Article

Not peer-reviewed version

Power Graphs as Hypergraphs and n^{th} Power Graphs as n-SuperHyperGraphs

Takaaki Fujita *

Posted Date: 2 July 2025

doi: 10.20944/preprints202507.0140.v1

Keywords: Superhypergraph; Hypergraph; Power Graph; Directed Power Graph; Group Theory

Article

Power Graphs as Hypergraphs and n^{th} Power Graphs as n-SuperHyperGraphs

Takaaki Fujita

Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan

Abstract

Graph theory studies the mathematical structures of vertices and edges to model relationships and connectivity [1,2]. A Power Graph has group elements as vertices, with an edge joining two elements whenever one is a power of the other. A Directed Power Graph uses group elements as vertices and places a directed edge $x \to y$ whenever $y = x^m$ for some $m \in \mathbb{N}$. Hypergraphs extend this framework by allowing hyperedges to connect arbitrarily many vertices at once [3], and SuperHyperGraphs further generalize hypergraphs via iterated powerset constructions to capture hierarchical linkages among edges [4,5]. In this paper, we prove that the Power Graph of a group can be realized as a hypergraph and that the Directed Power Graph is a directed hypergraph. Furthermore, we introduce the n^{th} Power Graph and the Directed n^{th} Power Graph, and show that they form subclasses of SuperHyperGraphs and Directed SuperHyperGraphs, respectively.

Keywords: SuperHyperGraphs; hypergraph; power graph; directed power graph; group theory

---

1. Preliminaries

We begin by reviewing the basic terminology and notation used throughout this paper. Unless specified otherwise, all graphs are assumed to be undirected, finite, and simple. For more extensive discussions of particular operations and concepts, the reader is referred to the literature.

1.1. Power Graph and Directed Power Graph

A Power Graph has group elements as vertices, with edge joining elements when one is a power of the other. Directed Power Graph uses group elements as vertices and a directed edge $x \to y$ whenever $y = x^m$ for some $m \in \mathbb{N}$.

Definition 1.1 (Group). [6–8] A *group* is a pair $(G, \cdot)$ consisting of a nonempty set G and a binary operation

$$\cdot \; : \; G \times G \longrightarrow G$$

satisfying the following axioms:

1. **Associativity:** For all $a, b, c \in G$,
$$(a \cdot b) \cdot c \;=\; a \cdot (b \cdot c).$$
2. **Identity:** There exists a unique element $e \in G$ such that for all $a \in G$,
$$e \cdot a = a \quad \text{and} \quad a \cdot e = a.$$
3. **Inverse:** For each $a \in G$, there exists an element $a^{-1} \in G$ such that
$$a \cdot a^{-1} = e \quad \text{and} \quad a^{-1} \cdot a = e.$$

Definition 1.2 (Power Graph). [9–11] Let G be a group. The *power graph* of G is the (undirected) graph

$$P(G) = (V, E), \qquad V = G,$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202507.0140.v1

whose edge set is

$$E = \{\{x,y\} \subseteq G \mid x \neq y, \exists m,n \in \mathbb{N}: x^m = y \text{ or } y^n = x\}.$$

Equivalently, $P(G)$ is the underlying simple graph of $\overrightarrow{P}(G)$.

Example 1.3 (Power Graph of the Cyclic Group of Order 4)**.** Let $G = \langle a \mid a^4 = e\rangle = \{e, a, a^2, a^3\}$. Its power graph $P(G) = (V, E)$ is given by

$$V = \{e, a, a^2, a^3\}, \quad E = \{\{e,a\}, \{e,a^2\}, \{e,a^3\}, \{a,a^2\}, \{a,a^3\}, \{a^2,a^3\}\}.$$

Definition 1.4 (Directed Power Graph)**.** [9,11] Let G be a group. The *directed power graph* of G is the digraph

$$\overrightarrow{P}(G) = (V, A), \qquad V = G,$$

where

$$A = \{(x,y) \in G \times G \mid x \neq y, y = x^m \text{ for some } m \in \mathbb{N}\}.$$

Example 1.5 (Directed Power Graph of the Cyclic Group of Order 4)**.** On the same group G, the directed power graph $\overrightarrow{P}(G) = (V, A)$ has

$$V = \{e, a, a^2, a^3\}, \quad A = \{(a,a^2), (a,a^3), (a,e), (a^2,e), (a^3,a^2), (a^3,a), (a^3,e)\}.$$

Here each arc (x,y) satisfies $y = x^m$ for some $m \in \mathbb{N}$ and $x \neq y$.

1.2. SuperHyperGraph

A *hypergraph* generalizes a standard graph by allowing *hyperedges* that can join any number of vertices simultaneously [3,12–14]. Extending this idea, a *SuperHyperGraph* incorporates iterated powerset constructions to capture hierarchical relationships among hyperedges, a topic of growing interest in recent studies [5,15–19]. Practical applications of SuperHyperGraphs include molecular modeling, network analysis, and signal processing [20–25]. In what follows, the integer parameter n in the nth powerset and in an n-SuperHyperGraph always denotes a nonnegative integer.

Definition 1.6 (Base Set)**.** A *base set* S is the underlying domain from which all further constructions are drawn. Formally,

$$S = \{ x \mid x \text{ belongs to the specified universe}\}.$$

Every element appearing in $\mathcal{P}(S)$ or in iterated powersets $\mathcal{P}_n(S)$ is an element of S.

Definition 1.7 (Powerset)**.** The *powerset* of a set S, written $\mathcal{P}(S)$, is the collection of all subsets of S, including $\emptyset$ and S itself:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S\}.$$

Definition 1.8 (Hypergraph)**.** [3,26] A *hypergraph* $H = (V(H), E(H))$ consists of

- A finite vertex set $V(H)$.
- A finite collection $E(H)$ of nonempty subsets of $V(H)$, called hyperedges.

Hypergraphs are well suited to model higher-order interactions among elements of $V(H)$.

Theorem 1.9. *Let G be a group and let*

$$P(G) = (V, E) \quad \text{with} \quad V = G, \quad E = \{\{x,y\} \subseteq G \mid x \neq y, \exists m,n \in \mathbb{N} : x^m = y \text{ or } y^n = x\}$$

be its power graph (Definition 1.2). Then

$$H = (G, E)$$

is a 2-uniform hypergraph (Definition 1.8), and its 2-section (the simple graph obtained by replacing each hyperedge by an undirected edge) is exactly $P(G)$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202507.0140.v1

Proof. We verify that $H = (G, E)$ satisfies the hypergraph axioms:

- $V(H) = G$ is finite (since G is finite or else this construction still formally applies).
- Each $e \in E$ is a nonempty subset of G, and by construction $|e| = 2$.

Hence H is a 2-uniform hypergraph. Its 2-section is formed by interpreting each hyperedge $\{x, y\}$ as an undirected edge between x and y. But exactly those pairs $\{x, y\}$ appear in E for which $x^m = y$ or $y^n = x$, so the resulting simple graph coincides with the power graph $P(G)$. □

Definition 1.10 (n-th Powerset)**.** [27–31] The n-th powerset of a set X, denoted $P_n(X)$, is defined by:

$$P_1(X) = \mathcal{P}(X), \qquad P_{n+1}(X) = \mathcal{P}\big(P_n(X)\big), \quad n \geq 1.$$

The corresponding nonempty powerset $P^*_n(X)$ is obtained by iterating $\mathcal{P}^*(\cdot)$, where $\mathcal{P}^*(Y) = \mathcal{P}(Y) \setminus \{\emptyset\}$.

Definition 1.11 (n-SuperHyperGraph)**.** [32–34] Let V_0 be a finite base set. Define iteratively

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$

An *n-SuperHyperGraph* is a pair

$$\mathrm{SuHyG}^{(n)} = (V, E), \quad V, E \subseteq \mathcal{P}^n(V_0),$$

where each element of V is called an n-supervertex and each element of E an n-superedge.

Example 1.12 (3-SuperHyperGraph: Global Project Organization)**.** We illustrate a 3-SuperHyperGraph by modeling the hierarchy of a multinational corporation's project structure.

Base set: employees.

$$V_0 = \{\text{Alice, Bob, Carol, Dave}\}.$$

First–level (teams). Elements of $\mathcal{P}^1(V_0)$:

$$T_1 = \{\text{Alice, Bob}\}, \quad T_2 = \{\text{Bob, Carol}\}, \quad T_3 = \{\text{Carol, Dave}\}.$$

Second–level (departments). Elements of $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}^1(V_0))$. Select:

$$D_1 = \{T_1, T_2\}, \quad D_2 = \{T_2, T_3\}.$$

Third–level (global divisions). Elements of $\mathcal{P}^3(V_0)$. Choose two representative supervertices:

$$v_1 = \{D_1, D_2\}, \quad v_2 = \{D_1, \{T_1, T_3\}\}.$$

Thus we set

$$V = \{v_1, v_2\} \subseteq \mathcal{P}^3(V_0), \quad E = \{\, \{v_1, v_2\} \} \subseteq \mathcal{P}(V).$$

Here v_1 models the "Asia–Europe Division" (linking Department 1 and Department 2), while v_2 models the "Global Integration Division" (combining Department 1 with the cross-team set $\{T_1, T_3\}$).

Interpretation. The 3-supervertices v_1, v_2 represent high–level divisions composed of lower–level departments and teams. The single superedge $e = \{v_1, v_2\}$ indicates a strategic collaboration between these two global divisions on a company-wide initiative.

Hence

$$\mathrm{SuHyG}^{(3)} = (V, E)$$

is a concrete instance of a 3-SuperHyperGraph reflecting a real-world organizational hierarchy.

1.3. Directed SuperHyperGraph

Directed SuperHyperGraphs are graph classes that extend SuperHyperGraphs, respectively, in a manner analogous to Directed Graphs. Below, we present their formal definitions and illustrative examples.

Definition 1.13 (Directed Hypergraph)**.** A *directed hypergraph* is a pair $H = (V, E)$ where

- V is a finite set of *vertices*, and
- E is a set of *directed hyperedges*.

Each hyperedge $e \in E$ is an ordered pair

$$e = (T_e,\, H_e),$$

where

$$T_e \subseteq V \quad (\text{the } \textit{tail} \text{ of } e), \qquad H_e \subseteq V \quad (\text{the } \textit{head} \text{ of } e),$$

and one typically requires $T_e \neq \emptyset$ and $H_e \neq \emptyset$. This structure generalizes a directed graph by allowing each hyperedge to connect multiple source vertices T_e to multiple target vertices H_e simultaneously.

Theorem 1.14. *Let G be a group and let $\overrightarrow{P}(G) = (G, A)$ be its directed power graph (Definition 1.4), where*

$$A = \{(x, y) \in G \times G \mid y = x^m \text{ for some } m \in \mathbb{N},\ x \neq y\}.$$

Define a directed hypergraph

$$H = (V, E_H), \quad V = G, \quad E_H = \Big\{(\{x\}, \{y\}) \mid (x, y) \in A\Big\}.$$

Then H is a directed hypergraph, and its underlying digraph is exactly $\overrightarrow{P}(G)$. Hence $\overrightarrow{P}(G)$ is realized as a directed hypergraph with all hyperedges of size one.

Proof. First, $V = G$ is finite and nonempty. Each hyperedge in E_H is of the form (T_e, H_e) with

$$T_e = \{x\} \subseteq V, \quad H_e = \{y\} \subseteq V,$$

and both T_e and H_e are nonempty by construction. Therefore H satisfies the requirements of the Definition.

Next, the underlying directed graph of H has an arc $x \to y$ precisely when there exists a hyperedge $(\{x\}, \{y\}) \in E_H$. By definition of E_H, this occurs if and only if $(x, y) \in A$, i.e. $y = x^m$ for some $m \geq 1$. This is exactly the arc-set of $\overrightarrow{P}(G)$. Hence the two directed graphs coincide. □

Definition 1.15 (Directed n-SuperHyperGraph)**.** (cf.[22,34,35]) Let S be a nonempty *base set* and let $n \geq 0$ be an integer. Define iterated powersets by

$$\mathcal{P}^0(S) = S, \quad \mathcal{P}^{k+1}(S) = \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 0).$$

A *directed n-SuperHyperGraph* is a pair

$$\mathrm{DSuHG}^{(n)} = (V,\, E),$$

where

$$V \subseteq \mathcal{P}^n(S), \quad E \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S),$$

and each directed n-superedge $e \in E$ is an ordered pair

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big), \quad \mathrm{Tail}(e), \mathrm{Head}(e) \subseteq \mathcal{P}^n(S),$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202507.0140.v1

typically both nonempty. Such an e carries "flow" from the entire set $\mathrm{Tail}(e)$ of n-supervertices into $\mathrm{Head}(e)$.

Example 1.16 (Directed 2-SuperHyperGraph: Corporate Workflow)**.** We model a simplified approval process in a company with two teams and a central review committee.

Base set (employees):

$$S = \{\text{Alice, Bob, Carol, Dave}\}.$$

First–level (teams):

$$T_1 = \{\text{Alice, Bob}\}, \quad T_2 = \{\text{Carol, Dave}\} \subseteq \mathcal{P}(S).$$

Second–level (departments):

$$D_1 = \{\, T_1\}, \quad D_2 = \{\, T_2\}, \quad D_{\text{all}} = \{\, T_1,\, T_2\} \subseteq \mathcal{P}^2(S).$$

Vertex set:

$$V = \{\, D_1,\, D_2,\, D_{\text{all}}\} \subseteq \mathcal{P}^2(S).$$

Directed superedges (approval flow):

$$E = \{\, (\{D_1\},\, \{D_{\text{all}}\}),\, (\{D_2\},\, \{D_{\text{all}}\})\} \subseteq \mathcal{P}^2(S) \times \mathcal{P}^2(S).$$

Interpretation. Each directed superedge $(\{D_i\},\, \{D_{\text{all}}\})$ represents the flow of deliverables (or approval requests) from department D_i into the central committee D_{all}. Thus

$$\mathrm{DSuHG}^{(2)} = (V, E)$$

is a concrete directed 2-SuperHyperGraph illustrating this corporate workflow.

Theorem 1.17. *Every directed hypergraph can be realized as a directed* 1*-SuperHyperGraph. Concretely, if*

$$H = (V_0, E), \quad E \subseteq \{(T_e, H_e) \mid T_e, H_e \subseteq V_0,\ T_e \neq \emptyset,\ H_e \neq \emptyset\},$$

then setting

$$S = V_0, \quad V = \{\{v\} \mid v \in V_0\} \subseteq \mathcal{P}^1(S),$$

and for each directed hyperedge $e = (T_e, H_e) \in E$ defining

$$\mathrm{Tail}(e) = \{\{v\} \mid v \in T_e\}, \quad \mathrm{Head}(e) = \{\{v\} \mid v \in H_e\},$$

yields a directed 1*-SuperHyperGraph* $\mathrm{DSuHG}^{(1)} = (V, E')$ *with*

$$E' = \{(\mathrm{Tail}(e),\, \mathrm{Head}(e)) \mid e \in E\} \subseteq \mathcal{P}^1(S) \times \mathcal{P}^1(S).$$

Proof. We verify that (V, E') satisfies Definition 1.15 for $n = 1$:

- By construction, $V = \{\{v\} \mid v \in V_0\} \subseteq \mathcal{P}^1(S)$ and is nonempty.
- Each $\mathrm{Tail}(e)$ and $\mathrm{Head}(e)$ is a nonempty subset of V, since $T_e, H_e \neq \emptyset$.
- Thus every $(\mathrm{Tail}(e), \mathrm{Head}(e)) \in E'$ is an element of $\mathcal{P}^1(S) \times \mathcal{P}^1(S)$.

Hence $\mathrm{DSuHG}^{(1)} = (V, E')$ is a directed 1-SuperHyperGraph. Since directed hyperedges (T_e, H_e) and directed 1-superedges $(\mathrm{Tail}(e), \mathrm{Head}(e))$ correspond bijectively, H is represented exactly by $\mathrm{DSuHG}^{(1)}$. □

Corollary 1.18. *Every directed hypergraph is a special case of a directed n-SuperHyperGraph for any $n \geq 1$, via the inclusion $\mathcal{P}^1(S) \subseteq \mathcal{P}^n(S)$.*

Proof. Embed each singleton $\{v\} \in \mathcal{P}^1(S)$ into $\mathcal{P}^n(S)$ by iterated singletoning: $\{v\} \mapsto \{\{\dots\{\{v\}\}\dots\}\}$. The same construction of tails and heads yields a directed n-SuperHyperGraph isomorphic to the directed 1-case. □

2. Main Results

As the main contributions of this paper, we show that:

- Every n-th power graph can be realized as a special case of an n-SuperHyperGraph.
- Every directed n-th power graph can be realized as a special case of a directed n-SuperHyperGraph.

2.1. *n-th Power Graphs*

Vertices are n-fold powersets of group G; two vertices are adjacent if one equals the m-th power-set image of the other.

Definition 2.1 (Iterated Exponentiation)**.** Let G be a group and $m \in \mathbb{N}$. Define maps

$$\varphi_m^{(1)} : G \to G, \quad x \mapsto x^m,$$

and recursively for $k \geq 1$,

$$\varphi_m^{(k+1)} : \mathcal{P}^{k+1}(G) \to \mathcal{P}^{k+1}(G), \quad A \mapsto \{\varphi_m^{(k)}(a) \mid a \in A\}.$$

Definition 2.2 (Undirected n-th Power Graph)**.** Fix $n \geq 1$. Let

$$V_n = \mathcal{P}^n(G), \quad E_n \;=\; \{\{A,B\} \subseteq V_n \mid A \neq B,\ B = \varphi_m^{(n)}(A) \text{ or } A = \varphi_m^{(n)}(B) \text{ for some } m \in \mathbb{N}\}.$$

Then the *n-th power graph* is the simple graph

$$P^{(n)}(G) \;=\; (V_n,\, E_n).$$

Example 2.3 (Rotating Shift Patterns as an Undirected 2nd Power Graph)**.** We model a weekly rotating shift system on a seven-day cycle as an undirected 2nd power graph.

Base group and daily blocks. Let

$$G = \mathbb{Z}_7 = \{0,1,2,3,4,5,6\},$$

with addition modulo 7 representing days of the week. Define two daily shift blocks:

$$A = \{0,1,2\} \quad (\text{Mon–Wed}), \quad B = \{3,4\} \quad (\text{Thu–Fri}).$$

Weekly patterns (V_2). Vertices are collections of these blocks in one week:

$$P = \{A,B\}, \quad Q = \{A+2, B+2\}, \quad R = \{A+3, B+3\},$$

where $A + m = \{x + m \pmod 7 \mid x \in A\}$. Thus

$$V_2 = \{\,P,\, Q,\, R\} \subseteq \mathcal{P}^2(G).$$

Edge relation (E_2). In the undirected 2nd power graph $P^{(2)}(G) = (V_2, E_2)$, two patterns $\{X,Y\} \subseteq V_2$ are adjacent precisely if one is obtained by applying the same shift m to the other:

$$E_2 = \{\{P,Q\},\, \{P,R\},\, \{Q,R\}\},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202507.0140.v1

since

$$Q = P + 2, \quad R = P + 3, \quad R = Q + 1.$$

Interpretation. Vertices represent entire weekly shift schedules; an (undirected) edge between two patterns indicates they differ by a uniform rotation of all daily blocks. Here $\{P, Q, R\}$ form a triangle, reflecting that any pattern can be reached from any other by some two-day rotation.

Theorem 2.4. *For any group G and integer $n \geq 1$, the graph $P^{(n)}(G)$ is a 2-uniform n-SuperHyperGraph on G. Concretely, if one sets*

$$V = \mathcal{P}^n(G), \quad E = E_n,$$

then (V, E) satisfies the definition of an n-SuperHyperGraph, and its underlying simple graph is exactly $P^{(n)}(G)$.

Proof. By construction, $V = \mathcal{P}^n(G) \subseteq \mathcal{P}^n(G_0)$ with $G_0 = G$, and $E \subseteq \mathcal{P}(V)$. Hence (V, E) is an n-SuperHyperGraph. Moreover, since every hyperedge in E has size 2 and corresponds precisely to the exponentiation relation $\varphi_m^{(n)}$, the simple graph obtained by replacing each $\{A, B\} \in E$ by an undirected edge is exactly $P^{(n)}(G)$. □

Theorem 2.5 (Functoriality)**.** *If $\psi\colon G \to H$ is a group isomorphism, then for each $n \geq 1$ there is a graph isomorphism*

$$\Psi^{(n)}\colon P^{(n)}(G) \xrightarrow{\cong} P^{(n)}(H), \qquad A \mapsto \psi(A) = \{\psi(a) \mid a \in A\}.$$

Proof. Since ψ is bijective, so is $\Psi^{(n)}\colon \mathcal{P}^n(G) \to \mathcal{P}^n(H)$. Moreover, if $\{A, B\}$ is an edge in $P^{(n)}(G)$, then $B = \varphi_m^{(n)}(A)$ or vice versa for some m. Applying $\Psi^{(n)}$ yields

$$\psi(B) = \{\psi(b) \mid b \in B\} = \{\psi(a^m) \mid a \in A\} = \{\psi(a)^m \mid a \in A\} = \varphi_m^{(n)}(\psi(A)),$$

showing $\{\psi(A), \psi(B)\}$ is an edge of $P^{(n)}(H)$. Thus $\Psi^{(n)}$ preserves adjacency and is a graph isomorphism. □

Theorem 2.6 (Vertex-Transitivity)**.** *For any finite group G and $n \geq 1$, the graph $P^{(n)}(G)$ is vertex-transitive under the action of G by conjugation on each level of the iterated powerset.*

Proof. Let $g \in G$. Conjugation by g induces a permutation

$$c_g^{(k)}\colon \mathcal{P}^k(G) \to \mathcal{P}^k(G), \quad A \mapsto gAg^{-1} = \{gag^{-1} \mid a \in A\},$$

for each $k = 1, 2, \ldots, n$. These maps are bijections and satisfy

$$c_g^{(k+1)}(\varphi_m^{(k+1)}(A)) = \{c_g^{(k)}(a^m) \mid a \in A\} = \{(c_g^{(k)}(a))^m \mid a \in A\} = \varphi_m^{(k+1)}(c_g^{(k+1)}(A)).$$

Hence conjugation commutes with iterated exponentiation, and so $\{A, B\} \in E_n$ if and only if $\{gAg^{-1}, gBg^{-1}\} \in E_n$. This shows the induced permutation on $V_n = \mathcal{P}^n(G)$ is an automorphism of $P^{(n)}(G)$. Since G acts transitively on itself by conjugation (and hence on its iterated powersets), $P^{(n)}(G)$ is vertex-transitive. □

Theorem 2.7 (Diameter Bound)**.** *Let G be a finite group of exponent e (so $g^e = 1$ for all $g \in G$). Then for any $n \geq 1$, the diameter of $P^{(n)}(G)$ is at most 2.*

Proof. Given any two vertices $A, B \in V_n = \mathcal{P}^n(G)$, if $\{A, B\} \in E_n$ we are done. Otherwise, consider the identity supervertex $\{1\} \in \mathcal{P}(G) \subset \mathcal{P}^n(G)$. Because each element of G has order dividing e,

$$\varphi_e^{(n)}(A) = \{a^e \mid a \in A\} = \{1\}, \quad \varphi_e^{(n)}(B) = \{1\}.$$

Thus $\{A, \{1\}\} \in E_n$ and $\{B, \{1\}\} \in E_n$, so there is a path A–$\{1\}$–B of length 2. Hence $\mathrm{diam}(P^{(n)}(G)) \leq 2$. □

doi:10.20944/preprints202507.0140.v1

2.2. Directed *n*-th Power Graphs

Directed n-th power graphs: vertices are n-fold powersets, with $A \to B$ if B equals the mth power of every element in A.

Definition 2.8 (Directed n-th Power Graph)**.** Under the same hypotheses, the *directed n-th power graph* of G is the digraph

$$\overrightarrow{P}^{(n)}(G) = (V_n, A_n), \qquad V_n = P^n(G),$$

where

$$A_n = \{ (A, B) \in V_n \times V_n \mid A \neq B, \ B = \{ x^m \mid x \in A \} \text{ for some } m \in \mathbb{N} \}.$$

Example 2.9 (Rotating Shift Schedules as a Directed 2-nd Power Graph)**.** We model a rotating shift schedule over a seven-day cycle using the directed 2-nd power graph.

Base group and first-level sets. Let

$$G = \mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\},$$

with addition modulo 7 representing days of the week (e.g. $0 =$Mon, $1 =$Tue,...). A *daily shift block* is any nonempty subset $A \subseteq G$. For example,

$$A_1 = \{0, 1, 2\} \quad \text{(Mon–Wed shift)}, \quad A_2 = \{3, 4\} \quad \text{(Thu–Fri shift)}.$$

Second-level: weekly roster patterns. Elements of $P^2(G) = \mathcal{P}(\mathcal{P}(G))$ are *weekly patterns*, i.e. collections of daily blocks. For instance,

$$P = \{ A_1, A_2 \} \quad \text{and} \quad Q = \{ A_1 + 2, A_2 + 2 \},$$

where $A + 2 = \{x + 2 \pmod 7 \mid x \in A\}$ denotes shifting every block forward by two days:

$$A_1 + 2 = \{2, 3, 4\}, \quad A_2 + 2 = \{5, 6\}.$$

Directed 2-nd power graph. Define

$$V_2 = P^2(G), \quad A_2 = \{ (P, Q) \in V_2 \times V_2 \mid Q = \{ A + m \mid A \in P \} \text{ for some } m \in \mathbb{N} \}.$$

Then $\overrightarrow{P}^{(2)}(G) = (V_2, A_2)$ is the directed 2-nd power graph. In particular, we have an arc (P, Q) because Q is obtained by applying the same shift $m = 2$ days to each daily block in P.

Interpretation. Vertices of $\overrightarrow{P}^{(2)}(G)$ represent entire weekly shift patterns, and a directed edge $P \to Q$ indicates that Q arises from P by uniformly advancing every daily block by m days. This captures the real-world operation of rotating shift schedules in a hierarchical (daily→weekly) framework.

Theorem 2.10. *For any group G and integer $n \geq 1$:*

1. *$P^{(1)}(G)$ coincides with the classical power graph $P(G)$ of Definition 1.2.*
2. *$\overrightarrow{P}^{(1)}(G)$ coincides with the directed power graph $\overrightarrow{P}(G)$ of Definition 1.4.*
3. *The vertex sets satisfy $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \cdots$, so $\{P^{(n)}(G)\}_{n \geq 1}$ is a nested family reflecting the iterated powerset $P^n(G)$.*

Proof.

1. By the Definition, $P^1(G) = G$. Hence the adjacency rule in $P^{(1)}(G)$ reduces exactly to "$x^m = y$ or $y^m = x$" for $x, y \in G$, which is the condition defining $P(G)$.
2. Likewise, $\overrightarrow{P}^{(1)}(G)$ has vertex set G and an arc (x, y) precisely when $y = x^m$ for some m, matching Definition 1.4.

doi:10.20944/preprints202507.0140.v1

3. From $P^k(G) = \mathcal{P}(P^{k-1}(G))$, every element of $P^{k-1}(G)$ is also a (singleton) element of $P^k(G)$. Thus $V_{k-1} \subseteq V_k$ for all k, and the sequence of graphs $P^{(n)}(G)$ grows strictly (or stabilizes) with n. This nesting mirrors the iterated powerset structure.

□

Theorem 2.11. *For any group G and $n \geq 1$, the digraph $\overrightarrow{P}^{(n)}(G)$ is a directed 2-uniform n-SuperHyperGraph. In particular, its arc-set A_n defines the directed superedges of an n-SuperHyperGraph structure on $V_n = \mathcal{P}^n(G)$.*

Proof. Since $A_n \subseteq V_n \times V_n$ and each (A, B) satisfies $B = \varphi_m^{(n)}(A)$, it endows (V_n, A_n) with the structure of a directed hypergraph whose hyperedges all have size two and carry a direction. By Definition (specialized to directed arcs), this coincides with a directed 2-uniform n-SuperHyperGraph. □

Theorem 2.12 (Transitivity)**.** *For any group G and $n \geq 1$, the directed n-th power graph $\overrightarrow{P}^{(n)}(G)$ is transitive: if (A, B) and (B, C) are arcs, then (A, C) is also an arc.*

Proof. Suppose $(A, B) \in A_n$ and $(B, C) \in A_n$. By definition there exist $m_1, m_2 \in \mathbb{N}$ such that

$$B = \varphi_{m_1}^{(n)}(A), \quad C = \varphi_{m_2}^{(n)}(B).$$

But then

$$C = \varphi_{m_2}^{(n)}\big(\varphi_{m_1}^{(n)}(A)\big) = \varphi_{m_1 m_2}^{(n)}(A),$$

so $(A, C) \in A_n$. Hence the adjacency relation is transitive. □

Theorem 2.13 (Reachability to the Identity Supervertex)**.** *Let G be a finite group of exponent e (i.e. $g^e = 1$ for all $g \in G$). In $\overrightarrow{P}^{(n)}(G)$, every vertex $A \in P^n(G)$ has a directed path of length one to the singleton supervertex $\{1\}$.*

Proof. For any $A \in V_n = \mathcal{P}^n(G)$, consider

$$B = \varphi_e^{(n)}(A) = \{ a^e \mid a \in A\} = \{1\}.$$

Since $A \neq B$ (unless $A = \{1\}$, in which case there is no self-loop), and $B = \varphi_e^{(n)}(A)$, the definition of arcs ensures $(A, \{1\}) \in A_n$. Thus every supervertex reaches $\{1\}$ by a single directed edge. □

3. Conclusion and Future Work

In this paper, we have shown that the Power Graph of a group can be realized as a hypergraph and that the Directed Power Graph is a directed hypergraph. Furthermore, we introduced the n^{th} Power Graph and the Directed n^{th} Power Graph, and demonstrated that they form subclasses of SuperHyperGraphs and Directed SuperHyperGraphs, respectively. In future work, we plan to extend the concepts developed here to various frameworks of uncertainty, including Fuzzy Sets [36,37], Intuitionistic Fuzzy Sets [38], Hyperfuzzy Sets [39–42], SuperHyperFuzzy Sets [43,44], Neutrosophic Sets [45,46], and Plithogenic Sets [47–49].

Funding: This study did not receive any financial or external support from organizations or individuals.

Data Availability Statement: This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Acknowledgments: We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

 doi:10.20944/preprints202507.0140.v1

Conflicts of Interest: The authors confirm that there are no conflicts of interest related to the research or its publication.

Ethical Approval: As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Disclaimer: This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

1. Diestel, R. *Graph theory*; Springer (print edition); Reinhard Diestel (eBooks), 2024.
2. Gross, J.L.; Yellen, J.; Anderson, M. *Graph theory and its applications*; Chapman and Hall/CRC, 2018.
3. Berge, C. *Hypergraphs: combinatorics of finite sets*; Vol. 45, Elsevier, 1984.
4. Smarandache, F. *Introduction to the n-SuperHyperGraph-the most general form of graph today*; Infinite Study, 2022.
5. Hamidi, M.; Smarandache, F.; Taghinezhad, M. *Decision Making Based on Valued Fuzzy Superhypergraphs*; Infinite Study, 2023.
6. Bogopolski, O. *Introduction to group theory*; 2008.
7. Mordeson, J.N.; Bhutani, K.R.; Rosenfeld, A. *Fuzzy group theory*; Vol. 182, Springer, 2005.
8. Lenz, R.; Bovik, A.C. Group Theory. 2007.
9. Bubboloni, D.; Pinzauti, N. Critical classes of power graphs and reconstruction of directed power graphs. *Journal of Group Theory* **2025**, *28*, 713–739.
10. Das, B.; Ghosh, J.; Kumar, A. The Isomorphism Problem of Power Graphs and a Question of Cameron. In Proceedings of the 44th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, 2024, p. 1.
11. Lucchini, A.; Stanojkovski, M. Independence and strong independence complexes of finite groups. *arXiv preprint arXiv:2503.19778* **2025**.
12. Fujita, T.; Singh, P.K. HyperFuzzy Graph and Hyperfuzzy HyperGraph. *Journal of Neutrosophic and Fuzzy Systems (JNFS)* **2025**, *10*, 01–13.
13. Gao, Y.; Zhang, Z.; Lin, H.; Zhao, X.; Du, S.; Zou, C. Hypergraph learning: Methods and practices. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **2020**, *44*, 2548–2566.
14. Feng, Y.; You, H.; Zhang, Z.; Ji, R.; Gao, Y. Hypergraph neural networks. In Proceedings of the Proceedings of the AAAI conference on artificial intelligence, 2019, Vol. 33, pp. 3558–3565.
15. Valencia, E.M.C.; Vásquez, J.P.C.; Lois, F.Á.B. Multineutrosophic Analysis of the Relationship Between Survival and Business Growth in the Manufacturing Sector of Azuay Province, 2020–2023, Using Plithogenic n-Superhypergraphs. *Neutrosophic Sets and Systems* **2025**, *84*, 341–355.
16. Bravo, J.C.M.; Piedrahita, C.J.B.; Bravo, M.A.M.; Pilacuan-Bonete, L.M. Integrating SMED and Industry 4.0 to optimize processes with plithogenic n-SuperHyperGraphs. *Neutrosophic Sets and Systems* **2025**, *84*, 328–340.
17. Fujita, T.; Smarandache, F. A Concise Study of Some Superhypergraph Classes. *Neutrosophic Sets and Systems* **2024**, *77*, 548–593.
18. Alqahtani, M. Intuitionistic Fuzzy Quasi-Supergraph Integration for Social Network Decision Making. *International Journal of Analysis and Applications* **2025**, *23*, 137–137.
19. Nalawade, N.B.; Bapat, M.S.; Jakkewad, S.G.; Dhanorkar, G.A.; Bhosale, D.J. Structural Properties of Zero-Divisor Hypergraph and Superhypergraph over $\mathbb{Z}_n$: Girth and Helly Property. *Panamerican Mathematical Journal* **2025**, *35*, 485.
20. Marcos, B.V.S.; Willner, M.F.; Rosa, B.V.C.; Yissel, F.F.R.M.; Roberto, E.R.; Puma, L.D.B.; Fernández, D.M.M. Using plithogenic n-SuperHyperGraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems* **2025**, *84*, 513–524.
21. Fujita, T. Exploration of Graph Classes and Concepts for SuperHypergraphs and n-th Power Mathematical Structures. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond* **2025**, *3*, 512.

22. Fujita, T. Review of some superhypergraph classes: Directed, bidirected, soft, and rough. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)* **2024**.
23. Fujita, T.; Ghaib, A.A. Toward a Unified Theory of Brain Hypergraphs and Symptom Hypernetworks in Medicine and Neuroscience. *Advances in Research* **2025**, *26*, 522–565.
24. Fujita, T. Hypergraph and Superhypergraph Approaches in Electronics: A Hierarchical Framework for Modeling Power-Grid Hypernetworks and Superhypernetworks. *Journal of Energy Research and Reviews* **2025**, *17*, 102–136.
25. Zhu, S. Neutrosophic n-SuperHyperNetwork: A New Approach for Evaluating Short Video Communication Effectiveness in Media Convergence. *Neutrosophic Sets and Systems* **2025**, *85*, 1004–1017.
26. Bretto, A. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer* **2013**, *1*.
27. Smarandache, F. Foundation of SuperHyperStructure & Neutrosophic SuperHyperStructure. *Neutrosophic Sets and Systems* **2024**, *63*, 21.
28. Jdid, M.; Smarandache, F.; Fujita, T. A Linear Mathematical Model of the Vocational Training Problem in a Company Using Neutrosophic Logic, Hyperfunctions, and SuperHyperFunction. *Neutrosophic Sets and Systems* **2025**, *87*, 1–11.
29. Fujita, T. An Introduction and Reexamination of Hyperprobability and Superhyperprobability: Comprehensive Overview. *Asian Journal of Probability and Statistics* **2025**, *27*, 82–109.
30. Smarandache, F. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*; Infinite Study, 2023.
31. Smarandache, F. SuperHyperStructure & Neutrosophic SuperHyperStructure, 2024. Accessed: 2024-12-01.
32. Fujita, T. Review of Probabilistic HyperGraph and Probabilistic SuperHyperGraph. *Spectrum of Operational Research* **2025**, *3*, 319–338.
33. Ghods, M.; Rostami, Z.; Smarandache, F. Introduction to Neutrosophic Restricted SuperHyperGraphs and Neutrosophic Restricted SuperHyperTrees and several of their properties. *Neutrosophic Sets and Systems* **2022**, *50*, 480–487.
34. Smarandache, F. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*; Infinite Study, 2020.
35. Smarandache, F. n-SuperHyperGraph and Plithogenic n-SuperHyperGraph. *Nidus Idearum* **2019**, *7*, 107–113.
36. Zadeh, L.A. Fuzzy sets. *Information and control* **1965**, *8*, 338–353.
37. Nishad, T.; Al-Hawary, T.A.; Harif, B.M. General Fuzzy Graphs. *Ratio Mathematica* **2023**, *47*.
38. Atanassov, K.T.; Atanassov, K.T. *Intuitionistic fuzzy sets*; Springer, 1999.
39. Fujita, T.; Smarandache, F. Examples of Fuzzy Sets, Hyperfuzzy Sets, and SuperHyperfuzzy Sets in Climate Change and the Proposal of Several New Concepts. *Climate Change Reports* **2025**, *2*, 1–18.
40. Smarandache, F. *Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics*; Infinite Study, 2017.
41. Fujita, T. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*; Biblio Publishing, 2025.
42. Nazari, Z.; Mosapour, B. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories* **2018**, *16*, 173–185.
43. Fujita, T.; Smarandache, F. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems* **2025**, *80*, 609–631.
44. Fujita, T. Short Survey on the Hierarchical Uncertainty of Fuzzy, Neutrosophic, and Plithogenic Sets. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, p. 285.
45. Smarandache, F. A unifying field in Logics: Neutrosophic Logic. In *Philosophy*; American Research Press, 1999; pp. 1–141.
46. Smarandache, F.; Kandasamy, W.; Ilanthenral, K. Neutrosophic graphs: A new dimension to graph theory **2015**.
47. Smarandache, F. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*; Infinite study, 2018.

 doi:10.20944/preprints202507.0140.v1

48. Azeem, M.; Rashid, H.; Jamil, M.K.; Gütmen, S.; Tirkolaee, E.B. Plithogenic fuzzy graph: A study of fundamental properties and potential applications. *Journal of Dynamics and Games* **2024**, pp. 0–0.
49. Sathya, P.; Martin, N.; Smarandache, F. Plithogenic Forest Hypersoft Sets in Plithogenic Contradiction Based Multi-Criteria Decision Making. *Neutrosophic Sets and Systems* **2024**, *73*, 668–693.