ON (q,r,w)-STIRLING NUMBERS OF THE SECOND KIND

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Abstract. In this paper, we introduce a new generalization of the r-Stirling numbers of the second kind based on the q-numbers via an exponential generating function. We investigate their some properties and derive several relations among q-Bernoulli numbers and polynomials, and newly defined (q,r,w)-Stirling numbers of the second kind. We also obtain q-Bernstein polynomials as a linear combination of (q,r,w)-Stirling numbers of the second kind and q-Bernoulli polynomials in w.

1. Introduction

The Stirling numbers of the second kind $S(n,k)$ defined by

$$\sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!} \qquad (n \geq k; k \in \mathbb{N}_0) \tag{1.1}$$

and their various generalizations have been studied by many mathematicians and physicists for a long time, see [2-8, 10-16]. For example, Broder [2] explored extensively the combinatorial and algebraic properties of the r-Stirling numbers, which is most cases generalize similar properties of the regular Stirling numbers. Carlitz [4] defined and studied an entirely different type of the generalization of the Stirling numbers, termed weighted Stirling numbers. Corcino *et al.* [6] established several properties for the q-analogue of the unified generalizations of Stirling numbers including the vertical and horizontal recurrence relations, and the rational generating function. Guo *et al.* [7] derived an explicit formula for computing Bernoulli polynomials at non-negative integers in terms of r-Stirling numbers of the second kind. Guo *et al.* [8] reviewed some explicit formulas and set a novel explicit formula for Bernoulli and Genocchi numbers in terms of Stirling numbers of the second kind. Kim [10] considered the degenerate Stirling polynomials of the second kind which are derived from the generating function and proved some new identities for these polynomials. Kim *et al.* [11] provided several expressions, identities and properties

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about the extended degenerate Stirling numbers of the second kind and the extended degenerate Bell polynomials. Mahmudov [12] introduced and investigated a class of generalized Bernoulli polynomials and Euler polynomials based on the q-integers. In addition, he derived the q-analogues of well-known formulas including the q-analogue of the Srivastava-Pintér addition theorem. Mangontarum *et al.* [13] introduced the (q,r)-Whitney numbers of the first and second kinds in terms of the q-Boson operator and acquired some fundamental properties including recurrence formulas, orthogonality and inverse relations, and other interesting identities. They also obtained a q-analogue of the r-Stirling numbers of the first and second kinds. Mansour *et al.* [14] gave an one-stop reference on the research activities and known results of normal ordering and Stirling numbers and also defined associated generalized Stirling numbers as normal ordering coefficients in analogy to the classical Stirling numbers. They discussed the Stirling numbers, closely related generalizations, and their role as normal ordering coefficients in the Weyl algebra. Qi *et al.* [16] investigated a closed form for the Stirling polynomials in terms of the Stirling numbers of the first and second kinds by virtue of the Faá di Bruno formula and two identities for the Bell polynomial of the second kind.

We use the following notations:

$$\mathbb{N} := \{1,2,3,\cdots\} \text{ and } \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

As usual, $\mathbb{R}$ denotes the set of all real numbers.

Based on the generating series (1.1), r-analogue of Stirling numbers of second kind is also given by the following generating function:

$$\sum_{n=k}^{\infty} S(n+r,k+r)\frac{t^n}{n!} = \frac{\left(e^t-1\right)^k}{k!}e^{rt} \quad \text{(see [2, 5, 7, 12, 13])}. \tag{1.2}$$

$$(n \geq k \text{ and } k \in \mathbb{N}_0, r \in \mathbb{N})$$

We now give the definitions of some notations of q-calculus which can be found in [3,12,13].

The q-analogue of any real number x is defined by

$$[x]_q = \frac{q^x-1}{q-1} \quad (q \neq 1).$$

The q-derivative operator D_q of a function f is considered as

$$(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x} \quad (q \neq 1 \text{ and } x \neq 0)$$

provided $f'(0)$ exists.

For any two functions $f(x)$ and $g(x)$, the product rule of the q-derivative operator is given by

$$D_q(f(x)g(x)) = g(qx)D_q f(x) + f(x) D_q g(x). \tag{1.3}$$

The q-binomial coefficients and q-factorial are defined, for positive integer n and k with $n \geq k$, by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!}$$

and

$$[n]_q! = [n]_q [n-1]_q [n-2]_q \cdots [1]_q \quad \left(n \in \mathbb{N}; [0]_q! = 1\right).$$

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The q-generalization of $(\lambda+\mu)^n$ is defined by

$$(\lambda\oplus\mu)_q^n=(\lambda+\mu)(\lambda+q\mu)\cdots\left(\lambda+q^{n-1}\mu\right)=\sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{n-k}{2}}\lambda^k\mu^{n-k}.$$

The two different type of q-exponential functions are given by

$$e_q(t)=\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\text{ and }E_q(t)=\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{t^n}{[n]_q!}\quad(t\in\mathbb{C};|t|<1),$$

which satisfy $e_q(x+y)=e_q(x)e_q(y)$ for the q-commuting variables x and y such as $yx=qxy$ and $e_q(x)E_q(y)=e_q\left((x\oplus y)_q\right)$ with $e_q(-x)E_q(x)=1$. We also have $D_qe_q(t)=e_q(t)$ and $D_qE_q(t)=E_q(qt)$.

The q-Stirling numbers of the second kind are defined by means of the following generating function to be (see [6,12,13])

$$\sum_{n=k}^{\infty}S_q(n,k)\frac{t^n}{[n]_q!}=\frac{(e_q(t)-1)^k}{[k]_q!}\tag{1.4}$$

which is a q-analogue of the generating series (1.1).

The generating series

$$\sum_{n=k}^{\infty}S_{q,r,w}(n+r+w,k+r+w)\frac{t^n}{[n]_q!}=\frac{(e_q(t)-1)^k}{[k]_q!}e_q(rt)E_q(wt)\tag{1.5}$$

is our main definition, called (q,r,w)-Stirling numbers of the second kind, to derive the results of the paper which seems to be a new generalization of Eq. (1.4).

By making use of Eq. (1.5), we investigate some new properties and derive some relations among q-Bernoulli numbers and polynomials, and newly defined (q,r)-Stirling numbers. We also obtain q-Bernstein polynomials as a linear combination of (q,r,w)-Stirling numbers of the second kind and q-Bernoulli polynomials in w.

2. (q,r,w)-Stirling Numbers of Second Kind

Recently, analogues and generalizations of the Stirling numbers of the second kind were studied by many mathematicians (see [2-8, 10-16]).

We begin with the following generating function, corresponding to q-calculus, of the (r,w)-Stirling numbers, which will have important role to derive the main results of this study.

For $n,k\in\mathbb{N}_0$ with $n\geq k\geq 0$, we introduce the (q,r,w)-Stirling numbers of the second kind

$$S_{q,r,w}(n+r+w,k+r+w)$$

by means of the following Taylor series about $t=0$:

$$\sum_{n=k}^{\infty}S_{q,r,w}(n+r+w,k+r+w)\frac{t^n}{[n]_q!}=\frac{(e_q(t)-1)^k}{[k]_q!}e_q(rt)E_q(wt).\tag{2.1}$$

Remark 1. *Setting $r=w=0$, the (q,r)-Stirling numbers of the second kind reduces to the numbers in (1.4).*

Remark 2. *Substituting q tends to 1 and taking $w=0$, the Eq. (2.1) turns into the Eq. (1.2).*

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Remark 3. *Taking $r=w=0$ and $q\to 1$ in (2.1), we obtain the classical Stirling numbers of the second kind (1.1).*

The (q,r,w)-Stirling numbers of the second kind satisfy the following properties.

Proposition 1. *The following relation holds true*

$$S_{q,r,w}(n+r+w,r+w)=(r\oplus w)_q^n \quad (n\in\mathbb{N}_0). \tag{2.2}$$

Because of $S_{q,r,w}(n+r+w,k+r+w)=0 \quad (k>n)$, we can write

$$\sum_{n=k}^{\infty} S_{q,r,w}(n+r+w,k+r+w)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty} S_{q,r,w}(n+r+w,k+r+w)\frac{t^n}{[n]_q!}.$$

The following theorem includes a relation between q-Stirling numbers of the second kind and (q,r)-Stirling numbers of the second kind.

Theorem 2. *For $n,k\in\mathbb{N}_0$ with $n\geq k\geq 0$, we have*

$$\begin{aligned} S_{q,r,w}(n+r+w,k+r+w) &= \sum_{j=k}^{n}\binom{n}{j}_q S_q(j,k)(r\oplus w)_q^{n-j} \\ &= \sum_{j=k}^{n}\binom{n}{j}_q S_{q,r,w}(n+w,k+w)\, r^{n-j} \\ &= \sum_{j=k}^{n}\binom{n}{j}_q S_{q,r,w}(n+r,k+r)\, q^{\binom{n-j}{2}} w^{n-j}. \end{aligned} \tag{2.3}$$

Proof. Using (2.1) and Cauchy product rule, we see

$$\begin{aligned} \sum_{n=k}^{\infty} S_{q,r,w}(n+r+w,k+r+w)\frac{t^n}{[n]_q!} &= \frac{(e_q(t)-1)^k}{[k]_q!}e_q(rt)E_q(wt) \\ &= \sum_{n=k}^{\infty} S_q(n,k)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(r\oplus w)_q^n\frac{t^n}{[n]_q!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{j=k}^{n}\binom{n}{j}_q S_q(j,k)(r\oplus w)_q^{n-j}\right)\frac{t^n}{[n]_q!}. \end{aligned}$$

Comparing the coefficients $\frac{t^n}{[n]_q!}$ of the both sides yields to the asserted result (2.3). The other equalities can be shown via the same proof technique above. □

In the case $w=0$, the formula (2.3) reduces to

$$S_{q,r}(n+r,k+r)=\sum_{j=k}^{n}\binom{n}{j}_q S_q(j,k)\, r^{n-j}. \tag{2.4}$$

Theorem 3. *The following identity holds*

$$S_{q,r,w}(n+r+w,k+l+r+w)=[l]_q!\sum_{j=0}^{n}\binom{n}{j}_q S_{q,r,0}(n+r,k+r)\,S_{q,0,w}(n+w,l+w).$$

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Proof. It is proved by making use of the following calculations

$$
\begin{aligned}
&\sum_{n=0}^{\infty} S_{q,r,w}(n+r+w, k+l+r+w) \frac{t^n}{[n]_q!} \\
= \; & \frac{(e_q(t)-1)^{k+l}}{[k]_q!} e_q(rt) E_q(wt) \\
= \; & [l]_q! \left(\frac{(e_q(t)-1)^k}{[k]_q!} e_q(rt) \right) \left(\frac{(e_q(t)-1)^l}{[l]_q!} E_q(wt) \right) \\
= \; & [l]_q! \left(\sum_{n=0}^{\infty} S_{q,r,0}(n+r, k+r) \frac{t^n}{[n]_q!} \right) \left(\sum_{n=0}^{\infty} S_{q,0,w}(n+w, k+w) \frac{t^n}{[n]_q!} \right) \\
= \; & \sum_{n=0}^{\infty} \left([l]_q! \sum_{j=0}^{n} \binom{n}{j}_q S_{q,r,0}(j+r, k+r) S_{q,0,w}(n-j+w, k+w) \right) \frac{t^n}{[n]_q!}.
\end{aligned}
$$

By comparing the coefficients $\frac{t^n}{[n]_q!}$ of the both sides of the series above, we complete the proof. □

The following theorem includes a recurrence relation for $S_{q,r,w}(n+r+w, k+r+w)$.

Theorem 4. *The following relationship holds for $q \in \mathbb{R}$ with $0 < q < 1$ and $n, k \in \mathbb{N}_0$ with $n \geq k \geq 0$:*

$$
\begin{aligned}
&\left([k]_q! - \frac{q^{n+1}}{q^{n+1}-1} \right) S_{q,r,w}(n+1+r+w, k+r+w) \qquad (2.5) \\
= \; & rS_{q,r,w}(n+r+w, k+r+w) + wS_{q,r,w}(n+qr+qw, k+qr+qw) \\
& - \frac{1}{q^{n+1}-1} S_{q,r,w}(n+1+qr+qw, k+qr+qw).
\end{aligned}
$$

Proof. Inspired by Mahmudov in [12], applying q-derivative operator D_q, with respect to t, to both sides of Eq. (2.1), the LHS (left hand side) becomes

$$
\begin{aligned}
D_q \left(\sum_{n=0}^{\infty} S_{q,r,w}(n+r+w, k+r+w) \frac{t^n}{[n]_q!} \right) &= \sum_{n=1}^{\infty} S_{q,r,w}(n+r+w, k+r+w) \frac{t^{n-1}}{[n-1]_q!} \\
&= \sum_{n=0}^{\infty} S_{q,r,w}(n+1+r+w, k+r+w) \frac{t^n}{[n]_q!},
\end{aligned}
$$

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and the RHS (right hand side), utilizing the product rule of the q-derivative (1.3), becomes

$$D_q\left(\frac{(e_q(t)-1)^k}{[k]_q!}e_q(rt)E_q(wt)\right)=e_q(qrt)E_q(qwt)\frac{\frac{(e_q(qt)-1)^k}{[k]_q!}-\frac{(e_q(t)-1)^k}{[k]_q!}}{(q-1)t}$$

$$+\frac{(e_q(t)-1)^k}{[k]_q!}D_q(e_q(rt)E_q(wt))$$

$$=e_q(rqt)E_q(wqt)\frac{(e_q(qt)-1)^k}{(q-1)t[k]_q!}-e_q(qrt)E_q(qwt)\frac{(e_q(t)-1)^k}{(q-1)t[k]_q!}$$

$$+r\frac{(e_q(t)-1)^k}{[k]_q!}e_q(rt)E_q(qwt)+w\frac{(e_q(t)-1)^k}{[k]_q!}e_q(rt)E_q(qwt)$$

$$=\frac{1}{(q-1)[k]_q!}\left(\sum_{n=0}^{\infty}S_{q,r,w}(n+r+w,k+r+w)q^n\frac{t^{n-1}}{[n]_q!}-\sum_{n=0}^{\infty}S_{q,r,w}(n+qr+qw,k+qr+qw)\frac{t^n}{[n]_q!}\right)$$

$$+\frac{r}{[k]_q!}\sum_{n=0}^{\infty}S_{q,r,w}(n+r+w,k+r+w)\frac{t^n}{[n]_q!}+\frac{w}{[k]_q!}\sum_{n=0}^{\infty}S_{q,r,w}(n+qr+qw,k+qr+qw)\frac{t^n}{[n]_q!}.$$

Matching the coefficients $\frac{t^n}{[n]_q!}$ of the both sides above, in conjunction with some elementary computations, the desired result (2.5) is derived. □

The higher order q-Bernoulli polynomials are defined by the following generating function to be

$$\sum_{n=0}^{\infty}B_{n,q}^{(k)}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{t}{e_q(t)-1}\right)^k e_q(xt)E_q(yt)\quad(|t|<2\pi),\tag{2.6}$$

where k is an integer and x is a complex number. When $x=y=0$ in (2.6), we have $B_{n,q}^{(k)}(0,0):=B_{n,q}^{(k)}$ called the higher order q-Bernoulli numbers. When $y=0$, we have $B_{n,q}^{(k)}(x,0):=B_{n,q}^{(k)}(x)$ called higher order q-Bernoulli polynomials in x. Obviously that

$$B_{n,q}^{(k)}(x)=\sum_{j=0}^{n}\binom{n}{j}_q B_{j,q}^{(k)}x^{n-j}.$$

We now give a correlation between the $(q,r,w=0)$-Stirling numbers of the second kind and the higher order q-Bernoulli polynomials via the following theorem.

Theorem 5. *For the case $w=0$, the following relationship holds true for $n,k\in\mathbb{Z}_{\geq 0}$ with $n\geq k\geq 0$:*

$$\sum_{j=0}^{n}\binom{n}{j}_q S_{q,r,0}(j+r,k+r)B_{n-j,q}^{(k)}(x)=\binom{n}{k}_q\sum_{j=0}^{n-k}\binom{n-k}{j}_q r^j x^{n-k-j}.\tag{2.7}$$

Proof. By utilizing the definitions in (2.3) and (2.6), we readily get

$$\sum_{n=k}^{\infty}S_{q,r,0}(n+r,k+r)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}B_{n,q}^{(k)}(x)\frac{t^n}{[n]_q!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}_q S_{q,r,0}(j+r,k+r)B_{n-j,q}^{(k)}(x)\right)\frac{t^n}{[n]_q!}$$

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and

$$\frac{\left(e_q(t)-1\right)^k}{[k]_q!}e_q(rt)\left(\frac{t}{e_q(t)-1}\right)^k e_q(xt)=\frac{t^k e_q(rt)\,e_q(xt)}{[k]_q!}$$
$$=\frac{1}{[k]_q!}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}_q r^j x^{n-j}\right)\frac{t^{n+k}}{[n]_q!}.$$

Therefore, the claimed result (2.7) is obtained. □

The following corollary including a relationship the higher order q-Bernoulli numbers $B_{n,q}^{(k)}$ and the (q,r)-Stirling numbers of the second kind is a direct consequence of the previous theorem.

Corollary 6. *Upon setting $x=0$ in (2.7), we get*

$$\sum_{j=0}^{n}\binom{n}{j}_q S_{q,r,0}(j+r,k+r)\,B_{n-j,q}^{(k)}=r^{n-k}\binom{n}{k}_q.$$

In the year 2010, Acikgoz and Araci [1] constructed the generating function of Bernstein polynomials $B_{k,n}(x)$ as follows:

$$\sum_{n=k}^{\infty}B_{k,n}(x)\frac{t^n}{n!}=\frac{(tx)^k}{k!}e^{t(1-x)}\qquad(t\in\mathbb{C};k=0,1,2,\cdots,n).$$

Based on this series, Gupta *et al.* [9] defined q-analogue of Bernstein polynomials as follows.

$$\sum_{n=k}^{\infty}b_{k,n}^q(x)\frac{t^n}{[n]_q!}=\frac{(tx)^k}{[k]_q!}e_q\left((1-q)(1-x)_q\,t\right). \tag{2.8}$$

From (2.8), we have

$$\begin{aligned}\sum_{n=0}^{\infty}b_{k,n}^q(x)\frac{t^n}{[n]_q!} &= x^k\left(\frac{\left(e_q(t)-1\right)^k}{[k]_q!}e_q\left((1-q)(1-x)_q\,t\right)E_q(wt)\right)\left(\left(\frac{t}{e_q(t)-1}\right)^k e_q(-wt)\right)\\ &= x^k\left(\sum_{n=0}^{\infty}S_{q,r,w}(n+r+w,k+r+w)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}B_{n,q}^{(k)}(-w)\frac{t^n}{[n]_q!}\right)\\ &= \sum_{n=0}^{\infty}\left(x^k\sum_{j=0}^{n}\binom{n}{j}_q S_{q,r,w}(j+r+w,k+r+w)\,B_{n-j,q}^{(k)}(-w)\right)\frac{t^n}{[n]_q!}\end{aligned}$$

where $r=(1-q)(1-x)_q$. Thus we conclude the paper with the following theorem.

Theorem 7. *Let $r=(1-q)(1-x)_q$. Then we have*

$$b_{k,n}^q(x)=x^k\sum_{j=0}^{n}\binom{n}{j}_q S_{q,r,w}(j+r+w,k+r+w)\,B_{n-j,q}^{(k)}(-w).$$

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