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Polynomial Identities For Binomial Sums Of Harmonic Numbers Of Higher Order

Takao Komatsu * and B. Sury

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Keywords: Polynomial identities; Harmonic numbers; determinant; Bell polynomials

Article

Polynomial Identities for Binomial Sums of Harmonic Numbers of Higher Order

Takao Komatsu [1,*] and B. Sury [2]

[1] Faculty of Education, Nagasaki University, Nagasaki 852-8521, Japan; komatsu@zstu.edu.cn
[2] Stat-Math Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore 560059 India; sury@isibang.ac.in
* Correspondence: komatsu@nagasaki-u.ac.jp

Abstract: We study the formulas for binomial sums of harmonic numbers of higher order

$$\sum_{k=0}^{n} H_k^{(r)} \binom{n}{k} (1-q)^k q^{n-k} = H_n^{(r)} - \sum_{j=1}^{n} \mathcal{D}_r(n,j) \frac{q^j}{j}.$$

Recently, Mneimneh proved that $\mathcal{D}_1(n,j) = 1$. In this paper, we find several different expressions of $\mathcal{D}_r(n,j)$ for $r \ge 1$.

Keywords: Polynomial identities; harmonic numbers; determinant; Bell polynomials

MSC: 11B65; 11A07; 05A10; 11B50; 11B73

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1. Introduction

For a positive integer r, define the n-th harmonic number of order r by

$$H_n^{(r)} := \sum_{i=1}^{n} \frac{1}{i^r}.$$

When $r = 1$, $H_n = H_n^{(1)}$ is the original harmonic number. In this paper, we study the formula

$$\sum_{k=0}^{n} H_k^{(r)} \binom{n}{k} (1-q)^k q^{n-k} = H_n^{(r)} - \sum_{j=1}^{n} \mathcal{D}_r(n,j) \frac{q^j}{j}. \tag{1}$$

In [1], for a positive integer n and $0 \le q \le 1$, it is shown that $\mathcal{D}_1(n,j) = 1$. Namely,

$$\sum_{k=0}^{n} H_k \binom{n}{k} (1-q)^k q^{n-k} = H_n - \sum_{j=1}^{n} \frac{q^j}{j}. \tag{2}$$

This relation is derived by the author from an interesting probabilistic analysis. The identity (2) is a generalization of the one

$$\sum_{k=0}^{n} H_k \binom{n}{k} = 2^n \left(H_n - \sum_{j=1}^{n} \frac{1}{j 2^j} \right),$$

which has been proved in [2] in the field of symbolic computation and in [3] in finite differences.

The main aim of this paper is to show several different expressions of $\mathcal{D}_r(n,j)$ as no simple form has been found.

In fact, more different generalizations of (1) or (2) can be considered. For example, recently in [4], the so-called hyperharmonic number generalizes harmonic number of order r in the formula. However, when we generalize too much, we often lose the fundamental properties that make us interesting.

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2. Observation

By using the harmonic numbers to express $\mathcal{D}_r(n,j)$, for $1 \le r \le 7$, we can manually get the following[1].

$$
\begin{aligned}
\mathfrak{D}_1(n,j) &= 1\,,\\
\mathfrak{D}_2(n,j) &= H_n - H_{n-j}\,,\\
\mathfrak{D}_3(n,j) &= \frac{(H_n - H_{n-j})^2}{2} + \frac{H_n^{(2)} - H_{n-j}^{(2)}}{2}\,,\\
\mathfrak{D}_4(n,j) &= \frac{(H_n - H_{n-j})^3}{6} + \frac{(H_n - H_{n-j})(H_n^{(2)} - H_{n-j}^{(2)})}{2} + \frac{H_n^{(3)} - H_{n-j}^{(3)}}{3}\,,\\
\mathcal{D}_5(n,j) &= \frac{(H_n - H_{n-j})^4}{4!} + \frac{(H_n - H_{n-j})^2(H_n^{(2)} - H_{n-j}^{(2)})}{4}\\
&\quad + \frac{(H_n - H_{n-j})(H_n^{(3)} - H_{n-j}^{(3)})}{3} + \frac{(H_n^{(2)} - H_{n-j}^{(2)})^2}{8} + \frac{H_n^{(4)} - H_{n-j}^{(4)}}{4}\,,\\
\mathcal{D}_6(n,j) &= \frac{(H_n - H_{n-j})^5}{5!} + \frac{(H_n - H_{n-j})^3(H_n^{(2)} - H_{n-j}^{(2)})}{12}\\
&\quad + \frac{(H_n - H_{n-j})^2(H_n^{(3)} - H_{n-j}^{(3)})}{6} + \frac{(H_n - H_{n-j})(H_n^{(2)} - H_{n-j}^{(2)})^2}{8}\\
&\quad + \frac{(H_n - H_{n-j})(H_n^{(4)} - H_{n-j}^{(4)}}{4} + \frac{(H_n^{(2)} - H_{n-j}^{(2)})(H_n^{(3)} - H_{n-j}^{(3)})}{6}\\
&\quad + \frac{H_n^{(5)} - H_{n-j}^{(5)}}{5}\,,\\
\mathcal{D}_7(n,j) &= \frac{(H_n - H_{n-j})^6}{6!} + \frac{(H_n - H_{n-j})^4(H_n^{(2)} - H_{n-j}^{(2)})}{48}\\
&\quad + \frac{(H_n - H_{n-j})^3(H_n^{(3)} - H_{n-j}^{(3)})}{18} + \frac{(H_n - H_{n-j})^2(H_n^{(2)} - H_{n-j}^{(2)})^2}{16}\\
&\quad + \frac{(H_n - H_{n-j})^2(H_n^{(4)} - H_{n-j}^{(4)})}{8} + \frac{(H_n - H_{n-j})(H_n^{(2)} - H_{n-j}^{(2)})(H_n^{(3)} - H_{n-j}^{(3)})}{6}\\
&\quad + \frac{(H_n - H_{n-j})(H_n^{(5)} - H_{n-j}^{(5)})}{5} + \frac{(H_n^{(2)} - H_{n-j}^{(2)})^3}{48}\\
&\quad + \frac{(H_n^{(2)} - H_{n-j}^{(2)})(H_n^{(4)} - H_{n-j}^{(4)})}{8} + \frac{(H_n^{(3)} - H_{n-j}^{(3)})^2}{18} + \frac{H_n^{(6)} - H_{n-j}^{(6)}}{6}\,.
\end{aligned}
$$

It is interesting to observe that the number of terms of each of the right-hand sides of $\mathcal{D}_r(n,j)$ is equal to the number of partitions of r ($1 \le r \le 7$), respectively. In addition, the same terms of generalized harmonic numbers appear in [6,7]:

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n}{(n+1)(n+2)} = 1\,,\\
&\sum_{n=1}^{\infty} \frac{(H_n)^2 - H_n^{(2)}}{2(n+1)(n+2)} = 1\,,\\
&\sum_{n=1}^{\infty} \frac{(H_n)^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}}{3!(n+1)(n+2)} = 1\,,
\end{aligned}
$$

[1] At first we calculated them one by one using the method in [5], but of course they can be calculated directly from the theorems described below.

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$$
\sum_{n=1}^\infty\frac{(H_n)^4-6(H_n)^2 H_n^{(2)}+8H_n H_n^{(3)}+3(H_n^{(2)})^2-6H_n^{(4)}}{4!(n+1)(n+2)}\,,
$$

$$
\begin{aligned}
&\sum_{n=1}^\infty\frac{1}{5!(n+1)(n+2)}\biggl((H_n)^5-10(H_n)^3 H_n^{(2)}+20(H_n)^2 H_n^{(3)})^2\\
&\quad+15H_n(H_n^{(2)}-30H_n H_n^{(4)}-20H_n^{(2)}H_n^{(3)}+24H_n^{(5)}\biggr)=1\,,\\
&\sum_{n=1}^\infty\frac{1}{6!(n+1)(n+2)}\biggl((H_n)^6-15(H_n)^4 H_n^{(2)}+40(H_n)^3 H_n^{(3)}\\
&\quad+45(H_n)^2(H_n^{(2)})^2-90(H_n)^2 H_n^{(4)}-120H_n H_n^{(2)}H_n^{(3)}+144H_n H_n^{(5)}\\
&\quad-15(H_n^{(2)})^3+90H_n^{(2)}H_n^{(4)}+40(H_n^{(3)})^2-120H_n^{(5)}\biggr)=1\,.
\end{aligned}
$$

3. Expressions (Main Results)

Let n, j, r be positive integers.

Theorem 1.

$$
\mathcal{D}_r(n,j)=\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l}\frac{1}{(n-l)^{r-1}}\,.
$$

Theorem 2. *For $r\ge 1$,*

$$
\mathcal{D}_{r+1}(n,j)=\sum_{j_1=1}^{j}\sum_{j_2=1}^{j_1}\cdots\sum_{j_r=1}^{j_{r-1}}\frac{1}{(n-j_1+1)(n-j_2+1)\cdots(n-j_r+1)}\,.
$$

$\mathcal{D}_r(n,j)$ $(r\ge 2)$ can be expressed in terms of the determinant ([8, Ch. I $\mathcal{S}$ 2]). See also [9,10].

Theorem 3.

$$
\mathcal{D}_{r+1}(n,j)
=\frac{1}{r!}\begin{vmatrix}
H_n-H_{n-j} & -1 & 0 & \cdots & 0\\
H_n^{(2)}-H_{n-j}^{(2)} & H_n-H_{n-j} & -2 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
H_n^{(r-1)}-H_{n-j}^{(r-1)} & H_n^{(r-2)}-H_{n-j}^{(r-2)} & H_n^{(r-3)}-H_{n-j}^{(r-3)} & \cdots & -r+1\\
H_n^{(r)}-H_{n-j}^{(r)} & H_n^{(r-1)}-H_{n-j}^{(r-1)} & H_n^{(r-2)}-H_{n-j}^{(r-2)} & \cdots & H_n-H_{n-j}
\end{vmatrix}.
$$

Remark 1. *By using the inversion formula (see, e.g., [11, Lemma 1],[12, Theorem 1],[8, p.28],) about (5) below, we also have*

$$
(-1)^{r-1}(H_n^{(r)}-H_{n-j}^{(r)})
=\begin{vmatrix}
\mathcal{D}_2(n,j) & 1 & 0 & \cdots & 0\\
2\mathcal{D}_3(n,j) & \mathcal{D}_2(n,j) & 1 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
(r-1)\mathcal{D}_r(n,j) & \mathcal{D}_{r-1}(n,j) & \mathcal{D}_{r-2}(n,j) & \cdots & 1\\
r\mathcal{D}_{r+1}(n,j) & \mathcal{D}_r(n,j) & \mathcal{D}_{r-1}(n,j) & \cdots & \mathcal{D}_2(n,j)
\end{vmatrix}.
$$

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$\mathcal{D}_r(n,j)$ ($r \ge 2$) can be expressed by a combinatorial sum ([7, Proposition 1 (17)]):

Theorem 4.

$$
\begin{aligned}
&\mathcal{D}_{r+1}(n,j)\\
&= \sum_{i_1+2i_2+3i_3+\cdots=r} \frac{1}{i_1!i_2!i_3!\cdots}\left(\frac{H_n-H_{n-j}}{1}\right)^{i_1}\left(\frac{H_n^{(2)}-H_{n-j}^{(2)}}{2}\right)^{i_2}\left(\frac{H_n^{(3)}-H_{n-j}^{(3)}}{3}\right)^{i_3}\cdots
\end{aligned}
$$

Remember that the (complete exponential) Bell polynomial $\mathbf{Y}_n(x_1,x_2,\dots,x_n)$ is defined by

$$
\exp\left(\sum_{m=1}^\infty x_m \frac{t^m}{m!}\right) = 1 + \sum_{n=1}^\infty \mathbf{Y}_n(x_1,x_2,\dots,x_n)\frac{t^n}{n!}\,.
$$

That is,

$$
\begin{aligned}
&\mathbf{Y}_n(x_1,x_2,\dots,x_n)\\
&= \sum_{k=1}^n \sum \frac{n!}{i_1!i_2!\cdots i_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{i_1}\left(\frac{x_2}{2!}\right)^{i_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{i_{n-k+1}}
\end{aligned}
$$

with $\mathbf{Y}_0 = 1$. Here, the second sum satisfies the conditions

$$
i_1 + 2i_2 + 3i_3 + \cdots + (n-k+1)i_{n-k+1} = n, \quad i_1 + i_2 + i_3 + \cdots = k\,.
$$

Theorem 5. *For $r \ge 1$, we have*

$$
\mathcal{D}_{r+1}(n,j) = \frac{1}{r!}\mathbf{Y}_r\big(H_n - H_{n-j}, 1!(H_n^{(2)} - H_{n-j}^{(2)}), 2!(H_n^{(3)} - H_{n-j}^{(3)}), \dots\big)\,.
$$

4. Proof

Proof of Theorem 1. We shall show

$$
\begin{aligned}
&\sum_{k=0}^n H_k^{(r)}\binom{n}{k}(1-q)^k q^{n-k}\\
&= H_n^{(r)} - \sum_{j=1}^n \binom{n}{j}\left(\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{j-1}{l}\frac{1}{(n-l)^r}\right)q^j\,. \qquad (3)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\sum_{k=0}^n H_k^{(r)}\binom{n}{k}(1-q)^k q^{n-k} = \sum_{k=0}^n H_k^{(r)}\binom{n}{k}\sum_{l=0}^k(-1)^{k-l}\binom{k}{l}q^{n-l}\\
&= \sum_{l=0}^n q^{n-l}\binom{n}{l}\sum_{k=l}^n(-1)^{k-l}\binom{n-l}{n-k}H_k^{(r)}\\
&= \sum_{j=0}^n \binom{n}{j}q^j\sum_{\nu=0}^j(-1)^{j-\nu}\binom{j}{\nu}H_{n-\nu}^{(r)}\\
&= H_n^{(r)} - \sum_{j=1}^n(-1)^{j-1}\binom{n}{j}q^j\sum_{\nu=0}^j(-1)^\nu\binom{j}{\nu}\sum_{l=0}^{n-1}\frac{1}{(n-l)^r}\,.
\end{aligned}
$$

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Since
$$\sum_{\nu=0}^{l}(-1)^{\nu}\binom{j}{\nu}=(-1)^{l}\binom{j-1}{l}\quad\text{(proved by induction on } l(\geq 0))$$
and
$$\sum_{\nu=0}^{j}(-1)^{\nu}\binom{j}{\nu}=(1-1)^{j}=0\,,$$
we have
$$\begin{aligned}&\sum_{\nu=0}^{j}(-1)^{\nu}\binom{j}{\nu}\sum_{l=0}^{n-1}\frac{1}{(n-l)^{r}}\\&=\sum_{l=0}^{j-1}\left(\sum_{\nu=0}^{l}(-1)^{\nu}\binom{j}{\nu}\right)\frac{1}{(n-l)^{r}}+\sum_{l=j}^{n-1}\left(\sum_{\nu=0}^{j}(-1)^{\nu}\binom{j}{\nu}\right)\frac{1}{(n-l)^{r}}\\&=\sum_{l=0}^{j-1}(-1)^{l}\binom{j-1}{l}\frac{1}{(n-l)^{r}}\,.\end{aligned}$$
By (3),
$$\begin{aligned}\mathcal{D}_{r}(n,j)&=j!\binom{n}{j}\left(\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{j-1}{l}\frac{1}{(n-l)^{r}}\right)\\&=\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l}\frac{1}{(n-l)^{r-1}}\,.\end{aligned}$$
□

Proof of Theorem 2. By Theorem 1,
$$\begin{aligned}&\mathcal{D}_{r}(n,j)-\mathcal{D}_{r}(n,j-1)\\&=\sum_{l=0}^{j-1}(-1)^{j-l-1}\left(\binom{n-l-1}{n-j}+\binom{n-l-1}{n-j+1}\right)\binom{n}{l}\frac{1}{(n-l)^{r-1}}\\&=\frac{1}{n-j+1}\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l}\frac{1}{(n-l)^{r-2}}\\&=\frac{\mathcal{D}_{r-1}(n,j)}{n-j+1}\,.\end{aligned}$$
Hence, by $\mathcal{D}_{r}(n,0)=0$ we have
$$\begin{aligned}\mathcal{D}_{r+1}(n,j)&=\mathcal{D}_{r+1}(n,j-1)+\frac{\mathcal{D}_{r}(n,j)}{n-j+1}\\&=\mathcal{D}_{r+1}(n,j-2)+\frac{\mathcal{D}_{r}(n,j-1)}{n-j+2}+\frac{\mathcal{D}_{r}(n,j)}{n-j+1}\\&=\cdots=\sum_{j_1=1}^{j}\frac{\mathcal{D}_{r}(n,j_1)}{n-j_1+1}\\&=\sum_{j_1=1}^{j}\frac{1}{n-j_1+1}\sum_{j_2=1}^{j_1}\frac{\mathcal{D}_{r-1}(n,j_2)}{n-j_2+1}\\&=\sum_{j_1=1}^{j}\frac{1}{n-j_1+1}\sum_{j_2=1}^{j_1}\frac{1}{n-j_2+1}\sum_{j_3=1}^{j_2}\frac{\mathcal{D}_{r-2}(n,j_3)}{n-j_3+1}\\&=\cdots\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{j_1=1}^{j} \frac{1}{n-j_1+1} \sum_{j_2=1}^{j_1} \frac{1}{n-j_2+1} \cdots \sum_{j_r=1}^{j_{r-1}} \frac{\mathcal{D}_1(n,j_r)}{n-j_r+1} \\
&= \sum_{j_1=1}^{j} \frac{1}{n-j_1+1} \sum_{j_2=1}^{j_1} \frac{1}{n-j_2+1} \cdots \sum_{j_r=1}^{j_{r-1}} \frac{1}{n-j_r+1} \,.
\end{aligned}
$$

□

In order to prove Theorem 4 amd Theorem 3, we need the following relations.

Lemma 1. *For the sequences* $\{p_n\}_{n\ge 1}$ *and* $\{h_n\}_{n\ge 1}$*, we have*

$$
(-1)^{n-1} p_n = \begin{vmatrix} h_1 & 1 & 0 & \cdots & 0 \\ 2h_2 & h_1 & 1 & & \vdots \\ 3h_3 & h_2 & & \ddots & \\ \vdots & \vdots & & & 1 \\ nh_n & h_{n-1} & \cdots & h_2 & h_1 \end{vmatrix}
$$

$$
\Longleftrightarrow \; n!h_n = \begin{vmatrix} p_1 & -1 & 0 & \cdots & 0 \\ p_2 & p_1 & -2 & & \vdots \\ p_3 & p_2 & & \ddots & 0 \\ \vdots & \vdots & & & -n+1 \\ p_n & p_{n-1} & \cdots & p_2 & p_1 \end{vmatrix}
= \sum_{\substack{i_1+2i_2+\cdots+ni_n \\ i_1,i_2,\dots,i_n\ge 0}} \frac{n!}{i_1!i_2!\cdots i_n!} \left(\frac{p_1}{1}\right)^{i_1} \left(\frac{p_2}{2}\right)^{i_2} \cdots \left(\frac{p_n}{n}\right)^{i_n} .
$$

Proof. The last identity is a simple modification of Trudi's formula ([13, Vol.3, p.214],[14]):

$$
\begin{vmatrix} a_1 & a_0 & \cdots & & 0 \\ a_2 & a_1 & \cdots & & \\ \vdots & \vdots & \ddots & & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix}
= \sum_{i_1+2i_2+\cdots+nt_n=n} \frac{(i_1+\cdots+i_n)!}{i_1!\cdots i_n!} (-a_0)^{n-i_1-\cdots-i_n} a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} .
$$

Notice that the expansion of the second determinant is equivalant to the relation

$$
nh_n = \sum_{i=1}^{n} p_i h_{n-1} \quad \text{with} \quad h_0 = 1 \,. \tag{4}
$$

By applying the inversion formula (see, e.g. [11, Lemma 1],[12, Theorem 1]), we can get the first identity. □

Proof of Theorem 3. The determinant in Theorem 3 is equivalent to the recurrence relation:

$$
\mathcal{D}_{r+1}(n,j) = \frac{1}{r} \sum_{i=1}^{r} (H_n^{(r-i+1)} - H_{n-j}^{(r-i+1)}) \mathcal{D}_i(n,j) \,. \tag{5}
$$

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By applying the relation (4) in the first identity of the second part of Lemma 1 to (5), we can get the desired determinant identity. The identity of Remark can be given from the first part of Lemma 1. □

Proof of Theorem 4. The result follows from the second part of Lemma 1 by setting $h_r = \mathcal{D}_{r+1}(n,j)$ and $p_i = H_n^{(i)} - H_{n-j}^{(i)}$, satisfying (5). □

Proof of Theorem 5. Since Bell polynomials satisfy the recurrence relation

$$\mathbf{Y}_r(x_1, x_2, \dots, x_r) = \sum_{i=1}^{r} \binom{r-1}{i-1} x_{r-i+1} \mathbf{Y}_{i-1}(x_1, x_2, \dots, x_{i-1})$$

(see, e.g., [15]), by setting $x_\ell = (\ell-1)!(H_n^{(\ell)} - H_{n-j}^{(\ell)})$, we have

$$\begin{aligned}
&\frac{1}{r!}\mathbf{Y}_r\big(H_n - H_{n-j}, 1!(H_n^{(2)} - H_{n-j}^{(2)}), 2!(H_n^{(3)} - H_{n-j}^{(3)}), \dots\big) \\
&= \frac{1}{r}\sum_{i=1}^{r}(H_n^{(r-i+1)} - H_{n-j}^{(r-i+1)}) \\
&\quad \times \frac{\mathbf{Y}_{i-1}\big(H_n - H_{n-j}, 1!(H_n^{(2)} - H_{n-j}^{(2)}), 2!(H_n^{(3)} - H_{n-j}^{(3)}), \dots\big)}{(i-1)!}\,.
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{D}_2(n,j) &= H_n - H_{n-j} \\
&= \mathbf{Y}_1\big(H_n - H_{n-j}, 1!(H_n^{(2)} - H_{n-j}^{(2)}), 2!(H_n^{(3)} - H_{n-j}^{(3)}), \dots\big)\,,
\end{aligned}$$

for $r \ge 1$, we can write the form in Theorem 5. □

5. Some Reductions

In particular, when $r = 1$ in Theorem 1, we find the following relation.

Corollary 1.

$$\sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-l-1}{n-j} \binom{n}{l} = 1\,. \tag{6}$$

When $r = 2$ in Theorem 1, we find the following relation. Here, $(n)_j = n(n-1)\cdots(n-j+1)$ ($j \ge 1$) is the falling factorial with $(n)_0 = 1$, and $\left[{n \atop k}\right]$ denotes the (unsigned) Stirling number of the first kind, arising from the relation $(x)_n = \sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k}\right] x^k$.

Corollary 2.

$$\begin{aligned}
\sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-l-1}{n-j} \binom{n}{l} \frac{1}{n-l} &= H_n - H_{n-j} \\
&= \frac{1}{(n)_j} \sum_{\nu=0}^{j-1} (-1)^{j-\nu-1} (\nu+1) \left[{j \atop \nu+1}\right] n^{\nu}\,.
\end{aligned} \tag{7}$$

Remark 2. *Note that*

$$\binom{n-l-1}{n-j} \binom{n}{l} \frac{1}{n-l} \ne \frac{l+1}{(n)_j} \left[{j \atop l+1}\right] n^{l}\,.$$

Proof of Corollary 2. The formula (7) is yielded from the definition of the Stirling numbers of the first kind:

$$
\begin{aligned}
(x)_j &= \sum_{k=0}^{j} (-1)^{j-k} \begin{bmatrix} j \\ k \end{bmatrix} x^k \\
&= \sum_{\nu=0}^{j-1} (-1)^{j-\nu-1} \begin{bmatrix} j \\ \nu+1 \end{bmatrix} x^{\nu+1} \quad (\text{if } j \geq 1) .
\end{aligned}
$$

Differentiating both sides with respect to x gives

$$
(x)_j \sum_{l=0}^{j-1} \frac{1}{x-l} = \sum_{\nu=0}^{j-1} (-1)^{j-\nu-1} (\nu+1) \begin{bmatrix} j \\ \nu+1 \end{bmatrix} x^{\nu} .
$$

Thus, the right-hand side of (7) is equal to

$$
\sum_{l=0}^{j-1} \frac{1}{n-l} = H_n - H_{n-j} .
$$

□

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