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Generalized Quasi-Continuities of Multifunctions on Bitopological Spaces

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Abstract

The aim of the article is to introduce a few variants of generalized quasi-continuity of multifunctions defined on a bitopological space and to study their mutual relationship. The results known for functions are extended to multifunctions which provide a wider range of relationships, mainly in terms of upper and lower semi continuities and corresponding continuities with respect to a dual bitopology. The proof procedures are based on a notion of pseudo refinement of two topologies and the Baire property in a bitopological space. A characterization of some continuities depending on two topologies by continuities depending only on one topology and the structure of the sets of semi discontinuity points are given. The end of the article is dedicated to several interpretations that facilitate and clarify orientation in the achieved results.

Keywords: bitopological space; dual bitopological space; multifunction; ideal topology; semi continuity; quasi-continuity; Baire continuity; sectional quasi-continuity; weak sectional quasi-continuity

MSC: 26A15; 54A10; 54C08; 54C60

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1. Introduction

This article is dedicated to the theory of continuous multifunctions which has been intensively developing in recent decades. The theory of multifunctions was first codified by Kuratowski [17] and it has advanced in a variety of ways and applications of this theory can be found for example, in economic theory, noncooparative games, artificial intelligence, medicine, information sciences and decision theory [2].

Many different types of generalized continuities have been introduced for functions, e.g., quasi-continuity [16], α-continuity [30], β-continuity [1], [32], semi continuity [19], B-continuity, Br-continuity [20,21], B^*-continuity [7–9], some further generalizations of B^*-continuity, namely, contra B^*-continuity, slight B^*-continuity, weak B^*-continuity [15], somewhat continuity [10], cliquishness [29] and many more. Similar continuities can also be introduced for multifunctions, for which each generalized type of continuity can be introduced in a lower and an upper variant, see the comprehensive article [28].

The definition of quasi-continuity was introduced in [16]. Nevertheless, a funtion of two variables being quasi-continuous under the assumption that it is continuous in each variable separately was mentioned by Volterra [3]. Many results have been obtained for functions as well as for multifunctions. The most studied issues include the structure of continuity points of quasi-continuous functions (quasi-continuous multifunctions), joint quasi-continuity of separately quasi-continuous functions, decomposition theorems, selection theorems, various applications of quasi-continuity in other areas of mathematics and last but not least quasi-continuity variants in a bitopological space. Relevant results can be found in earlier published works [4–6,12,20,21,24,26,27,31] and recently published works [14,22,25] document the continued interest in quasi-continuity research. We bring to the reader's attention an overview article [28] and a recently published book [13] which summarizes the results devoted to the quasi-continuity of functions and multifunctions from various aspects.

As for the bitopological case, a typical example is the strong quasi-continuity of real functions for the density topology and the Euclidean topology on the real numbers [11], [18], [35,36]. For two arbitrary topologies, where one is finer than the other, the strong quasi-continuity of functions was studied in [22]. Another motivation is the work of Rychlewicz [33], where in a bitopological space two types of generalized notion of quasi-continuity of multifunctions are defined, namely the upper (lower) intersection quasi-continuity and the upper (lower) inclusion quasi-continuity.

The objectives of the article are following:

(1) to investigate the basic relations between tree defined continuities with respect to two topologies (Definition 2, Remark 1),
(2) to investigate the relations between quasi-continuity, Baire continuity and continuities with respect to a given ideal (Theorem 1, 2),
(3) to characterize continuities depending on two topologies by continuities depending on one topology (Theorem 7, 8),
(4) to investigate the structure of the sets of upper and lower semi discontinuity points (Theorem 3, 5, 9, Corollary 7, Corollary 9),
(5) to investigate the relations between the continuities with respect to a given bitopological space and the continuities with respect to its dual bitopological space (Theorem 10, Corollary 9, Corollary 12).
(6) to provide a unifying interpretation of the results using diagrams and formal symbolism (Chapter 5).

2. Definitions and Basic Observations

This chapter is a survey of some basic notions concerning semi continuity, quasi-continuity, and Baire continuity. We also introduce tree variants of quasi-continuity of multifunctions with respect to two topologies that are the focus of our attention.

Let (X,τ), Y be two topological spaces. The closure (the interior) of $A \subset X$, the closure (the interior) of $B \subset Y$ is denoted by $\mathrm{cl}_\tau(A)$ $(\mathrm{int}_\tau(A))$, $\mathrm{cl}(B)$ $(\mathrm{int}(B))$, respectively. By $\mathcal{I}_\tau$ we denote the set of all subsets of X that are of τ-first category and a set is called τ-residual if its complement is of τ-first category. By $\mathbb{R}$ we denote the real numbers.

A multifunction $F : X \to Y$ is any set-valued mapping from X to $2^Y \setminus \{\emptyset\}$. For any set $W \subset Y$ the upper and lower inverse images are defined as

$$F^+(W) = \{x \in X : F(x) \subset W\},\ F^-(W) = \{x \in X : F(x) \cap W \neq \emptyset\}.$$

Let us note, that

$$X \setminus F^-(W) = F^+(Y \setminus W),$$
$$X \setminus F^+(W) = F^-(Y \setminus W).$$

A function $f : X \to Y$ is understood as a strictly nonempty single-valued multifunction with values $\{f(x)\}, x \in X$.

Definition 1. Let (X,τ) be a topological space and $F : X \to Y$ be a multifunction. Then, F is said to be

(1) lower (upper) semi τ-continuous (for short, τ-*lsc* (τ-*usc*)) [17], [28] at $x \in X$ if for any open set V for which $F(x) \cap V \neq \emptyset$ $(F(x) \subset V)$ there is a τ-open set U containing x such that $F(u) \cap V \neq \emptyset$ $(F(u) \subset V)$ for any $u \in U$. F is τ-*lsc* (τ-*usc*) if it is so at any point $x \in X$. That means, $F^-(V)$ $(F^+(V))$ is τ-open for any open set $V \subset Y$. In case of a single-valued mapping the lower (upper) semi τ-continuity coincides with τ-continuity.
By $C_\tau(f)$ $(C^l_\tau(F), C^u_\tau(F))$ we denote the set of all points in which a function f is τ-continuous (a multifunction F is τ-*lsc*, τ-*usc*) and $D_\tau(f) = X \setminus C_\tau(f)$ $(D^l_\tau(F) = X \setminus C^l_\tau(F), D^u_\tau(F) = X \setminus C^u_\tau(F))$.
(2) lower (upper) quasi τ-continuous [28], [31] (lower (upper) Baire τ-continuous [21]) at $x \in X$ if for any open set V for which $F(x) \cap V \neq \emptyset$ $(F(x) \subset V)$ and any τ-open set H containing x there is a nonempty τ-open set (a set of τ-second category with the τ-Baire property) $U \subset H$ such that

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$F(u) \cap V \neq \emptyset$ ($F(u) \subset V$) for any $u \in U$. In case of a single-valued mapping the lower (upper) quasi τ-continuity (the lower (upper) Baire τ-continuity) coincides with quasi τ-continuity (Baire τ-continuity). Note in the case of the lower (upper) Baire τ-continuity the set U can be replaced by a set $G \setminus A$, where G is a set of τ-second category and τ-open and A is of τ-first category.

By $Q_\tau(f)$ ($Q^l_\tau(F)$, $Q^u_\tau(F)$) we denote the set of all points in which a function f is quasi τ-continuous (a multifunction F is lower quasi τ-continuous, upper quasi τ-contunuous).

By $B_\tau(f)$ ($B^l_\tau(F)$, $B^u_\tau(F)$) we denote the set of all points in which a function f is Baire τ-continuous (a multifunction F is lower Baire τ-continuous, upper τ-Baire contunuous).

A bitopological space (X, τ, σ) is a set X with two topologies τ, σ on X. A bitopological space (X, σ, τ) is called a dual of (X, τ, σ). A typical example of a bitopological space is the space $(\mathbb{R}, \tau, \sigma)$ with the upper topology $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ and the lower topology $\sigma = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ on the real line $\mathbb{R}$. The usual Euclidean topology on $\mathbb{R}$ is the join topology $\tau \vee \sigma = \{H \cap G : H \in \tau, G \in \sigma\}$ of τ and σ.

Denote

$\mathcal{I}_\tau = \{A : A$ is of τ-first category$\}$,

$\mathcal{I}_\sigma = \{A : A$ is of σ-first category$\}$,

$\beta(\tau, \mathcal{I}) = \{G \setminus A : G \in \tau$ and $A \in \mathcal{I}\}$, where $\mathcal{I}$ is an ideal on X,

$\beta(\sigma, \mathcal{I}) = \{G \setminus A : G \in \sigma$ and $A \in \mathcal{I}\}$, where $\mathcal{I}$ is an ideal on X,

$\tau_\mathcal{I}$ - an ideal topology generated by the base $\beta(\tau, \mathcal{I})$,

$\sigma_\mathcal{I}$ - an ideal topology generated by the base $\beta(\sigma, \mathcal{I})$,

τ^* - an ideal topology generated by $\beta(\tau, \mathcal{I}_\tau)$,

$\sigma_{\mathcal{I}_\tau}$ - an ideal topology generated by $\beta(\sigma, \mathcal{I}_\tau)$.

Note $H \in \tau^*$ if and only if $H = G \setminus A$, where $G \in \tau$ and $A \in \mathcal{I}_\tau$.

The following definition introduces six variants of generalized quasi-continuity of multifunction with respect to a bitopological space (X, τ, σ), namely

-lower/upper quasi-continuity (denoted by $q^l(\tau, \sigma)/q^u(\tau, \sigma)$),

-sectional lower/upper quasi-continuity (denoted by $sq^l(\tau, \sigma)/sq^u(\tau, \sigma)$),

-weak sectional lower/upper quasi-continuity (denoted by $wsq^l(\tau, \sigma)/wsq^u(\tau, \sigma)$).

Definition 2. Let (X, τ, σ) be a bitopological space and $F : X \to Y$ be a multifunction. Then, F is said to be

(1) $q^l(\tau, \sigma)$-continuous ($q^u(\tau, \sigma)$-continuous),
(2) $sq^l(\tau, \sigma)$-continuous ($sq^u(\tau, \sigma)$-continuous),
(3) $wsq^l(\tau, \sigma)$-continuous ($wsq^u(\tau, \sigma)$-continuous)

at $a \in X$ if for any open set V for which $V \cap F(a) \neq \emptyset$ ($F(a) \subset V$) and any τ-open set U containing a there is a nonempty σ-open set G such that

(1) $G \subset U$ and $F(x) \cap V \neq \emptyset$ ($F(x) \subset V$) for any $x \in G$,
(2) $G \cap U \neq \emptyset$ and $F(x) \cap V \neq \emptyset$ ($F(x) \subset V$) for any $x \in G$,
(3) $G \cap U \neq \emptyset$ and $F(x) \cap V \neq \emptyset$ ($F(x) \subset V$) for any $x \in G \cap U$, respectively.

The global definitions are given by the local ones at each point. In case of a single-valued mapping both the upper and the lower versions defined above coincide with $q(\tau, \sigma)$-continuity, $sq(\tau, \sigma)$-continuity, $wsq(\tau, \sigma)$-continuity, respectively [23]. By $Q^l_{\tau,\sigma}(F)$, $sQ^l_{\tau,\sigma}(F)$, $wsQ^l_{\tau,\sigma}(F)$, $Q^u_{\tau,\sigma}(F)$, $sQ^u_{\tau,\sigma}(F)$, $wsQ^u_{\tau,\sigma}(F)$ we denote the corresponding sets of continuity points.

In the sequel we will consider the same notions with respect to a dual bitopological space (X, σ, τ).

In Definition 2, the item (1) is a natural generalization of the lower (upper) quasi τ-continuity (for $\tau = \sigma$). The item (1) and (2) are motivated by Richlewicz definition of lower (upper) inclusion and lower (upper) intersection quasi-continuity in a bitopological space (X, τ, σ) [33]. The item

(3) is a generalization of a lower (upper) strong quasi-continuity that was introduced for function (multifunction) defined on $(\mathbb{R},\tau,\sigma)$, where τ is the density topology and σ is the Euclidean topology on the real line $\mathbb{R}$ [11], [18], [35], [36].

Remark 1. Let (X,τ,σ) be a bitopological space. The next inclusions and equalities are clear. Note, all inclusions can be strict [23].

(1) (a)
$$Q^l_{\tau,\sigma}(F)\subset sQ^l_{\tau,\sigma}(F)\subset wsQ^l_{\tau,\sigma}(F),$$
$$Q^u_{\tau,\sigma}(F)\subset sQ^u_{\tau,\sigma}(F)\subset wsQ^u_{\tau,\sigma}(F).$$
(b)
$$C^l_{\sigma}(F)\subset sQ^l_{\tau,\sigma}(F),\quad C^l_{\tau}(F)\subset sQ^l_{\sigma,\tau}(F),$$
$$C^u_{\sigma}(F)\subset sQ^u_{\tau,\sigma}(F),\quad C^u_{\tau}(F)\subset sQ^u_{\sigma,\tau}(F).$$
(2) If $\sigma_1\subset\sigma_2$ and $\tau_1\subset\tau_2$, then
$$wsQ^l_{\tau_2,\sigma_1}(F)\subset wsQ^l_{\tau_1,\sigma_2}(F),\quad sQ^l_{\tau_2,\sigma_1}(F)\subset sQ^l_{\tau_1,\sigma_2}(F),\quad Q^l_{\tau_2,\sigma_1}(F)\subset Q^l_{\tau_1,\sigma_2}(F),$$
$$wsQ^u_{\tau_2,\sigma_1}(F)\subset wsQ^u_{\tau_1,\sigma_2}(F),\quad sQ^u_{\tau_2,\sigma_1}(F)\subset sQ^u_{\tau_1,\sigma_2}(F),\quad Q^u_{\tau_2,\sigma_1}(F)\subset Q^u_{\tau_1,\sigma_2}(F).$$
(3) If $\tau=\sigma$, then
$$C^l_{\tau}(F)\subset Q^l_{\tau,\tau}(F)=sQ^l_{\tau,\tau}(F)=wsQ^l_{\tau,\tau}(F)=Q^l_{\tau}(F),$$
$$C^u_{\tau}(F)\subset Q^u_{\tau,\tau}(F)=sQ^u_{\tau,\tau}(F)=wsQ^u_{\tau,\tau}(F)=Q^u_{\tau}(F).$$
If $\sigma\subset\tau$, then
$$Q^l_{\sigma,\tau}(F)=sQ^l_{\sigma,\tau}(F)=wsQ^l_{\sigma,\tau}(F),\quad Q^u_{\sigma,\tau}(F)=sQ^u_{\sigma,\tau}(F)=wsQ^u_{\sigma,\tau}(F),$$
$$Q^l_{\sigma}(F)\cup Q^l_{\tau}(F)\subset Q^l_{\sigma,\tau}(F),\quad wsQ^l_{\tau,\sigma}(F)\subset Q^l_{\tau}(F)\cap Q^l_{\sigma}(F),$$
$$Q^u_{\sigma}(F)\cup Q^u_{\tau}(F)\subset Q^u_{\sigma,\tau}(F),\quad wsQ^u_{\tau,\sigma}(F)\subset Q^u_{\tau}(F)\cap Q^u_{\sigma}(F).$$
(4) A multifunction F is lover σ-somewhat continuous (upper σ-somewhat continuous) [10] if for any open set $V\subset Y$, $\mathrm{int}_\sigma(F^-(V))$ $(\mathrm{int}_\sigma(F^+(V)))$ is nonempty, provided $F^-(V)$ $(F^+(V))$ is nonempty. If F is $wsq^l(\tau,\sigma)$-continuous ($wsq^u(\tau,\sigma)$-continuous), then F is lover σ-somewhat continuous (upper σ-somewhat continuous). It is clear, since F is $wsq^l(\tau,\sigma)$-continuous ($wsq^u(\tau,\sigma)$-continuous), then for any open set $V\subset Y$ such that $F^-(V)$ $(F^+(V))$ is nonempty and for $X\in\tau$ there is a nonempty σ-open set U such that $\emptyset\neq U\cap X=U$ and $F(u)\cap V\neq\emptyset$ $(F(u)\subset V)$ for any $u\in U\cap X=U$. That means, $U\subset F^-(V)$ $(U\subset F^+(V))$, so $\mathrm{int}_\sigma(F^-(V))$ $(\mathrm{int}_\sigma(F^+(V)))$ is nonempty.
(5) In [23] we defined another type of continuity for a function called $asq(\tau,\sigma)$-continuity. Two variants of this notion can be introduced for a multifunction. It is easy te see, F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous) if and only if F is $asq^l(\tau,\sigma)$-continuous ($asq^u(\tau,\sigma)$-continuous). The $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuity ($wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuity) is the most general type of continuity for which it is still possible to achieve relevant results, see Chapter 4.

Lemma 1. *Let (X,τ,σ) be a bitopological space and $F:X\to Y$ be a multifunction. Then,*

(1) F is $sq^l(\tau,\sigma)$-continuous ($sq^u(\tau,\sigma)$-continuous) at a if and only if $a\in\mathrm{cl}_\tau(\mathrm{int}_\sigma(F^-(V)))$ $(a\in\mathrm{cl}_\tau(\mathrm{int}_\sigma(F^+(V))))$ for any open set V intersecting (containing) $F(a)$.
(2) F is $sq^l(\tau,\sigma)$-continuous ($sq^u(\tau,\sigma)$-continuous) if and only if $F^-(V)\subset\mathrm{cl}_\tau(\mathrm{int}_\sigma(F^-(V)))$ $(F^+(V)\subset\mathrm{cl}_\tau(\mathrm{int}_\sigma(F^+(V))))$ for any open set $V\subset Y$.

Proof. (1) Suppose F is $sq^l(\tau,\sigma)$-continuous ($sq^u(\tau,\sigma)$-continuous) at a. Let $a \in F^-(V)$ $(a \in F^+(V))$, V be open and $a \in U \in \tau$. From $sq^l(\tau,\sigma)$-continuity ($sq^u(\tau,\sigma)$-continuity) of F at a there is a set $G \in \sigma$ such that $G \cap U \neq \emptyset$ and $G \subset F^-(V)$ $(G \subset F^+(V))$. Since $G \subset \mathrm{int}_\sigma(F^-(V))$ $(G \subset \mathrm{int}_\sigma(F^+(V)))$, $\emptyset \neq G \cap U \subset \mathrm{int}_\sigma(F^-(V)) \cap U$ $(\subset \mathrm{int}_\sigma(F^+(V)) \cap U)$. So, $a \in \mathrm{cl}_\tau(\mathrm{int}_\sigma(F^-(V)))$ $(a \in \mathrm{cl}_\tau(\mathrm{int}_\sigma(F^+(V))))$.

Suppose $a \in \mathrm{cl}_\tau(\mathrm{int}_\sigma(F^-(V)))$ $(a \in \mathrm{cl}_\tau(\mathrm{int}_\sigma(F^+(V))))$ for any open set V intersecting (containing) $F(a)$. Let $a \in U \in \tau$. Then, $U \cap \mathrm{int}_\sigma(F^-(V)) \neq \emptyset$ $(U \cap \mathrm{int}_\sigma(F^+(V)) \neq \emptyset)$. Put $G = \mathrm{int}_\sigma(F^-(V))$ $(G = \mathrm{int}_\sigma(F^+(V)))$. Then, $U \cap G \neq \emptyset$ and for any $x \in G \subset F^-(V)$ $(x \in G \subset F^+(V))$, $F(x) \cap V \neq \emptyset$ $(F(x) \subset V)$, so F is $sq^l(\tau,\sigma)$-continuous ($sq^u(\tau,\sigma)$-continuous) at a.

(2) follows from (1). □

3. Ideal Topological Setting and Baire Continuity

This chapter discusses the Baire continuity, which is closely related to quasi-continuity and it can be understood as a special case of the Baire property (Theorem 4). For the functions, the Baire continuity is equivalent to the quasi-continuity (Corollary 2), but for multifunctions, the upper/lower Baire continuity is a more general type of continuity than the upper/lower quasi-continuity (Example 1(3)). Although it is more general than the upper/lower quasi-continuity, the upper Baire continuity guarantees the existence of a quasi-continuous selector [20], [21] and the set of semi continuity points of upper/lower Baire continuous multifunction is a residual set (Corollary 3).

The main objectives of this section are the equivalence between the upper and lower quasi-continuity, the upper and lower Baire continuity, and other pairs of continuities with respect to a given ideal. Also, the points of continuity of upper and lower Baire continuous multifunction are investigated. The results of this section will be used in the next chapter.

Let $\mathcal{I}$ be an ideal on X. Recall, $\mathcal{I}$ is called τ-codence if any nonempty τ-open set is not from $\mathcal{I}$. The family $\beta(\tau,\mathcal{I}) = \{G \setminus A : G \in \tau \text{ and } A \in \mathcal{I}\}$ is a base for an ideal topology $\tau_\mathcal{I}$ finer than τ. Similar notions can be considered with respect to a topolofy σ. The next remark is clear.

Remark 2. Let $F : X \to Y$ be a multifunction. The next conditions (1), (2), (3) are equivalent.

(1) F is $q^l(\tau,\tau_\mathcal{I})$-continuous $(q^u(\tau,\tau_\mathcal{I}))$-continuous at a,

(2) F is $sq^l(\tau,\tau_\mathcal{I})$-continuous $(sq^u(\tau,\tau_\mathcal{I}))$-continuous at a,

(3) F is $wsq^l(\tau,\tau_\mathcal{I})$-continuous $(wsq^u(\tau,\tau_\mathcal{I}))$-continuous at a.

(4) F is $wsq^l(\tau,\sigma_\mathcal{I})$-cotinuous, $sq^l(\tau,\sigma_\mathcal{I})$-continuous, $q^l(\tau,\sigma_\mathcal{I})$-continuous ($wsq^u(\tau,\sigma_\mathcal{I})$-continuous, $sq^l(\tau,\sigma_\mathcal{I})$-continuous, $q^l(\tau,\sigma_\mathcal{I})$-continuous) at a if and only if for any open set V and any τ-open set U such that $V \cap F(a) \neq \emptyset$ $(F(a) \subset V)$ and $a \in U$ there is a nonempty set $E \in \beta(\sigma,\mathcal{I})$ such that $E \cap U \neq \emptyset$, $E \cap U \neq \emptyset$, $E \subset U$ and $V \cap F(e) \neq \emptyset$ $(F(e) \subset V)$ for any $e \in E \cap U, E, E$, respectively.

(5) It is clear, if F is upper Baire τ-continuous, lower Baire τ-continuous, then F is $q^u(\tau,\tau^*)$-continuous, $q^l(\tau,\tau^*)$-continuous, respectively. Suppose (X,τ) is τ-Baire. Then F is $q^u(\tau,\tau^*)$-continuous, $q^l(\tau,\tau^*)$-continuous if and only if F is upper Baire τ-continuous, lower Baire τ-continuous, respectively.

(6) If X is finite and (X,τ) is τ-Baire, then F is upper Baire τ-continuous, lower Baire τ-continuous if and only if F is upper, lower quasi τ-continuous, respectively.
Proof: Suppose X is finite. The implication "$\Leftarrow$" is clear. Suppose F is lower (upper) Baire τ-continuous at a. Let V be open intersecting $F(a)$ (containing $F(a)$) and $H \in \tau$, $a \in H$. Since F is lower Baire τ-continuousat a, there is a set $H_0 \in \tau$ and $A_0 \in \mathcal{I}_\tau$ such that $\emptyset \neq H_0 \setminus A_0 \subset H$ and $F(x) \cap V \neq \emptyset$ $(F(x) \subset V)$ for any $x \in H_0 \setminus A_0$. Since X is finite, A_0 is τ-nowhere dense and $\mathrm{cl}_\tau(A_0)$ is also τ-nowhere dense. So, $H_0 \setminus \mathrm{cl}_\tau(A_0)$ is τ-open nonempty and for any $x \in H_0 \setminus \mathrm{cl}_\tau(A_0) \subset H_0 \setminus A_0 \subset H$, $F(x) \cap V \neq \emptyset$ $(F(x) \subset V)$. That means F is lower quasi τ-continuous at a.

Theorem 1. *Let Y be regular, $\mathcal{I}$ be τ-codense and $F : X \to Y$ be a compact valued multifunction. The next conditions are equivalent.*

(1) *F is upper and lower quasi τ-continuous,*
(2) *F is $q^u(\tau, \tau_{\mathcal{I}})$-continuous and $q^l(\tau, \tau_{\mathcal{I}})$-continuous,*
(3) *F is $sq^u(\tau, \tau_{\mathcal{I}})$-continuous and $sq^l(\tau, \tau_{\mathcal{I}})$-continuous,*
(4) *F is $wsq^u(\tau, \tau_{\mathcal{I}})$-continuous and $wsq^l(\tau, \tau_{\mathcal{I}})$-continuous.*

Proof. (1)⇒(2) is trivial.

(2)⇒(1) Let $a \in X$, V be open such that $F(a) \subset V$ $(V \cap F(a) \neq \emptyset)$ and $a \in U \in \tau$. Since Y is regular and F is compact valued, there is a nonempty open set $V_0 \subset V$ such that $\mathrm{cl}(V_0) \subset V$ and $F(a) \subset V_0$ $(V_0 \cap F(a) \neq \emptyset)$.

Since F is $q^u(\tau, \tau_{\mathcal{I}})$-continuous ($q^l(\tau, \tau_{\mathcal{I}})$-continuous) at a, there is a nonempty set $E = G \setminus A \in \beta(\tau, \mathcal{I})$, $E \subset U$, $G \in \tau$, $A \in \mathcal{I}$ (see Remark 2(4)) such that $F(e) \subset V_0$ $(F(e) \cap V_0 \neq \emptyset)$ for any $e \in E$. We will show $F(e) \subset \mathrm{cl}(V_0)$ $(F(e) \cap \mathrm{cl}(V_0) \neq \emptyset)$ for any $e \in G$. Suppose there is $e_0 \in G$ such that $F(e_0) \cap (Y \setminus \mathrm{cl}(V_0)) \neq \emptyset$ $(F(e_0) \subset X \setminus \mathrm{cl}(V_0))$. Since F is $q^l(\tau, \tau_{\mathcal{I}})$-continuous ($q^u(\tau, \tau_{\mathcal{I}})$-continuous), there is a nonempty set $E_0 = G_0 \setminus A_0$, $E_0 \subset G$, $G_0 \in \tau$, $A_0 \in \mathcal{I}$ (see Remark 2(4)) such that $F(e) \cap (Y \setminus \mathrm{cl}(V_0)) \neq \emptyset$ $(F(e) \subset X \setminus \mathrm{cl}(V_0))$ for any $e \in E_0$. It is clear $G \cap G_0 \neq \emptyset$ (if $G \cap G_0 = \emptyset$, then $G \cap (G_0 \setminus A_0) = \emptyset$, a contradiction with $\emptyset \neq E_0 = G_0 \setminus A_0 \subset G$). Since $\mathcal{I}$ is τ-codense, $\emptyset \neq (G \cap G_0) \setminus (A \cup A_0) = (G \setminus A) \cap (G_0 \setminus A_0) = E \cap E_0$. So, for $e \in E \cap E_0$, we have $F(e) \subset V_0$ and $F(e) \cap (Y \setminus \mathrm{cl}(V_0)) \neq \emptyset$ $(F(e) \cap V_0 \neq \emptyset$ and $F(e) \subset X \setminus \mathrm{cl}(V_0))$, a contradiction. So, $F(e) \subset \mathrm{cl}(V_0)$ $(F(e) \cap \mathrm{cl}(V_0) \neq \emptyset)$ for any $e \in G$. Since $\emptyset \neq E = G \setminus A \subset U, \emptyset \neq G \cap U \subset U$. So, F is upper (lower) quasi τ-continuous at a.

The equivalences (2) ⇔ (3) ⇔ (4) follow from the items (1) (2), (3) of Remark 2. □

Corollary 1. *Let Y be regular, $\mathcal{I}$ be τ-codense and $f : X \to Y$ be a function. The next conditions are equivalent.*

(1) *f is quasi τ-continuous,*
(2) *f is $q(\tau, \tau_{\mathcal{I}})$-continuous,*
(3) *f is $sq(\tau, \tau_{\mathcal{I}})$-continuous,*
(4) *f is $wsq(\tau, \tau_{\mathcal{I}})$-continuous.*

Example 1. The assumption "$\mathcal{I}$ is τ-codense", the regularity of Y and one variant of continuity in Theorem 1 can not be omitted.

(1) Let $X = \{a, b\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$, $\tau = \{X, \emptyset, \{a\}\}$. Then, $\mathcal{I}$ is not τ-codense and $\tau_{\mathcal{I}} = \{X, \emptyset, \{a\}, \{b\}\}$. Put $f : X \to \mathbb{R}$ ($\mathbb{R}$ with the Euclidean topology) defined by $f(a) = 1$ and $f(b) = 0$. Then, f is $q(\tau, \tau_{\mathcal{I}})$-continuous but it is not quasi τ-continuous.
(2) Let $X = \mathbb{R}$ with the Euclidean topology τ and $Y = \{0, 1\}$ with topology $\{\emptyset, Y, \{0\}\}$. It is clear Y is not regular. Then, a function $f : \mathbb{R} \to Y$ defined as $f(x) = 0$ if x is irrational and $f(x) = 1$ otherwise is $q(\tau, \tau_{\mathcal{I}})$-continuous, where $\mathcal{I} = \{A \subset \mathbb{R} : A$ is of τ-first category$\}$ but it is not quasi τ-continuous.
(3) A multifunction $F : \mathbb{R} \to \mathbb{R}$ $(G : \mathbb{R} \to \mathbb{R})$ defined as $F(x) = [0, 1]$ if x is rational and $F(x) = \{0\}$ otherwise $(G(x) = \{0\}$ if x is rational and $G(x) = [0, 1]$ otherwise) is $q^u(\tau, \tau_{\mathcal{I}})$-continuous ($q^l(\tau, \tau_{\mathcal{I}})$-continuous) but F (G) is not upper nor lower quasi τ-continuous, where τ is the Euclidean topology on $\mathbb{R}$ and $\mathcal{I} = \{A \subset \mathbb{R} : A$ is of τ-first category$\}$.

Theorem 1 holds for any τ-codense ideal. So, also for an ideal $\mathcal{I}_\tau = \{A \subset X : A$ is of τ-first category$\}$, provided (X, τ) is τ-Baire. If (X, τ) is τ-Baire, then the upper Baire τ-continuity, the lower Baire τ-continuous is equivalent to the $q^u(\tau, \tau^*)$-continuity, the $q^l(\tau, \tau^*)$-continuity, respectively, see Remark 2(5). So, we have the next result that is a spacial case of Theorem 1, since $\tau_{\mathcal{I}_\tau} = \tau^*$. Note $\mathcal{I}_\tau$ is τ-codense if and only if (X, τ) is τ-Baire.

Theorem 2. *Let Y be regular, (X, τ) be τ-Baire and $F : X \to Y$ be a compact valued multifunction. The next conditions are equivalent.*

(1) *F is upper Baire τ-continuous and lower Baire τ-continuous,*
(2) *F is upper and lower quasi τ-continuous.*
(3) *F is $q^u(\tau,\tau^*)$-continuous and $q^l(\tau,\tau^*)$-continuous,*
(4) *F is $sq^u(\tau,\tau^*)$-continuous and $sq^l(\tau,\tau^*)$-continuous,*
(5) *F is $wsq^u(\tau,\tau^*)$-continuous and $wsq^l(\tau,\tau^*)$-continuous.*

Note, under conditions Y is regular, (X,τ) is τ-Baire, $\mathcal{I}$ is a τ-codense ideal and $F : X \to Y$ is a compact valued multifunction, all items from Theorem 1 and Theorem 2 are equivalent. It is a special case when the second topology is finer then the first one and further combinations will be solved in the next chapter

In function setting, by Theorem 1 and Theorem 2, we have the next corollary.

Corollary 2. *Let Y be regular, (X,τ) be τ-Baire, $\mathcal{I}$ be a τ-codense ideal and $f : X \to Y$ be a function. Then, we can express all equivalences by the next diagram.*

$$
\begin{array}{ccccccc}
 & q(\tau,\tau_{\mathcal{I}}) & & & & q(\tau,\tau^*) & \\
\swarrow\!\!\nearrow & & \nwarrow\!\!\searrow & & \swarrow\!\!\nearrow & & \nwarrow\!\!\searrow \\
sq(\tau,\tau_{\mathcal{I}}) & & Q_\tau(f)=X & \longleftrightarrow & B_\tau(f)=X & & sq(\tau,\tau^*) \\
\nwarrow\!\!\searrow & & \swarrow\!\!\nearrow & & \nwarrow\!\!\searrow & & \swarrow\!\!\nearrow \\
 & wsq(\tau,\tau_{\mathcal{I}}) & & & & wsq(\tau,\tau^*) &
\end{array}
$$

Theorem 3. *Let (X,τ) be a topological space, Y be a regular second countable topological space and $F : X \to Y$ be a multifunction.*

(1) *If F is $q^u(\tau,\tau^*)$-continuous ($sq^u(\tau,\tau^*)$-continuous, $wsq^u(\tau,\tau^*)$-continuous), then F is τ-lsc except for a set of τ-first category ($D^l_\tau(F) \in \mathcal{I}_\tau$).*
(2) *If F is $q^l(\tau,\tau^*)$-continuous ($sq^l(\tau,\tau^*)$-continuous, $wsq^l(\tau,\tau^*)$-continuous) and compact valued, then F is τ-usc except for a set of τ-first category ($D^u_\tau(F) \in \mathcal{I}_\tau$).*

Proof. Let $\mathcal{B} = \{B_n : n = 1,2,\dots\}$ be a base of Y.

(1) For a set $A \subset X$, put $D_\tau(A) = \{x \in X : A \cap G$ is of τ-second category for any $G \in \tau$ containing $x\}$. Let $S^l_\tau(F) = \{x \in X : x \in \mathrm{int}_\tau(D_\tau(F^-(V)))$ for any open set V such that $V \cap F(x) \neq \emptyset\}$.

Let $a \in S^l_\tau(F)$. We show F is τ-*lsc* at a. Let V be open and $V \cap F(a) \neq \emptyset$. Since Y is regular, there is an open set V_0 such that $\mathrm{cl}(V_0) \subset V$ and $V_0 \cap F(a) \neq \emptyset$. Denote $H = \mathrm{int}_\tau(D_\tau(F^-(V_0)))$. It is clear $a \in \mathrm{int}_\tau(D_\tau(F^-(V_0)))$. We will show $\mathrm{cl}(V_0) \cap F(x) \neq \emptyset$ for any $x \in H$. Suppose there is $x_0 \in H$ such that $F(x_0) \subset Y \setminus \mathrm{cl}(V_0)$. Since F is $q^u(\tau,\tau^*)$-continuous, there is a nonempty set $G \setminus A \subset H$, where $G \in \tau$, $A \in \mathcal{I}_\tau$ (see Remark 2(4)) and $F(x_0) \subset Y \setminus \mathrm{cl}(V_0)$ for any $x \in G \setminus A$. $(\star)$

It is clear $H \cap G \neq \emptyset$. Since $H \cap G \subset \mathrm{int}_\tau(D_\tau(F^-(V_0))) \subset D_\tau(F^-(V_0))$, there is a set $S \subset H \cap G$ which is of τ-second category and $F(x) \cap V_0 \neq \emptyset$ for any $x \in S$. That means, there is $s \in S \setminus A \subset (H \cap G) \setminus A \subset G \setminus A$, for which $F(s) \cap V_0 \neq \emptyset$ and $F(s) \subset Y \setminus \mathrm{cl}(V_0)$ (see $(\star)$), a contradiction.

Now it is sufficient to show that $X \setminus S^l_\tau(F)$ is of τ-first category. It is clear

$$X \setminus S^l_\tau(F) \subset \bigcup_{n=1}^{\infty} [F^-(B_n) \setminus \mathrm{int}_\tau(D_\tau(F^-(B_n)))] =: R.$$

Denote
$A_n = F^-(B_n) \setminus D_\tau(F^-(B_n))$,
$B_n = [D_\tau(F^-(B_n)) \setminus \mathrm{int}_\tau(D_\tau(F^-(B_n)))] \cap F^-(B_n)$.
Since the sets A_n and B_n are of τ-first category, the set

$$R = \bigcup_{n=1}^{\infty} A_n \cup B_n$$

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is of τ-first category, so $X \setminus S^l_\tau(F)$ is of τ-first category.

(2) Let $S^u_\tau(F) = \{x \in X : x \in \mathrm{int}_\tau(D_\tau(F^+(V))$ for any open set V such that $F(x) \subset V\}$. Similar like in the case (1) we can show that F is τ-*usc* at x if $x \in S^u_\tau(F)$. It is clear

$$X \setminus S^u_\tau(F) \subset \bigcup_{n=1}^{\infty} [F^+(G_n) \setminus \mathrm{int}_\tau(D_\tau(F^+(G_n)))] =: R$$

where $\{G_1, G_2, G_3, \dots\}$ is a sequence of all finite unions of sets from $\mathcal{B}$. Similar like in the case (1), we can show R is of τ-first category, so $X \setminus S^u_\tau(F)$ is of τ-first category.

The other cases in brackets follow from the equations $Q^u_{\tau,\tau^*} = sQ^u_{\tau,\tau^*} = wsQ^u_{\tau,\tau^*}$ and $Q^l_{\tau,\tau^*} = sQ^l_{\tau,\tau^*} = wsQ^l_{\tau,\tau^*}$, see Remark 1(3). □

Theorem 3 has two applications. It can be applied for the upper (lower) τ-quasi continuity and the upper (lower) Baire τ-continuity (see Remark 2(5)).

Corollary 3. *([4], [20]) Let Y be a regular second countable topological space and $F : X \to Y$ be a multifunction.*

(1) If F is upper τ-quasi continuous, upper Baire τ-continuous, then F is τ-lsc except for a set of τ-first category ($D^l_\tau(F) \in \mathcal{I}_\tau$), respectively.

(2) If F is compact valued and lower τ-quasi continuous, lower Baire τ-continuous, then F is τ-usc except for a set of τ-first category ($D^u_\tau(F) \in \mathcal{I}_\tau$), respectively.

Proof. If F is upper τ-quasi continuous (lower τ-quasi continuous and compact valued), then F is $q^u(\tau,\tau^*)$-continuous ($q^l(\tau,\tau^*)$-continuous) and by Theorem 3, F is τ-*lsc* (τ-*usc*) except for a set of τ-first category.

If F is upper Baire τ-continuous (lower Baire τ-continuous and compact valued), then F is $q^u(\tau,\tau^*)$-continuous ($q^l(\tau,\tau^*)$-continuous) and (X,τ)-is Baire. By Remark 2(5), F is $q^u(\tau,\tau^*)$-continuous ($q^l(\tau,\tau^*)$-continuous) and by Theorem 3, F is τ-*lsc* (τ-*usc*) except for a set of τ-first category. □

By Corollary 2, for a function, we have the next result.

Corollary 4. *Let Y be a regular second countable topological space, X be τ-Baire, $\mathcal{I}$ be a τ-codense ideal and $f : X \to Y$ be a function. If f is τ-quasi continuous, Baire τ-continuous, $q(\tau,\tau^*)$-continuous, $wsq(\tau,\tau_\mathcal{I})$-continuous, $sq(\tau,\tau_\mathcal{I})$-continuous, $q(\tau,\tau_\mathcal{I})$-continuous, then f is τ-continuous except for a set of τ-first category ($D_\tau(f) \in \mathcal{I}_\tau$), respectively.*

Next theorem shows that the lower and upper τ-Baire continuities are very closed to a multifunction having the τ-Baire property.

Theorem 4. *Let (X,τ) be a τ-Baire topological space, Y be a regular second countable topological space and $F : X \to Y$ be a compact valued multifunction. Suppose any open subset of Y can be written as a union of closed sets.*

(1) F is lower Baire τ-continuous if and only if for any open set V, $F^-(V) = (G \setminus A) \cup B$, where $G \in \tau$, $A, B \in \mathcal{I}_\tau$ and $B \subset \mathrm{cl}_\tau(G)$.

(2) F is upper Baire τ-continuous if and only if for any open set V, $F^+(V) = (G \setminus A) \cup B$, where $G \in \tau$, $A, B \in \mathcal{I}_\tau$ and $B \subset \mathrm{cl}_\tau(G)$.

Proof. (1) $\Rightarrow$: Let F be lower Baire τ-continuous. By Corollary 3, F is τ-*usc* except for a set A which is of τ-first category. Then, for any open set V, $F^+(V) \cap (X \setminus A)$ has the τ-Baire property, so $F^+(V)$ has the τ-Baire property. Let $V = \bigcup_{n=1}^{\infty} F_n$, where $F_n \subset Y$ is closed for any $n = 1, 2, \ldots$. Then

$$F^-(V) = \bigcup_{n=1}^{\infty} F^-(F_n) = X \setminus \bigcap_{n=1}^{\infty} F^+(X \setminus F_n)$$

has the τ-Baire property. Let $F^-(V) = (G \setminus A) \cup B$, where $G \in \tau$ and A, B are of τ-first category. We show $B \subset \mathrm{cl}_\tau(G)$. Suppose there is $x \in B \setminus \mathrm{cl}_\tau(G)$. Since F is lower Baire τ-continuous, there is a set C which is of τ-second category with the τ-Baire property and $C \subset F^-(V) \cap (X \setminus \mathrm{cl}_\tau(G)) = ((G \setminus A) \cup B) \cap (X \setminus \mathrm{cl}_\tau(G)) \subset B$. So, C is of τ-first category, a contradiction.

$\Leftarrow$: Let $a \in X$, U be τ-open containing a and V be open intersecting $F(a)$. Then, $a \in F^-(V) \cap U = [(G \setminus A) \cup B] \cap U = [(G \setminus A) \cap U] \cup [B \cap U]$, where $G \in \tau$, $A, B \in \mathcal{I}_\tau$ and $B \subset \mathrm{cl}_\tau(G)$. We show $(G \setminus A) \cap U$ is of τ-second category. If $a \in (G \setminus A) \cap U$, then $(G \setminus A) \cap U$ is of τ-second category, since (X, τ) is τ-Baire. If $a \in B \cap U$, then $G \cap U \neq \emptyset$, since $B \subset \mathrm{cl}_\tau(G)$. So, $(G \setminus A) \cap U$ is of τ-second category. Moreover, $(G \setminus A) \cap U$ has the τ-Baire property. Since $(G \setminus A) \cap U \subset F^-(V) \cap U$, F is lower Baire τ-continuousat a.

Item (2) is similar. □

4. Bitopological Setting

This section presents the main results of the article. In particular, we study the continuities depending on two topologies and their properties depending on one topology. Namely, let F be $q^l(\tau, \sigma)$-continuous ($q^u(\tau, \sigma)$-continuous), $sq^l(\tau, \sigma)$-continuous ($sq^u(\tau, \sigma)$-continuous), $wsq^l(\tau, \sigma)$-continuous ($wsq^u(\tau, \sigma)$-continuous), respectively. All continuities depend on two topologies. The following questions are of interest.

(1) Is F lower/upper quasi-continuous or lower/upper Baire contnuous with respect to τ or σ? (Theorem 6, Corollary 7)
(2) What is the structure of the sets of discontinuity points $D^l_\tau(F)$, $D^u_\tau(F)$, $D^l_\sigma(F)$, $D^u_\sigma(F)$ of F? (Theorem 5, Theorem 9, Corollary 9)
(3) Is there a characterization of a continuity depending on two topologies by continuity/continuities depending only on one topology ? (to find one topological decomposition theorem, Theorem 7, Theorem 8)
(4) What are the continuity properties of F with respect to the corresponding dual bitopological space ? (Theorem 10, Corollary 9, Corollary 12)

We will focus mainly on the following combinations of two continuities.

(5) $wsq^l(\tau, \sigma)$ and $wsq^u(\tau, \sigma)$ / $sq^l(\tau, \sigma)$ and $sq^u(\tau, \sigma)$ (the same topological order and the different versions of continuities, Corollary 7, Corollary 10)
(6) $wsq^u(\tau, \sigma)$ and $wsq^u(\sigma, \tau)$ / $sq^l(\tau, \sigma)$ and $sq^l(\sigma, \tau)$ (the different topological order and the same versions of continuities, see two last diagrams in Chapter 5)
(7) $wsq^l(\tau, \sigma)$ and $wsq^u(\sigma, \tau)$ / $sq^l(\tau, \sigma)$ and $sq^u(\sigma, \tau)$ (the different topological order and different versions of continuities, Corollary 9, Corollary 12)

Let us note a few known facts in a bitopological setting, provided $\sigma \subset \tau$.

Lemma 2. *Let (X, τ, σ) be a bitopolocal space, $\sigma \subset \tau$ and $F : X \to Y$ be a multifunction.*

(1) If F is $wsq^l(\tau, \sigma)$-continuous at a, then F is lower quasi τ-continuous at a and lower quasi σ-continuous at a.
(2) If F is $wsq^u(\tau, \sigma)$-continuous at a, then F is upper quasi τ-continuous at a and upper quasi σ-continuous at a.

Suppose Y is second countable.

(3) *If F is $wsq^l(\tau,\sigma)$-continuous, then*

$$D^l_\tau(F) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma, \quad D^l_\sigma(F) \in \mathcal{I}_\sigma.$$

Moreover, if F is compact valued and Y is regular, then

$$D^u_\tau(F) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma, \quad D^u_\sigma(F) \in \mathcal{I}_\sigma.$$

(4) *If F is $wsq^u(\tau,\sigma)$-continuous and compact valued, then*

$$D^u_\tau(F) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma, \quad D^u_\sigma(F) \in \mathcal{I}_\sigma.$$

Moreover, if Y is regular, then

$$D^l_\tau(F) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma, \quad D^l_\sigma(F) \in \mathcal{I}_\sigma.$$

Proof. Since $\sigma \subset \tau$, (1) and (2) hold, see Remark 1(3).

(3) Suppose F is $wsq^l(\tau,\sigma)$-continuous and Y is second countable. By (1), F is lower quasi τ-continuous and lower quasi σ-continuous and by [28], $D^l_\tau(F)$ is of τ-first category and $D^l_\sigma(F)$ is of σ-first category. Since $C^l_\sigma(F) \subset C^l_\tau(F)$, $D^l_\tau(F)$ is also of σ-first category. Moreover, if F is compact valued and Y is regular, then (by [28]) $D^u_\tau(F)$ is of τ-first category and $D^u_\sigma(F)$ is of σ-first category. Since $C^u_\sigma(F) \subset C^u_\tau(F)$, $D^u_\tau(F)$ is also of σ-first category.

(4) Suppose F is compact valued $wsq^u(\tau,\sigma)$-continuous and Y is second countable. By (2), F is upper quasi τ-continuous and upper quasi σ-continuous and by [28], $D^u_\tau(F)$ is of τ-first category and $D^u_\sigma(F)$ is of σ-first category. Since $C^u_\sigma(F) \subset C^u_\tau(F)$, $D^u_\tau(F)$ is also of σ-first category. Moreover, if Y is regular, then (by [28]) $D^l_\tau(F)$ is of τ-first category and $D^l_\sigma(F)$ is of σ-first category. Since $C^l_\sigma(F) \subset C^l_\tau(F)$, $D^l_\tau(F)$ is also of σ-first category. □

A question is if in Lemma 2, $wsq^l(\tau,\sigma)$-continuity ($wsq^u(\tau,\sigma)$-continuity) can be changed by a dual continuity, namely by $wsq^l(\sigma,\tau)$-continuity ($wsq^u(\sigma,\tau)$-continuity). The next example shows it is not possible.

Example 2. The function F (G) in Example 1(3) is $wsq^u(\tau,\tau^*)$-continuous (G is $wsq^l(\tau,\tau^*)$-continuous) but $C^u_\tau(F)$ ($C^l_\tau(G)$) is the set of all rational numbers.

In Lemma 2, the structure of the sets $D^u_\tau(F)$, $D^l_\tau(F)$, $D^u_\sigma(F)$, $D^l_\sigma(F)$ follows from the condition $\sigma \subset \tau$ which guarantees the lower (upper) quasi τ-continuity and the lower (upper) quasi σ-continuity. When the condition $\sigma \subset \tau$ is omitted, the structure of the sets $D^u_\tau(F)$, $D^l_\tau(F)$, $D^u_\sigma(F)$, $D^l_\sigma(F)$ may vary, as the next example shows.

Example 3.

(1) Let $X = \{a,b,c,d\}$. Put $\sigma = \{\emptyset, X, \{d\}, \{a,b\}, \{a,b,d\}\}$, $\tau = \{\emptyset, X, \{b\}, \{c,d\}, \{b,c,d\}\}$, $\sigma \not\subset \tau$, $\tau \not\subset \sigma$. The sets of σ-first category: $\emptyset, \{c\}$. The sets of τ-first category: $\emptyset, \{a\}$. A multifunction $F : X \to \mathbb{R}$ ($\mathbb{R}$ with the Euclidean topology) defined as $F(a) = F(c) = \{1,2\}$ and $F(b) = F(d) = \{1\}$ is $wsq^u(\tau,\sigma)$-continuous but $D^u_\sigma(F) = \{b\}$, $D^l_\sigma(F) = \{a,c\}$, $D^u_\tau(F) = \{d\}$, $D^l_\tau(F) = \{a,c\}$ are not of τ-first category nor σ-first category.

A multifunction $G : X \to \mathbb{R}$ defined as $G(a) = G(c) = \{1\}$ and $G(b) = G(d) = \{1,2\}$ is $wsq^l(\tau,\sigma)$-continuous but $D^u_\sigma(G) = \{a,c\}$, $D^l_\sigma(G) = \{b\}$, $D^u_\tau(G) = \{a,c\}$, $D^l_\tau(G) = \{d\}$ are not of τ-first category nor σ-first category.

(2) Even for a $sq(\tau,\sigma)$-continuous function the set $D_\sigma(f)$ doesn't have to be from $\mathcal{I}_\tau$. Let $X = \{a,b,c\}$. Put $\sigma = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}$, $\tau = \{\emptyset, X, \{a,b\}\}$, $\sigma \not\subset \tau$, $\tau \not\subset \sigma$. The sets of σ-first category: $\emptyset, \{b\}$. The sets of τ-first category: $\emptyset, \{c\}$. A function $f : X \to \mathbb{R}$ ($\mathbb{R}$ with the Euclidean

topology) defined as $f(a) = f(b) = 1$ and $f(c) = 2$ is $sq(\tau,\sigma)$-continuous but $D_\sigma(f) = \{b\}$ is not of τ-first category.

(3) Let $X = \{a,b,c,d\}$. Put $\sigma = \{\emptyset, X, \{a\}, \{b,c\}, \{d\}, \{a,b,c\}, \{b,c,d\}, \{a,d\}\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{d\}, \{a,b\}, \{b,d\}, \{a,d\}, \{a,b,d\}\}$, $\sigma \not\subset \tau$, $\tau \not\subset \sigma$. The sets of τ-first category: $\emptyset$, $\{c\}$. The sets of σ-first category: $\emptyset$. Let $f : X \to \mathbb{R}$, $f(a) = f(c) = 1$, $f(b) = f(d) = 2$ ($\mathbb{R}$ with the Euclidean topology). Then, f is $wsq(\tau,\sigma)$-continuous, f is quasi τ-continuous, $D_\tau(f) = \{c\}$ is of τ-first category, $D_\sigma(f) = \{b,c\}$ is of τ-second category. Note, the conditions $\tau \prec \sigma$, $B(\tau,\sigma)$ hold (see Definition 3 below) and this example shows the sets $D^u_\sigma(F)$ and $D^l_\sigma(F)$ in Corollary 7(3) are not necessary from $\mathcal{I}_\tau$.

Question is whether $D^u_\sigma(F)$ and $D^l_\sigma(F)$ in Lemma 2 are of τ-first category, provided $\sigma \subset \tau$. The next example shows it does not apply.

Example 4. Let $X = \{a,b,c\}$. Put $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$, $\tau = 2^X$, $\sigma \subset \tau$.

The sets of σ-first category: $\emptyset$, $\{c\}$.

The sets of τ-first category: $\emptyset$.

A multifunction $F_1 : X \to \mathbb{R}$ ($\mathbb{R}$ with the Euclidean topology) defined as $F_1(a) = F_1(b) = \{0,1\}$ and $F_1(c) = \{0\}$ is $wsq^l(\tau,\sigma)$-continuous but $D^l_\sigma(F_1) = \{c\}$ is not of τ-first category. Note, F_1 is not $wsq^u(\tau,\sigma)$-continuous at c.

A multifunction $F_2 : X \to \mathbb{R}$ defined as $F_2(a) = F(b) = \{1\}$ and $F_2(c) = \{1,2\}$ is $wsq^u(\tau,\sigma)$-continuous but $D^u_\sigma(F_2) = \{c\}$ is not of τ-first category. Note, F_2 is not $wsq^l(\tau,\sigma)$-continuous at c. A case when F is $wsq^l(\tau,\sigma)$-continuous and $wsq^u(\tau,\sigma)$-continuous is solved in Corollary 10, provided $\sigma \subset \tau$ and in Corollary 7(3), provided $\tau \prec \sigma$.

In the following we will show that the condition $\sigma \subset \tau$ can be replaced by a more general condition (see Definition 3 below) that guarantees satisfactory results concerning the set of discontinuity points and the mutual relations between the studied continuities.

Definition 3. ([23]) Let (X,τ,σ) be a bitopological space. A set A is τ-pseudo open if $A = G \cup Z$, where $G \in \tau$ and Z is of τ-first category. A topology τ pseudo refines σ, denoted by $\tau \prec \sigma$ if any σ-closed set is τ-pseudo open. By $B(\tau,\sigma)$ we denote the next property: for any $H \in \tau$ and any $G \in \sigma$ the intersection $H \cap G$ is of τ-second category, provided $H \cap G \neq \emptyset$.

Lemma 3. *([23]) Let (X,τ,σ) be a bitopological space.*

(1) If $\tau \prec \sigma$, then $H \cap G$ has the τ-Baire property for any $H \in \tau$ and $G \in \sigma$. (Since any σ-closed set is τ-pseudo open (so, it has τ-Baire property), any σ-open set has the τ-Baire property, so $H \cap G$ has τ-Baire property).

(2) Since any τ-closed set F can be expressed as $F = \operatorname{int}_\tau(F) \cup (F \setminus \operatorname{int}_\tau(F))$, where $F \setminus \operatorname{int}_\tau(F)$ is τ-nowhere dense, $\tau \prec \tau$.

(3) Let $\sigma \subset \tau$. Then, $B(\tau,\sigma)$ holds if and only if (X,τ) is τ-Baire.
If F is σ-closed, then F is also τ-closed. Since $F \setminus \operatorname{int}_\tau(F)$ is τ-nowhere dense, F can be expressed as $F = \operatorname{int}_\tau(F) \cup (F \setminus \operatorname{int}_\tau(F))$. So, $\tau \prec \sigma$. As a special case we get $\tau_{\mathcal{I}} \prec \tau$ for any ideal $\mathcal{I}$ and for $\mathcal{I}_\tau$ we have $\tau^* \prec \tau$.
Suppose F is τ^*-closed in (X,τ^*). Then $F^* \subset F$, where F^* is a set of all points at which F is of τ-second category and F^* is τ-closed. So, $F = \operatorname{int}_\tau(F^*) \cup [F^* \setminus \operatorname{int}_\tau(F^*)] \cup [F \setminus F^*]$. Since $F \setminus F^*$ is of τ-first category and $F^* \setminus \operatorname{int}_\tau(F^*)$ is τ-first category, F is τ-pseudo open. That means $\tau \prec \tau^*$.

(4) If $B(\tau,\sigma)$ holds, then (X,τ) is τ-Baire and any nonempty σ-open set is of τ-second category. The condition $B(\tau,\tau)$ holds if and only if (X,τ) is τ-Baire.

Theorem 5. *Let (X,τ,σ) be a bitopological space, $\tau \prec \sigma$, Y be a second countable topological space and $F : X \to Y$ be a multifunction.*

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(1) *If F is $sq^l(\tau,\sigma)$-continuous except for a set of τ-first category (F is $sq^l(\tau,\sigma)$-continuous), then F is σ-lsc except for a set of τ-first category ($D^l_\sigma(F) \in \mathcal{I}_\tau$).*

(2) *Suppose Y is regular. If F is compact valued $sq^u(\tau,\sigma)$-continuous except for a set of τ-first category (F is compact valued $sq^u(\tau,\sigma)$-continuous), then F is σ-usc except for a set of τ-first category ($D^u_\sigma(F) \in \mathcal{I}_\tau$).*

Proof. We will prove the item (1).

Let $\mathcal{B} = \{B_n : n = 1,2,\dots\}$ be a base of Y. Then, the set of all points at which F is not σ-*lsc* can be expressed as

$$\bigcup_{n=1}^{\infty}[F^-(B_n) \setminus \mathrm{int}_\sigma(F^-(B_n))].$$

We will prove that $A_n = F^-(B_n) \setminus \mathrm{int}_\sigma(F^-(B_n))$ is of τ-first category for any $n = 1,2,\dots$ Assume there is n such that A_n is of τ-second category. Denote $G = X \setminus \mathrm{int}_\sigma(F^-(B_n))$. Since $\tau \prec \sigma$ and G is σ-closed, there are $H \in \tau$ and I which is of τ-first category such that $G = H \cup I$. Since A_n ($A_n \subset H \cup I$) is of τ-second category, there is a point $a \in H \cap A_n$ and F is $sq^l(\tau,\sigma)$-continuous at a. Moreover, $H \cap \mathrm{int}_\sigma(F^-(B_n)) = \emptyset$.

Since F is $sq^l(\tau,\sigma)$-continuous at a, there is a nonempty set $U \in \sigma$ such that $U \cap H \neq \emptyset$ and $F(x) \cap B_n \neq \emptyset$ for any $x \in U$. That means, $U \subset \mathrm{int}_\sigma(F^-(B_n))$. Since $H \cap \mathrm{int}_\sigma(F^-(B_n)) = \emptyset$, $U \cap H = \emptyset$, a contradiction.

(2) is similar. Since Y is regular and F is compact valued, the set of all points at which F is not σ-*usc* can be expressed as $\bigcup_{n=1}^{\infty}[F^+(V_n) \setminus \mathrm{int}_\sigma(F^+(V_n))]$, where $\{V_1, V_2, V_3, \dots\}$ is a sequence of all finite unions of sets from $\mathcal{B}$. □

The assumption $\tau \prec \sigma$ in Theorem 5 cannot be omitted, see Example 3, where f is $sq(\tau,\sigma)$-continuous but $D_\sigma(f) = \{b\}$ is not of τ-first category. The set $\{b\}$ is σ-closed but it is not τ-pseudo open, so $\tau \not\prec \sigma$.

Theorem 5 does not hold for a $wsq^l(\tau,\sigma)$-continuous or $wsq^u(\tau,\sigma)$-continuous multifunction. In Example 4, F_1 (F_2) is $wsq^l(\tau,\sigma)$-continuous ($wsq^u(\tau,\sigma)$-continuous) but $D^l_\sigma(F_1)$ ($D^u_\sigma(F_2)$) is of τ-second category (it holds in some special cases, see next corollary). A case when a multifunction F is $wsq^l(\tau,\sigma)$-continuous and $wsq^u(\tau,\sigma)$-continuous is solved in Corollary 10.

From Theorem 5, we can obtain two consequences. It is worth to compare the item (3) in Corollary 5 concerning (X,τ^*,τ) with Theorem 3 concerning (X,τ,τ^*). Note, Corollary 5 is a special case of Lemma 2 ($\tau \subset \tau_{\mathcal{I}}$) and a generalization is solved in Corollary 10.

Corollary 5. *Let $\mathcal{I}$ be an ideal on X, Y be a second countable topological space and $F : X \to Y$ be a multifunction.*

(1) *If F is $wsq^l(\tau_{\mathcal{I}},\tau)$-continuous except for a set of $\tau_{\mathcal{I}}$-first category (except for a set of τ-first category), then F is $\tau_{\mathcal{I}}$-lsc except for a set of $\tau_{\mathcal{I}}$-first category (τ-lsc except for a set of τ-first category).*

(2) *Suppose Y is regular. If F is compact valued $wsq^u(\tau_{\mathcal{I}},\tau)$-continuous except for a set of $\tau_{\mathcal{I}}$-first category (except for a set of τ-first category), then F is $\tau_{\mathcal{I}}$-usc except for a set of $\tau_{\mathcal{I}}$-first category (τ-usc except for a set of τ-first category).*

(3) *Suppose Y is regular and F is compact valued. If F is $wsq^l(\tau^*,\tau)$-continuous except for a set of τ-first category ($wsq^u(\tau^*,\tau)$-continuous except for a set of τ-first category), then $D^l_\tau(F) \in \mathcal{I}_\tau$ ($D^u_\tau(F) \in \mathcal{I}_\tau$).*

Proof. (1) Since F is $wsq^l(\tau_{\mathcal{I}},\tau)$-continuous except for a set of $\tau_{\mathcal{I}}$-first category (except for a set of τ-first category), F is $wsq^l(\tau_{\mathcal{I}},\tau_{\mathcal{I}})$-continuous except for a set of $\tau_{\mathcal{I}}$-first category ($wsq^l(\tau,\tau)$-continuous except for a set of τ-first category), by Remark 1(2). So, F is $sq^l(\tau_{\mathcal{I}},\tau_{\mathcal{I}})$-continuous except for a set of $\tau_{\mathcal{I}}$-first category ($sq^l(\tau,\tau)$-continuous except for a set of τ-first category), by Remark 1(3). Since $\tau_{\mathcal{I}} \prec \tau_{\mathcal{I}}$ ($\tau \prec \tau$), by Theorem 5, F is $\tau_{\mathcal{I}}$-*lsc* except for a set of $\tau_{\mathcal{I}}$-first category (τ-*lsc* except for a set of τ-first category).

The item (2) is similar.

(3) Since $\tau^* = \tau_{\mathcal{I}_\tau}$, it follows from (1) and (2). □

In fact, Corollary 5 involves the next known result.

Corollary 6. *([28]) Let Y be a second countable topological space and $F : X \to Y$ be a multifunction.*

(1) If F is lower quasi τ-continuous except for a set of τ-first category (F is lower quasi τ-continuous), then F is τ-lsc except for a set of τ-first category ($D^l_\tau(F) \in \mathcal{I}_\tau$).

(2) If F is compact valued upper quasi τ-continuous except for a set of τ-first category (F is upper quasi τ-continuous), then F is τ-usc except for a set of τ-first category ($D^u_\tau(F) \in \mathcal{I}_\tau$).

Proof. Consider $\tau = \sigma$. Then, the lower quasi τ-continuity (upper quasi τ-continuity) is equivalent to the $sq^l(\tau,\tau)$-continuity ($sq^u(\tau,\tau)$-continuity), by Remark 1(3). Since $\tau \prec \tau$, we can use Theorem 5. □

Recall, $\sigma_{\mathcal{I}_\tau}$ is a topology generated by a base $\{G \setminus A : G \in \sigma$ and A is of τ-first category$\}$.

Theorem 6. *Let $B(\tau,\sigma)$ hold, $\tau \prec \sigma$ and $F : X \to Y$ be a multifunction.*

(1) If F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^l(\tau,\sigma)$-continuous) at a point a, then F is lower Baire τ-continuous at a. Moreover, if X is finite, then F is lower quasi τ-continuous at a.

(2) If F is $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^u(\tau,\sigma)$-continuous) at a point a, then F is upper Baire τ-continuous at a. Moreover, if X is finite, then F is upper quasi τ-continuous at a.

Proof. We will prove the item (1), the second one is similar.

Recall, (X,τ) is τ-Baire and any nonempty σ-open set is of τ-second category, by Lemma 3 (4). Let $a \in H \in \tau$ and V be open, $F(a) \cap V \neq \emptyset$. Since F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous at a, there is a set $E = G \setminus A$ such that $G \setminus A \neq \emptyset$, $G \in \sigma$, A is of τ-first category, $H \cap E \neq \emptyset$ and $F(x) \cap V \neq \emptyset$ for all $x \in H \cap E$.

It is clear, $H \cap G \neq \emptyset$. Since $B(\tau,\sigma)$ holds and $\tau \prec \sigma$ (see Lemma 3(1)), $H \cap G$ is of τ-second category and it has the τ-Baire property. That means, $(H \cap G) \setminus A = H \cap (G \setminus A)$ is of τ-second category with the τ-Baire property, $H \cap (G \setminus A) \subset H$ and $F(x) \cap V \neq \emptyset$ for any $x \in H \cap (G \setminus A) = H \cap E$. So, F is lower Baire τ-continuousat a.

Suppose X is finite. Since F is lower Baire τ-continuous at a, F is lower quasi τ-continuous at a, by Remark 2(6).

The case in the brackets follows from the inclusion $\sigma \subset \sigma_{\mathcal{I}_\tau}$, see Remark 1(2). □

By Lemma 3(3), if (X,τ) is τ-Baire, then all conditions $B(\tau^*,\tau)$, $\tau^* \prec \tau$, $B(\tau,\tau^*)$, $\tau \prec \tau^*$ hold. It is trivial example. The next example shows there is a non-trivial bitopological space in which all conditions $B(\tau,\sigma)$, $\tau \prec \sigma$, $B(\sigma,\tau)$, $\sigma \prec \tau$ hold. It also shows that the lower (upper) Baire τ-continuuity in Theorem 6 can not be replaced by the lower (upper) quasi τ-continuity, provided X is infinite.

Example 5. Let $X_0 = \{\frac{1}{n} : n = 2,3,4,\dots\}$, $X = X_0 \cup \{0\}$,
$X_1 = \{\frac{1}{2n+1} : n = 1,2,3,\dots\}$, $X_2 = \{\frac{1}{2n} : n = 1,2,3,4,\dots\}$,
$G_1 = \{0\}$, $G_2 = \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\}$, $G_3 = \{0, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\}$, $G_4 = \{0, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\}$,
$G_5 = \{0, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\}$, $G_6 = \{0, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots\}$, $G_7 = \{0, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \dots\}, \dots$
$\tau = \{\emptyset, X, G_2, G_4, G_6, \dots\}$, $\sigma = \{\emptyset, X, G_1, G_3, G_5, G_7 \dots\}$.

All finite subsets of X_0 are τ- and σ-nowhere dense, any infinite subset of X_0 is of τ- and σ-first category. Any subset of X has the τ- and σ-Baire property.

Any set containing 0 is of τ- and σ-second category, both topological spaces are Baire. Moreover, the conditions $B(\tau,\sigma)$, $\tau \prec \sigma$, $B(\sigma,\tau)$, $\sigma \prec \tau$ hold.

Consider $Y = \mathbb{R}$ with the Euclidean topology. Define a multifunction $F : X \to \mathbb{R}$ ($G : X \to \mathbb{R}$) as $F(x) = \{0\}$ ($G(x) = \{0,1\}$), if $x \in X_1 \cup \{0\}$ and $F(x) = \{0,1\}$ ($G(x) = \{0\}$) otherwise. Then, F is $wsq^u(\tau,\sigma)$-continuous (even $sq^u(\tau,\sigma)$-continuous) (G is $wsq^l(\tau,\sigma)$-continuous (even $sq^l(\tau,\sigma)$-continuous)) but it is not upper quasi τ-continuous at any point $x \in X_1 \cup \{0\}$ (lower quasi τ-continuous at any point $x \in X_1 \cup \{0\}$). So, the lower (upper) Baire τ-continuity in Theorem 6 can not be replaced by the lower (upper) quasi τ-continuity, provided X is infinite. The sets of continuities are summarized in the next tables.

(X,τ,σ)

$C^l_\tau(F)$	$X_1\cup\{0\}$	$C^u_\tau(F)$	X_2
$Q^l_\tau(F)$	$X_1\cup\{0\}$	$Q^u_\tau(F)$	X_2
$B^l_\tau(F)$	$X_1\cup\{0\}$	$B^u_\tau(F)$	X
$Q^l_{\tau,\sigma}(F)$	$X_1\cup\{0\}$	$Q^u_{\tau,\sigma}(F)$	X
$sQ^l_{\tau,\sigma}(F)$	$X_1\cup\{0\}$	$sQ^u_{\tau,\sigma}(F)$	X
$wsQ^l_{\tau,\sigma}(F)$	$X_1\cup\{0\}$	$wsQ^u_{\tau,\sigma}(F)$	X

(X,σ,τ)

$C^l_\sigma(F)$	$X_1\cup\{0\}$	$C^u_\sigma(F)$	X
$Q^l_\sigma(F)$	$X_1\cup\{0\}$	$Q^u_\sigma(F)$	X
$B^l_\sigma(F)$	$X_1\cup\{0\}$	$B^u_\sigma(F)$	X
$Q^l_{\sigma,\tau}(F)$	$\emptyset$	$Q^u_{\sigma,\tau}(F)$	X_2
$sQ^l_{\sigma,\tau}(F)$	$X_1\cup\{0\}$	$sQ^u_{\sigma,\tau}(F)$	X_2
$wsQ^l_{\sigma,\tau}(F)$	$X_1\cup\{0\}$	$wsQ^u_{\sigma,\tau}(F)$	X

$C^l_\tau(G)$	X_2	$C^u_\tau(G)$	$X_1\cup\{0\}$
$Q^l_\tau(G)$	X_2	$Q^u_\tau(G)$	$X_1\cup\{0\}$
$B^l_\tau(G)$	X	$B^u_\tau(G)$	$X_1\cup\{0\}$
$Q^l_{\tau,\sigma}(G)$	X	$Q^u_{\tau,\sigma}(G)$	$X_1\cup\{0\}$
$sQ^l_{\tau,\sigma}(G)$	X	$sQ^u_{\tau,\sigma}(G)$	$X_1\cup\{0\}$
$wsQ^l_{\tau,\sigma}(G)$	X	$wsQ^u_{\tau,\sigma}(G)$	$X_1\cup\{0\}$

$C^l_\sigma(G)$	X	$C^u_\sigma(G)$	$X_1\cup\{0\}$
$Q^l_\sigma(G)$	X	$Q^u_\sigma(G)$	$X_1\cup\{0\}$
$B^l_\sigma(G)$	X	$B^u_\sigma(G)$	$X_1\cup\{0\}$
$Q^l_{\sigma,\tau}(G)$	X_2	$Q^u_{\sigma,\tau}(G)$	$\emptyset$
$sQ^l_{\sigma,\tau}(G)$	X_2	$sQ^u_{\sigma,\tau}(G)$	$X_1\cup\{0\}$
$wsQ^l_{\sigma,\tau}(G)$	X	$wsQ^u_{\sigma,\tau}(G)$	$X_1\cup\{0\}$

The assumption $B(\tau,\sigma)$ or $\tau \prec \sigma$ in Theorem 6 can not be omitted as the next two examples show.

Example 6. Let $X = \{a,b\}$, $\tau = \{\emptyset, X, \{b\}\}$, $\sigma = 2^X$. The intersection of $\{a\} \in \sigma$ and $X \in \tau$ is equal to $\{a\}$ that is of τ-first category, so $B(\tau,\sigma)$ does not hold and $\tau \prec \sigma$ is fulfilled. Define $F : X \to \mathbb{R}$ ($\mathbb{R}$ with the Euclidean topology) as $F(a) = \{1\}$, $F(b) = \{0,1\}$. Then F is $wsq^u(\tau,\sigma)$-continuous, but it is not upper quasi τ-continuous nor upper Baire τ-continuous at a.

Example 7. Let X be a set containing at least two points and τ, σ be the indiscrete, the discrete topology on X, respectively. A set A is of τ-first category (σ-first category) if and only if $A = \emptyset$, so the condition $B(\tau,\sigma)$ ($B(\sigma,\tau)$) holds. It is clear $\tau \not\prec \sigma$. Let $f : X \to Y$ be any bijection, where $Y = X$ with the discrete topology. Then, f is $wsq(\tau,\sigma)$-continuous (even $q(\tau,\sigma)$-continuous), but it is not quasi τ-continuous nor Baire τ-continuous at any point.

The opposite implications in Theorem 6 are not valid, as the next example shows.

Example 8. Let X be a set containing at least two points and τ, σ be the discrete, the indiscrete topology on X, respectively. The sets of σ-first category: $\emptyset$. The sets of τ-first category: $\emptyset$. It is clear $\tau \prec \sigma$ and $B(\tau,\sigma)$ hold.

Let $f : X \to Y$ be any bijection, where $Y = X$ with the discrete topology. Then f is Baire τ-continuous (even τ-continuous and quasi τ-continuous) but it is not $wsq(\tau,\sigma)$-continuous at any point.

Remark 3. Example 5 and Example 8 show in general there is no connection between the upper/lower quasi τ-continuity and $wsq^u(\tau,\sigma)$-continuity/$wsq^l(\tau,\sigma)$-continuity. Lemma 2 shows $wsq^l(\tau,\sigma)$-continuity ($wsq^u(\tau,\sigma)$-continuity) implies the lower quasi τ-continuity and the lower quasi σ-continuity (the upper quasi τ-continuity and the upper quasi σ-continuity), provided $\sigma \subset \tau$. The opposite implication does not hold. By [22], there is even a function $f : (\mathbb{R},\tau,\sigma) \to (\mathbb{R},\sigma)$ that is quasi τ-continuous and quasi σ-continuous but it is not $wsq(\tau,\sigma)$-continuous, where τ is the density topology and σ is the Euclidean topology on the real line $\mathbb{R}$, see also [23].

By Theorem 6 and Corollary 3, we have the next corollary.

Corollary 7. *Let $B(\tau,\sigma)$ hold, $\tau \prec \sigma$, Y be regular second countable and $F : X \to Y$ be a multifunction.*

(1) *If F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^l(\tau,\sigma)$-continuous) and compact valued, then F is lower Baire τ-continuous and $D^u_\tau(F) \in \mathcal{I}_\tau$. Consequently, F is $sq^u(\sigma,\tau)$-continuous on a τ-residual set, by Remark 1(1)(b).*
(2) *If F is $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^u(\tau,\sigma)$-continuous), then F is upper Baire τ-continuous and $D^l_\tau(F) \in \mathcal{I}_\tau$. Consequently, F is $sq^l(\sigma,\tau)$-continuous on a τ-residual set, by Remark 1(1)(b).*
(3) *If F is compact valued $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^l(\tau,\sigma)$-continuous) and $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^u(\tau,\sigma)$-continuous), then F is lower and upper quasi τ-continuous and $D^u_\tau(F)$, $D^l_\tau(F) \in \mathcal{I}_\tau$ (by (1), (2) and Theorem 2).*

Example 3(3) shows the sets $D^u_\sigma(F)$, $D^l_\sigma(F)$ in Corollary 7(3) are not necessary from $\mathcal{I}_\tau$. Compare with Corollary 10, where among other things, it is shown that F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous and $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous if and only if F is lower and upper quasi τ-continuous and $D^u_\sigma(F)$, $D^l_\sigma(F) \in \mathcal{I}_\tau$, provided $\sigma \subset \tau$.

By Theorem 6 and Corollary 2, in the function setting we have the next result.

Corollary 8. *Let $B(\tau,\sigma)$ hold, $\tau \prec \sigma$ and Y be regular. If a function $f : X \to Y$ is $wsq(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq(\tau,\sigma)$-continuous), then f is quasi τ-continuous.*

From Theorem 5 and Remark 1(1)(b) we have the next one topological decomposition theorem.

Theorem 7. *Let (X,τ,σ) be a bitopological space, $\tau \prec \sigma$, Y be a second countable topological space and $F : X \to Y$ be a multifunction.*

(1) *F is $sq^l(\tau,\sigma)$-continuous except for a set of τ-first category if and only if F is σ-lsc except for a set of τ-first category ($D^l_\sigma(F) \in \mathcal{I}_\tau$).*
(2) *Suppose F is compact valued. Then, F is $sq^u(\tau,\sigma)$-continuous except for a set of τ-first category if and only if F is σ-usc except for a set of τ-first category ($D^u_\sigma(F) \in \mathcal{I}_\tau$).*

The next theorem deals with the global characterization (the equivalence (a) ⇔ (b) in the item (1) and (2) represents one topological decomposition). The items (3) in Theorem 8 characterizes the $sq^l(\tau,\sigma)$-continuity and $sq^u(\tau,\sigma)$-continuity that generalizes a equivalence (3) ⇔ (4) given in Corollary 10.

Theorem 8. *Let $B(\tau,\sigma)$ hold, $\tau \prec \sigma$, Y be a second countable topological space and $F : X \to Y$ be a multifunction. Then, the next conditions (a), (b), (c), (d) in (1), (2) are equivalent, respectively.*

(1) (a) *F is $sq^l(\tau,\sigma)$-continuous,*
  (b) *F is lower Baire τ-continuous and $D^l_\sigma(F) \in \mathcal{I}_\tau$,*
  (c) *F is $wsq^l(\tau,\sigma)$-continuous and $D^l_\sigma(F) \in \mathcal{I}_\tau$.*
  (d) *F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous and $D^l_\sigma(F) \in \mathcal{I}_\tau$.*
(2) *Suppose, Y is regular and F is compact valued.*
  (a) *F is $sq^u(\tau,\sigma)$-continuous,*
  (b) *F is upper Baire τ-continuous and $D^u_\sigma(F) \in \mathcal{I}_\tau$,*
  (c) *F is $wsq^u(\tau,\sigma)$-continuous and $D^u_\sigma(F) \in \mathcal{I}_\tau$.*
  (d) *F is $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous and $D^u_\sigma(F) \in \mathcal{I}_\tau$.*
(3) *Suppose, Y is regular and F is compact valued. Then the next conditions are equivalent.*
  (a) *F is $sq^l(\tau,\sigma)$-continuous and $sq^u(\tau,\sigma)$-continuous,*
  (b) *F is lower and upper quasi τ-continuous and $D^l_\sigma(F) \in \mathcal{I}_\tau, D^u_\sigma(F) \in \mathcal{I}_\tau$.*
(4) *Suppose Y is regular. A function f is $sq(\tau,\sigma)$-continuous if and only if f is quasi τ-continuous and f is σ-continuous except for a set of τ-first category.*

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Proof. (1) $(a) \Rightarrow (b)$: Suppose F is $sq^l(\tau,\sigma)$-continuous. By Theorem 5(1), F is σ-*lsc* except for a set of τ-first category and by Theorem 6, F is lower Baire τ-continuous.

$(b) \Rightarrow (c)$: Let $a \in X$, $a \in H \in \tau$ and V be open intersecting $F(a)$. From the lower Baire τ-continuous, there is a set $G \in \tau$ and A that is of τ-first category such that $G \setminus A \subset H$ and $F(x) \cap V \neq \emptyset$ for any $x \in G \setminus A$. Since $G \setminus A$ is of τ-second category, there is $b \in G \setminus A$ and F is σ-*lsc* at b. That means, there is a set $G_0 \in \sigma$ containing b such that $F(x) \cap V \neq \emptyset$ for any $x \in G_0$. So, $H \cap G_0 \neq \emptyset$ and $F(x) \cap V \neq \emptyset$ for any $x \in H \cap G_0$. That means, F is $wsq^l(\tau,\sigma)$-continuous at a.

The $(c) \Rightarrow (d)$ follows from the inclusion $\sigma \subset \sigma_{\mathcal{I}_\tau}$, see Remark 1(2).

$(d) \Rightarrow (a)$: Let $a \in X$, $a \in H \in \tau$ and V be open intersecting $F(a)$. From the $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuity, there is a set $G \setminus A \in \beta(\sigma,\mathcal{I}_\tau)$ and such that $H \cap (G \setminus A) \neq \emptyset$ and $F(x) \cap V \neq \emptyset$ for any $x \in H \cap (G \setminus A)$. Since $H \cap (G \setminus A)$ is τ-second category, there is $b \in H \cap (G \setminus A)$ such that F is σ-*lsc* at b. So, there is a set $G_0 \in \sigma$ containing b such that $F(x) \cap V \neq \emptyset$ for any $x \in G_0$. Since $H \cap G_0 \neq \emptyset$, F is $sq^l(\tau,\sigma)$-continuous at a.

The item (2) is similar.

(3) It follows from (1), (2) and Theorem 2.

(4) It follows from (3). □

In Theorem 8, the lower (upper) Baire τ-continuous can not replaced by the lower (upper) τ-quasi continuity, see Example 5.

By Theorem 5 and Corollary 7, we have the next result.

Theorem 9. *Let $B(\tau,\sigma)$ hold, $\tau \prec \sigma$, Y be a regular second countable topological space and $F : X \to Y$ be a compact valued multifunction.*

(1) If F is $sq^l(\tau,\sigma)$-continuous, then F lower Baire τ-continuous and F is both σ-lsc and τ-usc except for a set of τ-first category ($D^l_\sigma(F), D^u_\tau(F) \in \mathcal{I}_\tau$).

(2) If F is $sq^u(\tau,\sigma)$-continuous, then F is upper Baire τ-continuous and F is both σ-usc and τ-lsc except for a set of τ-first category ($D^u_\sigma(F), D^l_\tau(F) \in \mathcal{I}_\tau$).

In the following we will study the connection between the properties of multifunction with respect to (X,τ,σ) and with respect to (X,σ,τ). To be as general as possible, let us define the next Denjoy property in a bitopological setting.

Definition 4. (see [34] for the Euclidean topology and the sets of positive measure) Let (X,τ,σ) be a bitopological space. A multifunction $F : X \to Y$ has the lower (upper) Denjoy (σ,τ)-property (denoted by $d^l(\sigma,\tau)$-property, ($d^u(\sigma,\tau)$-property) at a point x if for any open set V for which $V \cap F(x) \neq \emptyset$ ($F(x) \subset V$) and any σ-open set U containing x there is a τ-second category set $B \subset U$ such that $V \cap F(b) \neq \emptyset$ ($F(b) \subset V$) for any $b \in B$. The global definition of the lower (upper) Denjoy (σ,τ)-property is given by the local ones at each point.

Remark 4. The next two easy items will be used in Theorem 10.

(1) Let $B(\tau,\sigma)$ hold. It is clear if F is $wsq^l(\sigma,\tau)$-continuous ($wsq^u(\sigma,\tau)$-continuous) at a point a, then F has the lower (upper) Denjoy (σ,τ)-property at a.

(2) Suppose $\sigma \subset \tau$ and (X,τ) is τ-Baire. If F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous) at a point a, then F has the lower (upper) Denjoy (σ,τ)-property at a.

Proof. Let $a \in X$ be arbitrary, $U \in \sigma$ containing a and V be open intersecting (containing) $F(a)$. Since $\sigma \subset \tau$ and F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous) at a, there is a set $G \setminus A \in \beta(\sigma,\mathcal{I}_\tau)$ such that $U \cap (G \setminus A) \neq \emptyset$ and $U \cap (G \setminus A) \subset F^-(V)$ ($U \cap (G \setminus A) \subset F^+(V)$). So, $U \cap G \neq \emptyset$ and $U \cap G$ is of τ-second category (since (X,τ) is τ-Bare and $\sigma \subset \tau$), consequently $(U \cap G) \setminus A = U \cap (G \setminus A)$ is of τ-second category. Since $U \cap (G \setminus A) \subset U$, F has the lower (upper) (σ,τ)-Denjoy property at a. □

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Theorem 10. *Let (X,τ,σ) be a bitopological space, $B(\tau,\sigma)$ hold, $\tau \prec \sigma$, Y be a regular topological space and $F : X \to Y$ be a compact valued multifunction.*

(1) Let F have the upper Denjoy (σ,τ)-property (F be $wsq^u(\sigma,\tau)$-continuous). Then, F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^l(\tau,\sigma)$-continuous) at a if and only if F is $sq^l(\tau,\sigma)$-continuous at a.

(2) Let F have the lower Denjoy (σ,τ)-property (F be $wsq^l(\sigma,\tau)$-continuous). Then, F is $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^u(\tau,\sigma)$-continuous) at a if and only if F is $sq^u(\tau,\sigma)$-continuous at a.

Proof. (1) Case (i): Let F have the upper Denjoy (σ,τ)-property.

"$\Rightarrow$" (a) Suppose F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous at a. We prove F is $sq^l(\tau,\sigma)$-continuous at a. Let $a \in H_1 \in \tau$ and V be open intersecting $F(a)$. Since Y is regular, there is an open set V_0 intersecting $F(a)$ and $\mathrm{cl}(V_0) \subset V$. Put $A = F^+(Y \setminus \mathrm{cl}(V_0))$ and $H_0 = \mathrm{int}_\tau(D_\tau(A))$, where $D_\tau(A)$ is the set of all points in which A is of τ-second category. We will prove $G_0 \cap H_0 \cap A$ is of τ-second category, provided $H_0 \cap G_0 \neq \emptyset$ for some $G_0 \in \sigma$. (1)

Since $B(\tau,\sigma)$ holds and $\tau \prec \sigma$, $H_0 \cap G_0$ is of τ-second category with the τ-Baire property, by Lemma 3. Let $H_0 \cap G_0 = (U \setminus S) \cup T$, where $U \in \tau$ and S, T are of τ-first category. Then, $\emptyset \neq U \cap H_0 \in \tau$. So, $U \cap H_0 \cap A$ is of τ-second category, consequently $U \cap A$ is of τ-second category. Moreover, $H_0 \cap G_0 \cap A = [(U \setminus S) \cup T] \cap A = [(U \cap A) \setminus (S \cap A)] \cup (T \cap A)$, so $H_0 \cap G_0 \cap A$ is of τ-second category. We have proven (1).

Let $H := H_1 \cup H_0 \in \tau$. Since $a \in H$ and F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous at a, there is a set $G \setminus I_1 \in \beta(\sigma,\mathcal{I}_\tau)$ such that $H \cap (G \setminus I_1) \neq \emptyset$ and $F(x) \cap V_0 \neq \emptyset$ for $x \in H \cap (G \setminus I_1)$. (2)
Since $H \cap (G \setminus I_1) \neq \emptyset$, $H \cap G \neq \emptyset$. (3)

We will show $H_0 \cap G = \emptyset$. (4)

If $H_0 \cap G \neq \emptyset$, then $G \cap H_0 \cap A$ is of τ-second category, by (1). So, there is $b \in (G \setminus I_1) \cap H_0 \cap A \subset H \cap (G \setminus I_1)$ for which $F(b) \subset Y \setminus \mathrm{cl}(V_0)$, a contradiction with the fact that $F(x) \cap V_0 \neq \emptyset$ for any $x \in H \cap (G \setminus I_1)$ (see (2)). That means, $H_0 \cap G = \emptyset$.

By (3) and (4), $\emptyset \neq H \cap G = H_1 \cap G \cup H_0 \cap G = H_1 \cap G$. (5)

We will show $F(x) \cap \mathrm{cl}(V_0) \neq \emptyset$ for any $x \in G$. (6)

Suppose $F(x_0) \subset X \setminus \mathrm{cl}(V_0)$ for some $x_0 \in G$. Since F has the upper Denjoy (σ,τ)-property at x_0, there is a set $B \subset G$ that is of τ-second category such that $F(x) \subset Y \setminus \mathrm{cl}(V_0)$ for any $x \in B$. We show $\mathrm{int}_\tau(D_\tau(B)) \cap G \neq \emptyset$. If $\mathrm{int}_\tau(D_\tau(B)) \cap G = \emptyset$, then $B \setminus \mathrm{int}_\tau(D_\tau(B)) = B$, a contradiction, since $B \setminus \mathrm{int}_\tau(D_\tau(B))$ is of τ-first category. Since $B \subset A$, $\mathrm{int}_\tau(D_\tau(B)) \subset \mathrm{int}_\tau(D_\tau(A)) = H_0$. So, $\emptyset \neq \mathrm{int}_\tau(D_\tau(B)) \cap G \subset H_0 \cap G$, a contradiction with (4).

We have proved there is a set $G \in \sigma$ such that $H_1 \cap G \neq \emptyset$ (see (5)) and $\emptyset \neq F(x) \cap \mathrm{cl}(V_0) \subset F(x) \cap V$ for any $x \in G$ (see (6)). So, F is $sq^l(\tau,\sigma)$-continuous at a.

(b) Suppose F is $wsq^l(\tau,\sigma)$-continuous at a. Since $\sigma \subset \sigma_{\mathcal{I}_\tau}$, F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous at a. By (a), F is $sq^l(\tau,\sigma)$-continuous at a.

"$\Leftarrow$" is trivial.

Case (ii): Let F be F $wsq^u(\sigma,\tau)$-continuous. Then, by Remark 4(1), F has the upper Denjoy (σ,τ)-property. So, by Case (i), the item (1) holds.

Similarly we can prove (2). □

From Theorem 10, Theorem 8 and Corollary 3 we have the next results.

Corollary 9. *Let (X,τ,σ) be a bitopological space, $B(\tau,\sigma)$ hold, $\tau \prec \sigma$, Y be a regular second countable topological space and $F : X \to Y$ be a compact valued multifunction.*

(1) If F is $wsq^u(\sigma,\tau)$-continuous and $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^l(\tau,\sigma)$-continuous), then F is $sq^l(\tau,\sigma)$-continuous, lower Baire τ-continuous, σ-lsc except for a set of τ-first category and τ-usc except for a set of τ-first category ($D^l_\sigma(F), D^u_\tau(F) \in \mathcal{I}_\tau$).

(2) *If F is $wsq^l(\sigma,\tau)$-continuous and $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous ($wsq^u(\tau,\sigma)$-continuous), then F is $sq^u(\tau,\sigma)$-continuous, upper Baire τ-continuous, σ-usc except for a set of τ-first category and τ-lsc except for a set of τ-first category ($D^u_\sigma(F), D^l_\tau(F) \in \mathcal{I}_\tau$).*

At first glance, by Lemma 2, if Y is regular second countable and F is compact valued $wsq^u(\tau,\sigma)$-continuous and $wsq^l(\tau,\sigma)$-continuous, then $D^l_\tau(F), D^u_\tau(F) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma$ and $D^l_\sigma(F), D^u_\sigma(F) \in \mathcal{I}_\sigma$, provided $\sigma \subset \tau$. In fact, the $wsq^u(\tau,\sigma)$-continuity and $wsq^l(\tau,\sigma)$-continuity yield far stronger results, as the next corollary shows. Example 4 shows, item (4) in the next corollary does not hold for one of the continuities $wsq^u(\tau,\sigma)$-continuity or $wsq^l(\tau,\sigma)$-continuity.

Corollary 10. *Let (X,τ,σ) be a bitopological space, $\sigma \subset \tau$, (X,τ) be τ-Baire and $F : X \to Y$ be a compact valued multifunction.*

(a) *If Y is regular, then the next conditions (1), (2), (3) are equivalent.*
- (1) *F is $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous and $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous,*
- (2) *F is $wsq^u(\tau,\sigma)$-continuous and $wsq^l(\tau,\sigma)$-continuous,*
- (3) *F is $sq^u(\tau,\sigma)$-continuous and $sq^l(\tau,\sigma)$-continuous.*

From any conditions (1), (2), (3) follows F is upper and lower quasi τ-continuous and upper and lower quasi σ-continuous.

(b) *Consider the next two conditions.*
- (4) *F is upper and lower quasi τ-continuous and $D^l_\sigma(F), D^u_\sigma(F) \in \mathcal{I}_\tau$,*
- (5) *F is upper and lower Baire τ-continuous and $D^l_\sigma(F), D^u_\sigma(F) \in \mathcal{I}_\tau$.*

If Y is regular second countable, then the conditions (1) - (5) are equivalent and from any condition (1) - (5) follows $D^l_\tau(F)$, $D^u_\tau(F)$, $D^l_\sigma(F)$, $D^u_\sigma(F) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma$.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear.

(1) $\Rightarrow$ (3):

Since F is $wsq^u(\tau,\sigma_{\mathcal{I}_\tau})$-continuous, F has the upper Denjoy (σ,τ)-property, by Remark 4(2). Then, by Theorem 10(1), F is $sq^l(\tau,\sigma)$-continuous.

Since F is $wsq^l(\tau,\sigma_{\mathcal{I}_\tau})$-continuous, F has the lower Denjoy (σ,τ)-property, by Remark 4(2). Then, by Theorem 10(2), F is $sq^u(\tau,\sigma)$-continuous.

Suppose Y is regular second countable and F is compact valued. By Lemma 2 and Theorem 5, $D^l_\tau(F)$, $D^u_\tau(F)$, $D^l_\sigma(F)$, $D^u_\sigma(F) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma$. Moreover, by Theorem 8 and Theorem 2, the conditions (3) and (4) are equivalent. Finally, by Theorem 1 and Theorem 2, (4) and (5) are equivalent. □

For a function, from Corollary 10 we have the next corollary that confirms some continuities for functions are equivalent, compare with Corollary 1. It also confirms that introducing of lower and upper continuities in Definition 2 for multifunctions yields to more diversified results.

Corollary 11. *([23]) Let (X,τ,σ) be a bitopological space, $\sigma \subset \tau$, (X,τ) be τ-Baire and $f : X \to Y$ be a function.*

(a) *If Y is regular, then the next conditions (1), (2), (3) are equivalent.*
- (1) *f is $wsq(\tau,\sigma_{\mathcal{I}_\tau})$-continuous,*
- (2) *f is $wsq(\tau,\sigma)$-continuous,*
- (3) *f is $sq(\tau,\sigma)$-continuous.*

From any conditions (1), (2), (3) follows f is quasi τ-continuous and quasi σ-continuous.

(b) *Consider the next two conditions.*
- (4) *f is quasi τ-continuous and $D_\sigma(f) \in \mathcal{I}_\tau$,*
- (5) *f is Baire τ-continuous and $D_\sigma(f) \in \mathcal{I}_\tau$.*

If Y is regular second countable, then the conditions (1) - (5) are equivalent and from any condition (1) - (5) follows $D_\tau(f), D_\sigma(f) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma$.

Note, in Corollary 11 (also in Corollary 10), the inclusion $\sigma \subset \tau$ can not be omitted, see the function f from Example 3. Also the τ-Baireness of (X, τ) is necessary and it does not apply locally, see [23].

Corollary 10 deals with the continuities with respect to (X, τ, σ). The next corollary combines the continuities with respect to (X, τ, σ) and its dual and the condition $\sigma \subset \tau$ is replaced by the conditions $B(\tau, \sigma)$, $\tau \prec \sigma$, $B(\sigma, \tau)$, $\sigma \prec \tau$.

Corollary 12. *Let (X, τ, σ) be a bitopological space, Y be a regular second countable topological space and $F : X \to Y$ be a compact valued multifunction. Suppose all conditions $B(\tau, \sigma)$, $\tau \prec \sigma$, $B(\sigma, \tau)$, $\sigma \prec \tau$ hold. Then, in (a), (b) the conditions (1), (2), (3) are equivalent, respectively.*

(a) (1) *F is $wsq^u(\sigma, \tau)$-continuous and $wsq^l(\tau, \sigma)$-continuous,*
(2) *F is $sq^u(\sigma, \tau)$-continuous and $sq^l(\tau, \sigma)$-continuous,*
(3) *F is upper σ-Baire continuous, $D^u_\tau(F) \in \mathcal{I}_\sigma$, lower τ-Baire continuous, $D^l_\sigma(F) \in \mathcal{I}_\tau$.*

(b) (1) *F is $wsq^l(\sigma, \tau)$-continuous and $wsq^u(\tau, \sigma)$-continuous,*
(2) *F is $sq^l(\sigma, \tau)$-continuous and $sq^u(\tau, \sigma)$-continuous,*
(3) *F is lower σ-Baire continuous, $D^l_\tau(F) \in \mathcal{I}_\sigma$, upper τ-Baire continuous, $D^u_\sigma(F) \in \mathcal{I}_\tau$.*

(c) *If F is $wsq^u(\sigma, \tau)$-continuous and $wsq^l(\tau, \sigma)$-continuous, then $D^l_\sigma(F)$, $D^u_\tau(F) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma$.*
If F is $wsq^l(\sigma, \tau)$-continuous and $wsq^u(\tau, \sigma)$-continuous, then $D^u_\sigma(F)$, $D^l_\tau(F) \in \mathcal{I}_\tau \cap \mathcal{I}_\sigma$.

Proof. (a) (1) $\Rightarrow$ (2) : If F is $wsq^u(\sigma, \tau)$-continuous and $wsq^l(\tau, \sigma)$-continuous, then F is $sq^l(\tau, \sigma)$-continuous (by Theorem 10(1) for $B(\tau, \sigma)$, $\tau \prec \sigma$) and $sq^u(\sigma, \tau)$-continuous (by Theorem 10(2) for $B(\sigma, \tau)$, $\sigma \prec \tau$).
(2) $\Rightarrow$ (1) is trivial.
(2) $\Leftrightarrow$ (3) follows from Theorem 8.

(b) is similar.

(c) follows from items (a), (b) and Corollary 3. □

5. Conclusions, Summary of Results and Symbolic Interpretation

The problems of bitopological spaces bring many combinations how to generalize some types of sets and continuities. For example, a β-open set A, which is defined by inclusion $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$, can be defined in a bitopological space by three ways: $A \subset \mathrm{cl}_\tau(\mathrm{int}_\sigma(\mathrm{cl}_\tau(A)))$, $A \subset \mathrm{cl}_\sigma(\mathrm{int}_\sigma(\mathrm{cl}_\tau(A)))$, $A \subset \mathrm{cl}_\tau(\mathrm{int}_\sigma(\mathrm{cl}_\sigma(A)))$ and tree ones in a dual bitopological space. Similar problems arise in the case of generalized continuities defined in a bitopological space. Moreover, the situation is complicated by two variants of upper and lower continuity. The reader may feel that even in our article there are many results and complicated connections. In Definition 2 there are twelve continuities (six with respect to (X, τ, σ) and six with respect to (X, σ, τ)). Theoretically, studding the sets of discontinuity points $D^l_\tau(F)$, $D^u_\tau(F)$, $D^l_\sigma(F)$, $D^u_\sigma(F)$ of these twelve continuities leads to 48 results. If we assume two different continuities from twelve (66 pairs) we have 264 results (of course, some of them are duplicated and trivial). Also, the study of the relationships between continuities leads to many combinations. In general, the number of questions increases rapidly and the number of results is enormous. Building upon the results of this paper, the following avenues present directions for further work.

1. To transform bitopological notions into concepts in terms of one topology. For example, similar to Theorem 7 and Theorem 8, to find a characterization of other continuities (namely, the upper/lower $q(\tau, \sigma)$-continuity and the upper/lower $wsq(\tau, \sigma)$-continuity) by suitable continuities dependent on one topology.

2. To generalize the results of the work to a space (X, τ, σ) in which one topology is replaced by a more general structure, for example, by a generalized topology, a soft topology, a fuzzy topology or a cluster system, see [21].

3. To look for a selection of multifunction with suitable continuity properties. For example, if F is $wsq^u(\tau, \sigma)$-continuiou, is there a function f that is quasi τ-continuous, $D_\sigma(f) \in \mathcal{I}_\tau$ and $f(x) \in F(x)$ for any $x \in X$?

Symbolic interpretation

Let's try to summarize the results and to find a simple and comprehensive view how to get such large number of results by simple rules. We present a series of relevant implications and a symbolic interpretation of the sets of discontinuity points and the relations between twelve studied continuities.

To better understand the symbolic interpretation below, it is useful to make a few notes. Let

$$\bullet, \circ \in \{l, u\}, \bullet \neq \circ$$

$$\boxtimes, \square \in \{\tau, \sigma\}, \boxtimes \neq \square$$

Notation

$(X, \boxtimes, \square)$	- bitopological space, $\boxtimes$ is the dominant topology: (X, τ, σ), (X, σ, τ)
$B(\boxtimes, \square)$	- Baireness: $B(\tau, \sigma)$, $B(\sigma, \tau)$
$\boxtimes \prec \square$	- $\boxtimes$ pseudo refines $\square$: $\tau \prec \sigma, \sigma \prec \tau$
$\mathcal{I}_{\boxtimes}$	- the family of $\boxtimes$-first category sets: $\mathcal{I}_{\tau}$, $\mathcal{I}_{\sigma}$
$q^{\bullet}(\boxtimes, \square)$	- quasi-continuity: $q^u(\tau, \sigma), q^l(\tau, \sigma), q^u(\sigma, \tau), q^l(\sigma, \tau)$
$sq^{\bullet}(\boxtimes, \square)$	- sectional quasi-continuity: $sq^u(\tau, \sigma), sq^l(\tau, \sigma), sq^u(\sigma, \tau), sq^l(\sigma, \tau)$
$sq(\boxtimes, \square)$	- upper and lower sectional quasi-continuity: $sq(\tau, \sigma), sq(\sigma, \tau)$
$wsq^{\bullet}(\boxtimes, \square)$	- weak sectional quasi-continuity: $wsq^u(\tau, \sigma), wsq^l(\tau, \sigma), wsq^u(\sigma, \tau), wsq^l(\sigma, \tau)$
$wsq(\boxtimes, \square)$	- upper and lover weak sectional quasi-continuity: $wsq(\tau, \sigma), wsq(\sigma, \tau)$
$wsq^{\bullet}(\boxtimes, \square_{\mathcal{I}_{\boxtimes}})$	- weak sectional quasi-continuity: $wsq^u(\tau, \sigma_{\mathcal{I}_{\tau}}), wsq^l(\tau, \sigma_{\mathcal{I}_{\tau}}), wsq^u(\sigma, \tau_{\mathcal{I}_{\sigma}}), wsq^l(\sigma, \tau_{\mathcal{I}_{\sigma}})$
$d^{\bullet}(\boxtimes, \square)$	- Denjoy property: $d^u(\tau, \sigma), d^l(\tau, \sigma), d^u(\sigma, \tau), d^l(\sigma, \tau)$
$D^{\bullet}_{\boxtimes}(F)$	- the set of semi discontinuity points of F: $D^u_{\tau}(F), D^l_{\tau}(F), D^u_{\sigma}(F), D^l_{\sigma}(F)$
$Q^{\bullet}_{\boxtimes}(F)$	- the set of quasi continuity points of F: $Q^u_{\tau}(F), Q^l_{\tau}(F), Q^u_{\sigma}(F), Q^l_{\sigma}(F)$
$B^{\bullet}_{\boxtimes}(F)$	- the set of Baire continuity points of F: $B^u_{\tau}(F), B^l_{\tau}(F), B^u_{\sigma}(F), B^l_{\sigma}(F)$

If $B(\boxtimes, \square)$, $\boxtimes \prec \square$, then the first topology is called a dominant topology. This topology often determines how the topologies $\boxtimes, \square$ and the versions of continuities $\bullet, \circ$ from the assumptions are transferred to the resulting continuities and the sets of discontinuity points. The dominant topology is usually preserved. We can also observe changes of $\bullet$ and $\circ$.

For example: If $B(\boxtimes, \square)$, $\boxtimes \prec \square$ ($\boxtimes$ is the dominant topology), then

$$wsq^{\bullet}(\boxtimes, \square) \rightarrow \bullet \text{ Baire } \boxtimes\text{-continuity } (B^{\bullet}_{\boxtimes}(F) = X), D^{\circ}_{\boxtimes}(F) \in \mathcal{I}_{\boxtimes} \tag{C.7}$$

$$wsq^{\bullet}(\boxtimes, \square) + wsq^{\circ}(\boxtimes, \square) \rightarrow Q^{\bullet}_{\boxtimes}(F) = Q^{\circ}_{\boxtimes}(F) = X. \tag{C.7}$$

Sometimes we can use "a cansel rule".

If $B(\boxtimes, \square)$, $\boxtimes \prec \square$, then $\boxtimes$ is the dominant topology and

$$\cancel{w}sq^{\bullet}(\boxtimes, \square) + wsq^{\circ}(\square, \boxtimes) \rightarrow sq^{\bullet}(\boxtimes, \square). \tag{T.10}$$

On the other hand if $B(\square, \boxtimes)$, $\square \prec \boxtimes$, then $\square$ is the dominant topology and

$$wsq^{\bullet}(\boxtimes, \square) + \cancel{w}sq^{\circ}(\square, \boxtimes) \rightarrow sq^{\circ}(\square, \boxtimes). \tag{T.10}$$

If $B(\boxtimes, \square)$, $\boxtimes \prec \square$, $B(\square, \boxtimes)$, $\square \prec \boxtimes$ (both topologies are dominant), then

$$wsq^{\bullet}(\boxtimes, \square) + wsq^{\circ}(\square, \boxtimes) \rightarrow D^{\circ}_{\boxtimes}(F), D^{\bullet}_{\square}(F) \in \mathcal{I}_{\boxtimes} \cap \mathcal{I}_{\square} \tag{C.12}$$

$$wsq^{\bullet}(\boxtimes, \square) + wsq^{\circ}(\square, \boxtimes) \leftrightarrow sq^{\bullet}(\boxtimes, \square) \text{ and } sq^{\circ}(\square, \boxtimes). \tag{C.12}$$

There is one exceptional case when the dominant topology is not preserved.

$$sq^{\bullet}(\boxtimes, \square) \rightarrow D^{\bullet}_{\square}(F) \in \mathcal{I}_{\boxtimes}.$$

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Let us describe the content of the following diagrams. The most comprehensive of them is **Diagram A** and the others (**Diagram B1, B2, C1, C2**) visually better express the relationships between two continuities and their consequences for the structure of the sets of discontinuity points.

Under conditions $B(\boxtimes,\square)$, $\boxtimes \prec \square$, **Diagram A** below summarizes the series of implications from the $q^{\bullet}(\boxtimes,\square)$-continuity to the condition $D^{\circ}_{\boxtimes} \in \mathcal{I}_{\boxtimes}$ with respect to $(X,\boxtimes,\square)$ and further from the $sq^{\circ}(\square,\boxtimes)$-continuity on a $\boxtimes$-residual set to the $wsq^{\circ}(\square,\boxtimes_{\mathcal{I}_{\square}})$-continuity on a $\boxtimes$-residual set and the $d^{\circ}(\square,\boxtimes)$-property on a $\boxtimes$-residual set with respect to $(X,\square,\boxtimes)$.

It contains three characterizations of the $sq^{\bullet}(\boxtimes,\square)$-continuity and the equivalence between the $\bullet$ Baire $\boxtimes$-continuity ($B^{\bullet}_{\boxtimes}(F) = X$) and the $\bullet$ quasi $\boxtimes$-continuity ($Q^{\bullet}_{\boxtimes}(F) = X$), provided X is finite (the conditions $B(\boxtimes,\square)$, $\boxtimes \prec \square$ can be omitted). Also, the equivalences between two continuities with the same topological order and the different versions of continuities are given, provided $\square \subset \boxtimes$, **Diagram A1**.

Moreover, in **Diagram A** we can see the combinations of two continuities and their consequences. Namely, two continuities with the different topological order and the different versions of continuities, namely the implications starting with

$$wsq^{\bullet}(\boxtimes,\square) + wsq^{\circ}(\square_{\mathcal{I}_{\boxtimes}},\boxtimes) \to \cdots .$$

In **Diagram A2** and **Diagram A3** we can see two continuities with the same topological order and the different versions of continuities, namely the implications starting with

$$wsq^{\bullet}(\boxtimes,\square) + wsq^{\circ}(\boxtimes,\square) \to \cdots \quad \text{and} \quad sq^{\bullet}(\boxtimes,\square) + sq^{\circ}(\boxtimes,\square) \to \cdots$$

It is useful to mention that the first topology $\boxtimes$ in the triplet $(X,\boxtimes,\square)$ (called a dominant topology) determines the continuity properties of F and the dominant topology is preserved. For example (see (**Diagram A**, **Diagram A2**)

$$q^{\bullet}(\boxtimes,\square) \to sq^{\bullet}(\boxtimes,\square) \to wsq^{\bullet}(\boxtimes,\square) \to wsq^{\bullet}(\boxtimes,\square_{\mathcal{I}_{\boxtimes}}) \to B^{\bullet}_{\boxtimes}(F) \to D^{\circ}_{\boxtimes}(F) \in \mathcal{I}_{\boxtimes}$$

$$wsq^{\bullet}(\boxtimes,\square) + wsq^{\circ}(\boxtimes,\square) \to \cdots \to Q^{\bullet}_{\boxtimes}(F) = Q^{\circ}_{\boxtimes}(F) = X,\ D^{\circ}_{\boxtimes}(F), D^{\bullet}_{\boxtimes}(F) \in \mathcal{I}_{\boxtimes}$$

Diagram B1 deals with two weak sectional quasi-continuities under conditions $B(\boxtimes,\square)$, $\boxtimes \prec \square$ / $B(\square,\boxtimes)$, $\square \prec \boxtimes$. Namely, two continuities with the different topological order and the different versions of continuities

$$wsq^{\bullet}(\boxtimes,\square) + wsq^{\circ}(\square,\boxtimes),\ wsq^{\circ}(\boxtimes,\square) + wsq^{\bullet}(\square,\boxtimes)$$

and with the same topological order and the different versions of continuities

$$wsq^{\bullet}(\boxtimes,\square) + wsq^{\circ}(\boxtimes,\square).$$

Diagram B2 deals with the same continuities under all conditions $B(\boxtimes,\square)$, $\boxtimes \prec \square$, $B(\square,\boxtimes)$, $\square \prec \boxtimes$.

In **Diagram C1** and **Diagram C2** we can see the consequences of two sectional quasi continuities under conditions $B(\boxtimes,\square)$, $\boxtimes \prec \square$ / $B(\square,\boxtimes)$, $\square \prec \boxtimes$ and under all conditions $B(\boxtimes,\square)$, $\boxtimes \prec \square$, $B(\square,\boxtimes)$, $\square \prec \boxtimes$.

Finally, we conclude with an example of investigating the set of discontinuity points of multifunction.

To state all relevant results in a compact form, in the following diagrams we assume: Y is a regular second countable topological space and $F : X \to Y$ is a compact valued multifunction (R. = Remark, T. = Theorem, C. = Corollary).

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Diagram A $B(\boxtimes,\square),\ \boxtimes \prec \square$

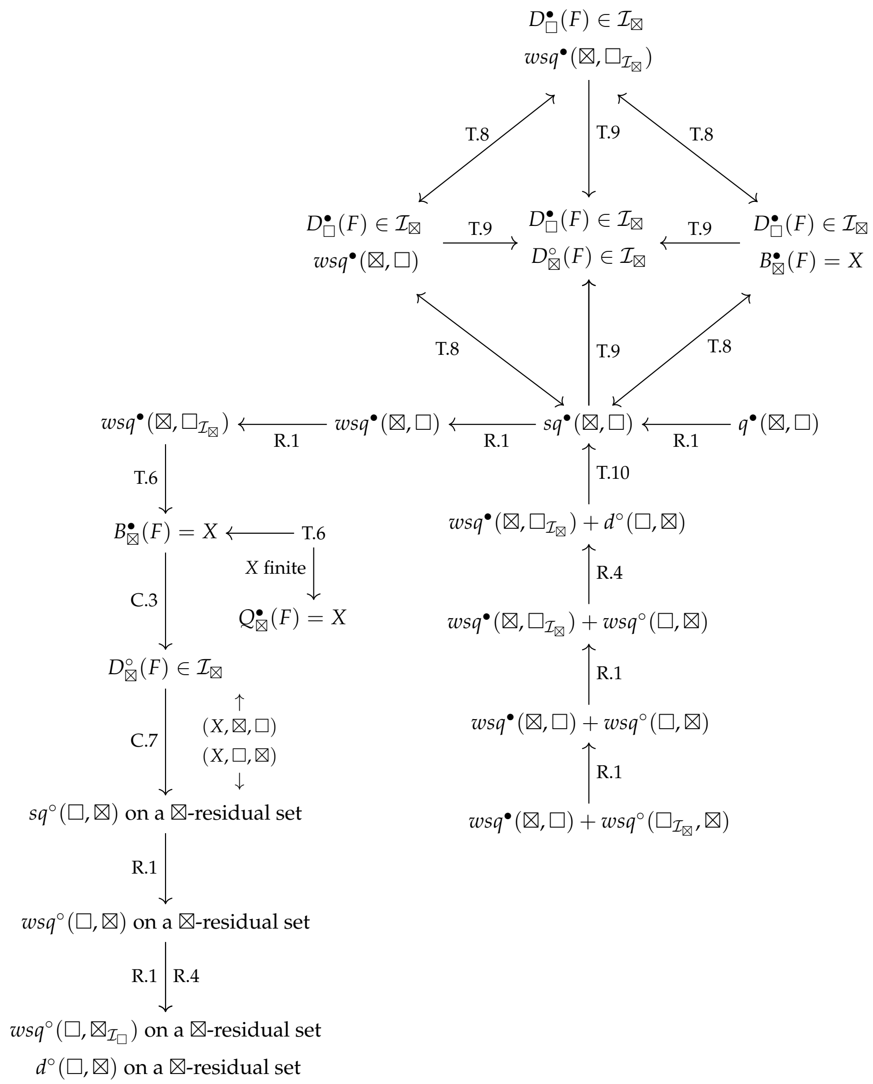

Diagram A1 $\square \subset \boxtimes$

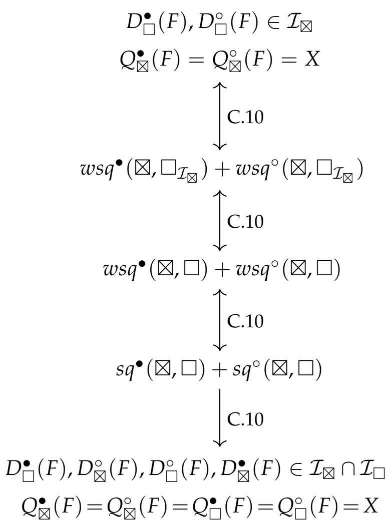

Diagram A2 $B(\boxtimes,\square),\ \boxtimes \prec \square$

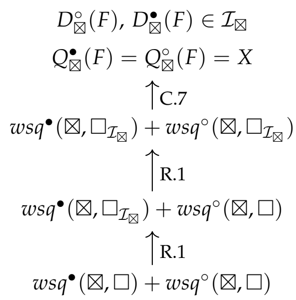

Diagram A3 $B(\boxtimes,\square),\ \boxtimes \prec \square$

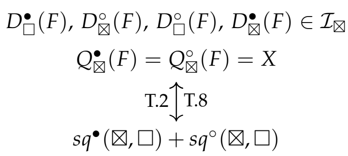

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Diagram B1

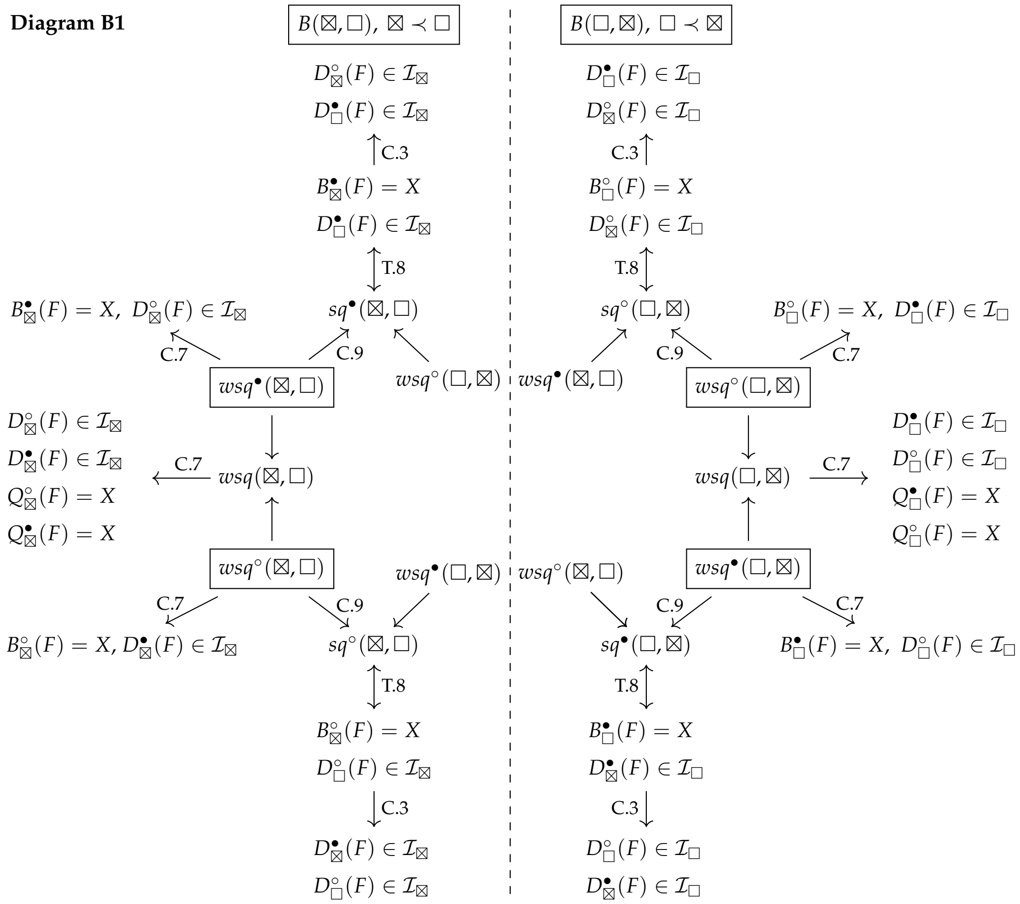


Diagram B2

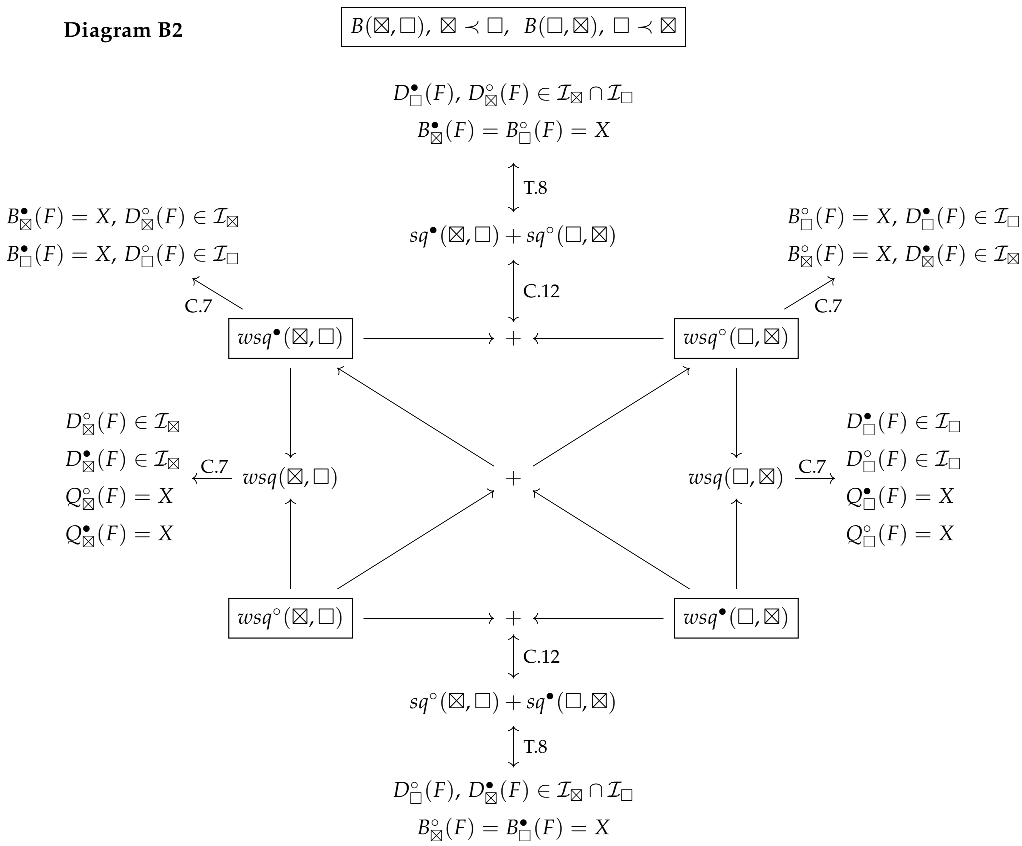


Diagram C1

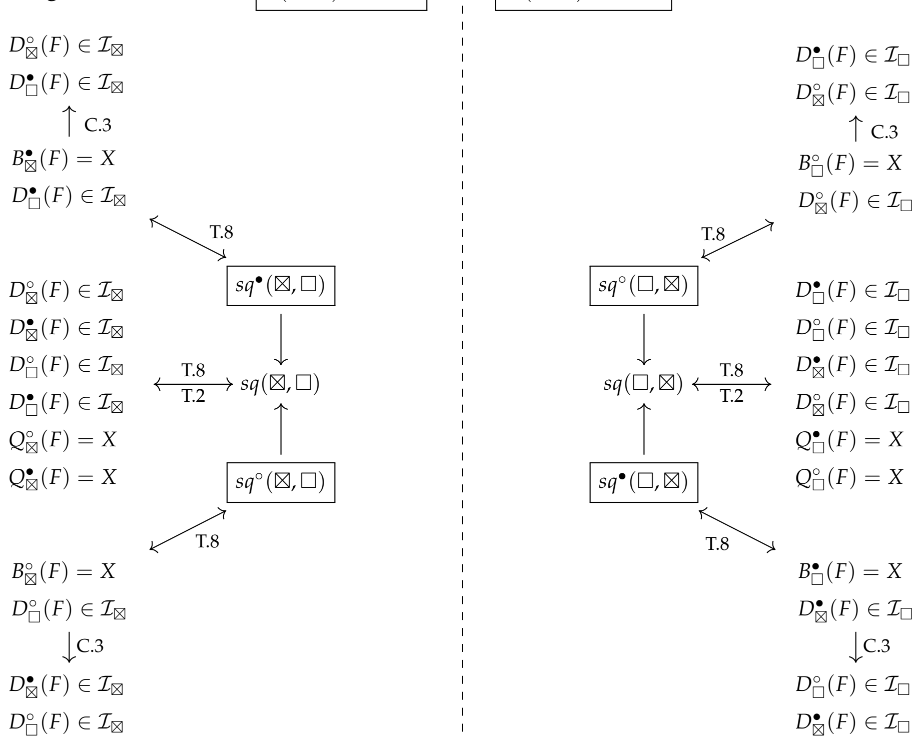
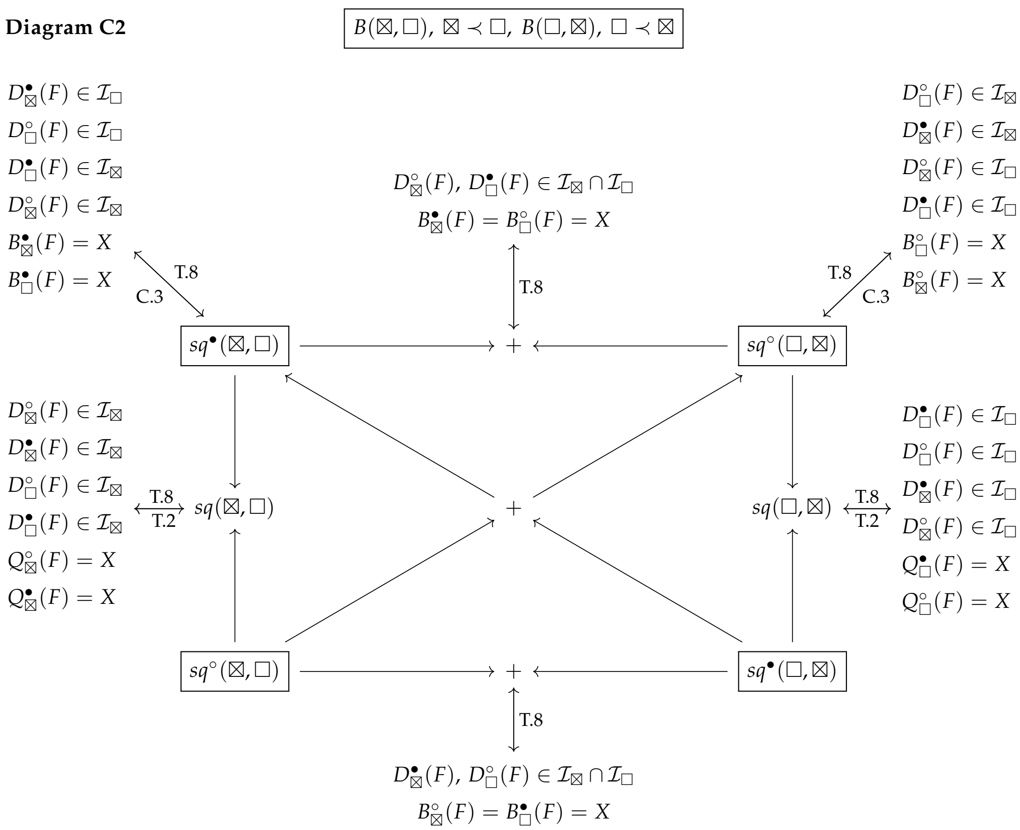


Diagram C2



Example 9. Find the structure of discontinuity points of a compact valued multifunction F that is $wsq^l(\sigma,\tau)$-, $wsq^l(\tau,\sigma)$-, $wsq^u(\tau,\sigma)$-, and $sq^l(\tau,\sigma)$-continuous, provided

a) $B(\tau,\sigma), \tau \prec \sigma$,
b) $B(\sigma,\tau), \sigma \prec \tau$,
c) $B(\tau,\sigma), \tau \prec \sigma$, $B(\sigma,\tau), \sigma \prec \tau$.

We will proceed strictly according to the rules of the symbolic interpretation even though some of them are duplicated.

a) $B(\tau,\sigma), \tau \prec \sigma$
By Symbolic interpretation,
1. $wsq^l(\tau,\sigma) \Rightarrow D^u_\tau(F) \in \mathcal{I}_\tau$,
2. $wsq^u(\tau,\sigma) \Rightarrow D^l_\tau(F) \in \mathcal{I}_\tau$,
3. $sq^l(\tau,\sigma) \Rightarrow D^u_\tau(F) \in \mathcal{I}_\tau, D^l_\sigma(F) \in \mathcal{I}_\tau$,
4. $wsq^u(\tau,\sigma)$ and $wsq^l(\sigma,\tau) \Rightarrow sq^u(\tau,\sigma) \Rightarrow D^l_\tau(F) \in \mathcal{I}_\tau$, $D^u_\sigma(F) \in \mathcal{I}_\tau$.
So, $D^u_\tau(F), D^l_\tau(F), D^u_\sigma(F), D^l_\sigma(F) \in \mathcal{I}_\tau$.
Moreover, $wsq^l(\tau,\sigma)$ and $wsq^u(\tau,\sigma) \Rightarrow F$ is upper and lower quasi τ-continuous.

b) $B(\sigma,\tau), \sigma \prec \tau$
By Symbolic interpretation,
1. $wsq^l(\sigma,\tau) \Rightarrow D^u_\sigma(F) \in \mathcal{I}_\sigma$.
2. $wsq^l(\sigma,\tau)$ and $wsq^u(\tau,\sigma) \Rightarrow sq^l(\sigma,\tau) \Rightarrow D^u_\sigma(F) \in \mathcal{I}_\sigma, D^l_\tau(F) \in \mathcal{I}_\sigma$.
So, $D^u_\sigma(F), D^l_\tau(F) \in \mathcal{I}_\sigma$.
Moreover, $wsq^l(\sigma,\tau) \Rightarrow F$ is lower Baire σ-continuous.

c) $B(\tau,\sigma), \tau \prec \sigma$, $B(\sigma,\tau), \sigma \prec \tau$
By a) and b), $D^u_\sigma(F), D^l_\tau(F) \in \mathcal{I}_\sigma \cap \mathcal{I}_\tau$ and $D^u_\tau(F), D^l_\sigma(F) \in \mathcal{I}_\tau$.

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