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Article

Several New Integral Inequalities of Hermite–Hadamard Type for Extended ϕ_{h-s} -Convex Functions

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[†] Dedicated to Professor Huan-Nan Shi in honor of his 77th birthday

Abstract

In the paper, the authors modify the definitions of ϕ_{h-s} -convex functions and extended ϕ_{h-s} -convex functions, establish two new integral identities and, by virtue of these two integral identities, present several new integral inequalities of the Hermite–Hadamard type for extended ϕ_{h-s} -convex functions.

Keywords: integral inequality; Hermite–Hadamard type; extended ϕ_{h-s} -convex function; Hölder inequality

MSC: 26A51; 26D15; 41A55

1. Introduction

In this paper, let $I \subseteq (-\infty, \infty)$ stand for an interval.

In the 2016 paper [7], the authors introduced the following concept of ϕ_{h-s} -convex functions and studied some properties of ϕ_{h-s} -convex functions. See also [6].

Definition 1 (Essentially modified version of [7, Definition 2.1]). Let $h : (0, 1) \rightarrow \mathbb{R}_+ = (0, \infty)$ and $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi(I) = \{\phi(x), x \in I\}$ is an interval with inner points. For $s \in [0, 1]$, a function $f : \phi(I) \rightarrow \mathbb{R}$ is said to be ϕ_{h-s} -convex on $\phi(I)$ if the inequality

$$f(t\phi(x) + (1-t)\phi(y)) \leq \left[\frac{t}{h(t)} \right]^s f(\phi(x)) + \left[\frac{1-t}{h(1-t)} \right]^s f(\phi(y)) \quad (1)$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

Remark 1. Under the condition of Definition 1,

1. if $s \in (0, 1]$, $\phi(x) = x$ for $x \in I$, and $h(t) = 1$ for $t \in (0, 1)$, then the ϕ_{h-s} -convex function becomes an s -convex function, see [1,4];
2. if $s \in (0, 1]$, $\phi(x) = x$ for $x \in I$, and $h(t) = \frac{1}{t^2}$ for $t \in (0, 1)$, then the ϕ_{h-s} -convex function is an s -Godunova-Levin convex function, see [2];
3. if $s = 1$, $\phi(x) = x$ for $x \in I$, and $h(t) = 2\sqrt{t(1-t)}$ for $t \in (0, 1)$, then the ϕ_{h-s} -convex function is an MT-convex function, see [8];
4. if $s = 1$, $\phi(x) = x$ for $x \in I$, and $h(t) = \frac{t}{h_1(t)}$ for $t \in (0, 1)$, then the ϕ_{h-s} -convex function is an h -convex function defined by [9, p. 304, Definition 4].

In [10], the concept of extended s -convex functions was innovated below.

Definition 2 ([10, Definition 2.1]). A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be extended s -convex on an interval I if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \quad (2)$$

holds for all $x, y \in I$ and $t \in (0, 1)$ and for some fixed $s \in [-1, 1]$.

In [11], the authors extended $s \in [0, 1]$ in Definition 1 to $s \in [-1, 1]$ and improved Definition 1 by introducing the following extended ϕ_{h-s} -convex functions.

Definition 3 (Slightly modified version of [11, Definition 2.1]). Let $h : (0, 1) \rightarrow \mathbb{R}_+$ and $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi(I) = \{\phi(x), x \in I\}$ is an interval with inner points. For some $s \in [-1, 1]$, a function $f : \phi(I) \rightarrow \mathbb{R}$ is said to be extended ϕ_{h-s} -convex on $\phi(I)$ if the inequality (1) holds for all $x, y \in I$ and $t \in (0, 1)$.

Remark 2. Under the condition of Definition 3,

1. if $s \in [0, 1]$, then the extended ϕ_{h-s} -convex function is a ϕ_{h-s} -convex function, see [7];
2. if $\phi(x) = x$ for $x \in I$ and $h(t) = 1$ for $t \in (0, 1)$, then the extended ϕ_{h-s} -function is an extended s -convex function, see [10].

Example 1. For some fixed $s \in [-1, 1] \setminus \{0\}$ and $0 < p < 1$, define $f(x) = x^{p/(p+1)}$ for $x \in (0, 1]$, $h(t) = t^{1-p/s(p+1)}$ for $t \in (0, 1)$, and

$$\phi(x) = \begin{cases} x^{p+1}, & x \in (0, 1) \setminus \left\{\frac{1}{2}\right\}; \\ 1, & x = \frac{1}{2}; \\ \left(\frac{1}{2}\right)^{p+1}, & x = 1. \end{cases}$$

Then $f(\phi(x)) = x^p$ and $\left[\frac{t}{h(t)}\right]^s = t^{p/(p+1)}$ for $x \in (0, 1) \setminus \left\{\frac{1}{2}\right\}$ and $t \in (0, 1)$, with $f(\phi(\frac{1}{2})) = 1$ and $f(\phi(1)) = \left(\frac{1}{2}\right)^p$.

Using the inequality $(u+1)^r \leq u^r + 1$ for $u > 0$ and $0 < r < 1$, for all $x, y \in (0, 1]$ and $t \in (0, 1)$, we obtain

$$[tx^{p+1} + (1-t)y^{p+1}]^{p/(p+1)} \leq t^{p/(p+1)}x^p + (1-t)^{p/(p+1)}y^p.$$

This means that, although ϕ is not continuous, but $\phi(I) = (0, 1]$ is an interval with inner points, the function $f(x) = x^{p/(p+1)}$ is still extended ϕ_{h-s} -convex on $(0, 1]$.

Example 2. When $\phi(I) = \{2, 5\}$, which is not an interval with inner points, the function f in (1) is defined on the set $\phi(I) = \{2, 5\}$ with two points. But the left hand side in (1) requires f to take values in the interval $(2, 5) = \{2t + 5(1-t), t \in (0, 1)\} = \{5 - 3t, t \in (0, 1)\}$. This is the reason why we required $\phi(I)$ in Definitions 1 and 3 to be an interval with inner points.

Example 3. For $0 < p < 1$, let

$$\phi_1(x) = 2x^{p+1}, \quad x \in I_1 = (0, 2^{-1/(p+1)}]$$

and

$$\phi_2(x) = x^{p+1}, \quad x \in I_2 = (0, 1].$$

It is easy to see that $\phi_1(I_1) = \phi_2(I_2) = (0, 1]$. If a function f were extended ϕ_{1h-s} -convex on $\phi_1(I_1) = (0, 1]$ and extended ϕ_{2h-s} -convex on $\phi_2(I_2) = (0, 1]$, we can still differentiate them naturally.

In [11], some integral inequalities for extended ϕ_{h-s} -convex functions were established. We now modify and restate Theorems 3.1 to 3.4 without any change of their proofs in [11] as follows.

Theorem 1 ([11, Theorem 3.1]). Let $h : (0, 1) \rightarrow \mathbb{R}_+$, let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $f : \phi(I) \rightarrow \mathbb{R}$ be differentiable, and let $|\phi'|$ be convex. For some $s \in [-1, 1]$ and $q \geq 1$, if $|f'|^q$ is an increasing extended ϕ_{h-s} -convex function on $\phi(I)$, $f'(\phi)\phi' \in L_1(I)$, and $(1-2x)\left[\frac{x}{h(x)}\right]^s \in L_1([0, 1])$, then for $a, b \in I^\circ$ with $a < b$, we have

$$\left| \frac{f(\phi(a)) + f(\phi(b))}{2} - \frac{1}{b-a} \int_a^b f(\phi(x)) dx \right| \leq \frac{(b-a)\|\phi'\|_\infty}{2^{2-1/q}} \left((|f'(\phi(a))|^q + |f'(\phi(b))|^q) \int_0^1 |1-2t| \left[\frac{t}{h(t)} \right]^s dt \right)^{1/q},$$

where $\|\phi'\|_\infty = \sup_{x \in [a,b]} |\phi'(x)|$.

Theorem 2 ([11, Theorem 3.2]). Let $h : (0, 1) \rightarrow \mathbb{R}_+$, let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $f : \phi(I) \rightarrow \mathbb{R}$ be differentiable, and let $|\phi'|$ be convex. For some $s \in [-1, 1]$ and $q > 1$, if $|f'|^q$ is an increasing extended ϕ_{h-s} -convex function on $\phi(I)$, $f'(\phi)\phi' \in L_1(I)$, and $\left[\frac{x}{h(x)}\right]^s \in L_1([0, 1])$, then for $a, b \in I^\circ$ with $a < b$, we have

$$\left| \frac{f(\phi(a)) + f(\phi(b))}{2} - \frac{1}{b-a} \int_a^b f(\phi(x)) dx \right| \leq \frac{(b-a)\|\phi'\|_\infty}{2} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left((|f'(\phi(a))|^q + |f'(\phi(b))|^q) \int_0^1 \left[\frac{t}{h(t)} \right]^s dt \right)^{1/q},$$

where $\|\phi'\|_\infty = \sup_{x \in [a,b]} |\phi'(x)|$.

Theorem 3 ([11, Theorem 3.3]). Let $h : (0, 1) \rightarrow \mathbb{R}_+$, let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $f : \phi(I) \rightarrow \mathbb{R}$ be differentiable, and let $|\phi'|$ be convex. For some $s \in [-1, 1]$ and $q \geq 1$, if $|f'|^q$ is an increasing extended ϕ_{h-s} -convex function on $\phi(I)$, $f'(\phi)\phi' \in L_1(I)$, and $\frac{x^{s+1}}{h^s(x)}, \frac{(1-x)x^s}{h^s(x)} \in L_1([0, 1])$, then for $a, b \in I^\circ$ with $a < b$, we have

$$\left| \frac{1}{b-a} \int_a^b f(\phi(x)) dx - f\left(\phi\left(\frac{a+b}{2}\right)\right) \right| \leq \frac{(b-a)\|\phi'\|_\infty}{2^{3(1-1/q)}} \left[\left(|f'(\phi(a))|^q \int_0^{1/2} t \left[\frac{t}{h(t)} \right]^s dt + |f'(\phi(b))|^q \int_{1/2}^1 (1-t) \left[\frac{t}{h(t)} \right]^s dt \right)^{1/q} + \left(|f'(\phi(a))|^q \int_{1/2}^1 (1-t) \left[\frac{t}{h(t)} \right]^s dt + |f'(\phi(b))|^q \int_0^{1/2} t \left[\frac{t}{h(t)} \right]^s dt \right)^{1/q} \right],$$

where $\|\phi'\|_\infty = \sup_{x \in [a,b]} |\phi'(x)|$.

Theorem 4 ([11, Theorem 3.4]). Let $h : (0, 1) \rightarrow \mathbb{R}_+$, let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $f : \phi(I) \rightarrow \mathbb{R}$ be differentiable, and let $|\phi'|$ be convex. For some $s \in [-1, 1]$ and $q > 1$, if $|f'|^q$ is an increasing extended ϕ_{h-s} -convex function on $\phi(I)$, $f'(\phi)\phi' \in L_1(I)$, and $\frac{x^{s+1}}{h^s(x)}, \frac{(1-x)x^s}{h^s(x)} \in L_1([0, 1])$, then for $a, b \in I^\circ$ with $a < b$, we have

$$\begin{aligned} \left| \frac{1}{b-a} \int_a^b f(\phi(x)) \, dx - f\left(\phi\left(\frac{a+b}{2}\right)\right) \right| &\leq \frac{(b-a)\|\phi'\|_\infty}{2^{(2q-1)/q}} \left(\frac{q-1}{2q-1}\right)^{1-1/q} \\ &\times \left\{ \left(|f'(\phi(a))|^q \int_0^{1/2} \left[\frac{t}{h(t)}\right]^s \, dt + |f'(\phi(b))|^q \int_{1/2}^1 \left[\frac{t}{h(t)}\right]^s \, dt \right)^{1/q} \right. \\ &\quad \left. + \left(|f'(\phi(a))|^q \int_{1/2}^1 \left[\frac{t}{h(t)}\right]^s \, dt + |f'(\phi(b))|^q \int_0^{1/2} \left[\frac{t}{h(t)}\right]^s \, dt \right)^{1/q} \right\}, \end{aligned}$$

where $\|\phi'\|_\infty = \sup_{x \in [a,b]} |\phi'(x)|$.

In this paper, we first establish two new integral identities under the condition that the range of the function ϕ is an interval, and then, by virtue of these two new integral identities and Hölder type integral inequalities, present some integral inequalities of the Hermite–Hadamard type for extended ϕ_{h-s} -convex functions.

2. Two New Integral Identities

In this section, we establish two new integral identities.

Lemma 1. Let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, let $\phi(I)$ be an interval with inner points, and let $f : \phi(I) \rightarrow \mathbb{R}_0$ be a differentiable function. If $f' \in L_1(\phi(I))$, then

$$\frac{f(\phi(a)) + f(\phi(b))}{2} - \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du = \frac{\phi(b) - \phi(a)}{2} \int_0^1 (1-2t)f'(t\phi(a) + (1-t)\phi(b)) \, dt \tag{3}$$

and

$$\begin{aligned} &\frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du - f\left(\frac{\phi(a) + \phi(b)}{2}\right) \\ &= [\phi(b) - \phi(a)] \left[\int_0^{1/2} t f'(t\phi(a) + (1-t)\phi(b)) \, dt + \int_{1/2}^1 (t-1) f'(t\phi(a) + (1-t)\phi(b)) \, dt \right] \end{aligned} \tag{4}$$

for $a, b \in I^\circ$ such that $\phi(a) \neq \phi(b)$

Proof. The identity (3) can be proved by using [3, Lemma 2.1].

The identity (4) follows from [5, Lemma 2.1]. $\square$

3. New Integral Inequalities of Hermite–Hadamard Type

In this section, we present several new integral inequalities of the Hermite–Hadamard type for extended ϕ_{h-s} -convex functions.

Theorem 5. Let $h : (0, 1) \rightarrow \mathbb{R}_+$, let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, and let $f : \phi(I) \rightarrow \mathbb{R}_0$ is an extended ϕ_{h-s} -convex function on $\phi(I)$ for some fixed $s \in [-1, 1]$. If $f \in L_1(\phi(I))$ and $\left[\frac{x}{h(x)}\right]^s \in L_1([0, 1])$, then for $a, b \in I^\circ$ such that $\phi(a) \neq \phi(b)$, we have

$$\frac{[2h(1/2)]^s}{2} f\left(\frac{\phi(a) + \phi(b)}{2}\right) \leq \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du \leq \frac{[2h(1/2)]^s [f(\phi(a)) + f(\phi(b))]}{2} \int_0^1 \left[\frac{t}{h(t)}\right]^s \, dt.$$

Proof. For $t \in (0, 1)$, using the extended ϕ_{h-s} -convexity of f , we have

$$\begin{aligned} f\left(\frac{\phi(a) + \phi(b)}{2}\right) &= f\left(\frac{t\phi(a) + (1-t)\phi(b) + (1-t)\phi(a) + t\phi(b)}{2}\right) \\ &\leq \left[\frac{1/2}{h(1/2)}\right]^s f(t\phi(a) + (1-t)\phi(b)) + \left[\frac{1/2}{h(1/2)}\right]^s f((1-t)\phi(a) + t\phi(b)) \quad (5) \\ &= \frac{f(t\phi(a) + (1-t)\phi(b)) + f((1-t)\phi(a) + t\phi(b))}{[2h(1/2)]^s}. \end{aligned}$$

Integrating with respect to $t \in (0, 1)$ on the very ends of the inequality (5) and making the variable transform $u = t\phi(a) + (1-t)\phi(b)$ for $t \in (0, 1)$ result in

$$\begin{aligned} f\left(\frac{\phi(a) + \phi(b)}{2}\right) &= \int_0^1 f\left(\frac{t\phi(a) + (1-t)\phi(b) + (1-t)\phi(a) + t\phi(b)}{2}\right) dt \\ &\leq \frac{1}{[2h(1/2)]^s} \int_0^1 [f(t\phi(a) + (1-t)\phi(b)) + f((1-t)\phi(a) + t\phi(b))] dt \\ &= \frac{2[2h(1/2)]^{-s}}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) du. \end{aligned}$$

Further, making the variable transform $u = t\phi(a) + (1-t)\phi(b)$ for $t \in (0, 1)$ and using the extended ϕ_{h-s} -convexity of f lead to

$$\begin{aligned} \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) du &= \int_0^1 f(t\phi(a) + (1-t)\phi(b)) dt \\ &\leq \int_0^1 \left(\left[\frac{t}{h(t)}\right]^s f(\phi(a)) + \left[\frac{1-t}{h(1-t)}\right]^s f(\phi(b)) \right) dt \\ &= [f(\phi(a)) + f(\phi(b))] \int_0^1 \left[\frac{t}{h(t)}\right]^s dt. \end{aligned}$$

Theorem 5 is thus proved. $\square$

Corollary 1. Under conditions of Theorem 5,

1. if $s = 1$, we have

$$h\left(\frac{1}{2}\right) f\left(\frac{\phi(a) + \phi(b)}{2}\right) \leq \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) du \leq h\left(\frac{1}{2}\right) [f(\phi(a)) + f(\phi(b))] \int_0^1 \frac{t}{h(t)} dt;$$

2. if $s = 0$, we have

$$\frac{1}{2} f\left(\frac{\phi(a) + \phi(b)}{2}\right) \leq \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) du \leq \frac{f(\phi(a)) + f(\phi(b))}{2};$$

3. if $s = -1$, we have

$$\frac{1}{4h(1/2)} f\left(\frac{\phi(a) + \phi(b)}{2}\right) \leq \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) du \leq \frac{f(\phi(a)) + f(\phi(b))}{4h(1/2)} \int_0^1 \frac{h(t)}{t} dt.$$

Corollary 2. Let $0 < p < 1$ and $a, b \in \mathbb{R}_+$ with $a < b$. Then

$$\frac{2^{1/(p+1)}}{4} \left(\frac{a^{p+1} + b^{p+1}}{2} \right)^{p/(p+1)} \leq \frac{(p+1)(b^{2p+1} - a^{2p+1})}{(2p+1)(b^{p+1} - a^{p+1})} \leq \frac{2^{1/(p+1)}(p+1)}{4(2p+1)} (a^p + b^p).$$

Proof. Let $\phi(x) = x^{p+1}$, $f(x) = x^{p/(p+1)}$ for $x \in \mathbb{R}_+$, and $h(t) = t^{1-p/(p+1)}$ for $t \in (0, 1)$ and for some fixed $s \in [-1, 1] \setminus \{0\}$. By virtue of [11, Example 2.1], we deduce that $f(x) = x^{p/(p+1)}$ is extended ϕ_{h-s} -convex on $\mathbb{R}_+$. Since $f(\phi(x)) = x^p$ and $\left[\frac{t}{h(t)}\right]^s = t^{p/(p+1)}$ for $x \in \mathbb{R}_+$ and $t \in (0, 1)$, with the help of Theorem 5, we arrive at

$$\begin{aligned} \frac{[2h(1/2)]^s}{2} f\left(\frac{\phi(a) + \phi(b)}{2}\right) &= \frac{1}{2^{1/(p+1)}} \left(\frac{a^{p+1} + b^{p+1}}{2}\right)^{p/(p+1)}, \\ \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du &= \frac{(p+1)(b^{2p+1} - a^{2p+1})}{(2p+1)(b^{p+1} - a^{p+1})}, \end{aligned}$$

and

$$\frac{[2h(1/2)]^s [f(\phi(a)) + f(\phi(b))]}{2} \int_0^1 \left[\frac{t}{h(t)}\right]^s dt = \frac{(p+1)(a^p + b^p)}{2^{1/(p+1)}(2p+1)}.$$

The proof of Corollary 2 is completed. $\square$

Theorem 6. Let $h : (0, 1) \rightarrow \mathbb{R}_+$, let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, and let $f : \phi(I) \rightarrow \mathbb{R}$ be differentiable. For some fixed $s \in [-1, 1]$, if $f' \in L_1(\phi(I))$ and $|f'|$ is an extended ϕ_{h-s} -convex function on $\phi(I)$, then for $a, b \in I^\circ$ such that $\phi(a) \neq \phi(b)$,

1. when $\frac{(1-2x)x^s}{h^s(x)} \in L_1([0, 1])$, we have

$$\begin{aligned} \left| \frac{f(\phi(a)) + f(\phi(b))}{2} - \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du \right| \\ \leq \frac{|\phi(b) - \phi(a)|}{2} [|f'(\phi(a))| + |f'(\phi(b))|] \int_0^1 |1 - 2t| \left[\frac{t}{h(t)}\right]^s dt; \quad (6) \end{aligned}$$

2. when $\frac{x^{s+1}}{h^s(x)} \in L_1([0, \frac{1}{2}])$ and $\frac{x^s(1-x)}{h^s(x)} \in L_1([\frac{1}{2}, 1])$, we have

$$\begin{aligned} \left| \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du - f\left(\frac{\phi(a) + \phi(b)}{2}\right) \right| \\ \leq |\phi(b) - \phi(a)| [|f'(\phi(a))| + |f'(\phi(b))|] \left(\int_0^{1/2} t \left[\frac{t}{h(t)}\right]^s dt + \int_{1/2}^1 (1-t) \left[\frac{t}{h(t)}\right]^s dt \right). \quad (7) \end{aligned}$$

Proof. Since $|f'|$ is a ϕ_{h-s} -convex function, we acquire

$$|f'(t\phi(x) + (1-t)\phi(y))| \leq \left[\frac{t}{h(t)}\right]^s |f'(\phi(x))| + \left[\frac{1-t}{h(1-t)}\right]^s |f'(\phi(y))| \quad (8)$$

for all $\phi(x), \phi(y) \in \phi(I)$ and any $t \in (0, 1)$.

From Lemma 1 and the inequality (8), it follows that

$$\begin{aligned} & \left| \frac{f(\phi(a)) + f(\phi(b))}{2} - \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du \right| \\ & \leq \frac{|\phi(b) - \phi(a)|}{2} \int_0^1 |1 - 2t| |f'(t\phi(a) + (1-t)\phi(b))| \, dt \\ & \leq \frac{|\phi(b) - \phi(a)|}{2} \left\{ \int_0^1 |1 - 2t| \left(\left[\frac{t}{h(t)} \right]^s |f'(\phi(a))| + \left[\frac{1-t}{h(1-t)} \right]^s |f'(\phi(b))| \right) \, dt \right\} \\ & = \frac{|\phi(b) - \phi(a)|}{2} [|f'(\phi(a))| + |f'(\phi(b))|] \int_0^1 |1 - 2t| \left[\frac{t}{h(t)} \right]^s \, dt. \end{aligned}$$

The inequality (6) is proved.

Utilizing Lemma 1 and the inequality (8), we gain

$$\begin{aligned} & \left| \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du - f\left(\frac{\phi(a) + \phi(b)}{2}\right) \right| \\ & \leq |\phi(b) - \phi(a)| \left[\int_0^{1/2} t |f'(t\phi(a) + (1-t)\phi(b))| \, dt + \int_{1/2}^1 (1-t) |f'(t\phi(a) + (1-t)\phi(b))| \, dt \right] \\ & \leq |\phi(b) - \phi(a)| \left\{ \int_0^{1/2} t \left(\left[\frac{t}{h(t)} \right]^s |f'(\phi(a))| + \left[\frac{1-t}{h(1-t)} \right]^s |f'(\phi(b))| \right) \, dt \right. \\ & \quad \left. + \int_{1/2}^1 (1-t) \left(\left[\frac{t}{h(t)} \right]^s |f'(\phi(a))| + \left[\frac{1-t}{h(1-t)} \right]^s |f'(\phi(b))| \right) \, dt \right\} \\ & = |\phi(b) - \phi(a)| [|f'(\phi(a))| + |f'(\phi(b))|] \left(\int_0^{1/2} t \left[\frac{t}{h(t)} \right]^s \, dt + \int_{1/2}^1 (1-t) \left[\frac{t}{h(t)} \right]^s \, dt \right). \end{aligned}$$

Thus, the inequality (7) is proved. Theorem 6 is thus proved. $\square$

Theorem 7. Let $h : (0, 1) \rightarrow \mathbb{R}_+$, let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, and let $f : \phi(I) \rightarrow \mathbb{R}$ be differentiable. For some fixed $s \in [-1, 1]$, $q > 1$, and $q > r \geq 0$, if $|f'|^q$ is an extended ϕ_{h-s} -convex function on $\phi(I)$, $f' \in L_1(\phi(I))$, and $\frac{|1-2x|^r x^s}{[h(x)]^s} \in L_1([0, 1])$, then for $a, b \in I^\circ$ such that $\phi(a) \neq \phi(b)$, we have

$$\begin{aligned} & \left| \frac{f(\phi(a)) + f(\phi(b))}{2} - \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du \right| \\ & \leq \frac{|\phi(b) - \phi(a)|}{2} \left(\frac{q-1}{2q-r-1} \right)^{1-1/q} \left([|f'(\phi(a))|^q + |f'(\phi(b))|^q] \int_0^1 |1-2t|^r \left[\frac{t}{h(t)} \right]^s \, dt \right)^{1/q}. \end{aligned}$$

Proof. As in the proof of Theorem 6, using Lemma 1, the inequality (8), and Hölder's integral inequality, we obtain

$$\begin{aligned} & \left| \frac{f(\phi(a)) + f(\phi(b))}{2} - \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du \right| \\ & \leq \frac{|\phi(b) - \phi(a)|}{2} \int_0^1 |1 - 2t| |f'(t\phi(a) + (1-t)\phi(b))| \, dt \\ & \leq \frac{|\phi(b) - \phi(a)|}{2} \left(\int_0^1 |1 - 2t|^{(q-r)/(q-1)} \, dt \right)^{1-1/q} \left(\int_0^1 |1 - 2t|^r |f'(t\phi(a) + (1-t)\phi(b))|^q \, dt \right)^{1/q} \\ & \leq \frac{|\phi(b) - \phi(a)|}{2} \left(\frac{q-1}{2q-r-1} \right)^{1-1/q} \left\{ \int_0^1 |1 - 2t|^r \left(\left[\frac{t}{h(t)} \right]^s |f'(\phi(a))|^q + \left[\frac{1-t}{h(1-t)} \right]^s |f'(\phi(b))|^q \right) \, dt \right\}^{1/q} \\ & = \frac{|\phi(b) - \phi(a)|}{2} \left(\frac{q-1}{2q-r-1} \right)^{1-1/q} \left([|f'(\phi(a))|^q + |f'(\phi(b))|^q] \int_0^1 |1 - 2t|^r \left[\frac{t}{h(t)} \right]^s \, dt \right)^{1/q}. \end{aligned}$$

The proof of Theorem 7 is completed. $\square$

Theorem 8. Let $h : (0, 1) \rightarrow \mathbb{R}_+$, let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, and let $f : \phi(I) \rightarrow \mathbb{R}$ be differentiable. For some fixed $s \in [-1, 1]$, $q > 1$, and $q > r \geq 0$, if $|f'|^q$ is an extended ϕ_{h-s} -convex function on $\phi(I)$, $f' \in L_1(\phi(I))$, $\frac{x^{s+r}}{h^s(x)} \in L_1([0, \frac{1}{2}])$, and $\frac{x^s(1-x)^r}{h^s(x)} \in L_1([\frac{1}{2}, 1])$, then for $a, b \in I^\circ$ such that $\phi(a) \neq \phi(b)$, we have

$$\begin{aligned} & \left| \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du - f\left(\frac{\phi(a) + \phi(b)}{2}\right) \right| \leq |\phi(b) - \phi(a)| \left(\frac{q-1}{2q-r-1} \right)^{1-1/q} \\ & \quad \times \left\{ \left(|f'(\phi(a))|^q \int_0^{1/2} t^r \left[\frac{t}{h(t)} \right]^s \, dt + |f'(\phi(b))|^q \int_{1/2}^1 (1-t)^r \left[\frac{t}{h(t)} \right]^s \, dt \right)^{1/q} \right. \\ & \quad \left. + \left(|f'(\phi(a))|^q \int_{1/2}^1 (1-t)^r \left[\frac{t}{h(t)} \right]^s \, dt + |f'(\phi(b))|^q \int_0^{1/2} t^r \left[\frac{t}{h(t)} \right]^s \, dt \right)^{1/q} \right\}. \end{aligned}$$

Proof. As argued in the proofs of Theorems 6 and 7, we deduce

$$\begin{aligned} & \left| \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(u) \, du - f\left(\frac{\phi(a) + \phi(b)}{2}\right) \right| \\ & \leq |\phi(b) - \phi(a)| \left[\left(\int_0^{1/2} t^{(q-r)/(q-1)} \, dt \right)^{1-1/q} \left(\int_0^{1/2} t^r |f'(t\phi(a) + (1-t)\phi(b))|^q \, dt \right)^{1/q} \right. \\ & \quad \left. + \left(\int_{1/2}^1 (1-t)^{(q-r)/(q-1)} \, dt \right)^{1-1/q} \left(\int_{1/2}^1 (1-t)^r |f'(t\phi(a) + (1-t)\phi(b))|^q \, dt \right)^{1/q} \right] \\ & \leq |\phi(b) - \phi(a)| \left(\frac{q-1}{2q-r-1} \right)^{1-1/q} \left\{ \left[\int_0^{1/2} t^r \left(\left[\frac{t}{h(t)} \right]^s |f'(\phi(a))|^q + \left[\frac{1-t}{h(1-t)} \right]^s |f'(\phi(b))|^q \right) \, dt \right]^{1/q} \right. \\ & \quad \left. + \left[\int_{1/2}^1 (1-t)^r \left(\left[\frac{t}{h(t)} \right]^s |f'(\phi(a))|^q + \left[\frac{1-t}{h(1-t)} \right]^s |f'(\phi(b))|^q \right) \, dt \right]^{1/q} \right\} \\ & = |\phi(b) - \phi(a)| \left(\frac{q-1}{2q-r-1} \right)^{1-1/q} \left\{ \left(|f'(\phi(a))|^q \int_0^{1/2} t^r \left[\frac{t}{h(t)} \right]^s \, dt + |f'(\phi(b))|^q \int_{1/2}^1 (1-t)^r \left[\frac{t}{h(t)} \right]^s \, dt \right)^{1/q} \right. \\ & \quad \left. + \left(|f'(\phi(a))|^q \int_{1/2}^1 (1-t)^r \left[\frac{t}{h(t)} \right]^s \, dt + |f'(\phi(b))|^q \int_0^{1/2} t^r \left[\frac{t}{h(t)} \right]^s \, dt \right)^{1/q} \right\}. \end{aligned}$$

Theorem 8 is thus proved. $\square$

4. Remarks

Finally, we give several remarks about related stuffs.

Remark 3. Definition 3 in this paper is a slightly modification of [11, Definition 2.1].

Remark 4. Remark 3.1 in [11] should be corrected as follows:

If ϕ has a derivative of the first order and $|\phi'|$ is concave, if for $q > 1$ the function $|f'|^q$ is decreasing extended ϕ_{h-s} -convex on $\phi(I)$, then Theorems 1 to 4 in this paper still hold.

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