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Relations between Newtonian and Relativistic Cosmology

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Abstract: The Schwarzschild metric emerges independent of Einstein's field equations, offering a straightforward derivation solely reliant on Newtonian mechanics and Minkowskian proper acceleration. This approach provides a clear path to understanding the gravitational field around a spherically symmetric mass without the need for the complexities of Einstein's full theory of General Relativity. Transitioning to our exploration of the cosmic Friedmann equations, we adopt a novel perspective rooted in a Lagrangian formulation grounded in Newtonian mechanics and the first law of thermodynamics. Our investigation operates under the assumption that the universe is populated by either a perfect fluid or a scalar field. By elucidating the intricate interplay between the Lagrangian formulation and the cosmic Friedmann equations, we uncover the fundamental principles governing the universe's dynamics within the framework of these elemental constituents. In our concluding endeavor, we embark on the task of harmonizing the classical equations—namely, the conservation, Euler, and Poisson equations—with the principles of General Relativity. This undertaking seeks to extend these foundational equations to encompass the gravitational effects delineated by General Relativity, thus providing a comprehensive framework for understanding the behavior of matter and spacetime in the cosmic context.

Keywords: Schwarzschild solution; Friedmann equations; Perturbation equations; Newtonian mechanics

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1. Introduction

Based on Einstein's calculations concerning the Perihelion motion of Mercury [1], where he utilized second-order approximations to determine the Christoffel symbols, K. Schwarzschild's subsequent publication of his renowned solution to the General Relativity (GR) equations [2] emerged as a pivotal moment. Schwarzschild's methodology stemmed from the assumptions laid out by Einstein in [1] regarding the metric's form. Employing the framework of unimodular gravity, a concept initially introduced by Einstein himself in a seminal presentation before the Prussian Academy of Science on November 4, 1915 [3], Schwarzschild elegantly solved the field equations, ultimately unveiling the metric produced by a singular point mass.

This solution proved instrumental in deepening our comprehension of gravitational fields surrounding spherically symmetric masses. Notably, it facilitated the anticipation and explication of gravitational phenomena, including the bending of light and the behaviors of objects within intense gravitational domains.

In our current investigation, we undertake a fresh perspective by delving into the relativistic implications of gravity on the motion of massive bodies. Herein, we derive the Schwarzschild solution sans reliance on Einstein's field equations, as elucidated in [4]. By commencing with Newtonian gravitational principles and assimilating the tenets of special relativity, we uncover the metric engendered by a point mass without entangling ourselves in the complexities of the entire General Theory of Relativity. This alternative route underscores the intrinsic link between gravity and spacetime geometry, offering a more accessible grasp of the Schwarzschild solution and accentuating the elegant and intuitive essence of Einstein's gravitational theory.

In the subsequent segment of our study, we engage with the Friedmann equations, initially conceived by Alexander Friedmann in the 1920s through the lens of General Relativity's field equations.

These equations furnish a comprehensive framework for comprehending the universe's evolutionary trajectory on cosmic scales, assuming homogeneity and isotropy while considering curvature and energy distribution.

It is noteworthy that Friedmann's groundbreaking contributions [5], depicting solutions encompassing both expanding and contracting universes, remained relatively obscure for a substantial duration [6]. Einstein himself initially met Friedmann's cosmological findings with skepticism, contending that they deviated from the tenets of General Relativity. However, Einstein later retracted his critique, albeit without immediately embracing the notion of an expanding cosmos. The eventual recognition of the significance of Friedmann's work ensued with the discovery of the Hubble-Lemaître law, which bridged cosmology with fundamental physics. Through the amalgamation of General Relativity and thermodynamics, a comprehensive understanding of the universe's evolution materialized.

Proceeding along this trajectory, we explore an alternative avenue to deducing the Friedmann equations, leveraging Newtonian mechanics and the first law of thermodynamics. While these equations can be derived from Newton's formulations when conceiving the universe as a homogeneous dust fluid, employing the classical Lagrangian formalism furnishes a more exhaustive portrayal of dynamic systems in terms of kinetic and potential energy. Moreover, this formalism enables the contemplation of general perfect fluids and scalar fields, from which the Friedmann equations also emerge within the realm of Newtonian mechanics, thereby enriching our comprehension of the universe's evolutionary dynamics.

Concluding our investigation, we turn our attention to the generalization of the classical trio of equations - the continuity, Euler, and Poisson equations. The aim is to ensure their congruence with the principles of General Relativity, particularly under first-order perturbations. This endeavor necessitates extending these foundational equations to encompass the gravitational effects delineated by General Relativity, thus furnishing a holistic framework for apprehending the interplay between matter and spacetime in the presence of perturbations.

2. Simple Derivation of the Schwarzschild Metric

We start with the second Newton's law for a radial trajectory under the influence of a potential $\Phi(r)$ with radial symmetry [7]:

$$r'' = -\frac{\partial \Phi}{\partial r}, \tag{1}$$

where the prime denotes the derivative with respect to Newtonian absolute time.

At this point, we replace the Newtonian acceleration by the proper acceleration of special relativity $r'' \to \ddot{r}$ [8], where the "dot" denotes the derivative with respect to proper time (that is, given a test particle with mass m we replace the inertial force $F_i = -mr''$ by $F_i = -m\ddot{r}$), and we impose that the test particle does not feel its weight, i.e., we impose the second Newtons law in the following form [9]:

$$F_i + F_g = 0 \Longrightarrow \ddot{r} = -\frac{\partial \Phi}{\partial r}, \tag{2}$$

where $F_g = -m\dfrac{\partial \Phi}{\partial r}$ is the gravitational force acting on the test particle.

This equation leads to the energy conservation:

$$\dot{r}^2 + 2\Phi = 2E. \tag{3}$$

Remark 2.1. *In a Lagrangian context, we consider the classical action* $S = \int \left(\frac{(r')^2}{2} - \Phi(r) \right) dt$ *and, we replace the Newtonian absolute time* t *by the proper time* τ*, obtaining* $S = \int \left(\frac{\dot{r}^2}{2} - \Phi(r) \right) d\tau$*.*

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Next, we consider the radially symmetric metric arising from the potential $\Phi(r)$, which in spherical coordinates can be written as:

$$d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

where τ is the proper time

Assuming that the potential is week at large distances, the metric will approach the Minkowski metric and, for large value of r, the functions $A(r)$ and $B(r)$ will approach 1.

On the other hand, in relativity, the world-line of a particle is determined by the principle of least action, where the action is given by [10]:

$$S = \int d\tau = \int \frac{d\tau^2}{d\tau^2} d\tau = \int (A(r)\dot{t}^2 - B(r)\dot{r}^2) d\tau \equiv \int L d\tau, \tag{5}$$

and we only consider radial trajectories.

Then, the Euler-Lagrange equation corresponding to the variation of the time coordinate t

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t}, \tag{6}$$

leads to the following conservation law:

$$A(r)\dot{t} = C, \tag{7}$$

where C is some constant.

Next, we consider the dynamical equation

$$1 = \frac{d\tau^2}{d\tau^2} = A(r)\dot{t}^2 - B(r)\dot{r}^2, \tag{8}$$

which leads to

$$\dot{r}^2 = \frac{A(r)\dot{t}^2 - 1}{B(r)} = \frac{C^2 - A(r)}{A(r)B(r)}, \tag{9}$$

where we have used the equation (7).

Equating (9) with (3), we get:

$$\frac{C^2 - A(r)}{A(r)B(r)} = 2E - 2\Phi, \tag{10}$$

and since for $r \to \infty$, the functions $A(r)$ and $B(r)$ approach to 1 and $\Phi(r)$ vanishes, one obtains $C^2 = 2E + 1$, and thus, the equation (9) becomes:

$$\dot{r}^2 = \frac{2E}{A(r)B(r)} + \frac{1 - A(r)}{A(r)B(r)}, \tag{11}$$

which must be compared with Eq.(3). Taking into account that $A(r)$ and $B(r)$ do not depend on the energy E, one obtains $A(r)B(r) = 1$, and thus, $\dot{r}^2 = 2E + 1 - A(r)$, which leads to:

$$A(r) = 1 + 2\Phi(r), \qquad B(r) = (1 + 2\Phi(r))^{-1}. \tag{12}$$

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Obviously, if one insert these functions into the relativistic Lagrangian (5), $L = A(r)\dot{t}^2 - B(r)\dot{r}^2$, and uses the Euler-Lagrange equation, one obtains:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Longrightarrow \ddot{r} = -\frac{\partial \Phi}{\partial r}. \tag{13}$$

Finally, when one considers the particular case of a point particle with mass M situated at the origin of coordinates, the potential is given by $\Phi(r) = -\frac{MG}{r}$, resulting in the well-known Schwarzschild metric:

$$A(r) = 1 - \frac{2MG}{r}, \qquad B(r) = \left(1 - \frac{2MG}{r}\right)^{-1}. \tag{14}$$

2.1. Schwarzschild's Derivation

Starting with the assumptions made by Einstein in [1] about the form of the metric and using unimodular gravity (recall that Einstein presented, for the first time -though it was incorrect in the general case but correct for a trace-less stress-energy tensor and thus correct in the vacuum case- his covariant field equations for gravity in unimodular form at the Prussian Academy of Science on November 4, 1915 [3]), Schwarzschild began with a metric in Cartesian coordinates (x, y, z) of the form [2]:

$$ds^2 = -F(r)dt^2 + G(r)(dx^2 + dy^2 + dz^2) + H(r)(xdx + ydy + zdz)^2, \tag{15}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, and the undetermined functions have to satisfy $\lim_{r\to\infty} F(r) = \lim_{r\to\infty} G(r) = 1$ and $\lim_{r\to\infty} H(r) = 0$.

After making a change to polar coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, the metric becomes

$$ds^2 = -F(r)dt^2 + (G(r) + r^2H(r))dr^2 + r^2G(r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{16}$$

In order to continue working in unimodular gravity, Schwarzschild introduced the new coordinates

$$x_1 = \frac{r^3}{3}, \quad x_2 = -\cos\theta, \quad x_3 = \phi, \quad x_4 = t, \tag{17}$$

obtaining the metric

$$ds^2 = -f_4(x_1)dx_4^2 + f_1(x_1)dx_1^2 + f_2(x_1)\frac{dx_2^2}{1 - x_2^2} + f_3(x_1)(1 - x_2^2)dx_3^2, \tag{18}$$

where the new undetermined functions f_i, which have to satisfy $f_1 f_2 f_3 f_4 = 1$, are related with the older ones by:

$$f_4 = F, \quad f_1 = \frac{G}{r^4} + \frac{H}{r^2}, \quad f_2 = f_3 = r^2G. \tag{19}$$

After imposing the vacuum field equations, Schwarzschild obtained:

$$f_2 = f_3 = (3x_1 + \sigma)^{2/3}, \quad f_4 = 1 - \alpha(3x_1 + \sigma)^{-1/3}, \quad f_1 = \frac{(3x_1 + \sigma)^{-4/3}}{1 - \alpha(3x_1 + \sigma)^{-1/3}}, \tag{20}$$

where α and σ are positive constants of integration.

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We can see that f_1 is singular at $x_1 = \frac{1}{3}(\alpha^3 - \sigma)$. Then, assuming that the field is produced by a point particle situated in the origin of coordinates, to move the discontinuity to the origin $r = 0$, one has to take $\sigma = \alpha^3$, and thus,

$$f_1 = \frac{1}{R_\alpha^4(r)}\left(1 - \frac{\alpha}{R_\alpha(r)}\right)^{-1}, \quad f_2 = f_3 = R_\alpha^2(r), \quad f_4 = 1 - \frac{\alpha}{R_\alpha(r)}, \tag{21}$$

with $R_\alpha(r) = R(\alpha^3, r)$ where $R^3(\sigma, r) = r^3 + \sigma$.

The final form of the metric becomes:

$$ds^2 = -\left(1 - \frac{\alpha}{R_\alpha(r)}\right)dt^2 + \left(1 - \frac{\alpha}{R_\alpha(r)}\right)^{-1} dR_\alpha^2(r) + R_\alpha^2(r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{22}$$

or in terms of the polar coordinates

$$ds^2 = -\left(1 - \frac{\alpha}{(r^3+\alpha^3)^{1/3}}\right)dt^2 + \frac{r^4}{(r^3+\alpha^3)^{4/3}}\left(1 - \frac{\alpha}{(r^3+\alpha^3)^{1/3}}\right)^{-1} dr^2 + (r^3+\alpha^3)^{2/3}(d\theta^2 + \sin^2\theta d\phi^2). \tag{23}$$

At the end of his work, Schwarzschild obtained the equation of motion of a planet in the equatorial plane $\theta = \pi/2$

$$\left(\frac{dx}{d\phi}\right)^2 = 1 - h + h\alpha x - x^2 + \alpha x^3, \tag{24}$$

where $x = 1/R$ and h is a constant of integration, and shown that approximately coincide with the one used by Einstein in [1] to calculate the advance of the Perihelion of Mercury.

Now, to identify the parameter α we use the week limit approximation $g_{00} \cong 1 + 2\Phi$, where $\Phi = -\frac{GM}{r}$ is the Newtonian potential corresponding with an star with mass M, for $r > r_0$, being r_0 the radius of the star. So, we have

$$\alpha \cong 2MG\frac{(r^3+\alpha^3)^{1/3}}{r} \Longrightarrow \alpha \cong 2MG\left(1 - \frac{(2MG)^3}{r^3}\right)^{-1/3} \cong 2MG, \tag{25}$$

where we have assumed that $r_0 \gg 2MG$.

This effectively holds in our solar system because the radius of the Sun is approximately $r_0 \cong 7 \times 10^5$ Km, and $2MG \cong 3$ Km. So, since α is a constant of integration, from the weak approximation, we can state that its value is $\alpha = 2MG$.

Two closing observation are warranted:

1. It is crucial to recognize from the metric (23) that the sole singularity arises at the coordinate origin $r = 0$.
2. It is evident that when we set the extraneous parameter σ to zero, we recover the conventional Schwarzschild metric form. This realization was promptly acknowledged by numerous scientists following Schwarzschild's publication [11,12].

3. Friedmann Equations from General Relativity

In this section, we will explore the Friedmann-Lemaître-Robertson-Walker (FLRW) metric (refer to [13] for an interpretation of this metric):

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left(\frac{dr}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{26}$$

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where $N(t)$ denotes the lapse function, k represents the spatial curvature, and $a(t)$ signifies the scale factor.

For this metric, the Ricci scalar is expressed as:

$$R = 6\left(\frac{1}{aN}\frac{d}{dt}\left(\frac{\dot{a}}{N}\right) + \frac{H^2}{N} + \frac{k}{a^2}\right), \tag{27}$$

where in this section, the "dot" signifies the derivative with respect to cosmic time, and $H = \frac{\dot{a}}{a}$ represents the Hubble rate.

Dealing with a homogeneous and isotropic universe filled with a perfect fluid (where pressure depends solely on energy density), the Einstein-Hilbert Lagrangian can be expressed as a function of the Ricci scalar, the space-time measure, and the matter content:

$$L_{EH} = \frac{1}{2}R\sqrt{-g} - 8\pi G\epsilon\sqrt{-g} = \frac{1}{2}Ra^3N - 8\pi G\epsilon a^3N, \tag{28}$$

where ϵ represents the energy density, and $\sqrt{-g} = Na^3$ serves as the measure for the FLRW metric.

It is noteworthy that this Lagrangian can be reformulated as:

$$L_{EH} = \frac{d}{dt}\left(\frac{\dot{a}a^2}{N}\right) - \frac{3\dot{a}^2a}{N} + 3kaN - 8\pi G\epsilon a^3N. \tag{29}$$

Therefore, given that the first term is a total derivative, this Lagrangian is equivalent to:

$$\bar{L}_{EH} = -\frac{3\dot{a}^2a}{N} + 3kaN - 8\pi G\epsilon a^3N. \tag{30}$$

Note that, the variation with respect the lapse leads to the so-called Hamiltonian constraint:

$$0 = \frac{\partial\bar{L}_{EH}}{\partial N} = \frac{3\dot{a}^2a}{N^2} + 3ka - 8\pi G\epsilon a^3, \tag{31}$$

where, after choosing $N(t) = 1$, one arrives to the first Friedmann equation:

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\epsilon. \tag{32}$$

To derive the dynamical equation, we perform a variation with respect to the scale factor. After setting $N = 1$, the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial\bar{L}_{EH}}{\partial\dot{a}}\right) = \frac{\partial\bar{L}_{EH}}{\partial a}, \tag{33}$$

leads to

$$6\ddot{a}a + 3\dot{a}^2 = 8\pi G\frac{d}{da}(\epsilon a^3) - 3ka, \tag{34}$$

where, due to the energy density's dependency solely on the scale factor, we replace $\frac{\partial}{\partial a}$ with $\frac{d}{da}$.

Next, assuming adiabatic evolution where the total entropy remains conserved, and utilizing the first law of thermodynamics

$$d(\epsilon a^3) = -pda^3 \Longrightarrow \frac{d}{da}(\epsilon a^3) = -3pa^2, \tag{35}$$

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where p represents pressure, we obtain:

$$\frac{\ddot{a}}{a} = -4\pi G p - \frac{1}{2}\left(H^2 + \frac{k}{a^2}\right). \tag{36}$$

From the first Friedmann equation (32), we arrive at the acceleration, or second Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \epsilon). \tag{37}$$

A final observation is warranted: In the FLRW space-time, the energy density and pressure of a perfect fluid solely depend on the scale factor, i.e., $\epsilon = \epsilon(a)$ and $p = p(a)$. Thus, we have:

$$\epsilon(a) = \rho(a) - \frac{3}{a^3}\int^a \bar{a}^2 p(\bar{a}) d\bar{a}, \tag{38}$$

where ρ represents mass density, assumed conserved, i.e., $d(a^3\rho) = 0 \Longrightarrow \rho = \frac{M}{a^3}$, with M being the mass contained within volume a^3. Indeed, from the first law of Thermodynamics, $\frac{d\epsilon}{da} = -\frac{3(\epsilon+p)}{a}$, the solution of which is given by (38).

Consequently, the Einstein-Hilbert Lagrangian, as a function of the scale factor and lapse function, reads:

$$\bar{L}_{EH}(a, \dot{a}, N) = -\frac{3\dot{a}^2 a}{N} + 3kaN - 8\pi G\epsilon(a)a^3 N, \tag{39}$$

with $\epsilon(a)$ given by (38).

Finally, note that the energy density can be expressed as a function of mass density as follows [14,15]:

$$\epsilon(\rho) = \rho\left(1 + \int^\rho \frac{p(\bar{\rho})}{\bar{\rho}^2} d\bar{\rho}\right), \tag{40}$$

and thus, as a function of ρ and N, the Einstein-Hilbert Lagrangian becomes:

$$\bar{L}_{EH}(\rho, \dot{\rho}, N) = -\frac{\dot{\rho}^2}{3N\rho^3} + 3k\left(\frac{M}{\rho}\right)^{1/3} N - 8\pi G\epsilon(\rho)\frac{MN}{\rho^3}. \tag{41}$$

3.1. Friedmann Equations for an Scalar Field

Scalar fields have proven to be highly advantageous in the study of cosmology. They have been instrumental in replicating the phenomenon of inflation [16], which explains the rapid expansion of the universe in its early stages. Additionally, scalar fields have also been utilized to model quintessence, a type of dark energy that is believed to be responsible for the accelerated expansion of the universe [17]. These applications demonstrate the versatility and effectiveness of scalar fields in advancing our understanding of the cosmos.

Hence, in this section, we explore a homogeneous scalar field, denoted as ϕ, minimally coupled with gravity, and derive the corresponding Friedmann equations within the framework of General Relativity. In this scenario, for the metric (26), the energy density and pressure are expressed as:

$$\epsilon = \frac{\dot{\phi}^2}{2N^2} + V(\phi), \quad p = \frac{\dot{\phi}^2}{2N^2} - V(\phi), \tag{42}$$

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and the corresponding Lagrangian is derived from (30) by substituting the energy density with minus the pressure, yielding:

$$\bar{L}_{EH}(a,\dot{a},\phi,\dot{\phi},N) = -\frac{3\dot{a}^2 a}{N} + 3kaN + 8\pi G a^3\left(\frac{\dot{\phi}^2}{2N} - NV(\phi)\right). \tag{43}$$

Consequently, upon performing the variation with respect to the lapse function, we obtain:

$$0 = \frac{\partial \bar{L}_{EH}}{\partial N} = \frac{3\dot{a}^2 a}{N^2} + 3ka - 8\pi G\left(\frac{\dot{\phi}^2}{2N^2} + V(\phi)\right), \tag{44}$$

which, upon selecting $N = 1$, transforms into the first Friedmann equation (32).

On the other hand, when $N = 1$, varying with respect to the scalar field yields the conservation equation:

$$\frac{d}{dt}\left(\frac{\partial \bar{L}_{EH}}{\partial \dot{\phi}}\right) = \frac{\partial \bar{L}_{EH}}{\partial \phi} \Longrightarrow \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \tag{45}$$

which equivalently represents the first law of thermodynamics.

Lastly, a straightforward computation demonstrates that the second Friedmann equation arises from the variation with respect to the scale factor.

4. Friedmann Equations from Newtonian Mechanics

We consider, in co-moving coordinates, a homogeneous large ball with a radius of $\bar{R}$ in Euclidean space (we can also consider $\bar{R} = +\infty$, but for finite radius, the total mass within the ball is finite $\bar{M} = \frac{4\pi}{3}\rho\bar{R}^3$, where ρ is the mass density). Assume that the ball expands radially. This means that if O is the center of the ball, a point P within the ball at t_0 transforms into point P_t at time t, and the distance from O to P_t is given by $d_{OP_t} \equiv d_{OP}(t) = a(t)d_{OP} = a(t)|\overrightarrow{OP}|$, where $a(t)$, with $a(t_0) = 1$, is the scale factor. Furthermore, at time t_0, we consider the triangle $\widehat{POQ}$, which transforms into the equivalent triangle $\widehat{P_tOQ_t}$ at time t. Therefore, as $d_{OP_t} = a(t)d_{OP}$ and $d_{OQ_t} = a(t)d_{OQ}$, using Thales' theorem we find that for any points P and Q within the ball, $d_{P_tQ_t} = a(t)d_{PQ}$.

The relation shows that any ball, at t_0, centered at a point P with radius $R \ll \bar{R}$, expands radially at the same rate as the original large ball. Additionally, the relative velocity between P_t and Q_t follows the Hubble-Lemaître law:

$$\frac{d}{dt}(d_{P_tQ_t}) = \dot{a}(t)d_{PQ} = H(t)d_{P_tQ_t}. \tag{46}$$

The equation of motion for the scale factor in Newtonian mechanics is derived by considering a ball centered at a given point P and initial radius R at time t_0. At time t, the radial force at a given point Q_t on the boundary of the ball is calculated to be $\mathbf{F}(Q_t) = f(a(t)R)\frac{\overrightarrow{P_tQ_t}}{a(t)R}$. To determine the function f, the flux entering the ball is computed as:

$$\Psi = \int \mathbf{F}.\mathbf{n}dS = 4\pi a^2(t)R^2 f(a(t)R), \tag{47}$$

where $\mathbf{n}$ is the external normal to the sphere surrounding the ball and dS is the measure of the sphere. On the other hand, from the Poisson equation $\nabla.\mathbf{F} = -4\pi G\rho$, and Gauss's theorem, the flux is also given by:

$$\Psi = \int \nabla.\mathbf{F}dV = -\frac{16\pi^2 G}{3}\rho a^3(t)R^3, \tag{48}$$

which leads to the expression for f as:

$$f(a(t)R) = -\frac{4\pi G}{3}\rho a(t)R \Longrightarrow \mathbf{F}(Q_t) = -\frac{4\pi G}{3}\rho\overrightarrow{P_tQ_t}. \tag{49}$$

Therefore, the acceleration experienced by a probe particle of mass m at the point Q due to the ball is determined by the second Newton's law as [4,18,19]:

$$m\frac{d^2}{dt^2}(\overrightarrow{P_tQ_t}) = -\frac{4\pi Gm}{3}\rho\overrightarrow{P_tQ_t} \Longrightarrow \ddot{a} = -\frac{4\pi G}{3}\rho a, \tag{50}$$

where we have used that $\overrightarrow{P_tQ_t} = a(t)\overrightarrow{PQ}$.

Here, it is important to recall that the constant κ appearing in Einstein's field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}, \tag{51}$$

where $R_{\mu\nu}$ denotes the Ricci tensor, $g_{\mu\nu}$ the metric, and $T_{\mu\nu}$ the energy-stress tensor, is obtained under the assumption that the background is flat. Specifically, by approximating $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the background $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ represents a small perturbation, Einstein's equations simplify to $R_0^0 = \frac{\kappa}{2}T_0^0$, where

$$R_0^0 = -\frac{1}{2}\Delta g_{00}, \quad \text{and} \quad T_0^0 = \rho. \tag{52}$$

Considering that in the Newtonian approximation, g_{00} and the Newtonian potential Φ are related by $g_{00} = -1 - 2\Phi$, and employing the Poisson equation $\Delta\Phi = 4\pi G\rho$, we obtain:

$$\Delta\Phi = \frac{\kappa\rho}{2} \Longrightarrow \kappa = 8\pi G. \tag{53}$$

Hence, it appears natural, as we have demonstrated, that the background is spatially flat and the volume of the ball is $\frac{4\pi}{3}a^3R^3$. However, as we shall see, the spatial curvature emerges in a natural manner.

Returning to Eq. (50), we eliminate the mass m to derive the second Friedmann equation for a dust field $(p = 0)$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho. \tag{54}$$

This equation can be derived from the Lagrangian:

$$L_N = \frac{R^2\dot{a}^2}{2} + \frac{GM}{aR} = \frac{R^2\dot{a}^2}{2} + \frac{4\pi G}{3}R^2a^2\rho, \tag{55}$$

where $M = \frac{4\pi}{3}a^3R^3\rho$ represents the mass inside the ball. Indeed, employing the Euler-Lagrange equation yields:

$$\frac{d}{dt}\left(\frac{\partial L_N}{\partial\dot{a}}\right) = \frac{\partial L_N}{\partial a} \Longrightarrow \ddot{a} = \frac{4\pi G}{3}\frac{\partial}{\partial a}(a^2\rho) = -\frac{4\pi G}{3}\rho a, \tag{56}$$

where we have utilized mass conservation:

$$\frac{\partial}{\partial a}(a^3\rho) = 0 \Longrightarrow \frac{\partial}{\partial a}(a^2\rho) = -a\rho. \tag{57}$$

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We observe that the radius R of the chosen ball does not affect the dynamical equations. Thus, we set $R = 1$.

To derive the second Friedmann equation for a general fluid field, we employ the relativistic equation $E = m$, relating the energy of a particle at rest to its mass. We substitute the mass density with the energy density in the Newtonian Lagrangian (55) with $R = 1$, resulting in:

$$\bar{L}_N = \frac{\dot{a}^2}{2} + \frac{4\pi G}{3} a^2 \epsilon. \tag{58}$$

Using the Euler-Lagrange equation and the first law of thermodynamics,

$$d(\epsilon a^3) = ad(\epsilon a^2) + \epsilon a^2 da \Longrightarrow -3pa^2 da = ad(\epsilon a^2) + \epsilon a^2 da \Longrightarrow d(\epsilon a^2) = -(3p + \epsilon) a da, \tag{59}$$

we readily derive (37).

The next step is to obtain the first Friedmann equation. This can be achieved by combining the second equation with the first law of Thermodynamics, expressed as follows:

$$\dot{\epsilon} = -3H(\epsilon + p). \tag{60}$$

Firstly, we rewrite (37) as:

$$\frac{\ddot{a}}{a} = -4\pi G(p + \epsilon) + \frac{8\pi G}{3}\epsilon. \tag{61}$$

Then, we calculate

$$\frac{dH^2}{dt} = 2H\frac{\ddot{a}}{a} - 2H^3. \tag{62}$$

Inserting (61) into it, we obtain:

$$\frac{d}{dt}\left(H^2 - \frac{8\pi G}{3}\epsilon\right) = -2H\left(H^2 - \frac{8\pi G}{3}\epsilon\right) \Longrightarrow \frac{d\left(H^2 - \frac{8\pi G}{3}\epsilon\right)}{H^2 - \frac{8\pi G}{3}\epsilon} = -2\frac{da}{a}, \tag{63}$$

whose solution is given by

$$H^2 - \frac{8\pi G}{3}\epsilon = \frac{C}{a^2}, \tag{64}$$

and by setting the constant of integration C equal to $-k$, we obtain the first Friedmann equation.

Hence, the Newtonian Lagrangian in terms of the scale factor is given by:

$$\bar{L}_N(a, \dot{a}) = \frac{\dot{a}^2}{2} + \frac{4\pi G}{3} a^2 \epsilon(a), \tag{65}$$

with $\epsilon(a)$ given by (38).

Finally, considering that the energy of a homogeneous ball of radius a is $E = \frac{4\pi}{3} a^3 \epsilon$, the Newtonian Lagrangian appears as $\bar{L}_N = E_{\text{kin}} - V$, where $E_{\text{kin}} = \frac{\dot{a}^2}{2}$ is the kinetic energy per unit mass and $V = -\frac{GE}{a}$ is the gravitational potential generated by the ball with rest mass E.

4.1. Friedmann Equations for an Scalar Field

In a manner analogous to relativistic cosmology, when dealing with a scalar field, we replace ϵ with $-p$ in the Newtonian Lagrangian (65).

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Let ϕ' denote the derivative of the scalar field with respect to the scale factor. We have $\dot{\phi} = \dot{a}\phi'$, and thus, the pressure takes the form $p = \frac{\dot{a}^2(\phi')^2}{2} - V(\phi)$, resulting in the Lagrangian:

$$\bar{L}_N(a,\dot{a}) = \frac{\dot{a}^2}{2} - \frac{4\pi G}{3}a^2\left(\frac{\dot{a}^2(\phi')^2}{2} - V(\phi)\right), \tag{66}$$

where the scalar field is now a function of the scale factor.

Firstly, the first law of thermodynamics, expressed as $\frac{d}{dt}(\epsilon a^3) = -3pa^2\dot{a}$, yields the conservation equation (45). Then, upon variation with respect to the scale factor, we obtain the second Friedmann equation (37). Essentially, this yields:

$$\frac{\partial \bar{L}_N}{\partial \dot{a}} = \dot{a} - \frac{4\pi G}{3}a^2\dot{a}(\phi')^2 = \dot{a} - \frac{4\pi G}{3}\frac{a^2}{\dot{a}}\dot{\phi}^2, \tag{67}$$

and thus,

$$\frac{d}{dt}\left(\frac{\partial \bar{L}_N}{\partial \dot{a}}\right) = \ddot{a} - \frac{4\pi G}{3}\frac{d}{dt}\left(\frac{a^2}{\dot{a}}\dot{\phi}^2\right) = \ddot{a} - \frac{4\pi G}{3}\left(a^2\dot{\phi}\frac{d}{dt}\left(\frac{\dot{\phi}}{\dot{a}}\right) + 2a\dot{\phi}^2 + \frac{a}{H}\dot{\phi}\ddot{\phi}\right). \tag{68}$$

On the other hand,

$$\frac{\partial \bar{L}_N}{\partial a} = -\frac{4\pi G}{3}\left(\dot{a}^2 a(\phi')^2 + \dot{a}^2a^2\phi'\phi'' - a^2\frac{\partial V}{\partial \phi}\phi' - 2aV\right) = -\frac{4\pi G}{3}\left(a\dot{\phi}^2 + a^2\dot{\phi}\frac{d}{dt}\left(\frac{\dot{\phi}}{\dot{a}}\right) - \frac{a^2}{\dot{a}}\frac{\partial V}{\partial \phi}\dot{\phi} - 2aV\right). \tag{69}$$

Then, applying the Euler-Lagrange equation, we arrive at:

$$\ddot{a} = -\frac{4\pi G}{3}\left(-a(\dot{\phi}^2 - 2V) - \frac{a}{H}\dot{\phi}\left(\frac{\partial V}{\partial \phi} + \ddot{\phi}\right)\right). \tag{70}$$

Utilizing the conservation equation (45), we derive the second Friedmann equation as:

$$\ddot{a} = -\frac{8\pi G}{3}(\dot{\phi}^2 - V)a = -\frac{4\pi G}{3}(\epsilon + 3p)a, \tag{71}$$

where we have employed the relation $\epsilon + 3p = 2(\dot{\phi}^2 - V)$.

4.2. Application to open systems

The Newtonian formulation extends to open systems, such as adiabatic systems where matter creation is permitted, conserving the total entropy. The first law of thermodynamics in such a scenario reads [20,21]:

$$d(a^3\epsilon) = -pda^3 + \frac{(\epsilon + p)a^3}{\bar{N}}d\bar{N}, \tag{72}$$

where $\bar{N}(t)$ represents the number of produced particles at time t. Equivalently, this equation can be expressed as

$$\dot{\epsilon} + 3H\left(1 - \frac{\dot{\bar{N}}}{3H\bar{N}}\right)(\epsilon + p) = 0. \tag{73}$$

Employing the Lagrangian $\bar{L}_N$, the second Friedmann equation becomes:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \epsilon) + \frac{4\pi G}{3}\frac{\dot{\bar{N}}}{H\bar{N}}(p + \epsilon) \Longleftrightarrow \frac{\ddot{a}}{a} = -4\pi G\left(1 - \frac{\dot{\bar{N}}}{3H\bar{N}}\right)(p + \epsilon) + \frac{8\pi G}{3}\epsilon, \tag{74}$$

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while the first one remains (32).

Moreover, combining both Friedmann equations provides insight into the evolution of the Hubble rate. Specifically, considering $\dot{H} = \frac{\ddot{a}}{a} - H^2$, we derive:

$$\dot{H} = -4\pi G\left(1 - \frac{\dot{\bar{N}}}{3H\bar{N}}\right)(p+\epsilon) + \frac{k}{a^2}. \tag{75}$$

This equation admits analytical solutions for linear Equations of State ($p = w\epsilon$, with w constant), particularly in spatially flat scenarios, across several open models.

For various functions Γ defining the particle production rate $\Gamma = \frac{\dot{\bar{N}}}{\bar{N}}$, one can analytically determine the universe's evolution [22,23]. For instance, in the case of a constant $\Gamma > 0$, the solution derived from the first Friedmann equation and (75) yields:

$$\dot{H} = -4\pi G\left(1 - \frac{\Gamma}{3H}\right)(1+w)\epsilon = -\frac{3(1+w)}{2}\left(1 - \frac{\Gamma}{3H}\right)H^2, \tag{76}$$

with the solution:

$$H(t) = \frac{\Gamma}{3}\exp\left(\frac{\Gamma(1+w)}{2}(t-t_s)\right)\left(\exp\left(\frac{\Gamma(1+w)}{2}(t-t_s)\right) - 1\right)^{-1}. \tag{77}$$

This solution indicates a big bang singularity at $t = t_s$ ($H(t_s) = +\infty$), transitioning to a de Sitter phase at late times, where $H(t) \cong \Gamma/3$.

In closing, it is worth noting that more generalized particle production rates, such as $\Gamma(H) = -\Gamma_0 + mH + n/H$, where Γ_0, m, and n are constants, have been extensively investigated [23,24]. These models predict early and late accelerated expansion phases for various parameter values.

5. Perturbations in Classical Mechanics: Perfect Fluids

This section endeavors to generalize the fundamental classical equations in fluid dynamics to align, at least to the first order of perturbations, with Einstein's field equations.

Expanding classical fluid dynamics to incorporate the principles of General Relativity marks a significant step in our understanding of the universe's behavior. By extending classical equations, we aim to capture the intricate dynamics of spacetime curvature influenced by fluid distributions.

In this pursuit, it becomes imperative to reconcile the robust framework of classical fluid dynamics with the profound insights offered by General Relativity. Achieving this alignment facilitates a deeper comprehension of how matter and energy interact with the fabric of spacetime.

5.1. First Law of Thermodynamics

Let $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the flow of a perfect fluid, with $\varphi_0 = \text{Id}$. We define the vector velocity $\mathbf{v}(\varphi_t(\mathbf{q}), t) = \frac{d\varphi_t(\mathbf{q})}{dt}$.

Then, we arrive at the crucial result, as outlined in [25]:

$$\left[\frac{d}{dt}\int_{\varphi_t(V)} f(\mathbf{q},t)dV\right]_{t=\bar{t}} = \int_{\varphi_{\bar{t}}(V)}\left(\frac{\partial f}{\partial t} + \nabla_{\mathbf{q}}\cdot(f\mathbf{v})\right)_{t=\bar{t}} dV, \tag{78}$$

where $\nabla \mathbf{q} \cdot \mathbf{u}$ denotes the divergence of the vector field $\mathbf{u}$.

Applying this result to the first law of thermodynamics:

$$\left[\frac{d}{dt}\int_{\varphi_t(V)} \epsilon(\mathbf{q},t)dV\right]_{t=\bar{t}} = -p(\mathbf{q},t)\left[\frac{d}{dt}\int_{\varphi_t(V)} 1dV\right]_{t=\bar{t}}. \tag{79}$$

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Here, once again, ϵ denotes the energy density of the fluid and p its pressure. This yields the conservation equation:

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\epsilon \mathbf{v}) = -p\nabla \cdot \mathbf{v}. \tag{80}$$

Next, we consider an expanding universe described by the flat FLRW metric $ds^2 = -dt^2 + a^2 d\mathbf{q}^2$. The element of volume is given by $dV = a^3 dq_1 dq_2 dq_3$, and in differential form, the first law of thermodynamics becomes:

$$\dot{\epsilon} + 3H(\epsilon + p) + \nabla_{\mathbf{q}} \cdot (\epsilon \mathbf{v}) + p\nabla_{\mathbf{q}} \cdot \mathbf{v} = 0, \tag{81}$$

which, up to first order ($\epsilon = \epsilon_0 + \delta\epsilon$ and $p = p_0 + \delta p$), leads to:

$$\dot{\epsilon}_0 + 3H(\epsilon_0 + p_0) = 0 \qquad \text{and} \qquad \dot{\delta\epsilon} + (\epsilon_0 + p_0)\nabla \mathbf{q} \cdot \mathbf{v} = 0. \tag{82}$$

We can also introduce gravity by considering the following metric in the weak field approximation $|\Phi| \ll 1$:

$$d\tau^2 = (1 + 2\Phi(\mathbf{q}))dt^2 - (1 - 2\Phi(\mathbf{q}))d\mathbf{q}^2, \tag{83}$$

which coincides with formula (106.3) of [10] (also obtained in Einstein's book "The Meaning of Relativity" [26]). In modern language, this is referred to as the "Newtonian gauge".

Then, applying the first law of thermodynamics to this metric including the expansion of the universe, i.e., to

$$d\tau^2 = (1 + 2\Phi(\mathbf{q}, t))dt^2 - a^2(t)(1 - 2\Phi(\mathbf{q}, t))d\mathbf{q}^2, \tag{84}$$

one obtains the first-order perturbed equation

$$\dot{\delta\epsilon} + (\epsilon_0 + p_0)\nabla_{\mathbf{q}} \cdot \mathbf{u} + 3H(\delta\epsilon + \delta p) - 3(\epsilon_0 + p_0)\dot{\Phi} = 0. \tag{85}$$

At this point, it is useful to use the notation $\mathbf{u} \equiv \frac{d\mathbf{q}}{d\eta} = a\mathbf{v}$ being η the conformal time and, $\nabla \equiv \frac{1}{a}\nabla_{\mathbf{q}}$, obtaining:

$$\dot{\delta\epsilon} + (\epsilon_0 + p_0)\nabla \cdot \mathbf{u} + 3H(\delta\epsilon + \delta p) - 3(\epsilon_0 + p_0)\dot{\Phi} = 0. \tag{86}$$

Remark 5.1. *It is important to recall that this equation is the same as the linearized equation* $\nabla_\mu T^{\mu 0} = 0$, *where*

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} \tag{87}$$

is the stress-energy tensor.

5.2. Euler's Equation

First of all, we recall that for a dust fluid, i.e., $|p| \ll \epsilon \cong \rho$, and the element of volume $dV = dq_1 dq_2 dq_3$, the classical Euler's equation can be written as:

$$\left[\frac{d}{dt}\int_{\varphi_t(V)} \rho \mathbf{v} dV\right]_{t=\bar{t}} = \int_{\partial\varphi_{\bar{t}}(V)} \mathcal{T}(\mathbf{n})dS - \int_{\varphi_{\bar{t}}(V)} \rho\nabla\Phi dV, \tag{88}$$

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where $\mathcal{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the stress tensor, dS is the element of area, and $\mathbf{n}$ is the external unit vector to the boundary. Taking into account that for a perfect fluid one has $\mathcal{T}(\mathbf{n}) = -p\mathbf{n}$ and, from the Gauss theorem, the Euler equation in the differential form becomes:

$$\partial_t(\rho\mathbf{v}) + \nabla_{\mathbf{q}}.(\rho\mathbf{v})\mathbf{v} + \rho\mathbf{v}.\nabla_{\mathbf{q}}\mathbf{v} + \nabla_{\mathbf{q}}p + \rho\nabla_{\mathbf{q}}\Phi = 0 \quad \text{or} \quad \dot{\mathbf{v}} + \mathbf{v}.\nabla_{\mathbf{q}}\mathbf{v} + \frac{1}{\rho}\nabla_{\mathbf{q}}p + \nabla_{\mathbf{q}}\Phi = 0, \tag{89}$$

where we have used the first law of thermodynamics for a dust fluid or the continuity equation $\dot{\rho} + \nabla_{\mathbf{q}}.(\rho\mathbf{v}) = 0$.

Note that the equation (89) is incompatible with special relativity. For this reason, we have to compare it with the conservation law $\partial_\mu T^{\mu k} = 0$ in the Minkowski spacetime:

$$\partial_t((\epsilon + p)\mathbf{v}) + \nabla\mathbf{q}.((\epsilon + p)\mathbf{v})\mathbf{v} + (\epsilon + p)\mathbf{v}.\nabla_{\mathbf{q}}\mathbf{v} + \nabla_{\mathbf{q}}p = 0. \tag{90}$$

Therefore, the Euler equation in an expanding universe compatible with special relativity is obtained by replacing the mass density ρ by the heat function per unit volume $(\epsilon + p)$ [27], the velocity $\mathbf{v}$ by $a^2\mathbf{v}$ and using the element of volume $dV = a^3 dq_1 dq_2 dq_3$ in (91):

$$\left[\frac{d}{dt}\int_{\varphi_t(V)} a^2(\epsilon + p)\mathbf{v}dV\right]_{t=\bar{t}} = \int_{\partial\varphi_{\bar{t}}(V)} \mathcal{T}(\mathbf{n})dS - \int_{\varphi_{\bar{t}}(V)} (\epsilon + p)\nabla_{\mathbf{q}}\Phi dV, \tag{91}$$

obtaining, using once again the notation $\mathbf{u} = a\mathbf{v}$ and, $\nabla = \frac{1}{a}\nabla_{\mathbf{q}}$, at the first order of perturbations:

$$\partial_t((\epsilon_0 + p_0)\mathbf{u}) + 4H(\epsilon_0 + p_0)\mathbf{u} + \nabla\delta p + (\epsilon_0 + p_0)\nabla\Phi = 0, \tag{92}$$

which is equivalent, up to linear terms, to the equation $\nabla_\mu T^{\mu k} = 0$.

5.3. Poisson's Equation

We start with the classical Poisson equation:

$$\Delta_{\mathbf{q}}\Phi = 4\pi G a^2 \rho \Longrightarrow \Delta\Phi = 4\pi G\rho, \tag{93}$$

where, once again, ρ denotes the mass density.

The last equation is the Hamiltonian constraint [28]:

$$\mathcal{I} + \mathcal{R} = 16\pi G E, \tag{94}$$

where $\mathcal{R}$ is the intrinsic curvature (the spatial curvature), $\mathcal{I} = K^2 - K_{ij}K^{ij}$, where $K_{ij} = \frac{1}{\sqrt{N}} g(\nabla_{\partial_i}\partial_j, \partial_t)$, is the extrinsic curvature and $E = T_{00}n^0 n^0$, with $n^\alpha = \frac{1}{\sqrt{1+2\Phi}}\partial_t$ as the unit time vector.

In our case, we have:

$$\mathcal{I} \cong 6(1 - 2\Phi)\big(H - \dot{\Phi}\big)^2, \qquad \mathcal{R} = 4\Delta\Phi \qquad \text{and} \qquad E \cong \epsilon. \tag{95}$$

Therefore, the third equation is:

$$2\Delta\Phi + 3\left(H^2 - 2H^2\Phi - 2H\dot{\Phi}\right) = 8\pi G\epsilon, \tag{96}$$

which leads to the perturbed equation:

$$\Delta\Phi - 3\left(H^2\Phi + H\dot{\Phi}\right) = 4\pi G\delta\epsilon. \tag{97}$$

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Remark 5.2. *One could understand $\frac{1}{16\pi G}\mathcal{I}$ as the kinetic energy of the field and $\frac{1}{16\pi G}\mathcal{R}$ as its potential energy.*

5.4. Generalization of the Three Classical Equations

The generalization of the three "classical" equations, for the volume element $dV = a^3(1 - 3\Phi)dq_1dq_2dq_3$, which, only up to linear order, are equivalent to the Einstein's field equations, are:

$$\begin{cases} \left[\frac{d}{dt}\int_{\varphi_t(V)} \epsilon dV\right]_{t=\bar{t}} & = & -p\left[\frac{d}{dt}\int_{\varphi_t(V)} 1dV\right]_{t=\bar{t}} \\ \left[\frac{d}{dt}\int_{\varphi_t(V)} a(\epsilon+p)\mathbf{u}dV\right]_{t=\bar{t}} & = & -\int_{\partial\varphi_{\bar{t}}(V)} p\mathbf{n}dS - \int_{\varphi_{\bar{t}}(V)} a(\epsilon+p)\nabla\Phi dV \\ 2\Delta\Phi + 3\left(H^2 - 2H^2\Phi - 2H\dot{\Phi}\right) & = & 8\pi G\epsilon. \end{cases} \tag{98}$$

In differential form, these "classical" equations can be approximated by:

$$\begin{cases} D_t\epsilon + (\epsilon+p)\left[3(H-\dot{\Phi}) + \nabla.\mathbf{u}\right] & = & 0 \\ D_t((\epsilon+p)\mathbf{u}) + (\epsilon+p)[4H\mathbf{u} + (\nabla.\mathbf{u})\mathbf{u} + \nabla\Phi] + \nabla p & = & 0 \\ 2\Delta\Phi + 3\left(H^2 - 2H^2\Phi - 2H\dot{\Phi}\right) & = & 8\pi G\epsilon, \end{cases} \tag{99}$$

where we have introduced the standard notation in fluid mechanics for the total time-derivative: $D_tf = \partial_tf + \mathbf{u}.\nabla f$.

We can observe that the first and second equations, i.e., the first law of thermodynamics and the generalization of the Euler's equation, up to linear order, correspond to the conservation of the energy-stress tensor: $\nabla_\mu T^{\mu\nu} = 0$. And the last one is the generalization of the Poisson equation.

In a static universe, for a dust fluid and a weak static potential, we recover the classical equations, namely, the continuity, Euler, and Poisson equations:

$$\begin{cases} D_t\rho + \rho\nabla.\mathbf{u} & = & 0 \\ D_t\mathbf{u} + \frac{1}{\rho}\nabla p + \nabla\Phi & = & 0 \\ \Delta\Phi & = & 4\pi G\rho. \end{cases} \tag{100}$$

And the linear order perturbed equations are the same as in General Relativity:

$$\begin{cases} \dot{\delta\epsilon} + (\epsilon_0+p_0)[\nabla.\mathbf{u} - 3\dot{\Phi}] + 3H(\delta\epsilon+\delta p) & = & 0 \\ \partial_t((\epsilon_0+p_0)\mathbf{u}) + (\epsilon_0+p_0)[4H\mathbf{u} + \nabla\Phi] + \nabla\delta p & = & 0 \\ \Delta\Phi - 3\left(H^2\Phi + H\dot{\Phi}\right) & = & 4\pi G\delta\epsilon. \end{cases} \tag{101}$$

To conclude this section, it is important to recognize that the last equation in (99) serves as a constraint, specifically the Hamiltonian constraint. However, by combining all three equations, we can derive the following dynamical equation for the Newtonian potential:

$$\frac{6}{a^5}\partial_t(a^5\dot{\Phi}) - 2\Delta\Phi - R_0(1-2\Phi) = 8\pi GT \Longleftrightarrow 6\ddot{\Phi} - 2\Delta\Phi + 30H\dot{\Phi} - R_0(1-2\Phi) = 8\pi GT, \tag{102}$$

where $T = 3p - \epsilon$ is the trace of the stress-energy tensor and $R_0 = 6(\dot{H} + 2H^2)$ the zero order Ricci scalar.

This equation encompasses both the second Friedmann equation and the Poisson equation. Hence, the three dynamical equations in Newtonian theory, which encapsulate the Friedmann equations and the perturbed equations of General Relativity, are:

$$\begin{cases} D_t\epsilon + (\epsilon + p)\left[3(H - \dot{\Phi}) + \nabla.\mathbf{u}\right] & = & 0 \\ D_t((\epsilon + p)\mathbf{u}) + (\epsilon + p)[4H\mathbf{u} + (\nabla.\mathbf{u})\mathbf{u} + \nabla\Phi] + \nabla p & = & 0 \\ 3\ddot{\Phi} - \Delta\Phi + 15H\dot{\Phi} - 3(\dot{H} + 2H^2)(1 - 2\Phi) & = & 4\pi G(3p - \epsilon). \end{cases} \tag{103}$$

A final remark is in order: By utilizing the conformal time $d\eta = \frac{dt}{a}$ and introducing the new variable $\bar{\mathbf{q}} = \sqrt{3}\mathbf{q}$, the dynamical equation for the Newtonian potential takes the form:

$$3\left[(a^4\Phi')' - a^4\Delta_{\bar{\mathbf{q}}}\Phi\right] + a^6R_0\Phi = 4\pi Ga^6\delta T, \tag{104}$$

where $\delta T = 3\delta p - \delta\epsilon$ represents the linear perturbation of the stress-energy tensor.

We can easily see that this equation can be obtained from the variation of the Lagrangian with respect to the Newtonian potential:

$$\mathcal{L} = a^4\left[\frac{(\Phi')^2}{2} - \frac{|\nabla_{\bar{\mathbf{q}}}\Phi|^2}{2} - \frac{a^2R_0}{6}\Phi^2 + \frac{4\pi Ga^2}{3}\delta T\Phi\right], \tag{105}$$

and we can recognize its similarity with the Lagrangian corresponding to a massless scalar field, ϕ, conformally coupled with gravity:

$$\mathcal{L} = \frac{a^2}{2}\left[(\phi')^2 - |\nabla_{\mathbf{q}}\phi|^2 - \frac{a^2R_0}{6}\phi^2\right]. \tag{106}$$

On the other hand, performing the transformation $\Phi = \dfrac{\bar{\Phi}}{a^2}$, the dynamical equation (104) becomes:

$$\bar{\Phi}'' - \Delta_{\bar{\mathbf{q}}}\bar{\Phi} - 2\mathcal{H}^2\bar{\Phi} = \frac{4\pi Ga^4}{3}\delta T, \tag{107}$$

which can be obtained from the variation of the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\bar{\Phi}')^2 - |\nabla_{\bar{\mathbf{q}}}\bar{\Phi}|^2 + 2\mathcal{H}^2\bar{\Phi}^2\right] + \frac{4\pi Ga^4}{3}\delta T\bar{\Phi}, \tag{108}$$

where the firsts three terms resemble those of a harmonic oscillator with a time-dependent frequency $\sqrt{2}\mathcal{H}$, and the last one represents the coupling between the potential and the stress-energy tensor. Additionally, in a static universe, the Newtonian potential satisfies the typical wave equation under the action of a mass source:

$$\ddot{\Phi} - \Delta_{\bar{\mathbf{x}}}\Phi = \frac{4\pi G}{3}\delta T, \tag{109}$$

where $\bar{\mathbf{x}} = a\bar{\mathbf{q}}$.

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Finally, we recast our equations in conformal time, coordinates $\bar{\mathbf{q}}$, velocity $\bar{\mathbf{u}} = \frac{d\bar{\mathbf{q}}}{d\eta}$, and potential $\bar{\Phi}$, i.e., using the metric $ds^2 = -(a^2 + 2\bar{\Phi})d\eta^2 + \frac{1}{3}(a^2 - 2\bar{\Phi})d\bar{\mathbf{q}}^2$:

$$\begin{cases} D_\eta \epsilon + (\epsilon + p)\left[3(\mathcal{H} - \bar{\Phi}' + 2\mathcal{H}\bar{\Phi}) + \nabla_{\bar{\mathbf{q}}}.\bar{\mathbf{u}}\right] & = & 0 \\ D_\eta((\epsilon + p)\bar{\mathbf{u}}) + (\epsilon + p)\left[4\mathcal{H}\bar{\mathbf{u}} + (\nabla_{\bar{\mathbf{q}}}.\bar{\mathbf{u}})\bar{\mathbf{u}} + \frac{3}{a^2}\nabla_{\bar{\mathbf{q}}}\bar{\Phi}\right] + 3\nabla_{\bar{\mathbf{q}}}p & = & 0 \\ \bar{\Phi}'' - \Delta_{\bar{\mathbf{q}}}\bar{\Phi} - 2\mathcal{H}^2\bar{\Phi} & = & \frac{4\pi G a^4}{3}\delta T, \end{cases} \tag{110}$$

where we continue using the total derivative $D_\eta f = \partial_\eta f + \bar{\mathbf{u}}.\nabla_{\bar{\mathbf{q}}} f$.

6. Conclusions

In our current investigation, we have successfully derived the Schwarzschild metric, a cornerstone in describing the curvature of spacetime around spherically symmetric masses. Our methodology involved replacing the conventional acceleration in Newton's second law with the proper acceleration as delineated in special relativity. This innovative approach elucidates how gravity influences the trajectories of objects within regions of intense gravitational fields. By integrating the tenets of special relativity into our analysis, we have deepened our comprehension of the intricate interplay between matter, energy, and the fabric of spacetime.

Moving forward, our exploration has extended to the derivation of the relativistic Friedmann equations, originating from the foundations of Newtonian mechanics and the first law of thermodynamics. Employing a Lagrangian formulation, we have expanded our inquiry to encompass the Friedmann equations for both fluid and scalar field scenarios. Noteworthy is the profound connection between the matter Lagrangian and energy density for a fluid, as well as its association with pressure in a universe housing a scalar field, a relationship persisting in its Newtonian counterpart as evidenced in our study. Moreover, our research has showcased the adaptability of this formulation by exploring its application to open systems, thus broadening the horizons of our investigation.

This alternative methodology for deriving the Friedmann equations furnishes a robust framework for probing the universe's dynamics and offers profound insights into the intricate interplay between diverse physical properties. By amalgamating Newtonian mechanics and thermodynamics, we stand poised to unravel the processes governing the evolution of our cosmos.

Lastly, our endeavors have extended to the derivation of perturbation equations in the Newtonian gauge within the realm of General Relativity, originating from the classical trio of equations governing a perfect fluid. These encompass the continuity equation, ensuring the conservation of mass, the Euler equation, governing the conservation of momentum, and the Poisson equation, delineating the gravitational potential's relationship with the energy density distribution. Through this comprehensive approach, we aim to bridge classical mechanics with the profound insights of General Relativity, thereby advancing our understanding of the universe's dynamic evolution.

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