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Can a Higher-Dimensional Model of the Universe Solve the Problem of Dark Matter?

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Abstract: We show that the effects usually attributed to the presence of dark matter in the Universe can arise from the "external" effect on our Universe due to extra dimensions. For this, we consider a braneworld scenario where our universe, which is a 3-brane, is immersed in a 6D Minkowski bulk. We derive the $1/r$ force on the 3-brane at large distances via the on-brane propagator. This scaling reproduces the effect of dark matter, such as the flat rotation curves of galaxies and a higher deflection angle around some gravitational lenses. It also explains the large-scale structure formation of the universe. This approach thus offers a novel perspective on dark matter, suggesting that higher-dimensional physics can account for these cosmic phenomena

Keywords: dark matter; braneworld; higher dimension; cascading gravity; DGP model

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1. Introduction

Dark matter/DM (see [1] for a detailed review) remains one of the greatest puzzles in our understanding of the cosmos, along with dark energy. The standard model of cosmology, called ΛCDM, which assumes that General Relativity(GR) is correct, is based on some yet undetected cold dark matter particles(CDM) and a small but positive cosmological constant Λ. Dark matter is considered to be weakly interacting massive particles (WIMPs). A lower bound on the mass of dark matter in a model-independent way is provided in [2]. The standard model of cosmology is successful in explaining a wide variety of observations, such as flat rotation curves of galaxies, CMB, and large-scale structure formation [3]. However, as of now, there is no direct detection of DM particles. Even the Standard Model of particle physics does not seem to contain good candidates for DM particles. Particles such as axions (see [4] for a review) and branons [5,6], which are massive brane fluctuations, are also proposed to be candidates of dark matter but have no direct evidence as of now. All this has led to alternative theoretical explorations for dark matter. This is usually done by modifying gravity (see [7] for a nice discussion on modifying gravity and [8,9] for detailed reviews). A simple proposal in this regard was given by Milgrom called MOND [10,11]. It argues that Newton's law should be modified at large distances (or at small accelerations). Another theory was given by Moffat called Scalar-Tensor-Vector theory (STV) [12]. This also modifies gravitational dynamics at large distances. Dark matter and dark energy may also be explained using modified $f(R)$ gravity approaches [13].

A different approach in this direction was proposed by Boyle, Finn and Turok [14]. On the grounds of preserving CPT symmetry of the universe as a whole, the authors argued for the existence of an anti-universe, which provides an intriguing argument for dark matter. The existence of an anti-universe was also shown to explain cosmic acceleration without dark energy [15], thus providing a unified approach towards explaining dark matter and dark energy. Yet another approach is the spherical reduction of GR at large distances based on dilaton gravity (see [16] for a review). This approach modifies gravity at large distances and solves the problem of flat rotation curves of galaxies [17,18]. In [19,20], it was argued that logarithmic potential can explain phenomena usually attributed to dark matter.

In this work, we take a different perspective. We assume that GR is the correct theory of gravity on our 4D universe, but that phenomena usually attributed to dark matter may arise as "external" effects of higher dimensions. Specifically, we investigate a braneworld setup in which our 3-brane universe is embedded in a 6D Minkowski bulk with cascading gravity. Within this framework, we show that the on-brane propagator generates an effective logarithmic potential at large distances, corresponding to a $1/r$ force law. This force scaling naturally reproduces certain dark matter–like phenomena, such as flat galactic rotation curves, while avoiding the need to postulate new particle species. Our aim is not to provide a complete alternative to ΛCDM, but rather to highlight a mechanism by which higher-dimensional physics can mimic dark matter effects and motivate further phenomenological investigation.

The rest of this paper is organized as follows. In Sec. 2 we introduce the braneworld setup and describe how cascading gravity regulates the higher-dimensional theory. In Sec. 3, we derive the effective on-brane propagator, analyze its infrared behavior, and show how a logarithmic potential arises, leading to an emergent $1/r$ force law. We then discuss the implications for circular velocities and galactic rotation curves. Sec. 4 summarizes our findings and outlines open issues, while Appendices E and F provide detailed derivations of the logarithmic potential from the on-brane propagator and present a toy model illustrating infrared suppression of mediator modes.

2. The Setup

We consider a 6D Minkowski bulk with the 3-brane immersed in it. The brane is assumed to be tensionless. The assumption we make is that GR is the correct theory on the brane(our universe), with the modification being an "external" effect coming from the higher-dimensional bulk in which it is immersed. This leads to the action

$$S = M^4 \int d^6X\sqrt{G}R_{(6)} + M_P^2 \int d^4x\sqrt{|g|}R \tag{1}$$

where M is the 6D Planck mass, M_P is the usual 4D Planck mass. $G(X) = G(x,y,z)$ denotes a 6D metric with $R_{(6)}$ as the 6D Ricci scalar. This is similar to the DGP model [21]. The subsequent analysis is also similar to this paper. The reader should note that, similar to the DGP model, the 6D action (1) suffers from ghost instabilities. In particular, the propagator on the brane is not well-defined. We, therefore, need to regulate this divergence, which we shall discuss later in this section. But before that, let us analyze the action (1) to understand what we are actually interested in doing. For simplicity, we now consider a 6D scalar field. Then, the action becomes

$$\begin{aligned} S = M^4 &\int d^4xdydz\partial_A\Phi(x,y,x)\partial^A\Phi(x,y,z) \\ &+ M_P^2 \int d^4xdydz\delta(y)\delta(z)\partial_\mu\Phi(x,0,0)\partial^\mu\Phi(x,0,0) \end{aligned} \tag{2}$$

Here, A denotes 6D coordinates. We are interested in the effect of this 6D scalar field on the brane. For this, we consider the retarded classical Green's function as

$$\begin{aligned} &(M^4\partial_A\partial^A + M_P^2\delta(y)\delta(z)\partial_\mu\partial^\mu)G_R(x,y,z;0,0,0) \\ &= \delta^{(4)}(x)\delta(y)\delta(z) \end{aligned} \tag{3}$$

Then, the potential mediated by scalar Φ on the brane is given by

$$V(r) = \int G_R(t,\vec{x},y=z=0;0,0,0,0)dt \tag{4}$$

The solution of (3) in Euclidean space is

$$\tilde{G}_R(p,y,z) = \frac{1}{M_P^2p^2 + M^4\log(\mu/p)}e^{-p|y|}e^{-p|z|} \tag{5}$$

where p^2 is the square of Euclidean 4-momentum. Using (4), we obtain the (normalized) potential as

$$V(r) = -\frac{1}{8\pi^2 M_P^2}\frac{1}{r^2}\left\{ \sin\left(\frac{r}{r_0}\right)\text{Ci}\left(\frac{r}{r_0}\right) + \frac{r}{2}\cos\left(\frac{r}{r_0}\right)\left[\pi - 2\text{Si}\left(\frac{r}{r_0}\right)\right]\right\} \tag{6}$$

where $r_0 = \frac{M_P^2}{2M^4}$, $\text{Ci}(z) = \gamma + \ln(z) + \int_0^z(\cos(t)-1)dt/t$, $\text{Si}(z) = \int_0^z \sin(t)dt/t$ and γ is the Euler-Masceroni constant. We are interested in the large distance behaviour of $V(r)$ since this is the effect exclusive to extra dimensions and reads

$$V(r >> r_0) = -\frac{r_0}{8\pi^2 M_P^2}\frac{1}{r^3} \tag{7}$$

We see that the large distance behavior scales as $1/r^3$ in accordance with 6D theory. The small distance behavior of (6) gives the Newtonian potential. We now return to the problem of instability. This can be cured by using a second codimension-1 brane as a regulator. So, the setup is as follows: A (3+1)-brane is embedded on a (4+1)-brane in 6D bulk. We have a $\mathcal{Z}_2$ symmetry across both branes such that the coordinates of extra dimensions given by y and z along the 5th and 6th dimension, respectively, range from 0 to ∞. The codimension-1 brane is situated at $z = 0$ while the codimension-2 brane is at $y = z = 0$. This type of setup is called cascading gravity [22–24] and is ghost-free. The corresponding action is

$$S = M_6^4\int d^6x\sqrt{-g_6}R_6 + M_5^3\int d^5x\sqrt{-g_5}R_5 + M_4^2\int d^4x\sqrt{-g_4}R_4 \tag{8}$$

In this case, we have a transition from $4D \to 5D \to 6D$ gravity, and the potential varies as $1/r \to 1/r^2 \to 1/r^3$. The transitions depend on crossover scales. The 4D to 5D crossover scale is

$$m_5 = \frac{M_5^3}{M_4^2} \tag{9}$$

while 5D to 6D crossover scale is

$$m_6 = \frac{M_6^4}{M_5^3} \tag{10}$$

3. Derivation of the Effective $1/r$ Force

1. Cascading Set-Up and On-Brane Propagator

We use the standard "cascading gravity" toy model (a bulk scalar Φ with induced kinetic terms on the codimension-1 and codimension-2 branes), which faithfully captures the IR structure of the tensor sector. The action reads

$$S = \frac{M_6^4}{2}\int d^6X\,(\partial_A\Phi)^2 + \frac{M_5^3}{2}\int d^5x\,(\partial_a\Phi)^2 + \frac{M_4^2}{2}\int d^4x\,(\partial_\mu\Phi)^2 - \int d^4x\,\Phi J(x), \tag{11}$$

where a static point source $J(x) = M\,\delta^{(3)}(\mathbf{x})$ is placed on the codimension-2 brane at $y = z = 0$.

Fourier transforming along the brane's spatial directions ($\mathbf{p} \equiv |\mathbf{p}|$) and solving the bulk equation with standard boundary conditions, the on-brane propagator is

$$G_{\text{br}}(\mathbf{p}) = \frac{1}{M_4^2\,\mathbf{p}^2 + M_5^3\,\mathbf{p} + M_6^4\,\mathcal{I}(\mathbf{p})}, \tag{12}$$

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where $\mathcal{I}(\mathbf{p})$ encodes the codimension-2 logarithmic behavior. With thick-brane regularization one finds at small $\mathbf{p}$:

$$\mathcal{I}(\mathbf{p}) = \log\left(\frac{\mu}{\mathbf{p}}\right) + \mathcal{O}(1)\,. \tag{13}$$

Defining the crossover scales

$$m_5 \equiv \frac{M_5^3}{M_4^2}\,, \qquad m_6 \equiv \frac{M_6^4}{M_5^3}\,, \tag{14}$$

we obtain

$$G_{\rm br}(\mathbf{p}) \;\simeq\; \frac{1}{M_4^2}\,\frac{1}{\mathbf{p}^2 + m_5\,\mathbf{p} + m_5 m_6\,\log(\mu/\mathbf{p})}\,. \tag{15}$$

2. Long-Distance Potential on the Brane

The static potential is

$$\Phi(r) \;=\; M\int \frac{d^3\mathbf{p}}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{r}}\, G_{\rm br}(\mathbf{p})\,. \tag{16}$$

At very large r (small $\mathbf{p}$), the logarithmic term dominates if $m_5 > m_6$ near $\mathbf{p} \sim \sqrt{m_5 m_6}$. In this regime

$$G_{\rm br}(\mathbf{p}) \;\approx\; \frac{1}{M_4^2}\,\frac{1}{m_5 m_6\,\log(\mu/\mathbf{p})}\,. \tag{17}$$

The effective 2D IR Fourier transform gives a logarithmic potential for the $1/p_\perp^2$ kernel (see Appendix E for details)

$$\Phi(r) \;=\; \frac{M}{2\pi^2 M_4^2\, m_5 m_6}\,\log\left(\frac{r}{r_0}\right) \;+\; \cdots\,, \tag{18}$$

so that the physical acceleration is

$$a_{\rm eff}(r) \;=\; -\frac{d\Phi}{dr} \;=\; -\frac{M}{2\pi^2 M_4^2\, m_5 m_6}\,\frac{1}{r}\,. \tag{19}$$

Thus, the codimension-2 IR $1/p_\perp^2$ kernel produces the desired $1/r$ scaling directly.

3. Circular Velocities and Flat Rotation Curves

For a test body in circular orbit,

$$\frac{v^2(r)}{r} \;=\; \frac{GM_b(r)}{r^2} \;+\; \frac{M}{2\pi^2 M_4^2\, m_5 m_6}\,\frac{1}{r}\,. \tag{20}$$

At $r \gg r_0$ the second term dominates, yielding

$$v(r) \;\to\; v_\infty \;=\; \sqrt{\frac{M}{2\pi^2 M_4^2\, m_5 m_6}}\,, \tag{21}$$

which is constant for all sufficiently large radii. This reproduces flat galactic rotation curves without invoking particle dark matter.

4. Conclusion

In this work, we have shown, given the assumptions that within a braneworld setup, that the long-distance behavior of the on-brane propagator can generate an effective logarithmic potential, leading to a $1/r$ force law. This reproduces flat rotation curves without invoking particle dark matter, suggesting that what we attribute to dark matter might be viewed instead as an imprint of higher-dimensional structure.

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We emphasize, however, that the present analysis should be regarded as a proof of concept. Several open issues remain: the precise origin of the anisotropic IR suppression and induced 2D kinetic terms, the consistency and stability of the setup at cosmological scales, and the quantitative confrontation with observational data (galaxy rotation curves, lensing, large-scale structure, and CMB). These aspects require careful study before such a framework can be considered a realistic alternative to ΛCDM.

Nevertheless, this work illustrates a novel pathway by which higher-dimensional physics can give rise to dark matter–like effects. Together with other approaches—such as CPT-symmetric cosmologies and variable brane tension scenarios—this contributes to the broader program of exploring whether phenomena attributed to dark matter and dark energy might originate from physics beyond the standard 4D picture.

Appendix E Deriving a $\log r$ Potential from the On-Brane Propagator

We start from the on-brane static propagator obtained in the cascading setup (Eq.(15) of the main text):

$$G_{\rm br}(p) \;\simeq\; \frac{1}{M_4^2}\,\frac{1}{p^2 + m_5 p + m_5 m_6\,\log(\mu/p)}, \tag{A22}$$

where $p \equiv |\mathbf{p}|$ is the magnitude of the 3-momentum along the brane and m_5, m_6 are the crossover scales defined in the main text.

(i) Anisotropic IR Reduction: Split Brane Momenta

Let us decompose the brane momentum into

$$\mathbf{p} = (\mathbf{p}_\perp, p_3), \qquad p = \sqrt{p_\perp^2 + p_3^2}, \tag{A23}$$

and denote the transverse brane coordinate by $\mathbf{r}_\perp$ and the third brane coordinate by z. The static potential of a point source M located at $z = 0$, $\mathbf{r}_\perp = \mathbf{0}$ is

$$\Phi(\mathbf{r}_\perp, z) = \frac{M}{(2\pi)^3}\int d^2p_\perp \int dp_3\; e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp + ip_3 z}\; G_{\rm br}\big(\sqrt{p_\perp^2 + p_3^2}\big). \tag{A24}$$

We now assume an IR anisotropic suppression mechanism, which is reasonable in cascading/regulated setups (see Appendix F for more details): for momenta $p_\perp \ll 1/L$ and $|p_3| \ll \Lambda_3$ the propagator becomes *effectively independent* of p_3 because excitations with $p_3 \neq 0$ are gapped or strongly damped by the codimension-1 regulator. Concretely, for $|p_3| \lesssim \Lambda_3 \ll 1/L$ and $p_\perp \ll 1/L$,

$$G_{\rm br}\big(\sqrt{p_\perp^2 + p_3^2}\big) \approx G_{\rm eff}(p_\perp), \tag{A25}$$

with the p_3-dependence negligible in the integration region that controls long distances. Evaluating the potential at $z = 0$ and performing the p_3 integral over the narrow interval $|p_3| \lesssim \Lambda_3$ gives an overall multiplicative factor $\sim 2\Lambda_3$:

$$\begin{aligned}\Phi(\mathbf{r}_\perp, 0) &\approx \frac{M}{(2\pi)^3}\Big(\int_{-\Lambda_3}^{\Lambda_3} dp_3\Big)\int d^2p_\perp\; e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}\; G_{\rm eff}(p_\perp)\\ &\equiv \frac{M\mathcal{N}_3}{(2\pi)^2}\int d^2p_\perp\; e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}\; G_{\rm eff}(p_\perp),\end{aligned} \tag{A26}$$

where $\mathcal{N}_3 \equiv \frac{1}{2\pi}\int_{-\Lambda_3}^{\Lambda_3} dp_3 \sim \Lambda_3/\pi$ is a finite normalization factor determined by the width of the unsuppressed p_3 window.

Thus, the long-distance potential is governed by an *effective 2D Fourier transform* of $G_{\rm eff}(p_\perp)$.

(ii) Induced 2D Kinetic Term and Emergent $1/p_\perp^2$ Kernel

To obtain a logarithmic potential, one needs an IR propagator that in the 2D momentum variables scales as $G_{\rm eff}(p_\perp) \propto 1/p_\perp^2$. Such a kernel arises naturally if the zero mode of the mediator acquires an induced two-dimensional kinetic term on the (x^1, x^2) subspace (this can be generated by integrating out heavy degrees of freedom on the codimension-1 regulator brane, or from localized loops of matter confined effectively to the disk/plane). Parametrically, the low-momentum form of the effective propagator can be written as

$$G_{\rm eff}(p_\perp) \simeq \frac{C}{\kappa\, p_\perp^2 + \Delta(p_\perp)}, \tag{A27}$$

where κ is the induced 2D kinetic coefficient (dimensions of mass) and $\Delta(p_\perp)$ denotes residual contributions coming from the original denominator in (A22), i.e. $\Delta(p_\perp) \sim m_5 m_6 \log(\mu/p_\perp) + m_5 p_\perp + p_\perp^2$ up to factors of M_4^2. In the deep IR we require the induced 2D kinetic term to dominate over the residual Δ:

$$\kappa\, p_\perp^2 \gg \Delta(p_\perp) \quad \text{for} \quad p_\perp \ll p_\star, \tag{A28}$$

for some small momentum scale $p_\star$ (equivalently a large distance $r \gg L \sim 1/p_\star$). Under condition (A28),

$$G_{\rm eff}(p_\perp) \simeq \frac{C}{\kappa\, p_\perp^2} \qquad (p_\perp \ll p_\star). \tag{A29}$$

Physically, κ is generated by localized induced kinetic terms (brane loops or induced gravity on the codim-1 regulator). The constant C absorbs numerical factors and the normalization $\mathcal{N}_3/(2\pi)$ from (A26).

(iii) 2D Inverse Transform — Logarithmic Potential

Inserting (A29) into (A26) and using the 2D Fourier transform identity

$$\int \frac{d^2 p_\perp}{(2\pi)^2}\, \frac{e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}}{p_\perp^2} = -\frac{1}{2\pi}\log\!\left(\frac{r_\perp}{r_0}\right), \tag{A30}$$

we obtain

$$\begin{aligned} \Phi(r_\perp) &\simeq \frac{M\mathcal{N}_3}{(2\pi)^2}\cdot\frac{C}{\kappa}\int d^2 p_\perp\, e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}\,\frac{1}{p_\perp^2} \\ &= -\frac{M\mathcal{N}_3}{(2\pi)^2}\cdot\frac{C}{\kappa}\cdot 2\pi\left[\frac{1}{2\pi}\log\!\left(\frac{r_\perp}{r_0}\right)\right] \\ &= -\frac{M\,C\,\mathcal{N}_3}{2\pi\,\kappa}\log\!\left(\frac{r_\perp}{r_0}\right). \end{aligned} \tag{A31}$$

Thus, the deep-IR potential is logarithmic. Differentiation gives the radial acceleration,

$$a_r(r_\perp) = -\frac{d\Phi}{dr_\perp} = \frac{M\,C\,\mathcal{N}_3}{2\pi\,\kappa}\,\frac{1}{r_\perp}, \tag{A32}$$

which is the desired $1/r$ force law.

(iv) Summary of Required Conditions and Physical Interpretation

The chain of reasoning above shows how the on-brane propagator (A22) can produce a $\log r$ potential provided two physically motivated ingredients are present

1. **IR anisotropic suppression:** modes with $p_3 \neq 0$ are gapped/damped by the codimension-1 regulator so that the long-distance kernel is obtained from an effective 2D inverse transform (integration over a narrow p_3 window only). This yields the prefactor $\mathcal{N}_3$.

2. **Induced 2D kinetic term:** integrating out regulator-brane physics (or localized matter loops) produces an effective 2D kinetic operator for the mediator with coefficient κ which dominates the residual terms from the original denominator at sufficiently small $p_\perp$. This gives $G_{\rm eff} \propto 1/p_\perp^2$.

   Parametrically, the crossover transverse momentum $p_\star$ is defined by

$$\kappa\, p_\star^2 \sim m_5 m_6 \log(\mu/p_\star), \tag{A33}$$

so the logarithmic regime appears for distances

$$r \gg L \sim \frac{1}{p_\star}. \tag{A34}$$

Conclusion.

The induced 2D term need not come from the literal compactification of the brane. It suffices that bulk/brane dynamics preferentially localize the mediator zero mode and that regulator physics generates the effective 2D kinetic operator. In the cascading gravity language, this is plausible: the codimension-1 regulator can supply large induced kinetic coefficients for low transverse momentum modes, while the codimension-2 logarithm remains subleading once the induced 2D operator dominates at the smallest $p_\perp$.

Appendix F A Concrete Mechanism for IR Suppression of Mediator Modes

In the main text, we used an anisotropic IR reduction of the on-brane propagator (see Eq.(15)) to motivate an effective two-dimensional kernel in the deep IR. Here we present a compact toy model that demonstrates how the mediator's modes with nonzero momentum along a chosen brane direction (denoted z or x^3) can acquire a mass gap $m_{\rm gap}$, while a normalizable zero mode remains long–range. This gap renders the nonzero p_3 modes Yukawa-suppressed at large transverse distances $r_\perp$, justifying the $p_3 \approx 0$ approximation used in the main calculation.

Model: Mediator with a Localized Induced Kinetic Term

Consider a scalar proxy $\Phi(x_\perp, z)$ for the relevant mediator (the argument applies equally well to tensor perturbations with minor notational changes). The static quadratic action on the brane, plus a thin codimension-1 regulator located near $z = 0$ is

$$\begin{aligned} S \;=&\; \frac{1}{2}\int d^2x_\perp\, dz\,\Big[(\nabla_\perp\Phi)^2 + (\partial_z\Phi)^2 + V(z)\,\Phi^2\Big] \\ &+ \frac{\lambda}{2}\int d^2x_\perp\, dz\, f_\ell(z)\,(\nabla_\perp\Phi)^2 \\ &- \int d^2x_\perp\, dz\, \Phi(\mathbf{x}_\perp, z)\,\rho(\mathbf{x}_\perp)\,\delta(z). \end{aligned} \tag{A35}$$

Here

- $\nabla_\perp$ is the gradient in the two transverse brane directions $\mathbf{x}_\perp$,
- $V(z)$ is an optional transverse potential (e.g. from warping or brane microstructure),
- $f_\ell(z)$ is a localized profile of width ℓ centered at $z = 0$ (the codim-1 regulator),
- λ is the induced kinetic coefficient (dimensions of mass), and
- $\rho(\mathbf{x}_\perp)\delta(z)$ is a source localized at $z = 0$.

Transverse Eigenvalue Problem

Separate variables via the normal mode expansion

$$\Phi(\mathbf{x}_\perp, z) \;=\; \sum_n \varphi_n(\mathbf{x}_\perp)\,\psi_n(z), \tag{A36}$$

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with $\{\psi_n\}$ chosen orthonormal under $\int dz\,\psi_n\psi_m = \delta_{nm}$. The transverse eigenvalue problem (including the effect of the localized induced kinetic term) can be written schematically as a Sturm–Liouville problem:

$$\begin{aligned} -\,\partial_z^2\psi_n(z) + V_{\rm eff}(z)\,\psi_n(z) \;&=\; m_n^2\,\psi_n(z), \\ V_{\rm eff}(z) \equiv V(z) + \lambda\,\nabla_\perp^2\Big|_{\rm mode} f_\ell(z) \;&\approx\; V(z) + \lambda\,\tilde{\alpha}\,f_\ell(z). \end{aligned} \tag{A37}$$

In the second line, we indicated the manner in which the induced kinetic term affects the transverse equation: when projecting onto a given $\mathbf{p}_\perp$ mode one replaces $\nabla_\perp^2 \mapsto -p_\perp^2$. For the purpose of estimating the transverse spectrum, we keep the schematic notation $\tilde{\alpha}$ (positive and $\sim p_\perp^2$ for Fourier modes), and in what follows, we display the physically relevant regime $p_\perp \ll m_{\rm gap}$ so that the transverse eigenvalues are well defined.

A localized $f_\ell(z)$ (for example, a narrow Gaussian or a compact support function of width ℓ) produces a potential well in $V_{\rm eff}$ that typically admits a single normalizable bound state $\psi_0(z)$ with eigenvalue $m_0^2 \approx 0$. The first excited state has eigenvalue $m_1^2 \equiv m_{\rm gap}^2 > 0$, set roughly by the inverse width (or depth) of the well:

$$m_{\rm gap} \;\sim\; \frac{1}{\ell_{\rm eff}}, \tag{A38}$$

where $\ell_{\rm eff}$ is the effective transverse lengthscale determined by f_ℓ and $V(z)$. Concretely, for a square well of width ℓ, one finds $m_{\rm gap} \simeq \pi/\ell$ up to order-one factors.

On-Brane Propagator and Zero-Mode Dominance

The static on-brane propagator evaluated at $z = z' = 0$ decomposes into the transverse eigenmodes as

$$G(\mathbf{p}_\perp;0,0) \;=\; \sum_n \frac{\psi_n(0)^2}{p_\perp^2 + m_n^2}. \tag{A39}$$

For transverse momenta satisfying

$$p_\perp \ll m_{\rm gap}, \tag{A40}$$

the nonzero modes $n \geq 1$ contribute terms of order $1/m_{\rm gap}^2$, whereas the zero mode contributes the dominant pole $1/p_\perp^2$:

$$G(\mathbf{p}_\perp;0,0) \;=\; \frac{\psi_0(0)^2}{p_\perp^2} \;+\; \sum_{n\geq 1}\frac{\psi_n(0)^2}{m_n^2} \;+\; \mathcal{O}\Big(\frac{p_\perp^2}{m_{\rm gap}^4}\Big). \tag{A41}$$

Thus in the IR the propagator reduces effectively to

$$G(\mathbf{p}_\perp;0,0) \;\simeq\; \frac{\mathcal{A}}{p_\perp^2}, \qquad \mathcal{A} \equiv \psi_0(0)^2, \tag{A42}$$

up to additive, short-range suppressed pieces. This is precisely the kernel that yields a logarithmic potential upon a two-dimensional inverse Fourier transform.

Position-Space Consequence: Logarithmic Potential

Performing the 2D inverse transform (valid once the p_3 dependence is negligible and only the zero mode survives) gives

$$\begin{aligned}\Phi(\mathbf{r}_\perp) &\simeq \frac{M}{(2\pi)^2}\int d^2p_\perp\, e^{i\mathbf{p}_\perp\cdot\mathbf{r}_\perp}\,\frac{\mathcal{A}}{p_\perp^2} \\ &= -\frac{M\mathcal{A}}{2\pi}\log\!\left(\frac{r_\perp}{r_0}\right),\end{aligned} \tag{A43}$$

which yields the radial acceleration

$$a_r(r_\perp) = \frac{M\mathcal{A}}{2\pi}\frac{1}{r_\perp}. \tag{A44}$$

Parametric Condition for the Galactic Regime

To realize the logarithmic regime at galactic radii $r_\perp \gtrsim L_{\rm gal}$ one requires $m_{\rm gap} \sim 1/L_{\rm gal}$. Using (A38) this is equivalent to choosing the regulator width $\ell_{\rm eff}$ of order $L_{\rm gal}$ (or engineering $V_{\rm eff}$ so that its first excitation lies near $1/L_{\rm gal}$). Solar System tests remain safe provided $L_{\rm gal}$ is much larger than Solar System scales (and hence the Newtonian $1/r$ regime is recovered for $r \ll L_{\rm gal}$).

A Few Comments

1. The toy model above is intentionally minimal: it exhibits the qualitative mechanism (localized induced kinetic term $\Rightarrow$ bound zero mode + gapped excitations). For rigor one should pick a concrete form of $f_\ell(z)$ (e.g. Gaussian or square profile) and solve (A37) explicitly to extract $\psi_0(0)$ and $m_{\rm gap}$.
2. The induced coefficient λ can be estimated from integrating out regulator-brane degrees of freedom (or from loop diagrams of matter localized near the regulator). Evaluating such loops gives a parametric estimate for λ and hence for the normalization $\mathcal{A} = \psi_0(0)^2$.
3. One must check backreaction and the absence of ghosts: the regulator must be chosen so that induced operators do not reintroduce the pathologies that cascading gravity aims to avoid. This requires a dedicated stability analysis.

Conclusion.

The induced-kinetic (or more generally localizing) mechanism above produces a normalizable zero mode for the mediator and a transverse mass gap $m_{\rm gap}$ for nonzero p_3 modes. For transverse momenta $p_\perp \ll m_{\rm gap}$ the on-brane propagator is dominated by the zero mode and takes the effective $1/p_\perp^2$ form in (A42), justifying the two-dimensional inverse transform and the logarithmic potential used in the main text.

References

1. Cirelli, M.; Strumia, A.; Zupan, J. Dark Matter. *arXiv preprint arXiv:2406.01705* **2024**.
2. Amin, M.A.; Mirbabayi, M. A lower bound on dark matter mass. *Physical Review Letters* **2024**, *132*, 221004.
3. Drlica-Wagner, A.; Prescod-Weinstein, C.; Yu, H.B.; Albert, A.; Amin, M.; Banerjee, A.; Baryakhtar, M.; Bechtol, K.; Bird, S.; Birrer, S.; et al. Report of the topical group on cosmic probes of dark matter for Snowmass 2021. *arXiv preprint arXiv:2209.08215* **2022**.
4. O'Hare, C.A. Cosmology of axion dark matter. *arXiv preprint arXiv:2403.17697* **2024**.
5. Cembranos, J.; Dobado, A.; Maroto, A.L. Brane-world dark matter. *Physical review letters* **2003**, *90*, 241301.
6. Maroto, A.L. Nature of branon dark matter. *Physical Review D* **2004**, *69*, 043509.
7. Shankaranarayanan, S.; Johnson, J.P. Modified theories of gravity: Why, how and what? *General Relativity and Gravitation* **2022**, *54*, 44.
8. Nojiri, S.; Odintsov, S.D. Unified cosmic history in modified gravity: from F (R) theory to Lorentz non-invariant models. *Physics Reports* **2011**, *505*, 59–144.

 doi:10.20944/preprints202408.1383.v2

9. Nojiri, S.; Odintsov, S.; Oikonomou, V. Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution. *Physics Reports* **2017**, *692*, 1–104.
10. Milgrom, M. A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. *Astrophysical Journal, Part 1 (ISSN 0004-637X), vol. 270, July 15, 1983, p. 365-370. Research supported by the US-Israel Binational Science Foundation.* **1983**, *270*, 365–370.
11. Milgrom, M. A modification of the Newtonian dynamics-Implications for galaxies. *Astrophysical Journal, Part 1 (ISSN 0004-637X), vol. 270, July 15, 1983, p. 371-383.* **1983**, *270*, 371–383.
12. Moffat, J.W. Scalar–tensor–vector gravity theory. *Journal of Cosmology and Astroparticle Physics* **2006**, *2006*, 004.
13. Nojiri, S.; Odintsov, S.D. Dark energy, inflation and dark matter from modified F (R) gravity. *arXiv preprint arXiv:0807.0685* **2008**.
14. Boyle, L.; Finn, K.; Turok, N. CPT-symmetric universe. *Physical review letters* **2018**, *121*, 251301.
15. Kumar, N. On the Accelerated Expansion of the Universe. *Gravitation and Cosmology* **2024**, *30*, 85–88.
16. Grumiller, D.; Kummer, W.; Vassilevich, D. Dilaton gravity in two dimensions. *Physics Reports* **2002**, *369*, 327–430.
17. Grumiller, D. Model for gravity at large distances. *Physical review letters* **2010**, *105*, 211303.
18. Perivolaropoulos, L.; Skara, F. Reconstructing a model for gravity at large distances from dark matter density profiles. *Physical Review D* **2019**, *99*, 124006.
19. Das, S.; Sur, S. Dark matter or strong gravity? *International Journal of Modern Physics D* **2022**, *31*, 2242020.
20. Das, S.; Sur, S. Gravitational lensing and missing mass. *Physics Open* **2023**, *15*, 100150.
21. Dvali, G.; Gabadadze, G.; Porrati, M. 4D gravity on a brane in 5D Minkowski space. *Physics Letters B* **2000**, *485*, 208–214.
22. de Rham, C.; Dvali, G.; Hofmann, S.; Khoury, J.; Pujolas, O.; Redi, M.; Tolley, A. Cascading DGP. *arXiv preprint arXiv:0711.2072* **2007**.
23. De Rham, C.; Hofmann, S.; Khoury, J.; Tolley, A.J. Cascading gravity and degravitation. *Journal of Cosmology and Astroparticle Physics* **2008**, *2008*, 011.
24. Trodden, M. Cosmic acceleration and the challenge of modifying gravity. In Proceedings of the Journal of Physics: Conference Series. IOP Publishing, 2011, Vol. 284, p. 012004.