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Article

Recurrence and Entropy for Discrete-Time Deterministic Dynamical Systems

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Abstract

We investigate discrete-time deterministic systems on finite state spaces equipped with symmetry groups, extending the analysis to actions of arbitrary countable linearly ordered groups. Under the assumption of strong recurrence, characterized by the absence of weakly wandering sets of positive measure, we establish the structural constraints governing dynamical invariants. For systems associated with amenable groups, we employ Følner sequences to rigorously define asymptotic frequencies and demonstrate that maximal Shannon entropy emerges naturally from the system's architecture rather than stochastic assumptions. We show that the interplay of strong recurrence and symmetry enforces specific distribution patterns; while transitive symmetry leads to a uniform stationary distribution and maximal entropy, we provide a generalized formula for non-transitive cases based on orbit decomposition. These results bridge classical recurrence theory and ergodic decomposition with modern measure-theoretic entropy, illustrated through concrete examples for both finite and infinite countable settings.

Keywords: discrete-time systems; recurrence; entropy; symmetry

1. Introduction

Symmetry and recurrence play a central role in the behavior of finite dynamical systems [6]. Classical work on recurrence and wandering sets was developed by Hajian and Kakutani [8] and later extended in the infinite-measure setting by Aaronson [1].

The interplay of these properties underlies phenomena ranging from cellular automata [17] to deterministic Markov chains [6], yet a comprehensive treatment connecting structural symmetry to emergent information-theoretic bounds remains lacking. Prior work has explored orbit decomposition under symmetry [11], period distributions in recurrent dynamics [12], and maximal entropy in stochastic settings [3], but little is known about deterministic systems where these features coexist. Understanding how deterministic recurrence and symmetry constrain long-term behavior is crucial not only for theoretical dynamical systems but also for applications in cryptography, combinatorial design, and the study of chaotic one-dimensional maps [15].

The current paper is a continuation of our recent paper on strongly recurrent dynamics [2]. In this paper, we consider discrete-time systems $(X_t)_{t \in \mathbb{Z}}$ evolving on finite state spaces Ω , equipped with a finite symmetry group G acting on Ω . Our focus is on systems exhibiting *strong recurrence*: every state in the effective state space Ω_{eff} is visited infinitely often by its own trajectory. This notion generalizes classical Poincaré and Furstenberg recurrence [6] and draws inspiration from studies of symmetric cellular automata [17] and recurrent combinatorial structures [12]. Strong recurrence, unlike ordinary recurrence, allows us to extract structural information about the orbit organization and dynamical invariants without invoking any stochastic assumptions.

A deeper motivation comes from the theory of wandering sets developed by Hajian and Ito [10]. In measure-preserving transformations, wandering sets represent states that are never revisited,

obstructing recurrence. By contrast, in strongly recurrent deterministic systems, such wandering phenomena are absent in Ω_{eff} , allowing the dynamics to fully explore all symmetry-invariant structures. This connection provides a bridge between classical ergodic theory and our finite deterministic setting, highlighting how recurrence and symmetry jointly enforce global structural constraints.

The central challenge lies in quantifying the joint effect of symmetry and recurrence on the system's invariant structure and information-theoretic properties. While previous studies analyzed symmetry-induced orbit decompositions [11] or recurrence in isolation [6], [12], a unified framework addressing both aspects and their impact on entropy has remained absent. In particular, deterministic systems with strong recurrence and symmetry can exhibit unexpectedly rich behavior: the deterministic trajectories, constrained solely by the structure of the system, may achieve information-theoretic maximality akin to uniform randomness in stochastic models [3]. The emergence of maximal Shannon entropy in this context is especially surprising because it arises purely from structural and combinatorial constraints, without introducing any stochastic assumptions.

Our contributions address these gaps through a detailed analysis of orbit structures, invariant measures, and emergent entropy in deterministic systems. We analyze the distribution of orbit lengths and establish tight bounds on periods. Unlike previous results that treat period distributions statistically, our analysis shows that the combination of strong recurrence and symmetry enforces strict regularities. These bounds are crucial for subsequent entropy calculations, as they determine the maximum unpredictability attainable within the system's deterministic constraints. In effect, the structure mimics key features of entropy-maximizing stochastic systems [3], providing a deterministic analogue of maximal uncertainty.

We demonstrate that the stationary distribution induced by the dynamics attains maximal Shannon entropy permitted by the orbit structure in the transitive amenable case. This result reveals a surprising phenomenon: deterministic systems can be as informationally rich as their stochastic counterparts purely through structural constraints. In contrast to classical entropy results for stochastic processes [3], here maximal entropy emerges without introducing any randomness, highlighting an intrinsic connection between deterministic symmetry, recurrence, and uncertainty. In a natural class of systems, strong recurrence and symmetry alone guarantee that the maximal entropy bound is exactly achieved. This structural emergence of maximal entropy contrasts sharply with prior approaches requiring uniformity or randomness assumptions, and it provides new insight into the relationship between determinism and unpredictability. Moreover, this result illuminates connections to dynamical invariants and classical recurrence theory: the system's invariant measures, orbit lengths, and return-time distributions align in such a way that entropy is maximized across all accessible states. By linking our results to Furstenberg recurrence [6] and Hajian–Ito wandering sets [10], we show that these entropy-maximizing properties are not incidental but are enforced by the fundamental structure of the dynamics.

These results collectively demonstrate that deterministic recurrence and symmetry are not merely constraints but potent mechanisms shaping the global information content of a system. The analysis also sheds light on counterintuitive phenomena, such as deterministic systems that are effectively as unpredictable as purely random processes, and the sensitivity of maximal entropy to symmetry-breaking perturbations. Our framework therefore bridges classical recurrence theory [6,10], deterministic dynamical systems [17], and information-theoretic measures [3], providing both theoretical insights and practical guidance for constructing systems with controlled yet highly unpredictable behavior.

2. Materials and Methods

The study of recurrence and wandering sets represents a fundamental pillar in the classification of measure-preserving transformations, particularly within infinite measure spaces where classical Poincaré recurrence often fails. The evolution of this field reflects a transition from the intuitive, linear dynamics of the integers to the more sophisticated framework of linearly ordered groups, a shift championed significantly by Vidhu Prasad [16] and recently refined by the authors in [2]. In

the classical setting of $\mathbb{Z}$ -actions, the notion of "moving forward" is naturally dictated by the set of natural numbers. This inherent order allows for the definition of weakly wandering sets, i.e., sets whose images under a sequence of transformations are pairwise disjoint. To extend these concepts to a general countable group, the notion of moving forward in time must be explicitly constructed. This is achieved through the introduction of a positive cone. The positive cone defines a (linear) order on the group, providing the necessary "direction" to identify sequences that wander away from their origin. The order serves as a mathematical "compass" that ensures transformations propagate through the space rather than cycling redundantly. Without a linear order, the very definition of a wandering sequence would collapse into algebraic noise.

The landmark contribution of Prasad [16] was demonstrating that the deep connection between recurrence and wandering sets is not a quirk of the integers, but a structural property of ordered groups. He established that strong recurrence—the property that a system eventually returns to every set of positive measure—is strictly equivalent to the absence of weakly wandering sets of positive measure. He also showed that for certain ergodic actions, strong and strict recurrence are not equivalent, providing counterexamples within these specific group structures. Prasad’s work was pivotal because it proved that the Hopf Decomposition (dividing a space into a "conservative" part that recurs and a "dissipative" part that wanders) remains robust in any group that admits a linear order. Furthermore, he introduced a critical nuance by proving that strong recurrence and strict recurrence (a bounded return time) are not equivalent, providing counterexamples that highlighted the complexity of infinite measure systems. The growth rates of wandering sequences—how fast a set "spreads" without overlapping—could distinguish between transformations that are algebraically identical but dynamically distinct. This progression reached its zenith in the monograph by Eigen et al [5], where the interaction between a group’s order and its measure-preserving actions provided a lens for classifying infinite ergodic systems.

Building upon Prasad’s foundation, the authors in [2] further clarified the boundaries of these recurrence properties. While Prasad focused on the existence of weakly wandering sets, we explored the implications of these properties for the existence of finite invariant measures. Our results in [2] suggested that while strong recurrence prevents "drifting", it does not alone guarantee a finite measure and yet utilize the geometry of the positive cone to define more stringent conditions to bridge the gap between mere recurrence and statistical stability.

In our setup, we let $(\Omega, \mathcal{B}, m)$ be a σ -finite measure space, and let G be a countable ordered group with positive cone G^+ . While the classical setup of strongly recurrent systems mainly deals with *non-atomic* measure spaces, we intentionally do not impose non-atomic assumption on m to have more flexible setting, unless it is needed (like in Proposition 4). We refer the reader to Remark 7 for details on how this assumption is reflected in our study. Throughout this paper, we assume G is a *linearly* ordered group; that is, for any $g, h \in G$, either $g \leq h$ or $h \leq g$. While the semigroup structure of G^+ is all that is required for the proofs of the main results, the linear order provides a natural directed time evolution and simplifies the formulation of Definition 1. A typical example is $G = \mathbb{Z}$ with its canonical positive cone. A less trivial, but yet standard example is $G = \mathbb{Z}^2$ with lexicographic order

$$G^+ = \{(a, b) \in \mathbb{Z}^2 : a > 0 \text{ or } (a = 0, b \geq 0)\},$$

that provides a nontrivial positive cone structure distinct from $\mathbb{Z}^+$. We consider measurable actions of G on Ω such that m is quasi-invariant. The order structure on G is used primarily to define a directed time evolution via G^+ . However, as the proofs of the main results require only that G^+ is an upward-directed semigroup, our results extend to semigroup actions without requiring full bi-orderability.

Definition 1 (Deterministic Dynamical System). *A deterministic discrete-time system on a finite state space Ω is defined by a trajectory mapping $X : G^+ \rightarrow \Omega$, where G^+ is the positive cone of a countable ordered group G . For any time index $t \in G^+$, the state of the system is given by $X(t) = X_t$. The sequence $(X_t)_{t \in G^+}$ is generated by*

a transition map $T : \Omega \rightarrow \Omega$ such that $X_{t+1} = T(X_t)$ for all $t \in G^+$. A trajectory is the sequence of values (X_t) starting from an initial state x_0 . The effective state space is defined as $\Omega_{\text{eff}} = \{x \in \Omega : \exists t \in G^+, X_t = x\}$.

Remark 1. (i) The definition of Ω_{eff} depends a priori on the initial condition x_0 : it is the forward orbit $\{T^t(x_0) : t \in G^+\}$ of x_0 under the transition map T . In general, different initial conditions may yield different effective state spaces. Under the transitivity assumption however, every state is reachable from every other state, and $\Omega_{\text{eff}} = \Omega$ for every initial condition. In the non-transitive case, $\Omega_{\text{eff}} = \Omega_{\text{eff}}(x_0)$ is the T -orbit of x_0 , and the entropy result (Corollary 6) is stated relative to this orbit. The cardinality $|\Omega_{\text{eff}}|$ is therefore a property of the initial condition in general, and a global invariant of the system only under transitivity. Under the ergodicity assumption, Ω_{eff} is well-defined up to null sets: for m -almost every initial condition, the resulting effective space coincides modulo null sets (see, e.g., [12]).

(ii) In strongly recurrent systems (Definition 4), the notation X_t refers to the realizations that visit every state in the effective state space Ω_{eff} infinitely often. Note that the random variable X is not the trajectory itself but a random variable whose distribution equals the time-average frequency of state visits along the trajectory. This is analogous to the way one defines an empirical distribution from a deterministic sequence: given a sequence $(x_1, x_2, \dots, x_n)$, the empirical measure $(1/n) \sum_i \delta_{x_i}$ is a deterministic object, but it can be regarded as the distribution of a random variable that selects a time uniformly at random. The 'randomness' is entirely encoded in the time-selection. Under unique ergodicity the empirical measure converges to the unique invariant measure.

Definition 2 (Symmetry of the Dynamics). A group G is a symmetry group of the deterministic system if the action of G on Ω commutes with the transition map T :

$$T(g \cdot x) = g \cdot T(x),$$

for all $g \in G, x \in \Omega$. The action is transitive on Ω_{eff} if for every $x, y \in \Omega_{\text{eff}}$, there exists $g \in G$ such that $g \cdot x = y$.

The foundational study of wandering sets in measure-preserving transformations goes back to [9,10].

Definition 3 (Weakly Wandering Set). A measurable set $W \subset \Omega$ is weakly wandering if there exists a subset $S \subset G$ such that the translates $\{sW : s \in S\}$ are pairwise disjoint.

Definition 4 (Strong Recurrence). We say that (G, Ω, m) is strongly recurrent if for every measurable set B with $m(B) > 0$, there exists a finite set $F \subset G^+$ such that for every $t \in G$, there exists $s \in F$ with $m(tB \cap sB) > 0$.

Remark 2. Requiring the finite recurrency set $F \subset G^+$ to cover all $t \in G^+$ is equivalent to requiring coverage for all $t \in G$, for any ordered group G with a quasi-invariant measure. Indeed, for $t \in G^-$ and $s \in G^+$, we have $st^{-1} \in G^+$ and $m(tB \cap sB) > 0$ iff $m(B \cap st^{-1}B) > 0$ by quasi-invariance. Hence coverage of t by $s \in F$ is equivalent to coverage of e by $st^{-1} \in F' \subset G^+$ for a suitable finite set F' . The two formulations of Definition 4 are therefore equivalent and could be used interchangeably.

Strong recurrence can be viewed as a condition excluding weakly wandering behavior and guaranteeing recurrence along algebraic structures in G^+ . By a result of Hajian and Ito, a nonsingular transformation on a non-atomic measure space admits no weakly wandering set of positive measure if and only if it admits an equivalent invariant σ -finite measure [10]. More generally, let (G, Ω, m) be strongly recurrent with m non-atomic. Then no measurable subset of positive measure can be weakly wandering. Indeed, by a slight modification of Theorem 3.2 of [2], strong recurrence is equivalent to the absence of weakly wandering sets. The proof of this equivalence given in [2] works for quasi-invariant measures as well, as it does not require full measure invariance. Therefore, every positive-measure set must intersect infinitely many of its translates.

The notion of strong recurrence strengthens classical recurrence by requiring a uniform finite recurrency set. To see that strong recurrence implies Poincaré recurrence, let B be strongly recurrent set of strictly positive measure and let $F \subset G^+$ be the corresponding finite set, then there exists $s \neq e$ such that $m(B \cap sB) > 0$, implying Poincaré recurrence. The converse is false in general, as in infinite measure-preserving systems, one can construct sets that return infinitely often but do not admit a finite recurrency set. A standard counterexample is a conservative infinite-measure-preserving transformation that admits a set B whose return times exist but do not admit a finite recurrency set F (take an irrational rotation on an infinite covering space). Strong recurrence also implies the Furstenberg recurrence (requiring multiple return times). This follows from the definition, since the finite recurrency set F guarantees return to within a bounded time window.

The relationship between conservativity and minimal recurrence is further explored in [19], where explicit conditions for the existence of minimal recurrency sets are established in the finite-measure setting. This has motivated the following notion of strict recurrency.

Definition 5 (Strictly Recurrent Action). *An action (G, Ω, m) is strictly recurrent if it is strongly recurrent and there exists $\epsilon > 0$ such that for all B with $m(B) > 0$, the finite set $F \subset G^+$ can be chosen to satisfy*

$$\sup_{s \in F} m(tB \cap sB) \geq \epsilon m(B),$$

for all $t \in G$.

In our current framework, the definition of Shannon entropy $H(X)$ is not merely a statistical measure but an emergent property of the system's structural constraints. By considering X as a random variable taking values in the effective state space Ω_{eff} , we move beyond simple deterministic trajectories to characterize the global distribution of states under the long-term dynamics. This transition is motivated by the fact that for strongly recurrent systems on arbitrary countable ordered groups, the absence of weakly wandering sets ensures that every accessible state is visited with a well-defined relative frequency. Consequently, the entropy quantifies the information-theoretic maximality that arises when the symmetry group G acts transitively on Ω_{eff} , forcing an equalization of these frequencies. Thus, $H(X)$ serves as a rigorous bridge between the deterministic symmetry of the transformation group and the uniform unpredictability typically associated with stochastic processes.

Definition 6 (Deterministic Shannon Entropy). *Let $(X_t)_{t \in G^+}$ be a deterministic system on a finite state space Ω with effective state space Ω_{eff} . Let μ be the unique G -invariant probability measure on Ω_{eff} induced by the strong recurrence and symmetry of the system. The random variable X is defined to follow the distribution $p(x) = \mu(\{x\})$. The Shannon entropy of the system is:*

$$H(X) = - \sum_{x \in \Omega_{eff}} p(x) \log p(x)$$

where $p(x)$ is the weight assigned to each state by the G -invariant measure.

Remark 3. (i) If the action of G on Ω is not transitive, $\Omega_{eff} = \Omega_{eff}(x_0)$ and so $H(X)$ may depend on the initial point x_0 , yet $H(X)$ is a well-defined invariant of the orbit of x_0 under T . Even in this case, we still use the same notation $H(X)$ instead of less conventional notation $H(X|x_0)$. This is a typical feature of non-transitive systems, as also noted by the authors in [4]. Note however that, While both the current paper and that of Daza et al. [4] address the emergence of entropy in deterministic systems, they diverge sharply in how they interpret the "uncertainty" of a non-transitive phase space. Here, non-transitivity is a structural partition where entropy remains a local invariant of the orbit and quantifies the "internal" richness of a specific trajectory as it is algebraically compelled to explore its own basin of attraction, whereas Daza et al. view non-transitivity through the lens of final-state unpredictability, shifting the focus from the orbit's internal path to the "external" geometry of the basin boundaries. To Daza et al., the entropy of a point deep within a basin is effectively zero because its

destination is certain, whereas uncertainty (and thus entropy) peaks at the fractal edges where basins collide. Consequently, while we aim at providing a tool to measure the complexity of the destination itself, Daza et al. provide a metric for the difficulty of reaching it, making the former a study of dynamical architecture and the latter a study of phase-space topology.

(ii) Under the assumption of a transitive G -action on Ω_{eff} , $p(x) = \frac{1}{|\Omega_{\text{eff}}|}$ for all $x \in \Omega_{\text{eff}}$. Note that in the latter case, X_t visits every point at least once, and so the denominator of the last equality is always nonzero. When G is finite, while the phase space (Ω, m) is non-atomic, the structural constraints of a finite symmetry group G imply that the effective state space Ω_{eff} consists of a finite set of points. In this case, $m(\Omega_{\text{eff}}) = 0$, and we lift the dynamics to an atomic probability measure μ supported on Ω_{eff} for the calculation of entropy. As a typical example, let (X_t) be a deterministic system on a finite state space Ω satisfying strong recurrence and admitting an amenable symmetry group G acting transitively on Ω_{eff} . Then the stationary distribution is uniform and $H(X) = \log |\Omega_{\text{eff}}|$. Note that by transitivity, $|\Omega_{\text{eff}}| \geq 1$. We restate this result and prove it in Section 3.3. The rationale for starting from the non-atomic ambient space, rather than working on Ω_{eff} directly, is that the hypotheses of strong recurrence and quasi-invariance are most naturally stated for the full space (Ω, m) . The finite effective space is a derived object, and the reduction to it is what allows us to apply finite combinatorics in orbit enumeration (see Theorem 5).

3. Results

The information-theoretic properties of a system are inseparable from its topological and group-theoretic constraints. By modeling the system through a trajectory mapping $X : G^+ \rightarrow \Omega$ indexed by a countable ordered group G , we ensure that the asymptotic behavior of the system is well-defined via Følner averaging. The transition from strong recurrence to strict recurrence provides the necessary bridge between qualitative return properties and quantitative stability. The strong recurrence guarantees that the system visits every measurable set infinitely often, while strict recurrence ensures these returns occur with a uniform lower bound, precluding the existence of weakly wandering sets. Crucially, by defining the effective state space Ω_{eff} as the empirical range of the mapping, we resolve the measure-theoretic conflict inherent in non-atomic background spaces (Ω, m) . This allows us to prove that for any amenable group acting transitively, the symmetry of the architecture crystallizes the dynamics into a uniform distribution. The deterministic Shannon entropy $H(X)$ reflects a fundamental symmetry between the group's architecture and the state space, where the information content is forced to represent the total capacity of the accessible configuration space. In this view, uncertainty is not a lack of knowledge about the system's path, but a measure of the structural complexity required to satisfy the symmetry constraints imposed by the group action.

3.1. Entropy

The emergence of maximal information content in deterministic dynamics is a central result of this study, demonstrating that intrinsic structural constraints alone can produce a state of maximal uncertainty. In a strongly recurrent system equipped with a symmetry group G acting transitively on the effective state space Ω_{eff} , the deterministic trajectories are forced to explore the available configurations in a manner that mimics uniform randomness. This phenomenon arises from the joint action of strong recurrence, which eliminates weakly wandering behavior to ensure every accessible state contributes to the long-term dynamics, and symmetry, which requires all states within an orbit to appear with identical asymptotic frequency. Consequently, the stationary distribution of the system becomes uniform, and the Shannon entropy $H(X)$ naturally achieves its theoretical maximum of $\log |\Omega_{\text{eff}}|$ without the need for any stochastic assumptions.

Example 1. (i) Let $\Omega = \mathbb{Z}_n$ with cyclic shift $X_{t+1} = X_t + 1 \pmod n$ and $G = \mathbb{Z}_n$ acting by shifts. Strong recurrence holds trivially. Each orbit is the full cycle and stationary measure is uniform, while $H_{\text{max}} = \log n$ is achieved.

(ii) Perturbing the above system by fixing one state under all dynamics reduces the effective orbit. Maximal entropy is no longer achieved, illustrating sensitivity to structural assumptions.

(iii) We construct a finite deterministic map on $\Omega = \{0, 1\}^3$ with symmetry group $G = \mathbb{S}_3$ acting by permutation of coordinates. Define the transition map $T : \Omega \rightarrow \Omega$ by

$$T(x_1, x_2, x_3) = (x_2, x_3, x_1),$$

which corresponds to the cyclic permutation $(123) \in \mathbb{S}_3$. The orbits of T are: $\mathcal{O}_1 = \{(0, 0, 0)\}$ (length 1), $\mathcal{O}_2 = \{(1, 1, 1)\}$ (length 1), $\mathcal{O}_3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ (length 3), and $\mathcal{O}_4 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ (length 3). This gives $|\Omega_{\text{eff}}| = 8$ states. Under the assumption that the symmetry group $G = \mathbb{S}_3$ acts transitively on Ω_{eff} (which holds here since any two elements of Ω_{eff} are related by a coordinate permutation in $\mathbb{S}_3$), we get a uniform stationary distribution $p(x) = 1/8$ for all $x \in \Omega_{\text{eff}}$, and $H_{\text{max}} = \log 8$.

(iv) (Non-uniform orbit structure) Let $\Omega = \{a_1, a_2, a_3, b_1, b_2, c_1\}$ with $|\Omega| = 6$. Define three G -orbits

$$\mathcal{O}_1 = \{a_1, a_2, a_3\}, \quad \mathcal{O}_2 = \{b_1, b_2\}, \quad \mathcal{O}_3 = \{c_1\},$$

so $|\mathcal{O}_1| = 3$, $|\mathcal{O}_2| = 2$, $|\mathcal{O}_3| = 1$. Let $G = \mathbb{Z}_6$ act by cyclic shift along the ordering $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_1$, $b_1 \rightarrow b_2 \rightarrow b_1$, $c_1 \rightarrow c_1$, and define the transition map T to coincide with the generator of this action. Let μ be the unique G -invariant probability measure on Ω . Strong recurrence holds on $\Omega_{\text{eff}} = \Omega$ since every state lies in a finite cycle. The G -action is not transitive on Ω , but since μ is G -invariant and G acts transitively on each $\mathcal{O}_i$, for any $x, y \in \mathcal{O}_i$, there exists $g \in G$ with $gx = y$, hence $\mu(\{x\}) = \mu(\{y\})$. Together with $\sum_{x \in \mathcal{O}_i} \mu(\{x\}) = \mu(\mathcal{O}_i)$, this gives $\mu(\{x\}) = \mu(\mathcal{O}_i)/|\mathcal{O}_i|$ for all $x \in \mathcal{O}_i$. The Shannon entropy is then calculated as follows

$$\begin{aligned} H &= \sum_{i=1}^3 \mu(\mathcal{O}_i) \log |\mathcal{O}_i| = \frac{3}{6} \log 3 + \frac{2}{6} \log 2 + \frac{1}{6} \log 1 \\ &= \frac{1}{2} \log 3 + \frac{1}{3} \log 2 \approx 0.781 + 0.231 = 1.012, \end{aligned}$$

which is strictly less than $\log 6 \approx 1.792$, illustrating that non-transitivity strictly reduces entropy below the global maximum.

(v) Let $\Omega = \mathbb{Z}$ and let the symmetry group be the countable group $G = \mathbb{Z}$ acting by translation. Define the deterministic dynamics $X_{t+1} = X_t + 1$. Each orbit equals the entire state space. The system is strongly recurrent modulo periodic boundary conditions, illustrating the infinite counterpart of the finite orbit decomposition in the previous example.

(vi) (Strongly recurrent but non-periodic system) Let $G = \mathbb{Z}$ act on a Sturmian shift (Ω_α, σ) , where $\alpha \in (0, 1)$ is irrational and Ω_α is the orbit closure of the Sturmian sequence $s_\alpha = (\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor)_{n \in \mathbb{Z}}$ under the shift σ . This system is not periodic since α is irrational. Strong recurrence holds because Sturmian shifts are uniformly recurrent: for every word w appearing in s_α and every $\ell \geq 1$, there exists $N = N(w)$ such that w appears in every block of length N of s_α (see [12], Chapter 13). Uniform recurrence implies strong recurrence in our sense: for any cylinder set B of positive measure, the finite recurrency set $F = \{0, 1, \dots, N(w)\}$ satisfies $m(tB \cap sB) > 0$ for some $s \in F$ and every $t \in \mathbb{Z}^+$.

(vii) (Non-cyclic strongly recurrent system) Let $G = \mathbb{Z}^2$ with lexicographic order and let $\alpha, \beta \in (0, 1)$ be rationally independent irrationals. Define the action on $\Omega = \mathbb{T}^2 = [0, 1)^2$ by

$$(n_1, n_2) \cdot (x_1, x_2) = (x_1 + n_1\alpha \bmod 1, x_2 + n_2\beta \bmod 1).$$

The system is not periodic: a trajectory $(x_1 + n_1\alpha, x_2 + n_2\beta)$ returns to its starting point only if $n_1\alpha \in \mathbb{Z}$ and $n_2\beta \in \mathbb{Z}$, which is impossible for irrational α, β . Strong recurrence holds by the two-dimensional equidistribution Weyl theorem [18]: for any rectangle $B = [a_1, b_1) \times [a_2, b_2) \subset \mathbb{T}^2$ of positive measure, the sequence of visits to B is syndetic, and the finite recurrency set F can be taken to be any sufficiently large finite subset of $\mathbb{Z}_+^2$ determined by the simultaneous Diophantine approximation properties of α and β .

Remark 4. (i) In above examples we have encountered cases where maximal entropy is achieved or missed. It is worth noting that the maximal entropy arises due to orbit decomposition and recurrence, without invoking randomness. The absence of wandering sets guarantees that every accessible state contributes, structurally enforcing $H_{\max}$. It demonstrates that intrinsic system constraints alone can produce maximal information content. The maximal entropy value $H_{\max} = \log |\Omega_{\text{eff}}|$ is therefore achieved without introducing any stochastic assumption. Instead it arises purely from two structural properties of the system: strong recurrence, which eliminates wandering states and ensures that every accessible state contributes to the long-term dynamics; symmetry with transitive action, which forces all states to appear with equal asymptotic frequency. More precisely, the claim is not that the uniform distribution is surprising in itself. The uniform distribution over n states maximizes Shannon entropy among all distributions on n states, by the standard concavity argument. The non-trivial point of our examples (and subsequent results) is that a purely deterministic, non-random system is forced by its algebraic structure (strong recurrence plus transitive symmetry) to visit every state with equal long-run frequency, thereby realizing the uniform distribution without any probabilistic assumption. In a generic deterministic system this is false: the system might be periodic with period shorter than $|\Omega_{\text{eff}}|$, or might visit some states more often than others. Our contribution is to identify the exact structural conditions (strong recurrence and transitive G -action) that are jointly necessary and sufficient to force uniformity for the action of finiter (or more generally amenable) groups (see Theorem 5 and Corollary 6).

(ii) The entropy $H(X)$ is the Shannon entropy of the stationary distribution on the state space, not the Kolmogorov–Sinai (metric) entropy of the dynamical system (which would be zero for a periodic deterministic system). One needs to carefully distinguish these two notions. While the Kolmogorov–Sinai entropy of a deterministic permutation of a finite set is always zero (since the partition entropy of any partition grows at most polynomially), the Shannon entropy of the time-stationary distribution is non-trivial and captures the uncertainty in the state at a randomly chosen time. The analogy with stochastic systems is that a uniformly ergodic Markov chain on n states also has stationary Shannon entropy $\log n$, whereas our result will demonstrate that a deterministic system can achieve the time-averaged uncertainty.

When the state space are infinite, the structure changes substantially.

Proposition 1 (Infinite Orbit Structure). *Let G be a countable group acting on an infinite state space Ω . If the system is strongly recurrent and conservative, then every measurable subset of positive measure intersects infinitely many orbit points.*

Proof. Let $B \subset \Omega$ with $m(B) > 0$. Strong recurrence implies that for every $t \in G$ there exists $s \in G^+$ such that

$$m(tB \cap sB) > 0.$$

Iterating this construction yields infinitely many return times. Hence infinitely many orbit elements intersect B . $\square$

We postpone the maximal entropy result for the case of amenable groups to Section 3.3.

3.2. Strong recurrence

In this section we study strong recurrence for finite and infinite countable symmetry groups. We demonstrate that strong recurrence and symmetry enforce deep structural constraints in finite deterministic systems, leading to maximal Shannon entropy in a purely structural manner. Connections to Furstenberg recurrence, Hajian–Ito wandering sets, and dynamical invariants provide a rigorous foundation linking deterministic dynamics to information-theoretic measures. Future work will explore infinite-state analogs and applications to combinatorial design and cryptography.

To start the discussion, let us to the relation between weakly wandering sets and conservativity by showing that strong recurrence implies conservativity (compare with [7]).

Proposition 2. *If (G, Ω, m) is strongly recurrent, then it is conservative: for every A with $m(A) > 0$, there exists $g \neq e$ in G with $m(A \cap gA) > 0$. Conversely, if (G, Ω, m) is conservative and admits an equivalent finite invariant measure, then it is strongly recurrent.*

Proof. Fix A with $m(A) > 0$. By strong recurrence, there exists finite $F \subset G^+$ such that for all $t \in G$, some $s \in F$ satisfies $m(tA \cap sA) > 0$. Taking $t = e$ yields some $s \in F$, $s \neq e$, with $m(A \cap sA) > 0$, proving conservativity.

Conversely, let ν be a finite G -invariant measure equivalent to m , and fix $B \subset \Omega$ with $m(B) > 0$. Since $\nu \sim m$, also $\nu(B) > 0$. Set

$$N := \left\lfloor \frac{\nu(\Omega)}{\nu(B)} \right\rfloor + 1.$$

Since ν is G -invariant, every translate gB satisfies $\nu(gB) = \nu(B) > 0$. Since $\nu(\Omega) < \infty$, any collection of more than N translates of B cannot be pairwise ν -disjoint. Fix any N elements $g_1, \dots, g_N \in G^+$ and set $F = \{g_1, \dots, g_N\}$. For any $t \in G^+$, the $N+1$ translates $tB, g_1B, \dots, g_NB$ cannot all be pairwise disjoint, so tB must intersect some g_iB , giving $\nu(tB \cap g_iB) > 0$. Since $\nu \sim m$, also $m(tB \cap g_iB) > 0$. Thus F is a finite recurrency set for B in the sense of Definition 4, establishing strong recurrence of (G, Ω, m) . $\square$

Example 2 (Integer shift on the unit interval). Let $\Omega = [0, 1)$ with Lebesgue measure μ , and let $G = (\mathbb{Z}, +)$ act on Ω by the rotation

$$T_n(x) = x + n\alpha \pmod{1},$$

where $\alpha \in (0, 1)$ is irrational. Fix a measurable set $B = [0, \beta)$ with $0 < \beta < 1$. Since $\{n\alpha \pmod{1}\}$ is dense in $[0, 1)$, for any $t \in \mathbb{Z}$ there exists $s \in F$ such that $T_t(B) \cap T_s(B) \neq \emptyset$. Consider $F = \{0, 1, \dots, N\}$ with N such that $N\alpha$ approximates all elements of $[0, 1)$ within β . Let us check the minimality. Removing any s from F may leave some t without intersection, so F is minimal. For quantitative bounds, observe that $\sup_{s \in F} \mu(T_t(B) \cap T_s(B)) \geq \epsilon \mu(B)$ with $\epsilon = \min\{1, \beta/N\}$.

The above example generalizes to compact abelian groups with irrational rotations, where minimal recurrency sets are finite and explicitly computable. The notion of minimal recurrency sets was first studied in ordered groups by Prasad [16].

Definition 7 (Minimal Recurrency Set). For $B \subset \Omega$ with $m(B) > 0$, a finite set $F \subset G^+$ is a minimal recurrency set if for all $t \in G$, there exists $s \in F$ such that $m(tB \cap sB) > 0$, and no proper subset of F satisfies this condition.

Proposition 3. *Every strongly recurrent set B admits a minimal recurrency set.*

Proof. Let $B \subset \Omega$ with $m(B) > 0$. By strong recurrence, there exists at least one finite recurrency set $F_0 \subset G^+$. If F_0 is not minimal, remove elements one by one, checking whether the remaining subset still satisfies the intersection condition for all $t \in G$. Continue until no further removal is possible. This yields a minimal recurrency set $F \subset F_0$, as desired. The process terminates in finitely many steps since F_0 is finite. $\square$

Theorem 1 (Uniform Bound on Minimal Recurrency Sets). Let (G, Ω, m) be strongly recurrent and fix $\delta > 0$. There exists $N(\delta) \in \mathbb{N}$ such that for every measurable B with $m(B) \geq \delta$, any minimal recurrency set for B has cardinality at most $N(\delta)$.

Proof. Since Ω is σ -finite, we may cover Ω by countably many sets of measure at most 1. For B with $m(B) \geq \delta$, consider the collection of intersections $\{tB \cap sB : t \in G, s \in F\}$. By strong recurrence, each t has some s giving positive measure. Finiteness of F ensures that the number of distinct s needed cannot exceed a constant $N(\delta)$ depending only on δ and the quasi-invariance constants. This gives the uniform bound. $\square$

Definition 8 (δ -Strong Recurrence). An action (G, Ω, m) is δ -strongly recurrent if there exists finite $F \subset G^+$ such that for all $t \in G$, some $s \in F$ satisfies

$$m(tB \cap sB) \geq \delta m(B).$$

Theorem 2 (Uniform Recurrence and Density). Let (G, Ω, m) be a strongly recurrent transformation group with $m(\Omega) < \infty$. For each measurable set $B \subset \Omega$ with $m(B) > 0$, we have

- (i) There exists a finite set $F \subset G^+$ and $c > 0$ such that $\sup_{s \in F} m(tB \cap sB) \geq c$ for infinitely many $t \in G$,
- (ii) The set of return times $R_B = \{g \in G^+ : m(B \cap gB) > 0\}$ has positive lower density inside G^+ .

Proof. (i) Since $m(\Omega) < \infty$, strong recurrence implies finite overlaps cover Ω . Let F be a minimal recurrency set. By the pigeonhole principle, some $s \in F$ must intersect infinitely many tB , giving a uniform lower bound c depending on $m(B)$ and $|F|$.

(ii) If R_B had zero density, we construct a weakly wandering set, which contradicts the fundamental characterization of strong recurrence. We proceed in several steps.

Step 1 (Infinitely many intersections). By strong recurrence, for every $t \in G$ there exists $s \in G^+$ such that

$$m(tB \cap sB) > 0.$$

Thus infinitely many return times exist.

Step 2 (Exclusion of sparse returns). Suppose for contradiction that

$$\liminf_{n \rightarrow \infty} \frac{|R_B \cap F_n|}{|F_n|} = 0.$$

We construct $g_1, g_2, \dots \in G^+$ inductively such that $m(g_i B \cap g_j B) = 0$ for all $i \neq j$. Choose $g_1 \in G^+$ arbitrarily. Given $g_1, \dots, g_k$, define

$$G_k := \{g \in G^+ : m(gB \cap g_i B) > 0 \text{ for some } i \leq k\} = \bigcup_{i=1}^k g_i R_B,$$

where the equality holds because $m(gB \cap g_i B) > 0$ iff $g_i^{-1}g \in R_B$ iff $g \in g_i R_B$. By the Følner property, for each fixed i ,

$$\frac{|g_i R_B \cap F_n|}{|F_n|} \leq \frac{|R_B \cap F_n|}{|F_n|} + \frac{|g_i F_n \triangle F_n|}{|F_n|} \rightarrow 0,$$

hence $|G_k \cap F_n|/|F_n| \rightarrow 0$ for each fixed k . In particular $G_k \subsetneq G^+$, so we may choose $g_{k+1} \in G^+ \setminus G_k$. Continuing this process, the resulting family $(g_i B)_{i \geq 1}$ is pairwise essentially disjoint, so B is weakly wandering with witnessing set $\{g_i : i \geq 1\} \subset G^+$. This contradicts strong recurrence, so R_B must have positive lower density. $\square$

The next observation is that strong recurrence excludes orbit escape.

Proposition 4. Let (G, Ω, m) be strongly recurrent and non-atomic. Then no orbit of positive measure can escape to infinity, in the sense that for every measurable set B with $m(B) > 0$ there exist infinitely many group elements g such that

$$m(B \cap gB) > 0.$$

Proof. By Theorem 3.2 of [2], strong recurrence is equivalent to the absence of weakly wandering sets. Suppose by contradiction that there exists a set B with $m(B) > 0$ such that

$$m(B \cap gB) = 0,$$

for all but finitely many g . Then the translates gB would be essentially disjoint, which is precisely the definition of a weakly wandering set. This contradicts the characterization of strong recurrence. Hence infinitely many intersections must occur. $\square$

Next let us show that strong recurrence passes to subgroups and cofinal subsemigroups.

Lemma 1 (Inheritance to Subgroups). *Let $H \leq G$ be a subgroup. If (G, Ω, m) is strongly recurrent, then the restricted action (H, Ω, m) is strongly recurrent.*

Proof. Let $B \subset \Omega$ with $m(B) > 0$. By strong recurrence, there exists a finite $F \subset G^+$ such that for every $t \in G$, some $s \in F$ satisfies $m(tB \cap sB) > 0$. Restricting to $t \in H$, the same set F works. Intersect F with H^+ and, if necessary, replace elements by their conjugates in H to ensure all $s \in H^+$. This produces a finite recurrency set for (H, Ω, m) . $\square$

Definition 9 (Subsemigroup Recurrence). *Let $S \subset G^+$ be an infinite, upward-directed subsemigroup. A set B is S -strongly recurrent if there exists finite $F \subset S$ such that for every $t \in S$, some $s \in F$ satisfies $m(tB \cap sB) > 0$.*

Lemma 2. *If (G, Ω, m) is strongly recurrent, then it is S -strongly recurrent for every cofinal subsemigroup $S \subset G^+$.*

Proof. Let B be strongly recurrent. Since S is cofinal, for any $t \in S$, there exists $g \in G^+$ such that $g \geq t$ in the order. By strong recurrence, there exists finite $F \subset G^+$ covering all t . Taking $F_S = F \cap S$ gives a finite set ensuring S -strong recurrence. $\square$

Finally, we show the symmetry invariance of strong recurrence.

Proposition 5. *Let (G, Ω, m) be strongly recurrent and let $\mathcal{H}$ be a group of symmetries acting on Ω that preserves the measure m . Then the extended action of the semidirect product $G \rtimes \mathcal{H}$ is also strongly recurrent.*

Proof. Step 1 (preservation of measure). Each symmetry $h \in \mathcal{H}$ satisfies $m(hA) = m(A)$, for every measurable set A .

Step 2 (recurrence under the original action). Since (G, Ω, m) is strongly recurrent, for every set B with $m(B) > 0$ we have

$$\inf_{t \in G} \sup_{s \in G^+} m(tB \cap sB) > 0.$$

Step 3 (extension to the semidirect product). Let $(g, h) \in G \rtimes \mathcal{H}$. Then $(g, h)B = g(hB)$. Since h preserves measure, the recurrence property applies to hB . Hence, there exists $s \in G^+$ such that

$$m(g(hB) \cap s(hB)) > 0.$$

Applying h^{-1} gives $m((g, h)B \cap (s, e)B) > 0$. Thus the strong recurrence inequality remains valid for the enlarged group. $\square$

The results established throughout this paper reveal that strong recurrence imposes a remarkably rigid structural framework on transformation groups.

3.3. Ergodic Decomposition and Amenable Groups

The long-term behavior of a deterministic system is fundamentally governed by how trajectories distribute themselves across the state space. In this section, we transition from point-wise dynamics to a statistical description of the system by analyzing the asymptotic frequency of visits to different regions of Ω . By leveraging the symmetry group G , we show that the state space can be decomposed into disjoint orbits, each of which functions as an invariant unit under the transition map T . To rigorously

define these frequencies for arbitrary countable groups, we employ the framework of amenability, which provides the necessary averaging structure to ensure that the limit of visit counts is well-defined and independent of the choice of averaging sequence.

Let us first remind two basic definitions. We refer the reader to [13] for more details and illustrative examples.

Definition 10 (Amenability). *A countable group G is said to be amenable if it admits a left-invariant mean. Equivalently, G is amenable if there exists a sequence of finite subsets that stay approximately invariant under translation by any group element. This property ensures that we can consistently define the "average" value of a function over the group.*

Definition 11 (Følner Sequence). *Let G be a countable amenable group. A sequence of finite subsets $(F_n)_{n \in \mathbb{N}}$ of G is called a Følner sequence if for every $g \in G$:*

$$\lim_{n \rightarrow \infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0$$

where Δ denotes the symmetric difference.

A Følner sequence provides the standard exhaustion of the group required to define the asymptotic frequency of visits for a trajectory $(X_t)_{t \in G^+}$. Let (F_n) be a Følner sequence in an amenable group G . The probability $p(x)$ for each state $x \in \Omega_{eff}$ is defined by the Følner limit:

$$p(x) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \#\{t \in F_n \cap G^+ : X_t = x\}$$

The Shannon entropy is $H(X) = -\sum_{x \in \Omega_{eff}} p(x) \log p(x)$.

Proposition 6 (Strong Recurrence and Ergodic Components). *Let (G, Ω, m) be strongly recurrent, with ergodic decomposition*

$$m = \int_{\Omega} m_{\omega} dP(\omega).$$

Then for P -almost every ω , (G, Ω, m_{ω}) is strongly recurrent.

Proof. For measurable $B \subset \Omega$, consider its decomposition $B = \int B_{\omega} dP(\omega)$. Strong recurrence gives finite F such that $m(tB \cap sB) > 0$ for all t . Integrating, there exists a set of full P -measure Ω_0 such that $m_{\omega}(tB_{\omega} \cap sB_{\omega}) > 0$ for all t and some $s \in F$. Hence each ergodic component is strongly recurrent. $\square$

Proposition 7 (Følner Sequence Recurrence for Amenable Groups). *Let G be countable and amenable with Følner sequence (F_n) . If (G, Ω, m) is strongly recurrent, then for any measurable $A \subset \Omega$ with $m(A) > 0$,*

$$\liminf_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{t \in F_n} m(A \cap tA) > 0.$$

Proof. For each n , by strong recurrence, there exists finite $F \subset G^+$ such that each $t \in F_n$ intersects some $s \in F$. Counting overlaps gives

$$\sum_{t \in F_n} m(A \cap tA) \geq \sum_{t \in F_n} \min_{s \in F} m(A \cap sA),$$

and dividing by $|F_n|$ and taking $\liminf$ ensures positivity. $\square$

3.4. Strict Recurrence and Symmetry

In this section, we extend the analysis to the more restrictive framework of strict recurrence, exploring its interaction with the algebraic structure of the symmetry group. While strong recurrence ensures that trajectories return to every measurable set of positive measure, strict recurrence imposes a uniform lower bound on these return frequencies, establishing a higher degree of dynamical stability. We characterize this property through the lens of syndetic sets in G and demonstrate that strict recurrence is invariant under measure equivalence. By formalizing these relationships, we provide the necessary quantitative tools to analyze systems where the information-theoretic maximality is not just an asymptotic limit but is enforced by rigid, uniform structural constraints.

Theorem 3 (Strict Recurrence Characterization). *Let (G, Ω, m) be strictly recurrent. Then for every measurable set $B \subset \Omega$ with $m(B) > 0$, there exists a finite set $F \subset G^+$ and $\epsilon > 0$ such that*

$$\inf_{t \in G} \sup_{s \in F} m(tB \cap sB) \geq \epsilon m(B).$$

Proof. We proceed in several steps.

Step 1 (Syndeticity). By strict recurrence, for every measurable B with $m(B) > 0$, the set

$$R(B) := \{g \in G : m(gB \cap B) > 0\}$$

is syndetic in G , i.e., there exists a finite $F \subset G^+$ such that

$$\bigcup_{s \in F} s^{-1}R(B) = G.$$

Step 2 (Uniform recurrence). Fix $t \in G$. By syndeticity, there exists $s \in F$ such that $t \in s^{-1}R(B)$. Since $st \in R(B)$, we get $m((st)B \cap B) > 0$. Applying the group action property, we have

$$m(tB \cap s^{-1}B) = m(stB \cap B) > 0.$$

Step 3 (Quantitative lower bound). By finiteness of F and σ -finiteness of m , we can define

$$\epsilon := \min_{s \in F} \frac{m(sB \cap B)}{m(B)} > 0.$$

Then for each $t \in G$, there exists $s \in F$ such that $m(tB \cap sB) \geq \epsilon m(B)$, which establishes the desired uniform lower bound. Since F and ϵ are independent of t , we have

$$\inf_{t \in G} \sup_{s \in F} m(tB \cap sB) \geq \epsilon m(B),$$

as required. $\square$

Theorem 4 (Invariance under Measure Equivalence). *Let m and ν be equivalent σ -finite measures on Ω . Then (G, Ω, m) is strongly recurrent if and only if (G, Ω, ν) is strongly recurrent.*

Proof. **Step 1** (Radon-Nikodym derivative). Since m and ν are equivalent, there exists a Radon-Nikodym derivative

$$\frac{d\nu}{dm} = f > 0 \quad m\text{-a.e.}$$

Step 2 (Forward direction). Assume (G, Ω, m) is strongly recurrent. For any $B \subset \Omega$ with $\nu(B) > 0$, equivalence implies $m(B) > 0$. By strong recurrence for m , there exists a finite $F \subset G^+$ and $\epsilon > 0$ such that

$$\inf_{t \in G} \sup_{s \in F} m(tB \cap sB) \geq \epsilon m(B).$$

Since f is bounded away from zero on B (up to null sets),

$$\nu(tB \cap sB) = \int_{tB \cap sB} f \, dm \geq \inf_{x \in tB \cap sB} f(x) m(tB \cap sB) \geq \delta m(tB \cap sB)$$

for some $\delta > 0$. Taking sup over $s \in F$ and inf over $t \in G$ gives

$$\inf_{t \in G} \sup_{s \in F} \nu(tB \cap sB) \geq \delta \epsilon m(B) > 0.$$

Thus (G, Ω, ν) is strongly recurrent.

Step 3 (Reverse direction). The argument is symmetric by equivalence of measures: if (G, Ω, ν) is strongly recurrent, the same finite F and positive ϵ apply, using the Radon-Nikodym derivative $dm/d\nu$. $\square$

We end this section by proving the result on maximal entropy, as promised. In the next result it is essential to assume that the system admits a unique G -invariant probability measure (the system is uniquely ergodic) and let the averages be taken with respect to such an invariant measure.

Theorem 5 (Orbit Frequencies and Entropy of Amenable Systems). *Let $(\Omega, \mathcal{B}, m)$ be a finite measure space and let G be a countable amenable ordered group acting measurably on Ω via m -preserving transformations $(T_g)_{g \in G}$. Assume that the system (G, Ω, m) is ergodic and uniquely ergodic. Let $\Omega_{\text{eff}} \subset \Omega$ be a finite effective state space invariant under G with a unique G -invariant probability measure μ on Ω_{eff} .*

(i) *For m -almost every initial condition $x_0 \in \Omega_{\text{eff}}$ and every G -orbit $\mathcal{O} \subseteq \Omega_{\text{eff}}$,*

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \#\{t \in F_n \cap G^+ : T_t x_0 \in \mathcal{O}\} = \mu(\mathcal{O}).$$

(ii) *If additionally G acts transitively on Ω_{eff} , then the induced probability distribution on Ω_{eff} is uniform and the Shannon entropy satisfies*

$$H(X) = \log |\Omega_{\text{eff}}|.$$

Proof. Step 1 (Existence of G -invariant measure). The space $\mathcal{P}(\Omega_{\text{eff}})$ of probability measures on the finite set Ω_{eff} is compact and convex. By the Markov–Kakutani fixed point theorem, the G -action on $\mathcal{P}(\Omega_{\text{eff}})$ by pushforward admits at least one fixed point, i.e. a G -invariant probability measure. By assumption this measure is unique and equals μ .

Step 2 (Convergence of Følner averages). Let (F_n) be a Følner sequence in G and define empirical measures

$$\mu_n := \frac{1}{|F_n|} \sum_{t \in F_n} \delta_{T_t x_0}.$$

Since $\mathcal{P}(\Omega_{\text{eff}})$ is compact, every subnet of (μ_n) has a convergent subnet. By the Følner property [13, Chapter 6], for any $g \in G$,

$$\|g_* \mu_n - \mu_n\| = \frac{|gF_n \triangle F_n|}{|F_n|} \rightarrow 0,$$

so every limit point of (μ_n) is G -invariant. By unique ergodicity, every limit point equals μ , hence the full sequence converges: $\mu_n \rightarrow \mu$ weakly. By ergodicity of (G, Ω, m) and the pointwise ergodic theorem for amenable group actions [13, Theorem 6.2], this convergence holds for m -almost every $x_0 \in \Omega$.

Step 3 (Orbit frequencies). Since $\mu_n \rightarrow \mu$ weakly and $\mathcal{O}$ is a finite union of atoms of Ω_{eff} ,

$$\frac{1}{|F_n|} \#\{t \in F_n \cap G^+ : T_t x_0 \in \mathcal{O}\} \rightarrow \mu(\mathcal{O}),$$

which proves (i).

Step 4 (G-invariance of p). Define $p(x) := \mu(\{x\})$ for $x \in \Omega_{\text{eff}}$. Since m is G-invariant, for any $g \in G$ and $x \in \Omega_{\text{eff}}$,

$$p(gx) = \mu(\{gx\}) = \mu(g\{x\}) = \mu(\{x\}) = p(x).$$

Thus p is constant on each G-orbit.

Step 5 (Uniformity under transitivity). If G acts transitively on Ω_{eff} , then all states belong to a single orbit, so p is constant on all of Ω_{eff} . Since $\sum_{x \in \Omega_{\text{eff}}} p(x) = 1$,

$$p(x) = \frac{1}{|\Omega_{\text{eff}}|}, \quad x \in \Omega_{\text{eff}}.$$

Step 6 (Entropy).

$$\begin{aligned} H(X) &= - \sum_{x \in \Omega_{\text{eff}}} p(x) \log p(x) \\ &= - \sum_{x \in \Omega_{\text{eff}}} \frac{1}{|\Omega_{\text{eff}}|} \log \frac{1}{|\Omega_{\text{eff}}|} \\ &= \log |\Omega_{\text{eff}}|, \end{aligned}$$

which proves (ii). $\square$

Remark 5. (i) The Step 2 of the above proof fails for non-amenable groups: without a Følner sequence, the above argument breaks down, and limit points of (μ_n) need not be G-invariant. There could be clever ways to use alternative assumptions or arguments, and extending Theorem 5 to non-amenable groups remains as an interesting open problem.

(ii) The G-invariance of p is a property of the measure μ established in Step 4 of the above proof, and it does not rely on any individual trajectory being G-invariant. The connection between μ and single-trajectory empirical frequencies is the content of Step 3: under unique ergodicity, the Følner averages along m -almost every trajectory converge to μ , so $p(x) = \mu(\{x\})$ is the common limiting frequency.

(iii) The convergence used in Steps 1–3 of the above theorem is not the pointwise ergodic theorem (which can fail for individual trajectories) but rather the weak convergence of empirical measures along Følner sequences to the unique G-invariant measure. This follows from two ingredients: (a) compactness of the space of probability measures on the finite set Ω_{eff} , which guarantees that every subnet of (μ_n) has a convergent subsequence; and (b) the Følner property, which implies that every limit point is G-invariant (as shown in Step 2 of the proof). Unique ergodicity then pins down the limit as μ , and since the limit is unique, the full sequence also converges. This argument does not invoke the von Neumann Mean Ergodic Theorem and it is a self-contained compactness-and-uniqueness argument.

(iv) We have assumed both ergodicity and unique ergodicity of (G, Ω, m) in the above result. Ergodicity of the background system is used via the pointwise ergodic theorem to ensure convergence of the Følner averages for m -almost every $x_0 \in \Omega$. Unique ergodicity identifies the limit as μ rather than some other G-invariant measure. Without unique ergodicity, different initial conditions or Følner sequences can give different limits, one per ergodic component. In the transitive case (part (ii)), transitivity of the G-action on the finite set Ω_{eff} implies minimality, which implies unique ergodicity, so unique ergodicity is then automatic. Ergodicity of the full system (G, Ω, m) remains a separate assumption in all cases. There are three standard sufficient conditions for unique ergodicity in this setting: (a) the G-action on Ω_{eff} is minimal (every orbit is dense, hence equal to Ω_{eff} in the finite case), which implies unique ergodicity by a classical argument; (b) the system is strictly ergodic (i.e., the Cesaro averages along every Følner sequence converge uniformly, not just in the L^2 -mean), a

stronger but checkable condition; (c) The set Ω_{eff} is finite and G acts transitively (our main case of interest), in which the action is automatically minimal, and the uniform measure is the unique invariant probability measure, and unique ergodicity holds without further assumption.

(v) The recurrence results of Sections 3.1–3.2 are stated for σ -finite measures, which is the natural setting for wandering sets and conservativity. Theorem 5 requires the stronger assumption $m(\Omega) < \infty$ because in an infinite-measure system, the Cesàro averages $\frac{1}{|F_n|} \# \{t \in F_n : T_t x_0 \in \mathcal{O}\}$ converge to zero m -almost everywhere for any set $\mathcal{O}$ of finite measure (by the Hurewicz ratio ergodic theorem; see [1]), making asymptotic visit frequencies ill-defined in that setting. The passage to a finite invariant measure is therefore necessary for the entropy theory, and is consistent with the construction of μ as a probability measure on the finite set Ω_{eff} .

(vi) The assumption that Ω_{eff} is finite is essential for Theorem 5. It is used in two ways: first, to ensure that the uniform distribution $p(x) = 1/|\Omega_{\text{eff}}|$ is a well-defined probability measure (for infinite Ω_{eff} a uniform distribution does not exist); and second, to ensure that $H(X) = \log |\Omega_{\text{eff}}|$ is finite. The orbit-frequency result (part (i)) extends to infinite Ω_{eff} under appropriate topological conditions, but the entropy result (part (ii)) is specific to the finite case.

Let us examine the constructions in the above theorem in some illustrative examples.

Example 3. (i) (Cyclic shift) Let $\Omega = \mathbb{Z}_n = \{0, 1, \dots, n-1\}$ with the counting measure $m(\{k\}) = 1$ for each $k \in \mathbb{Z}_n$, so $m(\Omega) = n < \infty$. Let $G = \mathbb{Z}$ with positive cone $G^+ = \{0, 1, 2, \dots\}$, acting on Ω by the cyclic shift

$$T(k) = k + 1 \bmod n.$$

For any initial condition $x_0 \in \Omega$, the forward orbit is

$$\Omega_{\text{eff}}(x_0) = \{T^t(x_0) : t \in G^+\} = \{x_0, x_0 + 1, \dots, x_0 + n - 1\} \bmod n = \mathbb{Z}_n,$$

so $\Omega_{\text{eff}} = \Omega$ for every x_0 . The background measure m is finite and T -invariant: $m(T^{-1}(A)) = m(A)$ for every $A \subseteq \Omega$. Note that m is atomic here, with $m(\{k\}) = 1 > 0$ for every k . For any $B \subseteq \Omega$ with $m(B) > 0$, take $F = \{0, 1, \dots, n-1\} \subset G^+$. For any $t \in G^+$, the translate $T^t B$ intersects $T^s B$ for $s = t \bmod n \in F$, since $T^n = \text{id}$. Hence (G, Ω, m) is strongly recurrent with recurrency set F independent of B . The standard Følner sequence for $G = \mathbb{Z}$ is $F_N = \{0, 1, \dots, N-1\}$. Starting from any $x_0 \in \Omega$, the empirical measure is

$$\mu_N = \frac{1}{N} \sum_{t=0}^{N-1} \delta_{T^t x_0} = \frac{1}{N} \sum_{t=0}^{N-1} \delta_{x_0 + t \bmod n}.$$

Writing $N = qn + r$ with $0 \leq r < n$, each state $k \in \mathbb{Z}_n$ is visited either q or $q+1$ times, so

$$\mu_N(\{k\}) = \frac{q + \epsilon_k}{N}, \quad \epsilon_k \in \{0, 1\},$$

and as $N \rightarrow \infty$, $q/N \rightarrow 1/n$ and $r/N \rightarrow 0$, giving

$$\mu_N(\{k\}) \rightarrow \frac{1}{n} \quad \text{for every } k \in \mathbb{Z}_n.$$

This convergence holds for every $x_0 \in \Omega$, because the orbit is periodic with period n and every state is visited exactly once per period.

The limit μ is the uniform probability measure on $\mathbb{Z}_n$:

$$\mu(\{k\}) = \frac{1}{n} \quad \text{for all } k \in \mathbb{Z}_n.$$

This is the unique G -invariant probability measure on $\Omega_{\text{eff}} = \mathbb{Z}_n$, confirming unique ergodicity. We could also arrange that $\mu \neq m/m(\Omega)$, by taking a non-uniform m .

For any orbit $\mathcal{O} \subseteq \mathbb{Z}_n$ (here the only orbit is all of $\mathbb{Z}_n$ since G acts transitively):

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{t \in \{0, \dots, N-1\} : T^t x_0 \in \mathcal{O}\} = \mu(\mathcal{O}) = \frac{|\mathcal{O}|}{n}.$$

As for the entropy,

$$H(X) = - \sum_{k=0}^{n-1} \frac{1}{n} \log \frac{1}{n} = \log n = \log |\Omega_{\text{eff}}|,$$

confirming Theorem 5(ii).

If T is modified by fixing one state, say $T'(0) = 0$ and $T'(k) = k + 1 \bmod n$ for $k \neq 0$, then $\Omega_{\text{eff}}(x_0)$ depends on x_0 : starting from $x_0 = 0$ gives $\Omega_{\text{eff}} = \{0\}$ and $H(X) = 0$, while starting from $x_0 = 1$ gives $\Omega_{\text{eff}} = \{1, 2, \dots, n-1\}$ and $H(X) = \log(n-1)$ does not achieve the global maximum $\log n$, illustrating sensitivity to symmetry-breaking as noted in Example 3.1(ii).

(ii) (Periodic $\mathbb{Z}^2$ action) Let $\Omega = \mathbb{Z}_p \times \mathbb{Z}_q$ for two coprime integers $p, q \geq 2$, so $|\Omega| = pq$. Equip Ω with the counting measure $m(\{(j, k)\}) = 1$ for each $(j, k) \in \Omega$, so $m(\Omega) = pq < \infty$. Let $G = \mathbb{Z}^2$ with lexicographic positive cone

$$G^+ = \{(a, b) \in \mathbb{Z}^2 : a > 0, \text{ or } (a = 0, b \geq 0)\},$$

acting on Ω by

$$(a, b) \cdot (j, k) = (j + a \bmod p, k + b \bmod q).$$

Define the transition map corresponding to the generator $(1, 0) \in G^+$:

$$T(j, k) = (j + 1 \bmod p, k).$$

Starting from $x_0 = (0, 0)$, the forward orbit under T is

$$\Omega_{\text{eff}}((0, 0)) = \{T^t(0, 0) : t \in G^+\} = \{(t \bmod p, 0) : t \geq 0\} = \mathbb{Z}_p \times \{0\}.$$

So with generator T alone, only the first component cycles and Ω_{eff} has cardinality p , not pq . To reach all of Ω we must use the full $G = \mathbb{Z}^2$ action, including the generator $(0, 1)$ which shifts the second component. This illustrates that Ω_{eff} depends on which elements of G^+ are used to define the trajectory. For the full G -orbit starting from $(0, 0)$:

$$G \cdot (0, 0) = \{(a \bmod p, b \bmod q) : (a, b) \in \mathbb{Z}^2\} = \mathbb{Z}_p \times \mathbb{Z}_q = \Omega,$$

so G acts transitively on Ω .

For any $B \subseteq \Omega$ with $m(B) > 0$, take

$$F = \{(a, b) \in G^+ : 0 \leq a \leq p-1, 0 \leq b \leq q-1\},$$

so $|F| = pq$. For any $t = (t_1, t_2) \in G^+$, set $s = (t_1 \bmod p, t_2 \bmod q) \in F$. Then $(t - s) \in p\mathbb{Z} \times q\mathbb{Z}$, so $T^t = T^s$ on Ω , giving $m(T^t B \cap T^s B) = m(T^s B) = m(B) > 0$. Hence F is a finite recurrency set for every B , and (G, Ω, m) is strongly recurrent. The standard Følner sequence for $G = \mathbb{Z}^2$ is the sequence of rectangles

$$F_N = \{0, 1, \dots, N-1\}^2 \subset G^+, \quad |F_N| = N^2.$$

For any $g = (a, b) \in \mathbb{Z}^2$,

$$\frac{|gF_N \triangle F_N|}{|F_N|} \leq \frac{2|a|N + 2|b|N}{N^2} = \frac{2(|a| + |b|)}{N} \rightarrow 0,$$

confirming (F_N) is a Følner sequence.

Starting from $x_0 = (0, 0)$, the empirical measure is

$$\mu_N = \frac{1}{N^2} \sum_{(a,b) \in F_N} \delta_{(a \bmod p, b \bmod q)}.$$

For each $(j, k) \in \mathbb{Z}_p \times \mathbb{Z}_q$, the number of pairs $(a, b) \in F_N$ with $a \equiv j \pmod{p}$ and $b \equiv k \pmod{q}$ is

$$\#\{0 \leq a \leq N-1 : a \equiv j \pmod{p}\} \times \#\{0 \leq b \leq N-1 : b \equiv k \pmod{q}\}.$$

Each count equals $\lfloor N/p \rfloor$ or $\lfloor N/p \rfloor + 1$ (and similarly for q), so

$$\mu_N(\{(j, k)\}) = \frac{(\lfloor N/p \rfloor + \epsilon_j)(\lfloor N/q \rfloor + \eta_k)}{N^2}, \quad \epsilon_j, \eta_k \in \{0, 1\}.$$

As $N \rightarrow \infty$, $\lfloor N/p \rfloor / N \rightarrow 1/p$ and $\lfloor N/q \rfloor / N \rightarrow 1/q$, so

$$\mu_N(\{(j, k)\}) \rightarrow \frac{1}{p} \cdot \frac{1}{q} = \frac{1}{pq}$$

for every $(j, k) \in \Omega$, and the convergence holds for every $x_0 \in \Omega$ since the orbit structure is the same from any starting point. The limit is the uniform probability measure:

$$\mu(\{(j, k)\}) = \frac{1}{pq} \quad \text{for all } (j, k) \in \Omega.$$

This is the unique G -invariant probability measure on Ω , since G acts transitively. Unique ergodicity holds. Note that $\mu = m/pq$, i.e. μ is the normalised counting measure.

Since G acts transitively on $\Omega_{\text{eff}} = \Omega$, the only G -orbit is all of Ω , and

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \#\{(a, b) \in F_N \cap G^+ : (a \bmod p, b \bmod q) \in \mathcal{O}\} = \mu(\mathcal{O}) = \frac{|\mathcal{O}|}{pq}$$

for every $\mathcal{O} \subseteq \Omega$ and every $x_0 \in \Omega$, confirming Theorem 5(i). Also,

$$H(X) = - \sum_{(j,k) \in \Omega} \frac{1}{pq} \log \frac{1}{pq} = \log pq = \log |\Omega_{\text{eff}}|,$$

confirming Theorem 5(ii). For example, with $p = 2, q = 3$: $H(X) = \log 6 \approx 1.792$.

If p and q are not coprime, say $p = q = 2$, the dynamics can decompose. For instance, consider the modified transition map $T'(j, k) = (j + 1 \bmod 2, k + 1 \bmod 2)$. Then $T'(0, 0) = (1, 1)$, $T'^2(0, 0) = (0, 0)$, so the orbit of $(0, 0)$ is $\{(0, 0), (1, 1)\}$ and the orbit of $(0, 1)$ is $\{(0, 1), (1, 0)\}$. The state space decomposes into two orbits of length 2, the action is not transitive, and

$$H(X) = \sum_{i=1}^2 \mu(\mathcal{O}_i) \log |\mathcal{O}_i| = \frac{1}{2} \log 2 + \frac{1}{2} \log 2 = \log 2 < \log 4 = \log |\Omega|,$$

where $\mu(\mathcal{O}_1) = \mu(\mathcal{O}_2) = 1/2$ since both orbits have equal length. This illustrates how the entropy could be strictly below the global maximum when the action is not transitive, and the deficit measures the non-uniformity of the orbit decomposition.

(iii) (Rational and irrational $\mathbb{Z}^2$ -rotations) Let $G = \mathbb{Z}^2$ with lexicographic order and let $\alpha, \beta \in (0, 1)$ be rationally independent irrationals. Define the action on $\Omega = \mathbb{T}^2 = [0, 1)^2$ with Lebesgue measure m by

$$(n_1, n_2) \cdot (x_1, x_2) = (x_1 + n_1 \alpha \bmod 1, x_2 + n_2 \beta \bmod 1).$$

The system is not periodic since α, β are irrational. The action is minimal (every orbit is dense in $\mathbb{T}^2$) and uniquely ergodic with unique invariant measure equal to Lebesgue measure, by Weyl's theorem [18]. Strong recurrence holds because minimality implies that return times to any open set of positive measure are syndetic, which gives the finite recurrency set required by Definition 4. However, $\Omega_{\text{eff}}(x_0)$ is a countably infinite dense subset of $\mathbb{T}^2$ for every $x_0 \in \mathbb{T}^2$, so $|\Omega_{\text{eff}}| = \infty$ and the entropy result of Theorem 5 does not apply. This example illustrates strong recurrence on an infinite state space.

By contrast, we may consider the rational rotations. Let $\Omega = \mathbb{T}^2 = [0, 1)^2$ with Lebesgue measure m , on which $G = \mathbb{Z}^2$ acts by:

$$(n_1, n_2) \cdot (x_1, x_2) = \left(x_1 + \frac{n_1 p_1}{q_1} \bmod 1, x_2 + \frac{n_2 p_2}{q_2} \bmod 1 \right)$$

where $p_1/q_1, p_2/q_2 \in \mathbb{Q}$ with $\gcd(p_i, q_i) = 1$. Starting from $x_0 = (x_1, x_2)$,

$$\Omega_{\text{eff}}(x_0) = \left\{ \left(x_1 + \frac{k_1 p_1}{q_1} \bmod 1, x_2 + \frac{k_2 p_2}{q_2} \bmod 1 \right) : 0 \leq k_1 \leq q_1 - 1, 0 \leq k_2 \leq q_2 - 1 \right\},$$

which is finite with $|\Omega_{\text{eff}}| = q_1 q_2$. The background Lebesgue measure m is non-atomic with $m(\Omega_{\text{eff}}) = 0$ while μ is the uniform measure on the finite set Ω_{eff} . Theorem 5 applies and $H(X) = \log(q_1 q_2)$.

(iv) ($\mathbb{Z}$ -rotations) Let $\Omega = \mathbb{T} = [0, 1)$ equipped with Lebesgue measure m , which is compact and non-atomic. Let $G = \mathbb{Z}$ act by rational rotation

$$T(x) = x + \frac{p}{q} \bmod 1,$$

where $p/q \in \mathbb{Q}$ with $\gcd(p, q) = 1$. For any initial condition $x_0 \in \mathbb{T}$, the effective state space is the finite orbit

$$\Omega_{\text{eff}}(x_0) = \left\{ x_0 + \frac{kp}{q} \bmod 1 : k = 0, 1, \dots, q-1 \right\},$$

with $|\Omega_{\text{eff}}| = q$.

Strong recurrence holds with recurrency set $F = \{0, 1, \dots, q-1\}$, since $T^q = \text{id}$ on Ω_{eff} . $G = \mathbb{Z}$ acts transitively on Ω_{eff} , the system is uniquely ergodic, and Theorem 5 gives

$$H(X) = \log q = \log |\Omega_{\text{eff}}|.$$

This example illustrates the general setting of the paper: $\Omega = \mathbb{T}$ is compact and non-atomic, so $m(\{x\}) = 0$ for every $x \in \mathbb{T}$ and in particular $m(\Omega_{\text{eff}}) = 0$; Ω_{eff} is a finite subset of Ω , sitting inside the compact space with zero background measure; the measure μ on Ω_{eff} is not the restriction of m to Ω_{eff} (which would be the zero measure), but the atomic probability measure constructed from the dynamics:

$$\mu\left(\left\{x_0 + \frac{kp}{q}\right\}\right) = \frac{1}{q}, \quad k = 0, 1, \dots, q-1.$$

By contrast, the irrational rotation $T(x) = x + \alpha \bmod 1$ with $\alpha \notin \mathbb{Q}$ gives $|\Omega_{\text{eff}}(x_0)| = \infty$ for every x_0 : the orbit is dense in $\mathbb{T}$ but never returns to x_0 exactly, so Ω_{eff} is countably infinite and Theorem 5 does not apply. The contrast between rational and irrational rotations on the same compact space $\mathbb{T}$ makes precise the role of the finiteness assumption on Ω_{eff} , as being a property of Ω but of the dynamics.

Finally, we treat the non transitive case in the next result. Here both $\Omega_{\text{eff}} = \Omega_{\text{eff}}(x_0)$ and $H(X) = H(X|x_0)$ depend on the initial point x_0 . This being understood from the context, as noted before, for the entropy we keep using the notation $H(X)$ instead of less conventional notation $H(X|x_0)$.

Theorem 6 (Non-transitive case). Let (G, Ω, m) be a strongly recurrent amenable transformation group acting on a finite state space Ω , and suppose the system is uniquely ergodic with unique G -invariant probability measure μ . Let $x_0 \in \Omega$ and suppose $\Omega_{\text{eff}}(x_0)$ decomposes into $k \geq 1$ distinct G -orbits

$$\Omega_{\text{eff}}(x_0) = \mathcal{O}_1 \sqcup \cdots \sqcup \mathcal{O}_k.$$

Then we have the following:

- (i) The measure μ is constant on each orbit: $\mu(\{x\}) = \mu(\mathcal{O}_i)/|\mathcal{O}_i|$ for all $x \in \mathcal{O}_i$.
- (ii) For each $x_0 \in \Omega$, the Shannon entropy satisfies

$$H(X) = \sum_{i=1}^k \mu(\mathcal{O}_i) \log |\mathcal{O}_i| \leq \log |\Omega_{\text{eff}}(x_0)|,$$

with equality if and only if all orbits have equal length, i.e. $|\mathcal{O}_1| = \cdots = |\mathcal{O}_k|$.

- (iii) In the transitive case $k = 1$, Ω_{eff} is independent of the initial condition and $H(X) = \log |\Omega_{\text{eff}}|$.

Proof. (i) Since μ is G -invariant and G acts transitively on each $\mathcal{O}_i$ by definition of an orbit, for any $x, y \in \mathcal{O}_i$ there exists $g \in G$ with $gx = y$, hence $\mu(\{y\}) = \mu(\{gx\}) = \mu(\{x\})$. Together with $\sum_{x \in \mathcal{O}_i} \mu(\{x\}) = \mu(\mathcal{O}_i)$ this gives $\mu(\{x\}) = \mu(\mathcal{O}_i)/|\mathcal{O}_i|$.

- (ii) By direct computation:

$$\begin{aligned} H(X) &= - \sum_{x \in \Omega_{\text{eff}}(x_0)} \mu(\{x\}) \log \mu(\{x\}) \\ &= - \sum_{i=1}^k \sum_{x \in \mathcal{O}_i} \frac{\mu(\mathcal{O}_i)}{|\mathcal{O}_i|} \log \frac{\mu(\mathcal{O}_i)}{|\mathcal{O}_i|} \\ &= - \sum_{i=1}^k \mu(\mathcal{O}_i) (\log \mu(\mathcal{O}_i) - \log |\mathcal{O}_i|) \\ &= \sum_{i=1}^k \mu(\mathcal{O}_i) \log |\mathcal{O}_i| + H(\mu(\mathcal{O}_1), \dots, \mu(\mathcal{O}_k)), \end{aligned}$$

where $H(\mu(\mathcal{O}_1), \dots, \mu(\mathcal{O}_k)) = - \sum_i \mu(\mathcal{O}_i) \log \mu(\mathcal{O}_i) \geq 0$ is the Shannon entropy of the orbit-weight distribution. Since $H(\mu(\mathcal{O}_1), \dots, \mu(\mathcal{O}_k)) \leq \log k$ and $\sum_i \mu(\mathcal{O}_i) |\mathcal{O}_i| = |\Omega_{\text{eff}}(x_0)|$, the concavity of $\log$ gives

$$H(X) \leq \log |\Omega_{\text{eff}}(x_0)|.$$

Equality holds if and only if $\mu(\{x\})$ is constant across all $x \in \Omega_{\text{eff}}(x_0)$, which by (i) requires $\mu(\mathcal{O}_i)/|\mathcal{O}_i|$ to be the same for all i . Since μ is the unique G -invariant probability measure, $\mu(\mathcal{O}_i) = |\mathcal{O}_i|/|\Omega_{\text{eff}}(x_0)|$, and the condition reduces to $|\mathcal{O}_1| = \cdots = |\mathcal{O}_k|$.

- (iii) When $k = 1$ the sum has one term and $H(\mu(\mathcal{O}_1)) = 0$, giving $H(X; x_0) = \mu(\mathcal{O}_1) \log |\mathcal{O}_1| = \log |\Omega_{\text{eff}}(x_0)|$, which is Theorem 5. $\square$

Remark 6. Theorem 5 and Corollary 6 require only strong recurrence (Definition 4). On the other hand, strict recurrence (Definition 5) imposes a uniform quantitative lower bound on return frequencies, that is used in characterization of syndeticity of return-time sets (Theorem 3).

While we have discussed certain examples where the ambient space is atomic, the non-atomic case remains the canonical case, showcasing the significance of transition to discrete ergodic measures on the effective part of the dynamics. Let us close our discussion by a remark on the significance of the case non-atomic ambient space for further development of the the present work.

Remark 7. (i) The theory of wandering sets and recurrence developed by Hajian and Kakutani [8] and Prasad [16] is conventionally formulated within non-atomic standard Borel spaces. This setting allows the application

of established results regarding the conservative-dissipative decomposition without necessitating a separate reconstruction of the theory for discrete or atomic spaces. In further development of the theory presented here (along the lines of [2]), it is crucial to be aware of this convention while working within standard Borel spaces.

(ii) For atomic measure spaces, any set that is not part of a periodic orbit is simply part of a transient "tail," collapsing the measure-theoretic wandering phenomena into simple periodicity. By assuming a non-atomic background, strong recurrence acts as a rigorous "filter" that explicitly forbids the dissipation of measure into an infinite background, forcing the dynamics to remain "trapped" within the structure described by Ω_{eff} .

(iii) A non-atomic space (Ω, m) represents the total potential state space in a continuous or infinite background where individual points have measure zero. The transition to the finite set Ω_{eff} represents a discretization process where the "detectable" or "accessible" states are identified. Because $m(\{x\}) = 0$ for all $x \in \Omega$, the transition from the non-atomic measure m to the atomic probability measure μ (the uniform counting measure) used in the entropy calculations, which is a deliberate support-switch, shows its significance particularly when the ambient space is non-atomic. This highlights the significance of the entropy, since in this case, the dynamics while occurring in a large-scale measure-preserving system, can be characterized by discrete Shannon entropy.

(iv) The Halmos-Wightman decomposition partitions the measure space into a conservative part (where the Poincaré Recurrence Theorem holds) and a dissipative part (composed entirely of wandering sets). A non-atomic ambient Borel space ensures that this decomposition is rigorously valid and that the strong recurrence hypothesis effectively vanishes the dissipative component. In the non-atomic case, the formal measure-theoretic "scaffolding" is required to prove that the system is conservative before the analysis shift to the combinatorial properties of the orbits in Ω_{eff} .

4. Discussion

In this paper, we have provided a rigorous treatment of strongly and strictly recurrent transformation groups, specifically within the context of finite and infinite countable symmetry groups. By bridging classical recurrence theory, such as the Hajian-Ito theory of wandering sets, with information-theoretic measures, we have demonstrated that Shannon entropy can emerge as a purely structural property of a deterministic system. We have highlighted the fact that the use of a non-atomic ambient measure space is crucial to fully demonstrate the significance of strong recurrence. This prevents the 'leakage' of measure to infinity and justifies our suggested technique for discretization of dynamics onto the finite set Ω_{eff} in the amenable case. Our results show that the combination of strong recurrence and transitive symmetry enforces a uniform stationary distribution, thereby maximizing the system's information content without the need for stochastic assumptions. It is worth noting that our entropy result in the amenable case requires only strong recurrence. Strict recurrence imposes a uniform quantitative lower bound on return frequencies but plays no role in the derivation of maximal entropy. The role of strict recurrence in this paper is to connect recurrence to syndeticity. It could also potentially provide quantitative stability estimates, something which could be further pursued in future.

While we have established the basic dynamical implications of strong and strict recurrence, several avenues for future research remain open. Key problems to be addressed include the characterization of all groups that admit strictly recurrent actions and the extension of strong recurrence implications to uncountable or non-amenable group structures. Furthermore, obtaining explicit quantitative estimates for recurrence constants in infinite measure spaces would significantly enhance our understanding of the sensitivity of maximal entropy to structural perturbations. These theoretical developments provide a foundation for future applications in combinatorial design, cryptography, and the analysis of complex networks.

5. Conclusions

We have presented a comprehensive treatment of strongly and strictly recurrent transformation groups, bridging classical recurrence theory with information-theoretic measures. Our findings reveal

that the global behavior of deterministic systems is governed by a rigid structural framework that enforces specific information-theoretic bounds.

A primary innovation of this work is the derivation of maximal Shannon entropy as a purely structural, rather than stochastic, property. While traditional entropy results for stochastic processes rely on the assumption of random variables following a given distribution, we have demonstrated that entropy emerges naturally from the interplay of strong recurrence and symmetry in deterministic dynamics. The fundamental difference lies in the origin of the probability distribution:

- *Random view:* In stochastic models, entropy quantifies the uncertainty inherent in a probabilistic process, where transitions are governed by chance and the stationary distribution is an external assumption or a result of a Markovian process.
- *Deterministic view:* In our framework, the probability distribution $p(x)$ is not assumed but is forced by the system's architecture. The absence of weakly wandering sets, guaranteed by strong recurrence, ensures that every accessible state is visited infinitely often. When coupled with a transitive symmetry group action, the system is algebraically compelled to visit every state in the effective state space with equal asymptotic frequency. Thus, maximal entropy $H(X) = \log |\Omega_{eff}|$ becomes a necessary structural invariant rather than a statistical observation.

By establishing these connections to Furstenberg recurrence and Hajian-Ito wandering sets, we have shown that deterministic recurrence and symmetry are potent mechanisms that shape the global information content of a system. Future work will explore the extension of these innovations to non-amenable group structures and infinite-state systems, further illuminating the deterministic foundations of unpredictability.

While we have established a rigid structural framework for entropy in deterministic systems, several theoretical challenges remain to be addressed:

- *Characterization of strict recurrence:* A central open problem is to provide a complete classification of all countable ordered groups that admit strictly recurrent actions on finite or σ -finite measure spaces. While strong recurrence is well-characterized by the absence of weakly wandering sets, the tighter quantitative constraints of strict recurrence require further investigation.
- *Extension to non-amenable groups:* Our current derivation of emergent entropy relies on the averaging properties of Følner sequences in amenable groups. Extending these results to non-amenable or uncountable group structures would require a different approach to defining asymptotic frequencies and invariant measures.
- *Quantitative recurrence in infinite measure spaces:* In infinite measure spaces, the relationship between recurrence and entropy is more sensitive to structural perturbations. Establishing explicit quantitative estimates for recurrence constants in these settings remains a significant hurdle.
- *Sensitivity to symmetry-breaking:* The sensitivity of maximal entropy to structural changes—such as fixing a single state or perturbing the transition map—warrants a deeper study of topological and measurable interactions.

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