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C*-Metric Bourgain-Figiel-Milman, Enflo Type and Mendel-Naor Cotype Problems

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Abstract

We ask for C*-metric version of following three: (1) Bourgain-Figiel-Milman Theorem, (2) Enflo Type, (3) Mendel-Naor Cotype.

Keywords: Bourgain-Figiel-Milman theorem; Enflo Type; Mendel-Naor Cotype

MSC: 47B10; 47L20; 46L08; 4L05

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1. C*-Metric Bourgain-Figiel-Milman Problem

Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional Banach spaces such that $\dim(\mathcal{X}) = \dim(\mathcal{Y})$. The Banach-Mazur distance between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X}, \mathcal{Y}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{X} \to \mathcal{Y} \text{ is invertible linear operator}\}.$$

For $n \in \mathbb{N}$, let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the standard Euclidean Hilbert space. In 1960, Dvoretzky proved the following surprising result [1].

Theorem 1.1. *[1,2] **(Dvoretzky Theorem)** There is a universal constant $C > 0$ satisfying the following property: If $\mathcal{X}$ is any n-dimensional real Banach space and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq C \log n \frac{\varepsilon^2}{|\log \varepsilon|},$$

there exists a k-dimensional Banach subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

We refer [3–17] for more information on Dvoretzky theorem. Having Theorem 1.1, we ask about the metric space version of that. Let $\mathcal{M}$ and $\mathcal{N}$ be finite metric spaces with $o(\mathcal{M}) = o(\mathcal{N})$. Given a map $f : \mathcal{M} \to \mathcal{N}$, define

$$\|f\|_{\text{Lip}} := \sup\left\{\frac{d(f(x), f(y))}{d(x, y)} : x, y \in \mathcal{M}, x \neq y\right\}.$$

The Bourgain-Figiel-Milman distance between $\mathcal{M}$ and $\mathcal{N}$ is defined as

$$d_{BFM}(\mathcal{M}, \mathcal{N}) := \inf\{\|f\|_{\text{Lip}}\|f^{-1}\|_{\text{Lip}} : f : \mathcal{M} \to \mathcal{N} \text{ is injective}\}.$$

In 1986, Bourgain, Figiel and Milman proved the following nonlinear Dvoretzky theorem [18].

Theorem 1.2. *[18] (**Bourgain-Figiel-Milman Theorem**) For every $\varepsilon > 0$, there is a universal constant $C(\varepsilon) > 0$ satisfying the following: Every finite metric space $\mathcal{M}$ contains a subset $\mathcal{N}$ satisfying following conditions.*

(i) *There is an injective map $f : \mathcal{N} \to \ell^2(\mathbb{N}, \mathbb{R})$ such that*

$$d_{BFM}(\mathcal{N}, f(\mathcal{N})) \leq 1 + \varepsilon.$$

(ii)

$$o(\mathcal{N}) \geq C(\varepsilon) \log(o(\mathcal{M})).$$

Further, following holds: If $\varepsilon < 1$, then we can take

$$C(\varepsilon) = \frac{c_1 \varepsilon}{\log(\frac{c_2}{\varepsilon})} \quad \textit{for some } c_1 > 0, c_2 > 0.$$

We will formulate problem based on Theorem 1.2 to C*-algebra valued metric spaces. These are generalizations of metric spaces introduced in 2014 by Ma, Jiang and Sun [19].

Definition 1.3. *[19] Let $\mathcal{A}$ be a unital C*-algebra and $\mathcal{A}^+ := \{aa^* : a \in \mathcal{A}\}$ be the set of all positive elements in $\mathcal{A}$. Let $\mathcal{M}$ be a set. A map $d^* : \mathcal{M} \times \mathcal{M} \to \mathcal{A}^+$ is said to be a C*-valued metric if it satisfies following conditions.*

(i) *If $x, y \in \mathcal{M}$ are such that $d^*(x, y) = 0$, then $x = y$.*
(ii) *$d^*(x, y) = d^*(y, x)$ for all $x, y \in \mathcal{M}$.*
(iii) *$d^*(x, y) \leq d^*(x, z) + d^*(z, y)$ for all $x, y, z \in \mathcal{M}$.*

In this case, $(\mathcal{M}, d^*)$ is called as a C-metric space.*

Given a unital C*-algebra $\mathcal{A}$, we have the standard Hilbert C*-module over $\mathcal{A}$:

$$\ell^2(\mathbb{N}, \mathcal{A}) := \left\{ \{a_n\}_{n=1}^\infty : a_n \in \mathcal{A}, \forall n \in \mathbb{N}, \sum_{n=1}^\infty a_n^* a_n \in \mathcal{A} \right\}$$

equipped with the inner product

$$\langle \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \rangle := \sum_{n=1}^\infty a_n^* b_n, \quad \forall \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in \ell^2(\mathbb{N}, \mathcal{A}).$$

Hence the norm on $\ell^2(\mathbb{N}, \mathcal{A})$ is given by

$$\|\{a_n\}_{n=1}^\infty\| := \left\| \sum_{n=1}^\infty a_n^* a_n \right\|^{\frac{1}{2}}, \quad \forall \{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N}, \mathcal{A}).$$

Let Ω be a compact Hausdorff space. Let $\mathcal{A} := C(\Omega)$ be the C*-algebra of set of all continuous functions on Ω with standard involution and sup norm. Define

$$d^*(\{f_n\}_{n=1}^\infty, \{g_n\}_{n=1}^\infty) := \left(\sum_{n=1}^\infty |f_n - g_n|^2 \right)^{\frac{1}{2}} \in \mathcal{A}, \quad \forall \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in \ell^2(\mathbb{N}, \mathcal{A}).$$

Then $(\ell^2(\mathbb{N}, \mathcal{A}), d^*)$ is a C*-metric space. We now ask for C*-metric version of Bourgain-Figiel-Milman theorem.

Problem 1.4. *(**C*-metric Bourgain-Figiel-Milman Problem**) Let $\mathcal{A}$ be a unital C*-algebra. For every $\varepsilon > 0$, whether there is a universal constant $C(\varepsilon, \mathcal{A}) > 0$ satisfying the following: Every finite C*-metric space $\mathcal{M}$ contains a subset $\mathcal{N}$ satisfying following conditions.*

(i) *There is an injective map* $f : \mathcal{N} \to \ell^2(\mathbb{N}, \mathcal{A})$ *such that*

$$d_{BFM}(\mathcal{N}, f(\mathcal{N})) \leq 1 + \varepsilon.$$

(ii)

$$o(\mathcal{N}) \geq C(\varepsilon, \mathcal{A}) \log(o(\mathcal{M})).$$

2. C*-Metric Enflo Type and Mendel-Naor Cotype Problems

Let $\mathcal{H}$ be a Hilbert space, $n \in \mathbb{N}$. Recall that for any n points $h_1, \dots, h_n \in \mathcal{H}$, we have

$$\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^n \varepsilon_j h_j \right\|^2 = \sum_{j=1}^n \|h_j\|^2. \tag{1}$$

It is Equality (1) which motivated the definition of Type and Cotype for Banach spaces.

Definition 2.1. *[2] Let* $1 \leq p \leq 2$*. A Banach space* $\mathcal{X}$ *is said to be of* ***(Rademacher) Type*** p *if there exists* $T_p(\mathcal{X}) > 0$ *such that*

$$\left(\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \right)^{\frac{1}{p}} \leq T_p(\mathcal{X}) \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}}, \quad \forall x_1, \dots, x_n \in \mathcal{X}, \ \forall n \in \mathbb{N}.$$

Definition 2.2. *[2] Let* $2 \leq q < \infty$*. A Banach space* $\mathcal{X}$ *is said to be of* ***(Rademacher) Cotype*** q *if there exists* $C_q(\mathcal{X}) > 0$ *such that*

$$\left(\sum_{j=1}^n \|x_j\|^q \right)^{\frac{1}{q}} \leq C_q(\mathcal{X}) \left(\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^q \right)^{\frac{1}{q}}, \quad \forall x_1, \dots, x_n \in \mathcal{X}, \ \forall n \in \mathbb{N}.$$

There is a vast literature on Type-Cotype of Banach spaces, see [2,20–28].

In 1970, Enflo introduced the notion of Type for metric spaces [29,30].

Definition 2.3. *[29,30] Let* $p > 0$*. A metric space* $(\mathcal{M}, d)$ *is said to be of Enflo Type* p *if there exists* $E_p(\mathcal{M}) > 0$ *satisfying following conditions: For every* $n \in \mathbb{N}$ *and for every* $f : \{-1,1\}^n \to \mathcal{M}$ *it holds*

$$\left(\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}} d(f(\varepsilon_1, \dots, \varepsilon_n), d(-\varepsilon_1, \dots, -\varepsilon_n))^p \right)^{\frac{1}{p}} \leq$$

$$E_p(\mathcal{M}) \left(\sum_{j=1}^n \frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}} d(f(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n), d(\varepsilon_1, \dots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n))^p \right)^{\frac{1}{p}}.$$

In 2008, Mendel and Naor introduced the notion of Cotype for metric spaces [30].

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Definition 2.4. *[30] Let $q > 0$. A metric space $(\mathcal{M}, d)$ is said to be of Mendel-Naor Cotype q if there exists $\Gamma_q(\mathcal{M}) > 0$ satisfying following conditions: For every $n \in \mathbb{N}$, there exists an even integer $m \in \mathbb{N}$ such that for every $f : \mathbb{Z}_m^n \to \mathcal{M}$ it holds*

$$\begin{aligned}&\sum_{j=1}^{n} \frac{1}{m^n} \sum_{x \in \mathbb{Z}_m^n} d\left(f\left(x + \frac{m}{2}e_j, f(x)\right)\right)^q \\ &\leq \Gamma_q(\mathcal{M})^q m^q \frac{1}{m^n} \sum_{x \in \mathbb{Z}_m^n} \frac{1}{3^n} \sum_{\varepsilon \in \{-1,0,1\}^n} d(f(x+\varepsilon), f(x))^q, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n),\end{aligned}$$

where $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^n$.

We now ask for C*-metric versions of Enflo Type and Mendel-Naor Cotype.

Problem 2.5. ***(C*-metric Enflo Type Problem)*** *Whether there is a way to define C*-metric Enflo Type?*

Problem 2.6. ***(C*-metric Mendel-Naor Cotype Problem)*** *Whether there is a way to define C*-metric Mendel-Naor Cotype?*

Note that there are also notions of Type for metric spaces by Bourgain, Milman and Wolfson [31] and by Ball [32]. These lead to following problems.

Problem 2.7. ***(C*-metric Bourgain-Milman-Wolfson Type Problem)*** *Whether there is a way to define C*-metric Bourgain-Milman-Wolfson Type?*

Problem 2.8. ***(C*-metric Markov/Ball Type Problem)*** *Whether there is a way to define C*-metric Markov/Ball Type?*

Remark 2.9. *In 2023, we formulated Modular Dvoretzky and Type-Cotype problems which are still open [33].*

References

1. Dvoretzky, A. Some results on convex bodies and Banach spaces. In Proceedings of the Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960). Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961, pp. 123–160.
2. Albiac, F.; Kalton, N.J. *Topics in Banach space theory*; Vol. 233, *Graduate Texts in Mathematics*, Springer, [Cham], 2016; pp. xx+508. https://doi.org/10.1007/978-3-319-31557-7.
3. Milman, V.D.; Schechtman, G. *Asymptotic theory of finite-dimensional normed spaces*; Vol. 1200, *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1986; pp. viii+156.
4. Milman, V.D. A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies. *Funkcional. Anal. i Priložen.* **1971**, *5*, 28–37.
5. Milman, V. Dvoretzky's theorem—thirty years later. *Geom. Funct. Anal.* **1992**, *2*, 455–479. https://doi.org/10.1007/BF01896663.
6. Figiel, T. A short proof of Dvoretzky's theorem on almost spherical sections of convex bodies. *Compositio Math.* **1976**, *33*, 297–301.
7. Szankowski, A. On Dvoretzky's theorem on almost spherical sections of convex bodies. *Israel J. Math.* **1974**, *17*, 325–338. https://doi.org/10.1007/BF02756881.
8. Matousek, J. *Lectures on discrete geometry*; Vol. 212, *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2002; pp. xvi+481. https://doi.org/10.1007/978-1-4613-0039-7.
9. Matsak, I.; Plichko, A. Dvoretzky's theorem by Gaussian method. In *Functional analysis and its applications*; Elsevier Sci. B. V., Amsterdam, 2004; Vol. 197, *North-Holland Math. Stud.*, pp. 171–184. https://doi.org/10.1016/S0304-0208(04)80166-X.
10. Pisier, G. *The volume of convex bodies and Banach space geometry*; Vol. 94, *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1989; pp. xvi+250. https://doi.org/10.1017/CBO9780511662454.

11. Gordon, Y.; Guedon, O.; Meyer, M. An isomorphic Dvoretzky's theorem for convex bodies. *Studia Math.* **1998**, *127*, 191–200. https://doi.org/10.4064/sm-127-2-191-200.
12. Figiel, T. A short proof of Dvoretzky's theorem. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. No. XXIII*; 1975; pp. 6 pp. (erratum, p. 3).
13. Figiel, T. Some remarks on Dvoretzky's theorem on almost spherical sections of convex bodies. *Colloq. Math.* **1971/72**, *24*, 241–252. https://doi.org/10.4064/cm-24-2-241-252.
14. Figiel, T.; Lindenstrauss, J.; Milman, V.D. The dimension of almost spherical sections of convex bodies. *Acta Math.* **1977**, *139*, 53–94. https://doi.org/10.1007/BF02392234.
15. Schechtman, G. A remark concerning the dependence on ϵ in Dvoretzky's theorem. In *Geometric aspects of functional analysis (1987–88)*; Springer, Berlin, 1989; Vol. 1376, *Lecture Notes in Math.*, pp. 274–277. https://doi.org/10.1007/BFb0090061.
16. Gordon, Y. Gaussian processes and almost spherical sections of convex bodies. *Ann. Probab.* **1988**, *16*, 180–188.
17. Pisier, G. Probabilistic methods in the geometry of Banach spaces. In *Probability and analysis (Varenna, 1985)*; Springer, Berlin, 1986; Vol. 1206, *Lecture Notes in Math.*, pp. 167–241. https://doi.org/10.1007/BFb0076302.
18. Bourgain, J.; Figiel, T.; Milman, V. On Hilbertian subsets of finite metric spaces. *Isr. J. Math.* **1986**, *55*, 147–152. https://doi.org/10.1007/BF02801990.
19. Ma, Z.; Jiang, L.; Sun, H. C^*-algebra-valued metric spaces and related fixed point theorems. *Fixed Point Theory Appl.* **2014**, *2014*, 11. https://doi.org/10.1186/1687-1812-2014-206.
20. Latala, R.; Oleszkiewicz, K. On the best constant in the Khinchin-Kahane inequality. *Studia Math.* **1994**, *109*, 101–104.
21. Szarek, S.J. On the best constants in the Khinchin inequality. *Studia Math.* **1976**, *58*, 197–208. https://doi.org/10.4064/sm-58-2-197-208.
22. Johnson, W.B.; Lindenstrauss, J., Eds. *Handbook of the geometry of Banach spaces. Vol. I*; North-Holland Publishing Co., Amsterdam, 2001; pp. x+1005.
23. Johnson, W.B.; Lindenstrauss, J., Eds. *Handbook of the geometry of Banach spaces. Vol. 2*; North-Holland, Amsterdam, 2003; pp. xii+1866.
24. Li, D.; Queffelec, H. *Introduction to Banach spaces: analysis and probability. Vol. 1*; Vol. 166, *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2018; pp. xxx+431.
25. Li, D.; Queffelec, H. *Introduction to Banach spaces: analysis and probability. Vol. 2*; Vol. 167, *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2018; pp. xxx+374.
26. Diestel, J.; Jarchow, H.; Tonge, A. *Absolutely summing operators*; Vol. 43, *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1995; pp. xvi+474. https://doi.org/10.1017/CBO9780511526138.
27. Ivanisvili, P.; van Handel, R.; Volberg, A. Rademacher type and Enflo type coincide. *Ann. Math. (2)* **2020**, *192*, 665–678. https://doi.org/10.4007/annals.2020.192.2.8.
28. Maurey, B. Type, cotype and K-convexity. In *Handbook of the geometry of Banach spaces. Volume 2*; Amsterdam: North-Holland, 2003; pp. 1299–1332.
29. Enflo, P. On the nonexistence of uniform homeomorphisms between L^p-spaces. *Ark. Mat.* **1970**, *8*, 103–105. https://doi.org/10.1007/BF02589549.
30. Mendel, M.; Naor, A. Metric cotype. *Ann. Math. (2)* **2008**, *168*, 247–298. https://doi.org/10.4007/annals.2008.168.247.
31. Bourgain, J.; Milman, V.; Wolfson, H. On type of metric spaces. *Trans. Am. Math. Soc.* **1986**, *294*, 295–317. https://doi.org/10.2307/2000132.
32. Ball, K. Markov chains, Riesz transforms and Lipschitz maps. *Geom. Funct. Anal.* **1992**, *2*, 137–172. https://doi.org/10.1007/BF01896971.
33. Krishna, K.M. Modular Dvoretzky, Type-Cotype, Khinchin-Kahane and Grothendieck Inequality Problems. *Preprints: https://doi.org/10.20944/preprints202302.0181.v1* **2023**. https://doi.org/10.20944/preprints202302.0181.v1.