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K. Manesh Krishna *

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Functional Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Bound

K. Mahesh Krishna

School of Mathematics and Natural Sciences, Chanakya University Global Campus, NH-648, Haraluru Village, Devanahalli Taluk, Bengaluru Rural District, Karnataka State, 562 110, India; kmaheshak@gmail.com

Abstract: Pfender *[J. Combin. Theory Ser. A, 2007]* provided a one-line proof for a variant of the Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein upper bound for spherical codes, which offers an upper bound for the celebrated (Newton-Gregory) kissing number problem. Motivated by this proof, we introduce the notion of codes in pointed metric spaces (in particular on Banach spaces) and derive a nonlinear (functional) Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender upper bound for spherical codes. We also introduce nonlinear (functional) Kissing Number Problem.

Keywords: spherical code; kissing number; linear programming

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1. Introduction

Let $d \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. A set $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{R}^d$ is said to be (d, n, θ)**-spherical code** [1] in $\mathbb{R}^d$ if

$$\langle \tau_j, \tau_k \rangle \leq \cos\theta, \quad \forall 1 \leq j, k \leq n, j \neq k.$$

Fundamental problem associated with spherical codes is the following.

Problem 1. *Given d and θ, what is the maximum n such that there exists a (d, n, θ)-spherical code $\{\tau_j\}_{j=1}^n$ in $\mathbb{R}^d$?*

The case $\theta = \pi/3$ is known as the famous **(Newton-Gregory) kissing number problem**. With extensive efforts from many mathematicians, it is still not completely resolved in every dimension (but resolved in dimensions, 1, 2, 3, 4, 8 and 24) [2–21]. We refer [22–37] for more on spherical codes. Problem 1 has connection even with sphere packing [38]. Most used method for obtaining upper bounds on spherical codes is the Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein bound which we recall. Let $n \in \mathbb{N}$ be fixed. The Gegenbauer polynomials are defined inductively as

$$\begin{aligned}
G_0^{(n)}(r) &:= 1, \quad \forall r \in [-1, 1], \\
G_1^{(n)}(r) &:= r, \quad \forall r \in [-1, 1], \\
&\vdots \\
G_k^{(n)}(r) &:= \frac{(2k + n - 4)rG_{k-1}^{(n)}(r) - (k-1)G_{k-2}^{(n)}(r)}{k + n - 3}, \quad \forall r \in [-1, 1], \quad \forall k \geq 2.
\end{aligned}$$

Then the family $\{G_k^{(n)}\}_{k=0}^\infty$ is orthogonal on the interval $[-1, 1]$ with respect to the weight

$$\rho(r) := (1 - r^2)^{\frac{n-3}{2}}, \quad \forall r \in [-1, 1].$$

Theorem 1. *[32,33] (**Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein Linear Programming Bound**) Let $\{\tau_j\}_{j=1}^n$ be a (d,n,θ)-spherical code in $\mathbb{R}^d$. Let P be a real polynomial satisfying following conditions.*

(i) $P(r) \leq 0$ for all $-1 \leq r \leq \cos\theta$.
(ii) Coefficients in the Gegenbauer expansion

$$P = \sum_{k=0}^{m} a_k G_k^{(n)}$$

satisfy

$$a_0 > 0, \quad a_k \geq 0, \ \forall 1 \leq k \leq m.$$

Then

$$n \leq \frac{P(1)}{a_0}.$$

In 2007, Pfender gave a one-line proof for a variant of Theorem 1.

Theorem 2. *[19] (**Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Bound**) Let $\{\tau_j\}_{j=1}^n$ be a (d,n,θ)-spherical code in $\mathbb{R}^d$. Let $c > 0$ and $\phi : [-1,1] \to \mathbb{R}$ be a function satisfying following.*

(i)

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \phi(\langle \tau_j, \tau_k \rangle) \geq 0.$$

(ii) $\phi(r) + c \leq 0$ for all $-1 \leq r \leq \cos\theta$.

Then

$$n \leq \frac{\phi(1) + c}{c}.$$

In particular, if $\phi(1) + c \leq 1$, then $n \leq 1/c$.

Motivated from Theorem 2, we formulate the notion of codes in pointed metric spaces. We show that bound of Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender can be extended for pointed metric spaces (in particular, for Banach spaces).

2. Metric Codes

Let $(\mathcal{M}, 0)$ be a pointed metric space. The collection $\operatorname{Lip}_0(\mathcal{M}, \mathbb{R})$ is defined as $\operatorname{Lip}_0(\mathcal{M}, \mathbb{R}) := \{f : \mathcal{M} \to \mathbb{R} \text{ is Lipschitz and } f(0) = 0\}$. For $f \in \operatorname{Lip}_0(\mathcal{M}, \mathbb{R})$, the Lipschitz norm is defined as

$$\|f\|_{\operatorname{Lip}_0} := \sup_{x,y \in \mathcal{M}, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

We introduce metric codes as follows.

Definition 1. *Let $(\mathcal{M}, 0)$ be a pointed metric space with metric m. For $1 \leq j \leq n$, let $f_j \in \operatorname{Lip}_0(\mathcal{M}, \mathbb{R})$ and $\tau_j \in \mathcal{M}$. The pair $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is said to be a (n,θ)-**metric code** or (n,θ)-**nonlinear code** or (n,θ)-**Lipschitz code** in $\mathcal{M}$ if following conditions hold.*

(i) $\|f_j\|_{\operatorname{Lip}_0} = 1$ for all $1 \leq j \leq n$.
(ii) $m(\tau_j, 0) = 1$ for all $1 \leq j \leq n$.

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(iii) $f_j(\tau_j) = 1$ *for all* $1 \leq j \leq n$.
(iv) $f_j(\tau_k) \leq \cos\theta$ *for all* $1 \leq j, k \leq n, j \neq k$.

We call the case $\theta = \pi/3$ *as the* ***nonlinear kissing number problem****.*

For Banach spaces, we define (linear) functional codes as follows.

Definition 2. *Let* $\mathcal{X}$ *be a real Banach space. For* $1 \leq j \leq n$*, let* $f_j \in \mathcal{X}^*$ *and* $\tau_j \in \mathcal{X}$*. The pair* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ *is said to be a* (n, θ)***-functional code*** *in* $\mathcal{X}$ *if following conditions hold.*

(i) $\|f_j\| = 1$ *for all* $1 \leq j \leq n$.
(ii) $\|\tau_j\| = 1$ *for all* $1 \leq j \leq n$.
(iii) $f_j(\tau_j) = 1$ *for all* $1 \leq j \leq n$.
(iv) $f_j(\tau_k) \leq \cos\theta$ *for all* $1 \leq j, k \leq n, j \neq k$.

We call the case $\theta = \pi/3$ *as the* ***functional kissing number problem****.*

Proposition 1. *For the space* $\mathbb{R}^d$*, Definition 2 matches with the spherical codes (in particular with the kissing-number problem).*

Proof. Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ be a (n, θ)-functional code in $\mathbb{R}^d$. We need to show that $f_j(x) = \tau_j$ for all $x \in \mathbb{R}^d$ and for all $1 \leq j \leq n$. Let $1 \leq j \leq n$. From Riesz representation theorem, there exists a unique $w_j \in \mathbb{R}^d$ such that $f_j(x) = \langle x, \omega_j \rangle$ for all $x \in \mathbb{R}^d$ and $\|f_j\| = \|\omega_j\|$. Now we need to show that $\omega_j = \tau_j$. Since $\|f_j\| = 1$, we must have $\|\omega_j\| = 1$. But then

$$1 = f_j(\tau_j) = \langle \tau_j, \omega_j \rangle \leq \|\tau_j\| \|\omega_j\| = 1.$$

Therefore $\omega_j = \alpha\tau_j$ for some $\alpha \in \mathbb{R}$. The conditions $\|\omega_j\| = \|\tau_j\| = 1$ and $\langle \tau_j, \omega_j \rangle = 1$ then force $\omega_j = \tau_j$. $\square$

Following is a nonlinear generalization of Theorem 2.

Theorem 3. *(**Functional Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Bound**) Let* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ *be a* (n, θ)*-metric code in a pointed metric space* $\mathcal{M}$*. Let* $c > 0$ *and* $\phi : [-1, 1] \to \mathbb{R}$ *be a function satisfying following.*

(i)

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \phi(f_j(\tau_k)) \geq 0.$$

(ii) $\phi(r) + c \leq 0$ *for all* $-1 \leq r \leq \cos\theta$.

Then

$$n \leq \frac{\phi(1) + c}{c}.$$

In particular, if $\phi(1) + c \leq 1$*, then* $n \leq 1/c$*.*

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Proof. Define $\psi : [-1,1] \ni r \mapsto \psi(r) := \phi(r) + c \in \mathbb{R}$. Then

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{n}\psi(f_j(\tau_k)) &= \sum_{j=1}^{n}\psi(f_j(\tau_j)) + \sum_{1\leq j,k\leq n, j\neq k}\psi(f_j(\tau_k))\\ &= \sum_{j=1}^{n}\psi(1) + \sum_{1\leq j,k\leq n, j\neq k}\psi(f_j(\tau_k))\\ &= n(\phi(1)+c) + \sum_{1\leq j,k\leq n, j\neq k}(\phi(f_j(\tau_k))+c)\\ &\leq n(\phi(1)+c) + 0 = n(\phi(1)+c).\end{aligned}$$

We also have

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\psi(f_j(\tau_k)) = \sum_{j=1}^{n}\sum_{k=1}^{n}(\phi(f_j(\tau_k))+c) = \sum_{j=1}^{n}\sum_{k=1}^{n}\phi(f_j(\tau_k)) + cn^2.$$

Therefore

$$cn^2 \leq \sum_{j=1}^{n}\sum_{k=1}^{n}\phi(f_j(\tau_k)) + cn^2 \leq n(\phi(1)+c).$$

□

Following generalization of Theorem 3 is easy.

Theorem 4. *Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ be a (n,θ)-metric code in a pointed metric space $\mathcal{M}$. Let $c > 0$ and*

$$\phi : \{f_j(\tau_k) : 1 \leq j,k \leq n\} \to \mathbb{R}$$

be a function satisfying following.

(i)

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\phi(f_j(\tau_k)) \geq 0.$$

(ii) $\phi(r) + c \leq 0$ *for all* $r \in \{f_j(\tau_k) : 1 \leq j,k \leq n, j \neq k\}$.

Then

$$n \leq \frac{\phi(1)+c}{c}.$$

In particular, if $\phi(1) + c \leq 1$, then $n \leq 1/c$.

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