trans. Biomed. Eng.*, 2001, 48(1), pp. 55–65.
12. Chis O.-T., Banga J. R., and Balsa-Canto E. GenSSI: a software toolbox for structural identifiability analysis of biological models. *Bioinformatics*, 2011, 27(18), pp. 2610–2611,
13. Denis-Vidal L., Joly-Blanchard G., and Noiret C. Some effective approaches to check the identifiability of uncontrolled nonlinear systems. *Math. Comput. Simul*, 2001, 57(1), pp. 35–44.
14. X. Xia and C. H. Moog. Identifiability of nonlinear systems with application to HIV/AIDS models. *IEEE Trans. Autom. Control*, 2003, 48(2), pp. 330–336.
15. Stigter J. D. and Molenaar J. A fast algorithm to assess local structural identifiability. *Automatica*, 2015, 58, pp. 118–124.
16. Villaverde A. F. Observability and structural identifiability of nonlinear biological systems. *Complexity*, 2019, Article ID 8497093. https://doi.org/10.1155/2019/8497093
17. Hengl S., Kreutz C., Timmer J. and Maiwald T. Data-based identifiability analysis of non-linear dynamical models. *Bioinformatics*, 2007, 23(19), pp. 2612–2618.
18. Murphy S.A, van der Vaart AW: On profile likelihood. *Journal of the American Statistical Association*, 2000, 95, pp. 449-465, https://doi.org/10.2307/2669386
19. Raue A, Kreutz C, Theis F.J., Timmer J. Joining forces of Bayesian and frequentist methodology: a study for inference in the presence of non-identifiability. *Phil Trans R Soc A,* 2013, 371:20110544. https://doi.org/10.1098/rsta.2011.0544
20. Neale MC, Miller MB: The use of likelihood-based confidence intervals in genetic models. *Behav Genet*, 1997, 27, pp. 113–120. https://doi.org/10.1023/A:1025681223921.

doi:10.20944/preprints202506.1561.v1

21. Cedersund G: Prediction uncertainty estimation despite unidentifiability: an overview of recent developments. In *Uncertainty in biology*, 2016, pp. 449–466, https://doi.org/10.1007/978-3-319-21296-8_17.
22. Karabutov N. Identification of decentralized control systems. Preprints ID 143582. Preprints202412.1808.v1. doi:10.20944/preprints202412.1808.v1
23. Karabutov N. Adaptive observers for linear time-varying dynamic objects with uncertainty estimation. *International journal of intelligent systems and applications*, 2017, 9(6), pp. 1-15.
24. Karabutov N.N. Identifiability and Detectability of Lyapunov Exponents for Linear Dynamical Systems. *Mekhatronika, Avtomatizatsiya, Upravlenie,* 2022, 23(7), pp. 339-350. https://doi.org/10.17587/mau.23.339-350
25. Karabutov N. Identifiability and detectability of Lyapunov exponents in robotics. In *Design and control advances in robotics*, *Ed. Mohamed Arezki Mellal*, IGI Global Scientific Publishing, 2022, 10(11), pp. 152-174.
26. Bohr G. *Almost periodic functions*. Moscow: Librocom, 2009.
27. Karabutov N. Adaptive identification of decentralized systems. In *Advances in mathematics research. Volume 37. Ed. A. R. Baswell*. New York: Nova Science Publishers, Inc. 2025. 79-115.
28. Karabutov N.N., Shmyrin A.M. Application of adaptive observers for system identification with Bouc-Wen hysteresis. *Bulletin of the Voronezh State Technical University*, 2019, 15(6), pp. 7-13.
29. Barbashin E. A. Lyapunov functions. Moscow: Nauka Publ., 1970
30. Gantmakher F. R. Theory of matrices. Moscow, Nquka, 2010.