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Short Note

Non-Archimedean Cauchy Bound for Roots of Non-Archimedean Polynomials

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Abstract

Let $\mathbb{K}$ be a non-Archimedean valued field. Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n \in \mathbb{K}[z], \quad a_n \neq 0$. If $\lambda \in \mathbb{K}$ satisfies $p(\lambda) = 0$, then we show that $|\lambda| \leq \min\left\{1, \frac{1}{|a_n|^{\frac{1}{n}}}\left(\max_{0\leq j\leq n-1}|a_j|\right)^{\frac{1}{n}}\right\}$ or $1 < |\lambda| \leq \frac{1}{|a_n|}\max_{0\leq j\leq n-1}|a_j|$. This is the non-Archimedean version of the Cauchy upper bound for every root of a complex polynomial derived by Cauchy in 1829. Our bound is different from the non-Archimedean bound obtained by Nica and Sprague [*Am. Math. Mon., 2023*].

Keywords: Cauchy bound; non-Archimedean field

MSC: 30C15; 12J25; 26E30

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1. Introduction

In 1829, the father of Complex Analysis, Cauchy derived the following fundamental bound for each root of a complex polynomial [1].

Theorem 1. *[1–3] (**Cauchy Upper Bound**) Let*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n \in \mathbb{C}[z].$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| < 1 + \max_{0\leq j\leq n-1}|a_j|.$$

Corollary 1. *[1–3] (**Cauchy Upper Bound**) Let*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n \in \mathbb{C}[z], \quad a_n \neq 0.$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| < 1 + \max_{0\leq j\leq n-1}\left|\frac{a_j}{a_n}\right| = 1 + \frac{1}{|a_n|}\max_{0\leq j\leq n-1}|a_j|.$$

Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n \in \mathbb{C}[z]$ with $a_0 \neq 0, a_n \neq 0$. Define

$$q(z) := z^n p\left(\frac{1}{z}\right) = a_n + a_{n-1}z + \cdots + a_1 z^{n-1} + a_0 z^n \in \mathbb{C}[z].$$

Now by applying Corollary 1 to q one gets the following result.

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Theorem 2. *[2,3] (**Cauchy Lower Bound**) Let*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathbb{C}[z], \quad a_0 \neq 0, a_n \neq 0.$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$\begin{aligned} |\lambda| &> \frac{1}{1 + \max_{1 \leq j \leq n} \left| \frac{a_j}{a_0} \right|} = \frac{1}{1 + \frac{1}{|a_0|} \max_{1 \leq j \leq n} |a_j|} \\ &= \frac{|a_0|}{|a_0| + \max_{1 \leq j \leq n} |a_j|}. \end{aligned}$$

We naturally ask for versions of Theorems 1 and 2 for polynomials over non-Archimedean fields. Nica and Sprague derived following results in 2023 using non-Archimedean Gershgorin disk theorem and Frobenius companion matrices [4].

Theorem 3. *[4] (**Nica-Sprague Upper Bound**) Let $\mathbb{K}$ be a non-Archimedean valued field. Let*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathbb{K}[z], \quad a_n \neq 0.$$

If $\lambda \in \mathbb{K}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| \leq \max\left\{1, \frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}\right\}.$$

Theorem 4. *[4] (**Nica-Sprague Lower Bound**) Let $\mathbb{K}$ be a non-Archimedean valued field. Let*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathbb{K}[z], \quad a_0, a_n \neq 0.$$

If $\lambda \in \mathbb{K}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| \geq \frac{|a_0|}{\max\{|a_0|, |a_1|, \ldots, |a_n|\}}.$$

In this article, we give different bound than Nica-Sprague bound with direct proofs.

2. Non-Archimedean Cauchy Bound

Let $\mathbb{K}$ be a non-Archimedean valued field and $\mathbb{K}[z]$ be the set of all polynomials over $\mathbb{K}$. We derive a non-Archimedean version of Theorem 1 as follows.

Theorem 5. *(**Non-Archimedean Cauchy Upper Bound**) Let*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathbb{K}[z].$$

If $\lambda \in \mathbb{K}$ satisfies $p(\lambda) = 0$, then

$$\frac{|\lambda|^n}{\max\{1, |\lambda|, \ldots, |\lambda|^{n-1}\}} \leq \max_{0 \leq j \leq n-1} |a_j|.$$

Further, we have the following.

1. *If $|\lambda| \leq 1$, then*

$$|\lambda| \leq \left(\max_{0 \leq j \leq n-1} |a_j|\right)^{\frac{1}{n}}.$$

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2. *If* $|\lambda| > 1$, *then*

$$|\lambda| \leq \max_{0 \leq j \leq n-1} |a_j|.$$

In particular, we have

$$|\lambda| \leq \min\left\{1, \left(\max_{0 \leq j \leq n-1} |a_j|\right)^{\frac{1}{n}}\right\}$$

or

$$1 < |\lambda| \leq \max_{0 \leq j \leq n-1} |a_j|.$$

Proof. Define $M := \max_{0 \leq j \leq n-1} |a_j|$. Since $p(\lambda) = 0$, we have

$$\lambda^n = -(a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1}).$$

By taking the non-Archimedean value, we get

$$\begin{aligned}
|\lambda|^n = |\lambda^n| &= |-(a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1})| \\
&= |a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1}| \\
&\leq \max\{|a_0|, |a_1\lambda|, \ldots, |a_{n-1}\lambda^{n-1}|\} \\
&= \max\{|a_0|, |a_1||\lambda|, \ldots, |a_{n-1}||\lambda^{n-1}|\} \\
&= \max\{|a_0|, |a_1||\lambda|, \ldots, |a_{n-1}||\lambda|^{n-1}\} \\
&\leq \max\{M, M|\lambda|, \ldots, M|\lambda|^{n-1}\} \\
&= M \max\{1, |\lambda|, \ldots, |\lambda|^{n-1}\}.
\end{aligned}$$

□

Corollary 2. *(**Non-Archimedean Cauchy Upper Bound**) Let*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n \in \mathbb{K}[z], \quad a_n \neq 0.$$

If $\lambda \in \mathbb{K}$ *satisfies* $p(\lambda) = 0$, *then*

$$\frac{|\lambda|^n}{\max\{1, |\lambda|, \ldots, |\lambda|^{n-1}\}} \leq \frac{1}{|a_n|} \max_{0 \leq j \leq n-1} |a_j|.$$

Further, we have the following.

1. *If* $|\lambda| \leq 1$, *then*

$$|\lambda| \leq \frac{1}{|a_n|^{\frac{1}{n}}} \left(\max_{0 \leq j \leq n-1} |a_j|\right)^{\frac{1}{n}}.$$

2. *If* $|\lambda| > 1$, *then*

$$|\lambda| \leq \frac{1}{|a_n|} \max_{0 \leq j \leq n-1} |a_j|.$$

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In particular, we have

$$|\lambda| \leq \min\left\{1, \frac{1}{|a_n|^{\frac{1}{n}}}\left(\max_{0\leq j\leq n-1}|a_j|\right)^{\frac{1}{n}}\right\}$$

or

$$1 < |\lambda| \leq \frac{1}{|a_n|}\max_{0\leq j\leq n-1}|a_j|.$$

We note the remarkable difference between Corollary 2 and Theorem 3. The bound in Corollary 2 explicitly depends upon the degree of the polynomial, whereas it is not degree-dependent in Theorem 3.

Remark 1. *The bound in Theorem 2 is best possible. Following is an example. Let p be a prime number. Define*

$$p(z) := 1 + z \in \mathbb{Q}_p[z].$$

Then $\lambda = -1$ is the only root of p. We have

$$|\lambda| = 1 = \max\{|1|\}.$$

Following is non-Archimedean version of Theorem 2.

Theorem 6. *(**Non-Archimedean Cauchy Lower Bound**) Let*

$$p(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \in \mathbb{K}[z], \quad a_0 \neq 0, a_n \neq 0.$$

If $\lambda \in \mathbb{K}$ satisfies $p(\lambda) = 0$, then

$$|\lambda|\max\{1, |\lambda|, \ldots, |\lambda|^{n-1}\} \geq \frac{|a_0|}{\max_{1\leq j\leq n}|a_j|}.$$

Further, we have the following.

1. *If $|\lambda| \leq 1$, then*

$$|\lambda| \geq \frac{|a_0|}{\max_{1\leq j\leq n}|a_j|}.$$

2. *If $|\lambda| \geq 1$, then*

$$|\lambda| \geq \frac{|a_0|^{\frac{1}{n}}}{\left(\max_{1\leq j\leq n}|a_j|\right)^{\frac{1}{n}}}.$$

In particular, we have

$$\frac{|a_0|}{\max_{1\leq j\leq n}|a_j|} \leq |\lambda| \leq 1$$

or

$$|\lambda| \geq \max\left\{1, \frac{|a_0|^{\frac{1}{n}}}{\left(\max_{1\leq j\leq n}|a_j|\right)^{\frac{1}{n}}}\right\}.$$

We now give examples.

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Example 1. *Let $\mathbb{K}$ be a non-Archimedean field. Let $a, b \in \mathbb{K}$. Define $p(z) := a + bz + z^2 \in \mathbb{K}[z]$. Let λ be a root of p. Theorem 5 gives*

$$|\lambda|^2 \leq (\max\{|a|, |b|\})(\max\{1, |\lambda|\})$$

If $|\lambda| \leq 1$, then

$$|\lambda| \leq (\max\{|a|, |b|\})^{\frac{1}{2}}.$$

If $|\lambda| > 1$, then

$$|\lambda| \leq \max\{|a|, |b|\}.$$

Now Theorem 3 gives

$$|\lambda| \leq \max\{1, |a|, |b|\}.$$

Example 2. *Let p be a prime number. Let $n \in \mathbb{N}$ be such that $n < p$. Let $a \in \mathbb{Q}_p$. Define*

$$p(z) := a + z + 2z^2 + \cdots + (n-1)z^{n-1} + nz^n \in \mathbb{Q}_p[z].$$

Let λ be a root of p. Since $n < p$, using Theorem 5 we get

$$\begin{aligned}
|\lambda|^n &\leq \left(\max\left\{\frac{|a|}{|n|}, \frac{|1|}{|n|}, \frac{|2|}{|n|}, \dots, \frac{|n-1|}{|n|}\right\}\right)\left(\max\{1, |\lambda|, |\lambda|^2, \dots, |\lambda|^{n-1}\}\right) \\
&= (\max\{|a|, 1, 1, \dots, 1\})\left(\max\{1, |\lambda|, |\lambda|^2, \dots, |\lambda|^{n-1}\}\right) \\
&= (\max\{|a|, 1\})\left(\max\{1, |\lambda|, |\lambda|^2, \dots, |\lambda|^{n-1}\}\right).
\end{aligned}$$

If $|\lambda| \leq 1$, then

$$|\lambda| \leq (\max\{|a|, 1\})^{\frac{1}{n}}.$$

If $|\lambda| > 1$, then

$$|\lambda| \leq \max\{|a|, 1\} = |a|.$$

Now Theorem 3 gives

$$|\lambda| \leq \max\left\{1, \frac{|a|}{|n|}, \frac{|1|}{|n|}, \frac{|2|}{|n|}, \dots, \frac{|n-1|}{|n|}\right\} = \max\{1, |a|\}.$$

Example 3. *Let p be a prime number. Let $n \in \mathbb{N}$. Let $a \in \mathbb{Q}_p$. Define*

$$p(z) := a + pz + z^n \in \mathbb{Q}_p[z].$$

Let λ be a root of p. Using Theorem 5 we get

$$\begin{aligned}
|\lambda|^n &\leq \max\{|a|, |p|, |0|, \dots, |0|\}\left(\max\{1, |\lambda|, |\lambda|^2, \dots, |\lambda|^{n-1}\}\right) \\
&= \max\left\{|a|, \frac{1}{p}\right\}\left(\max\{1, |\lambda|, |\lambda|^2, \dots, |\lambda|^{n-1}\}\right).
\end{aligned}$$

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If $|\lambda| \leq 1$, *then*

$$|\lambda| \leq \max\left\{|a|, \frac{1}{p}\right\}^{\frac{1}{n}}.$$

If $|\lambda| > 1$, *then*

$$|\lambda| \leq \max\left\{|a|, \frac{1}{p}\right\}.$$

Theorem 3 gives

$$|\lambda| \leq \max\{1, |a|, |p|, |0|, \ldots, |0|\} = \max\left\{1, |a|, \frac{1}{p}\right\}.$$

3. Conclusions

1. In 1829, Cauchy derived an upper bound for the zeros of complex polynomials [1].
2. In 2023, Nica and Sprague derived an upper bound for the zeros of polynomials over non-Archimedean fields [4].
3. In this article, we derived non-Archimedean version of Cauchy bound.

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